Constitutive Hybrid Processes:
a Process-Algebraic Semantics
for Hybrid Bond Graphs

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Abstract

Models of physical systems have to be based on physical principles such as conservation of energy and continuity of power. These principles are inherently enforced by the *bond graph* modeling formalism. Often, however, physical components may be best modeled as piecewise continuous with discrete mode changes, which leads to a violation of continuity principles. To support such *hybrid* models, bond graphs can be extended by facilitating a dynamic model structure, resulting in *hybrid bond graphs*. Behavior generation then requires computing continuous-time evolution, detecting the occurrence of events, executing the discrete state changes, and re-initializing the continuous-time state. This paper presents a comprehensive representation of these different aspects of behavior using *hybrid process algebra*. The behavior of a hybrid bond graph can then be studied using a uniform representation while a direct correspondence with the elements of the bond graph is maintained. Additionally, nondeterminism can be included in hybrid bond graph semantics which may alleviate the modeling task without being detrimental to the required analyses.

1   Introduction

Computational modeling of physics is an important element in the design of engineered systems. For example, computational models allow a broad search in the design space for a physical system under design. As another example, because of their predictive power, models are an essential part in designing efficient control strategies.

Given that a good model is one that helps solve the particular task at hand, in this broad spectrum of applications, a given physical system may be represented by many different models. Each of these models is most efficient for solving the issue to be addressed. To achieve this, only phenomena observed in the physical system that are considered of importance are embodied by the model. All other phenomena are best abstracted away.

For some components, this may result in a model that captures the dynamics by differential equations that are active in a given mode. Switching logic is used to control the mode switches and this combination leads to models of a *hybrid* nature. These hybrid models combine behavior on a continuous time scale with discrete state transition behavior at given points in time. For example, an inelastic collision may be modeled by the independent movement of two masses before the collision and the movement of the combined masses after the collision.

The computational representation of a hybrid model can be called a *hybrid system*, and behavior generation, analysis, verification, and synthesis based on hybrid systems is an active area of research (e.g., [1, 2]). A popular hybrid system formalism for verification purposes is the so-called *hybrid automaton* [3]. This is a nondeterministic finite state machine with a system of ordinary differential equations (ODEs) associated with each state. Each state then represents the global mode of behavior of the system. The ODE that is associated with a mode, a state in the hybrid automaton, is evaluated during the time at which the state is active.

For modeling physical systems, the explicit global representation of the different modes is, however, cumbersome. Often, changes in the mode of behavior of a physical component are best specified at the local, component, level. As such, the mode changes correspond to changes in the structure of a model.

Bond graphs [4] are a formalism designed for modeling physical systems by capturing the exchange of energy between physical phenomena, embodied by bond graph elements. Traditionally, the structure that captures the energy exchange, however, is static. To extend bond graphs for hybrid modeling, the structure of energy exchange can be made dynamic. This has been documented in *ad hoc* cases (e.g., the *time junction* [5]) and later formalized in a general framework [6].

Though changing the energy exchange structure allows modeling of mode changes in a local sense, as desired, the principles that are inherent in the bond graph modeling formalism may be violated. For example, abruptly inhibiting the energy exchange between elements of a physical model may violate the *continuity of power* principle.

Furthermore, changing the structure of a model may lead to an abrupt exchange of energy between elements of the model that at a more detailed level is governed by continuous dynamics. This redistribution of energy can be computed from a global representation of the model as a system of differential and algebraic equations (DAEs) [7, 8, 9]. The drawback of such global and dissociate representations is that the required transformations make it difficult to reason about and understand how the modeled system behaves. However, to obtain deep insight into the system behavior, a representation directly linked to the bond graph topology is suited better.

Previous work [10, 11] developed a methodology to compute the energy redistribution based on local equations for each of the model element types. In [10] substitute values for storage elements and their stored energy are derived. In [11], an iteration scheme would attempt to find an energy distribution that was consistent with the new model structure while ensuring conservation laws (e.g., conservation of charge and conservation of momentum) would be satisfied. Part of the work presented here is a formalization of these approaches.

Related, but different, is the need to capture the entire hybrid behavior in a comprehensive formulation. The execution of a hybrid system, because of the combination of computational models, consists of a number of distinct phases: (i) continuous-time simulation, (ii) event detection, (iii) mode transition, and (iv) reinitialization of the continuous-time behavior [12]. To study the interaction between these, a formalism that allows capturing the respective parts in a single language is helpful, if not desired.

This work provides such a unifying representation using *hybrid process algebra*, which allows a representation of behavior at a local level. The syntax of the employed language contains elements to capture the behavior in each of the different phases of hybrid system simulation. Furthermore, the language supports nondeterminism while adhering to physical characteristics such as not to create energy spontaneously. This allows abstracting detailed continuous behavior in a gross sense, without the need for detailed analysis of the specific parameter abstraction or time-scale abstraction, as discussed in previous work [13].

First, in Section 2, principles of physics modeling are reviewed and bond graphs are introduced as a formalism for modeling the continuous-time dynamics of physical systems. In Section 3, bond

graphs are then extended with facilities to model discontinuities, resulting in hybrid bond graphs. Next, Section 4 gives an overview of the hybrid process algebra HyPA. In Section 5 it is shown how hybrid bond graphs can be represented in HyPA. In Section 6 a number of examples is given to illustrate the approach. Finally, Section 7 concludes this work.

2 Modeling of Physical Systems

When modeling physical systems at a macroscopic level (i.e. quantum effects do not play a significant role), two essential assumptions can be observed: conservation of energy and the possibility to use a *lumped approach*. These principles are inherently enforced by bond graphs. Mathematical languages like Modelica and Simulink are much broader in their conception and do not enforce conservation laws in models based on their syntax. Therefore, in order to create an understanding of the modeling principles involved, we study the semantics of bondgraphs and hybrid bondgraphs in this paper. In future research, our results can be lifted to broader mathematical frameworks, when practical ease of use starts playing a role.

2.1 Principles of Physics Modeling

Modeling the dynamics of physical systems often takes a *lumped parameter* approach. This is based on the notion of *reticulation*, which allows ideal physical behavior to be captured by a network of components, each of which represents an ideal behavior described by one or more parameters [14]. In contrast to *fields*, these components have well defined points of interaction, called *ports*.

The behavior of each component is then defined by its *constitutive relations* between external variables, parameters, and internal variables.

For example, in translational mechanics, three ideal phenomena can be identified: storage of momentum, storage of displacement, and dissipation. In translational mechanics, the components that embody these phenomena are typically referred to as masses, springs and dampers, respectively. A mass then gives rise to a constitutive relation between momentum p and velocity v. A spring gives a constitutive relation between displacement x and force F. A damper gives a constitutive relation between force F and velocity v. Often, these relations are linear that depend on the parameters m, k and b respectively

$$
\begin{aligned}
\text{Mass} &: \quad p = m \cdot v \\
\text{Spring} &: \quad F = k \cdot x \\
\text{Damper} &: \quad F = b \cdot v
\end{aligned}
$$

The variables that are employed by the constitutive relations in the translational mechanics domain then include momentum p, force F, displacement x and velocity v. All mechanical behavior, can be expressed in terms of relations between these variables, and the changes of their values over time. A change in momentum corresponds to a force, leading to the constitutive differential equation $F = \dot{p}$. A change in displacement corresponds to a velocity $v = \dot{x}$.

In addition to the constitutive relations, the connection semantics have to be defined. Since the behavior of each mechanical component can be captured by having pairs of a force and velocity variable, the connection relations have to be defined over such a pair. In case of mechanical components connected in series, the forces acting on each component are equal whereas the velocities of the individual components have to be balanced. This leads to the relationships in Eqn. 1:

$$
\left(\begin{array}{l} v_1 \pm \ldots \pm v_n = 0 \\ F_1 = \ldots = F_n \end{array} \right)
\tag{1}
$$

Likewise, in case of mechanical components connected in parallel, the forces on each of the components have to be balanced, while the velocities are equal. This leads to the relationships in Eqn. 2:

$$
\left(\begin{array}{l} F_1 \pm \ldots \pm F_n = 0 \\ v_1 = \ldots = v_n \end{array} \right)
\tag{2}
$$

Therefore, two basic connection semantics can be employed; a parallel and a series connection. These connection relations are the constitutive relations of those two types of connection components.

To illustrate, consider the mass-spring-damper system in Fig. 1 with a force F acting on the mass. Note that the separate ideal physical phenomena that have been identified such as the spring and damper effect may correspond to one physical artifact. The constitutive equations are presented on the right-hand side. All forces and directions are defined to be positive in the upward direction. This has to be taken into account when writing down the balance relations. The spring, k, and damper, b, are connected in parallel and so their respective forces, F_k and F_b, have to balance the difference between the force exerted by the mass, m, and the external force, F. This leads to Eqn. cont1: $0 = (F_m - F) + F_k + F_b$. The other equation, Eqn. cont2, states that the velocities of mass, spring and damper should be equal: $v_m = v_k = v_b$.

The constitutive equations of the mass are given by Eqn. cont3, for the momentum of this mass, p_m, and Eqn. cont4, for the time-derivative, $\dot{p}_m$. There are two constitutive equations for the spring as well. The first is Eqn. cont5, which computes the spring force, F_k, with x_k being the spring displacement and the displacement in rest chosen to be 0. The second is Eqn. cont6, which gives the time-derivative, $\dot{x}_k$, of the displacement. Finally, the constitutive equation of the damper, Eqn. cont7, is a linear viscous friction relation on the damper force, F_b.

$$\left\{ \begin{array}{ll} 0 = (F_\mathrm{m} - F) + F_\mathrm{k} + F_\mathrm{b} & (\mathrm{cont1}) \\ v_\mathrm{m} = v_\mathrm{k} = v_\mathrm{b} & (\mathrm{cont2}) \\ p_\mathrm{m} = m \cdot v_\mathrm{m} & (\mathrm{cont3}) \\ \dot{p}_\mathrm{m} = F_\mathrm{m} & (\mathrm{cont4}) \\ F_\mathrm{k} = k \cdot x_\mathrm{k} & (\mathrm{cont5}) \\ \dot{x}_\mathrm{k} = v_\mathrm{k} & (\mathrm{cont6}) \\ F_\mathrm{b} = b \cdot v_\mathrm{b} & (\mathrm{cont7}) \end{array} \right.$$
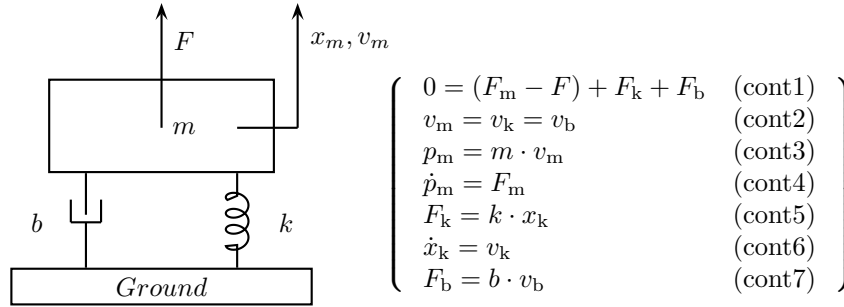
Figure 1: Continuous constitutive equations for a mass-spring-damper system

2.2 Bond Graphs

The dynamics of physical systems can be modeled in a number of *physical domains* such as the mechanical domain, the hydraulic domain, and the electrical domain. Two types of variables can be identified for each domain: (i) an intensive variable and (ii) an extensive variable [15].

Intensive and extensive variables refer to the two types of variables in Gibbs' free energy law: $dE = vdq + pdq + \mu dn + ...$ This states that the change of energy in a system is the change of the extensive variable in a physical domain against the intensive variable in that domain. With effort e the intensive variable and stored physical quantity q (also often referred to as generalized displacement) the extensive variable, this can be written as $dE = \Sigma_i e_i dq_i$ In terms of power as the derivative of energy with respect to time, this then becomes $P = \Sigma_i e_i f_i$ and the flow variable f is sometimes referred to as extensive variable as well. Strictly speaking this depends on the domain that is referenced as the intensive variable in one domain may be the time-derivative of the extensive variable in another domain. For example, in the electrical domain charge is the extensive variable according to Gibbs' free energy law. The derivative with respect to time of charge is current, which can be referred to as the time derivative of the extensive variable in

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the electrical domain, but which is the intensive variable in the magnetic domain with flux the extensive variable. Details on this duality can be found in [16, 17].

In the translational mechanical domain, the intensity corresponds to the force, F, and the extensity corresponds to the displacement, x. A mathematical function that is meaningful across many physical domains, and so provides a unifying notion, is *energy*. Energy can be defined as the stored extensity for given intensity. As such, it is the surface underneath the intensity per extensity function. Any change in energy, dE, corresponds to an area, for example in the mechanical domain $dE = F dx$ with F the intensity and x the extensity. To account for the temporal dependency of x, yields $dE = Fv dt$ with $v = \dot{x}$.

These notions are intrinsic to *bond graphs* [4], a formalism that identifies an *effort* and *flow* variable type in each domain, where the effort corresponds to the intensive variable and the flow variable to the time derivative of the extensive variable. The product of effort and flow then corresponds to the change of free energy over time. This provides a unifying approach to modeling the dynamics of physical systems because of the use of generic constitutive variables. For example, the bond graph effort variable (denoted e) corresponds to force in the mechanical domain while the bond graph flow variable (denoted f) corresponds to velocity.

The effort and flow variables are time-derivatives of variables called *generalized momentum* and *generalized displacement*, respectively. So, for the generalized momentum, denoted p, the constitutive relation $\dot{p} = e$ holds and for the generalized displacement, denoted q, the relation $\dot{q} = f$ holds. In the translational mechanical domain, this implies that generalized momentum corresponds to momentum and generalized displacement to displacement. In the electrical domain, generalized momentum corresponds to flux and and generalized displacement corresponds to charge. Table 1 summarizes the analogies between different domains, and their bond graph generalizations. The change in energy is determined by the effort and flow variables through the so-called *power relation* $\dot{E} = e \cdot f$.

Table 1: Bond graph variable analogies

	effort e	flow f	generalized momentum p	generalized displacement q
translational mechanics	force F $[\mathrm{N}] = [\frac{\mathrm{kg\ m}}{\mathrm{s}^2}]$	velocity v $[\frac{\mathrm{m}}{\mathrm{s}}]$	momentum p $[\mathrm{N\ s}]$	displacement x $[\mathrm{m}]$
rotational mechanics	torque τ $[\mathrm{N\ m}]$	ang. vel. ω $[\frac{\mathrm{rad}}{\mathrm{s}}]$	ang. mom. b $[\mathrm{N\ m\ s}]$	angle Θ $[\mathrm{rad}]$
electronics	voltage u $[\mathrm{V}] = [\frac{\mathrm{N\ m}}{\mathrm{C}}]$	current i $[\mathrm{A}] = [\frac{\mathrm{C}}{\mathrm{s}}]$	flux linkage λ $[\mathrm{V\ s}]$	charge q $[\mathrm{C}]$
hydraulics	pressure P $[\frac{\mathrm{N}}{\mathrm{m}^2}]$	volume-flow Q $[\frac{\mathrm{m}^3}{\mathrm{s}}]$	pressure p_{p} $[\frac{\mathrm{N\ s}}{\mathrm{m}^2}]$	volume V $[\mathrm{m}^3]$

Formally, a bond graph is a directed graph, with nodes representing basic physical phenomena of a system (called *elements*), and edges (called *bonds*) that represent the way in which the phenomena exchange energy and the direction in which energy flows. A bond graph fragment, potentially with unconnected bonds, or a complete bond graph model may be referred to as a *component*.

The semantics of a bond graph is formed by assigning differential equations and algebraic equations to each of the nodes and edges of the graph. These differential equations and algebraic equations describe the behavior of the four variables p, q, e and f, for each of the physical

components in the system. The bonds identify the variables of the nodes they connect.

The elements of a bond graph can be classified in energic elements, those that affect the stored energy in the model, and junction elements, those that distribute energy without energy storage, supply, or dissipation.

Energic elements The elements of a bond graph are described by giving a relation between p, q, e and f. The following list of energic elements are classified, based on the properties of their constitutive relation. Both the class of elements as well as the constitutive relations are identified by the same symbol, though in small capitals for the relation. Note that the functions may depend on other variables of the system than the ones that are explicitly mentioned, as long as the given constraints are met.

- A *resistance* (bond graph symbol R), is an element with a constitutive relation $e = \text{R}(f)$ such that the function R satisfies $x \cdot \text{R}(x) \geq 0$ for all x. Consequently, $\dot{E} = e \cdot f \geq 0$, which models that a resistance dissipates energy. This flow of energy into the thermal domain is often not explicitly shown.

- An *inductance* (bond graph symbol I), is an element with a constitutive relation $f = \text{I}(p)$. This models an element that stores energy in the form of generalized momentum. For intrinsic stability of the element, it is usually assumed that the function I satisfies $\frac{\partial \text{I}(p)}{\partial p} \geq 0$ (see [18]). The intuition behind this is that an effort e leads to an acceleration $\dot{f} = \frac{\partial}{\partial t}\text{I}(p) = \frac{\partial \text{I}(p)}{\partial p}\dot{p} = \frac{\partial \text{I}(p)}{\partial p}e$ in the same direction.

- A *capacitance* (bond graph symbol C), is an element with a constitutive relation $e = \text{C}(q)$. This models an element that stores energy in the form of generalized displacement. For intrinsic stability, it is usually assumed that the function C satisfies $\frac{\partial \text{C}(q)}{\partial q} \geq 0$.

- A *flow source* (bond graph symbol S_f), is an element with a constitutive relation $f = \text{S}_\text{f}$. A flow-source enforces a certain flow, and has an arbitrary effort at its disposition to achieve this.

- An *effort source* (bond graph symbol S_e), is an element with a constitutive relation $e = \text{S}_\text{e}$. An effort source enforces a certain effort and an arbitrary flow at its disposition to achieve this.

In this list, the effort and flow source are considered *active* elements because they can supply energy. The resistor, capacitor, and inductor are *passive* elements because they cannot supply energy. Table 2 summarizes the analogies between the passive elements, their represention in different physical domains, and standardized notations that are used in the different fields to denote variables and parameters that characterize the elements.

As mentioned before, a bond graph is a directed graph. The direction of a bond is depicted by a half arrow. A bond always points in the so-called *positive direction of power*. In other words, if the flow and effort are both positive, the arrow points towards the element if it stores or dissipates energy (as is the case with inertias, capacitances and resistances), and away from the element if it supplies energy to the system (as is the case with sources of effort and flow). This is depicted in Fig. 2.
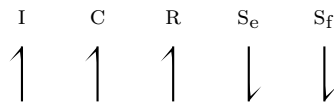

Figure 2: Power directions for standard bond graph elements

Table 2: Bond graph element analogies

	inductance I	capacitance C	resistance R
translational mechanics	mass m [kg]	spring $\frac{1}{k}$ $[\frac{m}{N}]$	damper b $[\frac{N\,s}{m}]$
rotational mechanics	inertia J [kg m^2]	spring $\frac{1}{k}$ $[\frac{rad}{N\,m}]$	damper c [N m s]
electronics	inductor L $[H] = [\frac{V\,s}{A}]$	capacitor C $[F] = [\frac{A\,s}{V}]$	resistor R $[\Omega] = [\frac{V}{A}]$
hydraulics	fluid-inertia I $[\frac{kg}{m^4}]$	reservoir C $[\frac{m^5}{N}]$	resistor R $[\frac{N\,s}{m^5}]$

Junctions Section 2.1 discussed that connection in mechanical systems is based on a balance between forces acting on components with equal velocities, or a balance of velocities of components on which equal forces act. In electrical systems, these connections (applied to currents and voltages) are known as Kirchoff's laws, and in bond graphs they are known as 1- and 0-juntions. A 1-junction satisfies the following equations, for a set of connected bond graph elements B (where $\pm$ is shorthand for $+$ or $-$, rather than 'approximately'):

$$\left(\begin{matrix} \sum_{b \in B} \pm e_b = 0 \\ \forall_{b,b' \in C} \; f_b = f_{b'} \end{matrix} \right) \tag{3}$$

Dually, a 0-junction is characterized by the relations:

$$\left(\begin{matrix} \sum_{b \in B} \pm f_b = 0 \\ \forall_{b,b' \in B} \; e_b = e_{b'} \end{matrix} \right) \tag{4}$$

This implies continuity of power, as there is no energy storage by the junctions:

$$\left(\sum_{b \in B} \dot{E}_b = \sum_{b \in B} e_b \cdot f_b = 0 \right) \tag{5}$$

When there are elements connected to a junction of which the power directions point outward, they have a negative effect on the exchange of generalized momentum or generalized position. The equations Eqn. 3, Eqn. ?? and Eqn. 5, are therefore for the case where all bond arrows point inwards, into the junction. Some possible configurations, and the associated constitutive equations, have been depicted in figure 3.

To facilitate the creation of a set of constitutive relations for a bond graph model, each bond is often furnished with a unique number that is employed as a subscript for all the variables associated with that bond. As an example, the mechanical system of Fig. 1 is turned into the bond graph of Fig. 4 as follows. Firstly, the position of the mass is defined relative to the position of the ground, while the position of the spring and damper refers to the amount with which they are stretched. The force F is connected to the mass, so their positions are equal. Secondly, a change of position in the upward direction is defined to be positive. From Fig. 1 it is then clear that a change in position of the mass, leads to an equal change in position of the force, the spring and the damper. This suggests a 1-junction between the bond graph elements representing these components. Thirdly, the power directions of the bond graph elements in Fig. 2 require the direction of the force F, and the forces generated by the spring and the damper, to be defined positive in the upward direction. Lastly, because everything is modeled relative to the ground, it

$$\left(\begin{array}{c} e_1 + e_2 + e_3 = 0 \\ f_1 = f_2 = f_3 \end{array} \right) \qquad \left(\begin{array}{c} -e_1 + e_2 - e_3 = 0 \\ f_1 = f_2 = f_3 \end{array} \right)$$
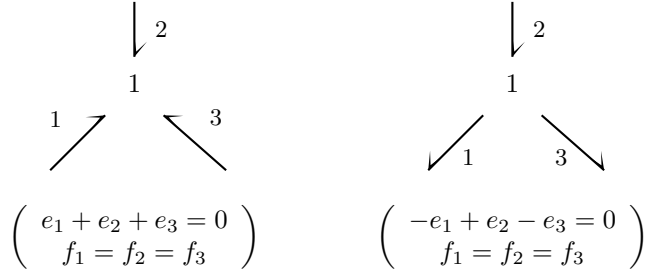
Figure 3: Constitutive equations for junctions with different power directions

is not necessary to model the ground explicitly. The constitutive relations of the components are considered linear and the parameters of these linear relations are given along with the component type, separated by a colon. The constitutive relations for this model were presented in Fig. 1.
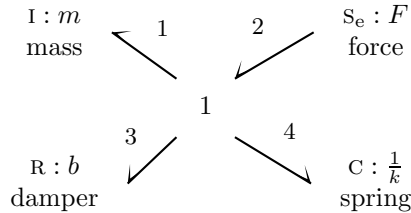

Figure 4: Bond graph for a mass-spring-damper system

3 Hybrid Bond Graphs

As presented in Section 2.2, bond graph models inherently embody principles of energy conservation and power continuity. Indeed, in classical physics, the Heaviside principle [19] captures the assumption that energy can only be transported from one point in time and space to another, by a continuous trajectory. Some models of physical systems, though, violate the principle of continuity of power, as they introduce instantaneous changes as abstractions from brief periods of time (e.g., models of collisions, switching behavior, and opening and closing of valves).

3.1 Discontinuities in Physical System Models

To illustrate a model of a physical system that violates continuity of power, consider the system of two bodies, m_1 and m_2, that engage in an ideal collision as shown in Fig. 5. When a variable x changes during a discontinuity, its initial value is denoted by x^- and its end value by x^+. Newton's collision rule now states that the difference in velocities of two bodies immediately after a collision, $v_1^+ - v_2^+$, can be expressed by the difference in velocities immediately before the collision, $v_1^- - v_2^-$, while accounting for a *coefficient of restitution*, ϵ, as $v_1^+ - v_2^+ = \epsilon(v_1^- - v_2^-)$ with $0 \le \epsilon \le 1$. In case $\epsilon < 1$, when the collision occurs, there is an instantaneous dissipation of energy in the system consisting of the two bodies, which violates the continuity of power assumption.

To restore adherence to the continuity of power principle, the change of momentum of the two bodies needs to be modeled in more detail so the dissipation occurs continuously over a short interval of time. For example, the bodies can be modeled as masses that, upon impact, are momentarily connected via a spring-damper system, as shown in Fig. 6. This reflects the elastic effects and the dissipation of energy that occur during impact.
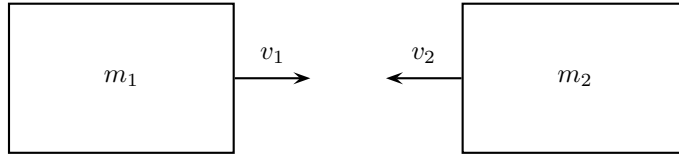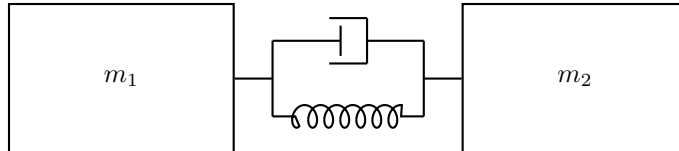
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Figure 5: Two colliding bodies

Figure 6: Power continuous model of two colliding bodies

Such a model that adheres to the principle of continuity of power is conceptually easier to design and provides valuable insight into detailed physical phenomena that may otherwise be overlooked. However, there are cases where the model that embodies an instantaneous change of velocities is preferred. For example, the behavior that comes about because of the detailed phenomena often operates on a time scale that is much smaller than the gross behavior of interest. Therefore, a numerical solver has to employ relatively small time steps, which slows down numerical simulation. This is especially problematic in case of real-time simulation, when a fixed upper bound exists for the computations that can be performed to simulate behavior over a fixed time interval. Therefore, real-time simulation typically employs a fixed step simulation, which prohibits the presence of behavior on such small time scales.

Another reason to allow instantaneous changes is because not all parameters of the detailed physical phenomena may be known. In the example of the impact between bodies, the elasticity of the bodies, and the precise damping factors that play a role at impact, are difficult to measure, while their exact value only influences the behavior of the system during a very short time. In such a situation, it may prove convenient to abstract from this exact behavior, and model it as a discontinuous change. Note, that in such a case non-determinism may be introduced into the model behavior, as for example the value of the restitution coefficient ϵ becomes a (bounded) unknown. In such a situation, simulators may have to be adapted to explore the different behaviors that result from taking different values for ϵ, but this is left as a topic for future research.

3.2 Computing Discontinuous Changes

In this section, the discontinuous changes in model variables are derived. For this, a systematic and thorough derivation is given in this section.

When computing discontinuous changes in bond graph models, the relation between intensive and extensive variables changes. In the continuous case, the relation is $\dot{p} = e$ for inductances and $\dot{q} = f$ for capacitances. In the discontinuous case, this time-derivative does not apply. Instead, the difference of the extensive variables between the start and end of the discontinuity is used as a starting point to derive a relation that does hold. To reflect the different nature of the relation, $\hat{e}$ and $\hat{f}$ is written for the intensive variables in the discontinuous equations. Note, that in the formal semantics that we are about to associate with hybrid bond graphs, e and $\hat{e}$ are treated as *different* variables.

Given the constitutive relations for continuous behavior, constitutive relations for discontinuous behavior can be derived. The discontinuous behavior, from a physical point of view, is an abstraction from behavior that happens over a very short time interval $[\tau^-, \tau^+]$. Integration of the constitutive relations for continuous behavior over this time interval may therefore provide

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insight as to what may happen during a discontinuity.

With some abuse of notation,[1] for inductances the following relation between discontinuous intensive variables and their extensive variables can be found: $\hat{e} = \int_{\tau^-}^{\tau^+} e(t)dt = p^+ - p^-$. For capacitances, the relation $\hat{f} = \int_{\tau^-}^{\tau^+} f(t) = q^+ - q^-$ can be found. Furthermore, the constitutive relation for conservation of energy, leads to the following definition regarding the discontinuous behavior:

$$\Sigma_{c\in C}\ E_c^+ - E_c^- = \Sigma_{c\in C} \int_{\tau^-}^{\tau^+}\ \dot{E}_c(t)\cdot dt = \Sigma_{c\in C} \int_{\tau^-}^{\tau^+}\ e(t)\cdot f(t)\cdot dt = 0 \tag{6}$$

For the derivation of the change of energy in inductances and capacitances, the descriptions of the continuous behavior is used as starting point again. Using partial integration, the changes in stored energy based on changes in generalized displacement and generalized momentum can be calculated.

- For an *inductance*, integrating the continuous relation $\dot{E} = e \cdot f = \dot{p} \cdot \text{I}(p)$ over the trajectory of p yields $E_c^+ - E_c^- = \int_{p^-}^{p^+} \text{I}(x)\ dx$ for the change in stored energy as a function of the change in generalized momentum.

- For a *capacitance*, dual to the inductance, the change is given by $E_c^+ - E_c^- = \int_{q^-}^{q^+} \text{C}(x)\ dx$.

Note, that the integration over a trajectory is only valid if I and C do not depend on other variables than p or q respectively. More precisely, if I and C depend on other variables, the change in these variables in the interval $[\tau^-, \tau^+]$ should be negligible. This should be verified by the modeler.

For a *flow-source*, *effort-source*, and *resistance*, the partial integration approach is not applicable, because there is no extensive variable to integrate over. The precise amount of energy that is dissipated or delivered by these elements cannot be derived from the state change of the system alone. This problem is solved by assuming that the interval $[\tau^-, \tau^+]$ that is used to abstract is sufficiently short to make the energy exchange of these components with their environment negligible. This indeed constrains the ability to model arbitrary discontinuous behavior, but as will be seen in Section 6, there are many interesting systems for which this assumption is applicable. For flow-sources, effort-sources and resistances the change in energy is *defined* as $E_c^+ - E_c^- = 0$.

For junctions, integration over $[\tau^-, \tau^+]$ does not change the nature of the equations. The only change is syntactic, turning continuous intensive variables into their discontinuous counterparts. For a the 1-junctions this yields:

$$\left(\begin{array}{l} \Sigma_{c\in C}\ \hat{e}_c = 0 \\ \forall_{c,c'\in C}\ \hat{f}_c = \hat{f}_{c'} \end{array} \right) \tag{7}$$

and for a 0-junction this yields:

$$\left(\begin{array}{l} \Sigma_{c\in C}\ \hat{f}_c = 0 \\ \forall_{c,c'\in C}\ \hat{e}_c = \hat{e}_{c'} \end{array} \right). \tag{8}$$

It is important to see, that the power relation does not allow a decomposition into a separate integration over the intensive variables. Formally, the inequality

$$\Sigma_{c\in C} (E_c^+ - E_c^-) = \Sigma_{c\in C} \left(\int_{\tau^-}^{\tau^+}\ e(t)\cdot f(t)\cdot dt \right)$$
$$\neq \tag{9}$$
$$\Sigma_{c\in C} \left(\int_{\tau^-}^{\tau^+}\ e(t)\cdot dt \right)\cdot \left(\int_{\tau^-}^{\tau^+}\ f(t)\cdot dt \right) = \Sigma_{c\in C} (p^+ - p^-)\cdot(q^+ - q^-) = \Sigma_{c\in C}\ \hat{e}\cdot\hat{f},$$

holds, and, therefore, the power relation and the energy conservation laws have to be made explicit in treating hybrid bond graphs.

---

[1] Technically, it should be written $\hat{e}^+ = p^+ - p^-$, as the intensive variable $\hat{e}$ obtains its value during the discontinuity. But, since the end-value of $\hat{e}$ is the only value that is used in the theory that is developed, the additional $+$ is omitted to avoid cluttering of symbols in the formulae.

Using the constitutive relations for discontinuous behavior, the impact of bodies shown in Fig. 5 can be modeled as a discontinuous behavior of two masses. If the linear constitutive differential equation $f = \frac{1}{m} \cdot p$ is used for the colliding bodies, with p the momentum, m the mass, and f the velocity (we use generalized variables here), the following discontinuous constitutive relations for the behavior at the point of time of impact are obtained:

$$\left\{ \begin{array}{rclcll} E_1^+ - E_1^- & + & E_2^+ - E_2^- & \leq & 0 & \text{(discr1)} \\ E_1^+ - E_1^- & = & \int_{p_1^-}^{p_1^+} (\frac{1}{m_1} \cdot x) dx & = & \frac{(p_1^+)^2 - (p_1^-)^2}{2 \cdot m_1} & \text{(discr2)} \\ E_2^+ - E_2^- & = & \int_{p_2^-}^{p_2^+} (\frac{1}{m_2} \cdot x) dx & = & \frac{(p_2^+)^2 - (p_2^-)^2}{2 \cdot m_2} & \text{(discr3)} \\ \hat{e}_1 & = & p_1^+ - p_1^- & & & \text{(discr4)} \\ \hat{e}_2 & = & p_2^+ - p_2^- & & & \text{(discr5)} \\ \hat{e}_1 & + & \hat{e}_2 & = & 0 & \text{(discr6)} \\ \hat{f}_1 & = & \hat{f}_2 & = & 0 & \text{(discr7)} \end{array} \right.$$

Note that, due to the inequality in Eqn. discr1, these equations may have multiple solutions. This means that the modeler is not required to determine specific traces corresponding to physical parameters that may be difficult to obtain. Instead, analysis of classes of behaviors is supported, alleviating the modeling task. If these classes need to be refined, the modeler has the opportunity to further constrain the behaviors that are possible.

In the next section, a formalism known as *hybrid process algebra* is discussed, which allows the description of systems with mixed continuous and discontinuous behavior. In Section 5, constitutive hybrid processes, rather than constitutive equations, are employed to describe the mixed continuous and discontinuous behavior that can occur when discontinuities in bond graph elements and junctions are allowed. Also, special hybrid junction elements that can be used to explicitly model switching behavior are discussed.

4 Hybrid Process Algebra

In this section, a short introduction to a subset of the hybrid process algebra HyPA is given. Only the relevant atoms and operations are discussed. A more complete treatment of the algebra is presented in other work [20, 21].

4.1 Syntax

HyPA models can capture both continuous and discontinuous behavior of a system. Continuous and discontinuous behavior are described using predicates over model variables $\mathcal{V}_m$. The following constants and function symbols need to be introduced for the proceedings:

1. Continuous behavior is described using flow predicates $\mathcal{P}_f$ on the model variables $\mathcal{V}_m$ and their derivatives with respect to time $\dot{\mathcal{V}}_m = \{\dot{x} \mid x \in \mathcal{V}_m\}$. A *flow clause*, $(V \mid P_f)$, is an atomic process that models continuous physical behavior. It is denoted as a pair $(V \mid P_f)$ of a set of model variables $V \subseteq \mathcal{V}_m$, and a flow predicate $P_f \in \mathcal{P}_f$. The set V identifies variables that are not allowed to change discontinuously while executing the flow clause. The flow predicate P_f models which flow behavior is executed by the clause.

2. Discontinuous behavior is described by re-initialization predicates $\mathcal{P}_r$ on model variables with a minus superscript, i.e. $\mathcal{V}_m^-$, to denote conditions that only hold at the start of a re-initialization, and on model variables with a plus superscript, i.e. $\mathcal{V}_m^+$, to denote conditions that hold only at the end of a re-initialization. A *process re-initialization* $[V \mid P_r] \gg p$ models the behavior of a process p where the model variables are first submitted to a discontinuous change. This change is specified by the set of model variables $V \subseteq \mathcal{V}_m$ and the re-initialization predicate P_r. In the case of process re-initializations, the set V models which variables are allowed to change. Note that this is the opposite of flow clauses. In this work, changes that

take place are explicitly modeled using re-initializations. Therefore, $V = \mathcal{V}_\mathrm{m}$ in all flow-clauses and re-initialization clauses that are used. To simplify notation, V is not written explicitly, instead $[P_\mathrm{r}]$ for $[\mathcal{V}_\mathrm{m} \,|\, P_\mathrm{r}]$ and (P_f) for $(\mathcal{V}_\mathrm{m} \,|\, P_\mathrm{f})$ are used.

3. The *alternative composition $p \oplus q$* models a (non-deterministic) choice between the processes p and q.

4. The *disrupt $p \blacktriangleright q$* models that process q may take over execution from process p at any moment. In other words, it models a mode switch from p to q.

5. The *parallel composition $p \parallel q$* models concurrent execution of p and q. For the restricted version of HyPA in this work, the intuition behind parallel composition is simply that processes synchronize the execution of their continuous and discontinuous behavior.

The binding order of the operators of HyPA is as follows: $\blacktriangleright$, $\gg$, $\parallel$, $\oplus$, where alternative composition binds weakest, and the disrupt binds strongest.

More complex processes can be defined using recursive specifications $X : p$, where X is a process variable and p is a term possibly containing X and other process variables. Amongst others, recursion is a powerful way to express repetition in a process.

4.2 Semantics

In addition to the syntax, the semantics of the hybrid process algebra needs to be defined. Two basic approaches to this are possible: (i) An operational type approach can be employed to capture how the state and structure of a model evolves using one syntax with transformation constructs. (ii) A denotational type approach can be employed to define how a model in one syntax can be transformed into a model in another syntax with a formal semantics.

Here a mathematical formalism called *labeled transition systems* is used as the target syntax. Process algebra (also called process theory or process calculus) studies these labeled transition systems in an algebraic manner, by defining ways to compose labeled transition systems from smaller ones. The atomic flow clauses in the hybrid process algebra give rise to very basic labeled transition systems, similar to Sontag-machines [22]. The other operators of the algebra may then be used to transform these basic transition systems into more complex systems.

Formally, a labeled transition system is defined as follows:

Definition 1 (Labeled Transition System) *A labeled transition system $< X, \Sigma, \leadsto >$ is a triple consisting of a set of states X, a set of observations Σ and a transition relation $\leadsto \subseteq X \times \Sigma \times X$.*

The transition relation models the dynamic behavior of the system. For example, a transition $(x, \sigma, x') \in \leadsto$, also denoted $< x > \overset{\sigma}{\leadsto} < x' >$, signifies that the system when in state x may evolve to state x', and that an outside observer sees an event σ when this happens. Note, that $\leadsto$ is a relation, not necessarily a function, and may hence model non-deterministic behavior as well.

As mentioned when explaining the syntax of the HyPA subset, the behavior of a physical system is described in terms of model variables $\mathcal{V}_\mathrm{m}$. With these variables, a time axis $\mathbb{T}$ and a valuation space $\mathbb{V}$ is associated (here, the real-line $\mathbb{T} = \mathbb{V} = \mathbb{R}$). Depending on whether continuous or discontinuous behavior is described, these variables take a value as either functions of an interval in time to the valuation space, or as a pair of valuations (before and after the discontinuity). What is observed, is only the behavior through time. This leads to the definition:

$$\Sigma = \mathcal{V}_\mathrm{m} \rightarrow (\mathbb{T} \mapsto \mathbb{V}),$$

abusing the notation $\mapsto$ for partial functions from a left- and right-closed interval. The state of the system depends on the valuation of the variables, as well as on the mode the system is in. These modes are described by process terms, and denoted by $\mathcal{P}$. The state-space of the labeled transition system is then defined as:

$$X = \mathcal{P} \times \mathcal{V}_\mathrm{m}.$$

What remains, is to define the transition relation. This is performed using a so-called *structured operational semantics* [23], a system of derivation rules in which a hypothesis is denoted above a line, and a conclusion regarding the transition system below that line. The partial semantics of HyPA, constrained to the operations and transition relations needed for this paper, is given in Table 3. Note that variables $p, p', q, q' \in \mathcal{P}$ are used ranging over the process terms, variables $\sigma, \sigma' \in \Sigma$ to denote flows, variables $\nu, \nu', \nu'' \in \mathbb{V}$ to denote static valuations, variables $t, t' \in \mathbb{T}$ denote times, and there is a variable X for recursion.

Table 3: Partial structured operational semantics of HyPA

Flow	$$\frac{\sigma \models P_\mathrm{f}, dom(\sigma) = [t, t'], \nu = \sigma(t), \nu' = \sigma(t')}{< \left(P_\mathrm{f} \right), \nu > \overset{\sigma}{\rightsquigarrow} < \left(P_\mathrm{f} \right), \nu' >}$$
Re-initialization	$$\frac{(\nu, \nu') \models P_\mathrm{r}, < p, \nu' > \overset{\sigma}{\rightsquigarrow} < p', \nu'' >}{< \left[P_\mathrm{r} \right] \gg p, \nu > \overset{\sigma}{\rightsquigarrow} < p', \nu'' >}$$
Alternative	$$\frac{< p, \nu > \overset{\sigma}{\rightsquigarrow} < p', \nu' >}{< p \oplus q, \nu > \overset{\sigma}{\rightsquigarrow} < p', \nu' >} \qquad \frac{< q, \nu > \overset{\sigma}{\rightsquigarrow} < q', \nu' >}{< p \oplus q, \nu > \overset{\sigma}{\rightsquigarrow} < q', \nu' >}$$
Disrupt	$$\frac{< p, \nu > \overset{\sigma}{\rightsquigarrow} < p', \nu' >}{< p \blacktriangleright q, \nu > \overset{\sigma}{\rightsquigarrow} < p' \blacktriangleright q, \nu' >} \qquad \frac{< q, \nu > \overset{\sigma}{\rightsquigarrow} < q', \nu' >}{< p \blacktriangleright q, \nu > \overset{\sigma}{\rightsquigarrow} < q', \nu' >}$$
Parallel	$$\frac{< p, \nu > \overset{\sigma}{\rightsquigarrow} < p', \nu' >, \ < q, \nu > \overset{\sigma}{\rightsquigarrow} < q', \nu' >}{< p \parallel q, \nu > \overset{\sigma}{\rightsquigarrow} < p' \parallel q', \nu' >}$$
Recursion	$$\frac{< p, \nu > \overset{\sigma}{\rightsquigarrow} < p', \nu' >}{< X, \nu > \overset{\sigma}{\rightsquigarrow} < p', \nu' >} \text{ whenever } X \text{ defined by } X : p$$

The *flow* semantics state that given a flow σ that complies with the flow predicate P_f and that holds on the time interval $[t, t']$, with boundary valuations ν and ν', the state $< \left(P_\mathrm{f} \right), \nu >$ evolves into $< \left(P_\mathrm{f} \right), \nu' >$ with σ being observed.

The *re-initialization* semantics state that a re-initialization $< \left[P_\mathrm{r} \right] p \gg, \nu >$ of process p at value ν evolves into $< p', \nu'' >$, if the process p itself can evolve into $< p', \nu'' >$ starting from the value ν', and the pair (ν, ν') satisfies the re-initialization predicate P_r.

The *alternative composition* semantics state that if a process p with valuation ν may evolve to a process p' with valuations ν', then the choice between p and q starting in ν can evolve into p' with valuations ν' as well, with the same observations. Similarly, the choice between p and q may evolve into q' if q can evolve into q' from the given valuation.

The *disrupt* semantics state that if a state $< p, \nu >$ can transition to a state $< p', \nu' >$ then, if the disrupt does not happen, the state $< p \blacktriangleright q, \nu >$ transitions to the state $< p' \blacktriangleright q, \nu' >$. In case the disrupt does happen and the state $< q, \nu >$ can transition to the state $< q', \nu' >$ then the disrupt state $< p \blacktriangleright q, \nu >$ transitions to $< q', \nu' >$. Note that the semantics are not urgent, and, therefore, behavior may be nondeterministic.

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The *parallel composition* semantics state that when two independent state transitions $< p, \nu >$ to $< p', \nu' >$ and $< q, \nu >$ to $< q', \nu' >$, with the same observation σ of flows, are composed, then the composition executes these two transitions synchronously.

The *recursion* semantics states that when a recursion on X is defined by p, then in case of a transition from $< p, \nu >$ to $< p', \nu' >$, a transition from $< X, \nu >$ can be found by substituting p to arrive at $< p', \nu' >$.

4.3 Related formalisms

In this paper, we have chosen the hybrid process algebra HyPA as a carrier for the semantics of hybrid bond graphs. To a certain degree, this was a choice of convenience, as one of the authors (Cuijpers) is also the main founder of HyPA. But from a process algebraic point of view, HyPA is in our opinion also the best suited formalism for the task.

Other hybrid process algebras have been developed, such as $\mathrm{ACP}_{\mathrm{hs}}^{\mathrm{srt}}$ [24], hybrid χ [25] and the ϕ-calculus [26], but they all have a property inherited from hybrid automata theory [27, 28], requiring that abstract actions are used to represent discontinuities in the physics. HyPA and BHPC [29] are the only hybrid process algebras that allow discontinuous behavior without explicit communication between actions. Finally, we chose HyPA over BHPC, because the latter was not available yet when the research presented in this paper started, but also because it allows us to prove statements about the general structure of constitutive processes using equational reasoning, which has not yet been developed for BHPC. (For example, in [21] we prove that a parallel composition of constitutive processes can always be written as a single constitutive process.)

From a non-process algebraic point of view, one might argue that a formalism like impuls differential inclusions [30] is in fact closer to what we actually need to describe the semantics of bondgraphs. This is in fact a valid argument, worthy of future research. However, it is the possibility of using abstract actions to synchronize between processes, that made us choose for HyPA over non-process algebraic formalisms. Although we do not need them in our semantics (we do not even introduce them formally in this paper), they are available in the full language, and they may be used to model concepts like switches in a bond graph that are controlled by some abstract algorithm. An simple example of this can be found in [31], where an action is used to model an impact sensor that triggers a control mechanism.

5 Constitutive Processes instead of Constitutive Equations

This section combines the constitutive equations for continuous and discontinuous behavior into hybrid constitutive processes. It also describes special bond graph elements, not discussed so far, that can be used for modeling a change in connection structure [32, 33, 34]. Ultimately, the parallel composition of constitutive hybrid processes leads to an algebraic description of an entire bond graph model. This is illustrated in a number of case studies in the next section.

5.1 Resistance

A resistance models dissipation of energy. The continuous behavior is described using the constitutive relation $e = \mathrm{R}(f)$, for which it is assumed that $x \cdot \mathrm{R}(x) \geq 0$ for every x. This ensures that a resistor models dissipation of energy $\dot{E} \geq 0$. As explained previously, during discontinuous behavior the dissipation of energy by resistors is assumed to be negligible. The index i denotes the bond to which the resistance is connected in the graph.

$$\mathrm{Resistance}_i(\mathrm{R}) : \left(\left[\begin{array}{l} E_i^+ - E_i^- = 0 \\ \hat{e}_i = 0 \\ \hat{f}_i = 0 \end{array} \right] \gg \left(\begin{array}{l} \dot{E}_i = e_i \cdot f_i \\ e_i = \mathrm{R}(f_i) \end{array} \right) \right) \blacktriangleright \mathrm{Resistance}_i(\mathrm{R}).$$

5.2 Inductance and Capacitance

The continuous behavior of an inductance is described by the equation $f_i = \mathrm{I}(p_i)$, where $\frac{\partial \mathrm{I}(p_i)}{\partial p_i} > 0$. The change in stored energy of an inductance is determined by the momentum-integral over this constitutive relation. The modeler should verify that the relation $\mathrm{I}(p_i)$ does not depend on other variables that change during discontinuous behavior. The index i denotes the bond to which the resistance is connected in the graph.

$$\text{Inductance}_i(\mathrm{I}) : \left(\begin{bmatrix} E_i^+ - E_i^- = \int_{p_i^-}^{p_i^+} \mathrm{I}(x)dx \\ \hat{e}_i = p_i^+ - p_i^- \\ \hat{f}_i = 0 \end{bmatrix} \gg \begin{pmatrix} \dot{E}_i = e_i \cdot f_i \\ \dot{p}_i = e_i \\ f_i = \mathrm{I}(p_i) \end{pmatrix} \right) \blacktriangleright \text{Inductance}_i(\mathrm{I}).$$

A capacitance is the dual of an inductance, and its constitutive process is:

$$\text{Capacitance}_i(\mathrm{C}) : \left(\begin{bmatrix} E_i^+ - E_i^- = \int_{q_i^-}^{q_i^+} \mathrm{C}(x)dx \\ \hat{f}_i = q_i^+ - q_i^- \; \hat{e}_i = 0 \end{bmatrix} \gg \begin{pmatrix} \dot{E}_i = e_i \cdot f_i \\ \dot{q}_i = f_i \\ e_i = \mathrm{C}(q_i) \end{pmatrix} \right) \blacktriangleright \text{Capacitance}_i(\mathrm{C}),$$

with $\frac{\partial \mathrm{C}(q_i)}{\partial q_i} > 0$.

5.3 Sources

A source of flow has the constitutive relation $f = \mathrm{s}_f$, while for a source of effort we have $e = \mathrm{s}_e$.

$$\text{Flow-Source}_i(\mathrm{s}_f) : \left(\begin{bmatrix} E_i^+ - E_i^- = 0 \\ \hat{e}_i = 0 \\ \hat{f}_i = 0 \end{bmatrix} \gg \begin{pmatrix} \dot{E}_i = e_i \cdot f_i \\ f_i = \mathrm{s}_f \end{pmatrix} \right) \blacktriangleright \text{Flow-Source}_i(\mathrm{s}_f).$$

$$\text{Effort-Source}_i(\mathrm{s}_e) : \left(\begin{bmatrix} E_i^+ - E_i^- = 0 \\ \hat{e}_i = 0 \\ \hat{f}_i = 0 \end{bmatrix} \gg \begin{pmatrix} \dot{E}_i = e_i \cdot f_i \\ e_i = \mathrm{s}_e \end{pmatrix} \right) \blacktriangleright \text{Effort-Source}_i(\mathrm{s}_e).$$

5.4 Junctions

The junctions represent the bond graph analogies for Kirchoff's laws. Bear in mind, that the positive direction of power determines the sign of the contribution of variables to the summations, as in the explanation of junctions in Section 2.2. This is also the case for the summation of energy. To emphasize this, we have used the notation $\pm$ (not to be confused with the notation for 'approximately') in front of every variable that should be positive when the power direction points inward and negative when outward.

$$0\text{-junction}_C \quad : \quad \left(\begin{bmatrix} \sum_{c \in C} \pm(E_c^+ - E_c^-) = 0 \\ \sum_{c \in C} \pm \hat{f}_c = 0 \\ \forall_{c,c' \in C} \; \hat{e}_c = \hat{e}_{c'} \end{bmatrix} \gg \begin{pmatrix} \sum_{c \in C} \pm f_c = 0 \\ \forall_{c,c' \in C} \; e_c = e_{c'} \end{pmatrix} \right) \blacktriangleright 0\text{-junction}_C,$$

$$1\text{-junction}_C \quad : \quad \left(\begin{bmatrix} \sum_{c \in C} \pm(E_c^+ - E_c^-) = 0 \\ \sum_{c \in C} \pm \hat{e}_c = 0 \\ \forall_{c,c' \in C} \; \hat{f}_c = \hat{f}_{c'} \end{bmatrix} \gg \begin{pmatrix} \sum_{c \in C} \pm e_c = 0 \\ \forall_{c,c' \in C} \; f_c = f_{c'} \end{pmatrix} \right) \blacktriangleright 1\text{-junction}_C.$$

5.5 Controlled junctions and switches

In this section, several hybrid bond graph elements are discussed, as they were proposed earlier in literature. The controlled junctions are based on the work of [32, 33]. The switching element, was proposed in [34]. As will be seen further on, the switch can be expressed in terms of controlled junctions and 0-value sources.

Controlled junctions and switches both serve to model changes in the connection structure between elements. When active, a controlled junction acts like the junction it is associated with. When inactive, it acts like a collection of 0-effort sources or a collection of 0-flow sources, depending on the specific type of the controlled junction. The 0/E, and 1/E type act as an effort source when inactive, while the 0/F and 1/F type act as a flow source.

When switching between modes, laws for the discontinuous behavior apply. The predicates Act and Inact, model when a controlled junction or switch is active or inactive, respectively, and hence control the switching. These can be predicates over all the model variables $\mathcal{V}_m$, and hence may include both extensive and intensive variables (see Section 6 for an example). To denote the predicate Act where all variables $x \in \mathcal{V}_m$ are replaced by x^-, the notation Act^- is employed. The notation Inact^- is handled analogously.

Note, once again, that the intensive variables as they are used to describe continuous behavior are different from the intensive variables as they are used to describe discontinuous behavior. In other words, e and $\hat{e}$ are different elements of $\mathcal{V}_m$. In particular, this means that the when a continuous intensive variable e occurs as e^- in Act^-, this should be understood as 'the last value the continuous intensive variable took before the switch occurred' rather than 'the value of the discontinuous intensive variable'.

A *change* in connection structure between components may give rise to a dissipation of energy. For example, the collision that was discussed in Section 3 can be modeled as a change in connection structure between two masses, but can also be modeled in more detail by including a spring-damper interaction model fragment. In this model fragment, the damper represents the loss of energy because of deformation of the masses in a continuous model. In our constitutive process for a controlled junction, this dissipation of energy is obtained from an inequality regarding the energy balance. When switching from Act to Inact, or *vice versa*, it is found that a controlled junction may dissipate energy (i.e., $\sum_{c \in C} (E_c^+ - E_c^-) \geq 0$). The discontinuous behavior of the other variables during this decrease is governed by either the equations for the discontinuous behavior of the associated junction or the equations for the 0-effort source or 0-flow source. In the equations below, the same notation $(\pm)$ as with junctions is employed, to emphasize the dependence of signs on the power direction of the bonds.

The following hybrid constitutive process for the (0/E)-junction is obtained:

$$
\begin{aligned}
(0/\text{E})_C(\text{Act,Inact}) \quad : \quad & \left(\begin{bmatrix} \text{Act}^- \\ \sum_{c \in C} \pm(E_c^+ - E_c^-) = 0 \\ \sum_{c \in C} \pm \hat{f}_c = 0 \\ \forall_{c,c' \in C} \ \hat{e}_c = \hat{e}_{c'} \end{bmatrix} \gg \begin{pmatrix} \text{Act} \\ \sum_{c \in C} \pm f_c = 0 \\ \forall_{c,c' \in C} \ e_c = e_{c'} \end{pmatrix} \right. \\[2em]
\oplus \quad & \begin{bmatrix} \text{Inact}^- \\ \forall_{c \in C} \left\{ \begin{array}{l} E_c^+ - E_c^- = 0 \\ \hat{e}_c = 0 \\ \hat{f}_c = 0 \end{array} \right. \end{bmatrix} \gg \begin{pmatrix} \text{Inact} \\ \forall_{c \in C} \ e_c = 0 \end{pmatrix} \\[2em]
\oplus \quad & \begin{bmatrix} \text{Inact}^- \\ \sum_{c \in C} \pm(E_c^+ - E_c^-) \geq 0 \\ \sum_{c \in C} \pm \hat{f}_c = 0 \\ \forall_{c,c' \in C} \ \hat{e}_c = \hat{e}_{c'} \end{bmatrix} \gg \begin{pmatrix} \text{Act} \\ \sum_{c \in C} \pm f_c = 0 \\ \forall_{c,c' \in C} \ e_c = e_{c'} \end{pmatrix} \\[2em]
\oplus \quad & \left. \begin{bmatrix} \text{Act}^- \\ \sum_{c \in C} \pm(E_c^+ - E_c^-) \geq 0 \\ \sum_{c \in C} \pm \hat{f}_c = 0 \\ \forall_{c,c' \in C} \ \hat{e}_c = \hat{e}_{c'} \end{bmatrix} \gg \begin{pmatrix} \text{Inact} \\ \forall_{c \in C} \ e_c = 0 \end{pmatrix} \right) \\[2em]
\blacktriangleright \quad & (0/\text{E})_C(\text{Act,Inact})
\end{aligned}
$$

It is straightforward to construct the dual definitions for 1/E, 0/F and 1/F.

A switch acts like a 0-effort source when active and as a 0-flow source when inactive. As before, when switching modes, energy may be dissipated while the other variables behave according to the equations of one of the sources. Because the power bond of a switch is always pointing outward,

the decrease in energy is associated with the constitutive equation $E_i' \leq 0$. The constitutive hybrid process of a switch has a similar structure as that of a controlled junction.

$$
\text{Switch}_i(\text{Act,Inact}) \quad : \quad \left(\begin{bmatrix} \text{Act}^- \\ E_i^+ - E_i^- = 0 \\ \hat{e}_i = 0 \\ \hat{f}_i = 0 \end{bmatrix} \gg \begin{pmatrix} \text{Act} \\ e_i = 0 \end{pmatrix} \right.
$$

$$
\oplus \begin{bmatrix} \text{Inact}^- \\ E_i^+ - E_i^- = 0 \\ \hat{e}_i = 0 \\ \hat{f}_i = 0 \end{bmatrix} \gg \begin{pmatrix} \text{Inact} \\ f_i = 0 \end{pmatrix}
$$

$$
\oplus \begin{bmatrix} \text{Inact}^- \\ E_i^+ - E_i^- \leq 0 \\ \hat{e}_i = 0 \\ \hat{f}_i = 0 \end{bmatrix} \gg \begin{pmatrix} \text{Act} \\ e_i = 0 \end{pmatrix}
$$

$$
\oplus \begin{bmatrix} \text{Act}^- \\ E_i^+ - E_i^- \leq 0 \\ \hat{e}_i = 0 \\ \hat{f}_i = 0 \end{bmatrix} \gg \begin{pmatrix} \text{Inact} \\ f_i = 0 \end{pmatrix} \right)
$$

$$
\blacktriangleright \quad \text{Switch}_i(\text{Act,Inact})
$$

Note, that a switch can also be represented as a 0-effort source acting on a controlled 1/F-junction. Dually, representing a switch as a 0-flow source acting on a controlled 0/E-junction (with the switching predicates reversed), is possible as well. If the variables associated with bonds 2 and 3 are abstracted, the three bond graphs depicted in Fig. 7 are found to be equivalent.
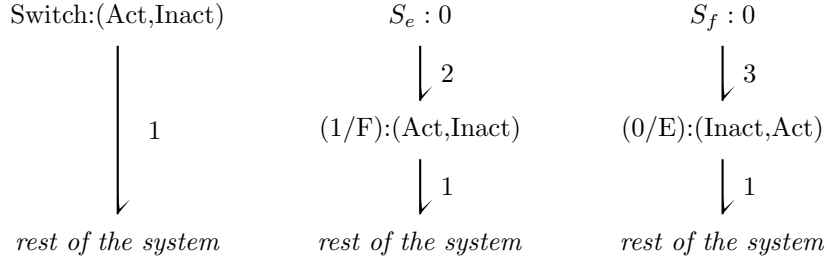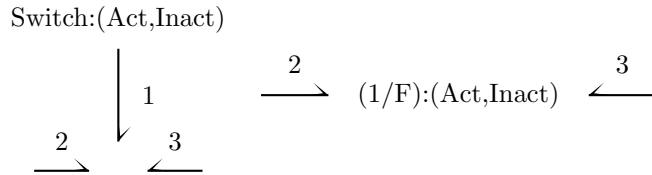

Figure 7: Three equivalent switches

Figure 8: Two non-equivalent switches

The two bond graphs depicted in Fig. 8 are *not* equivalent. In the left bond graph, there is a situation where the total energy of bonds 2 and 3 may decrease, while the switch acts like a 0-effort source. Still, some energy may go from 2 to 3, or *vice versa*. The constitutive hybrid

process of this partial bond graph contains the following subprocess as one of its alternatives:

$$\left[\begin{array}{l} \text{Act}^- \\ (E_2^+ - E_2^-) + (E_3^+ - E_3^-) = -(E_1^+ - E_1^-) \geq 0 \\ \hat{f}_1 = \hat{f}_2 = \hat{f}_3 = 0 \\ \hat{e}_1 = \hat{e}_2 = \hat{e}_3 = 0 \end{array} \right] \gg \left(\begin{array}{l} \text{Inact} \\ f_2 = f_3 = 0 \end{array} \right)$$

In the right bond graph, given that same situation, the energy of each of the bonds, separately, has to decrease, clearly indicating that the elements are disconnected. For this bond graph, the following subprocess is found as an alternative:

$$\left[\begin{array}{l} \text{Act}^- \\ (E_2^+ - E_2^-) \geq 0 \\ (E_3^+ - E_3^-) \geq 0 \\ \hat{f}_2 = \hat{f}_3 = 0 \\ \hat{e}_2 = \hat{e}_3 = 0 \end{array} \right] \gg \left(\begin{array}{l} \text{Inact} \\ f_2 = f_3 = 0 \end{array} \right)$$

This detail motivates a slight preference for the use of controlled junctions over switching elements. However, the physical consequences of each have to still be investigated in more detail, so no definitive answer can be given yet as to which is better. In the examples in Section 6, the controlled junction is used to model discontinuities.

Intermezzo For standard bond graph theory, there is a set of graph reduction rules, that lead to equivalent bond graphs, modulo elimination of variables associated with connections between junctions [35]. It is conjectured, that those rules are still valid for the hybrid case, perhaps with the exception that special elements, such as resistors of value 0, and infinite resistances, inductances and capacitances, need special treatment. A technical detail, however, prevents a formal proof at this point, since the notion of bisimulation $\Leftrightarrow$ that is used on the processes involved does not support the elimination of variables yet. For the formal derivation of bond graph reduction rules, a notion of abstraction from continuous variables is needed in HyPA. Such a notion of abstraction is currently being developed (see for example [36, 37]). Using this notion, also new rules for dealing with hybrid elements, like the informal rule of figure 7, can be developed. Even without these abstraction rules, it can be proven, for example, that 1-junction$_C \Leftrightarrow (1/\text{E})_C(\mathit{true},\mathit{false})$, which illustrates that a controlled junction behaves like an ordinary junction if it is always activated.

5.6 Transformers and gyrators

Two standard bond graph elements that have not been discussed so far, are the transformer and the gyrator. They are used to model transformation between different physical domains. Examples are motors, levers, pumps etc. Transformers and gyrators always define a certain ratio M between the flow and effort on the one, and on the other side of the element. As before, the modeler should verify that this ratio does not change significantly during discontinuous behavior. The sign contribution on variables used in transformers and gyrators, depends on the positive direction of power. In the case of transformers and gyrators, all variables are negated when the direction of power of a certain bond is outward.

Transformer$_{\{i,j\}}(\text{M})$:
$$\left(\left[\begin{array}{l} \pm(E_i^+ - E_i^-) \pm (E_j^+ - E_j^-) = 0 \\ \pm\hat{e}_i = \text{M} \cdot \pm\hat{e}_j \\ \pm\hat{f}_j = \text{M} \cdot \pm\hat{f}_i \end{array} \right] \gg \left(\begin{array}{l} \pm e_i = \text{M} \cdot \pm e_j \\ \pm f_j = \text{M} \cdot \pm f_i \end{array} \right) \right) \blacktriangleright \text{Transformer}_{\{i,j\}}(\text{M}),$$

Gyrator$_{\{i,j\}}(\text{M})$:
$$\left(\left[\begin{array}{l} \pm(E_i^+ - E_i^-) \pm (E_j^+ - E_j^-) = 0 \\ \pm\hat{e}_i = \text{M} \cdot \pm\hat{f}_j \\ \pm\hat{e}_j = \text{M} \cdot \pm\hat{f}_i \end{array} \right] \gg \left(\begin{array}{l} \pm e_i = \text{M} \cdot \pm f_j \\ \pm e_j = \text{M} \cdot \pm f_i \end{array} \right) \right) \blacktriangleright \text{Gyrator}_{\{i,j\}}(\text{M}).$$

6 Case Studies

To illustrate, first an impact control unit, as produced by Assembleon and Philips CFT (see also [21, 31]), in which the collision plays an important role, is presented. Next, a model of an electrical circuit containing diodes is studied. For this model, it is shown how so-called *implicit switching* leads to a model with less modes of operation than one might expect on first sight.

6.1 Impact Control at Assembleon

In the manufacturing of printed circuit boards (PCB), it is important to bring a component as fast as possible to the PCB, and then press it on the PCB for a sufficient time, with sufficient force, to make the adherent that is used for connection stick. In the actual mounting machine a number of components can be placed at the same time, using several impact modules. A simplified model of one such an impact module is drawn in Fig. 9. Because the nature of the components and the PCBs that are used varies significantly, it is not desirable to model the impact in too much detail. However, one of the requirements on the module is to prevent cracking of the component, which can be related to the energy that is absorbed upon impact. Therefore, modeling the amount of energy that is absorbed during the actual impact of the component on the PCB is a measure of likelihood that a crack occurs.
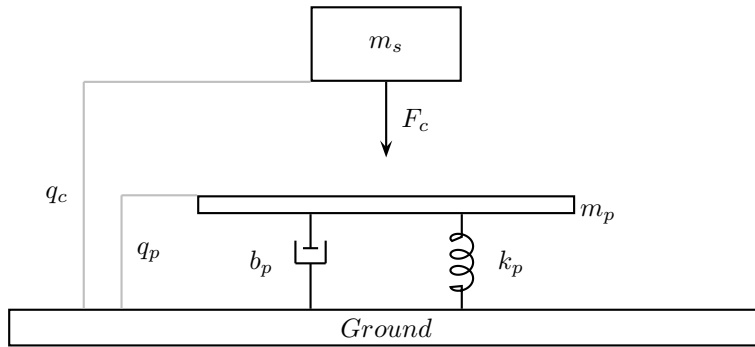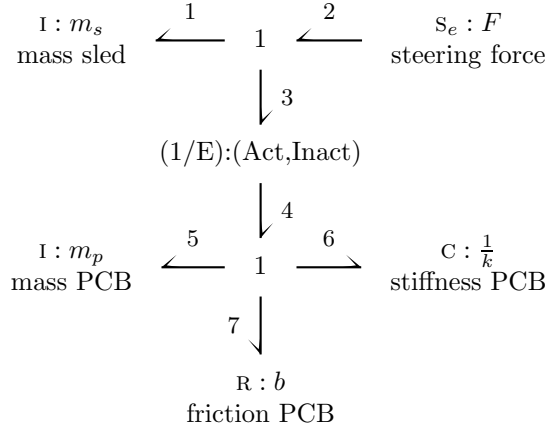

Figure 9: Schematic model of the impact module

The bond graph that is associated with the schematic model of Fig. 9 is presented in Fig. 10. This bond graph model is based on the assumption that the detailed dynamics of the impact behavior that could be modeled by a spring damper model fragment are not of interest. Furthermore, it is assumed that the effect of the external force on the detailed behavior is not of interest either. The actual validation of those assumptions is beyond the scope of this paper.

There are switching conditions in Fig. 10 that determine when two masses exchange momentum. The switching condition determining when the masses are in contact, $(q_1 = q_5 \wedge f_1 > f_5) \vee (q_1 = q_5 \wedge f_1 = f_5 \wedge \frac{1}{m_s} \cdot (e_1 + e_3) \geq \frac{1}{m_p} \cdot (e_2 - e_4))$, depends not only on the position of the masses, but also on their velocity and the normal forces between the masses [38]. The latter is formulated on the possible difference in acceleration of the masses (given by $a = \frac{e}{m}$). The reason for this, is that the two masses should not only connect when their position is equal, but they should also disconnect as soon as the interaction force becomes negative, which corresponds to accelerations that cause the masses to distance from each other.

From the bond graph in Fig. 10, the following parallel composition of constitutive processes is

Figure 10: Bond graph model for the impact module

constructed.

$$
\begin{aligned}
\text{Module} \quad &: \quad \text{1-Junction}_{\{1,2,3\}} \parallel (1/\text{E})_{\{3,4\}}(\text{Act,Inact}) \parallel \text{1-Junction}_{\{4,5,6,7\}} \\
&\parallel \quad \text{Inductance}_1(m_s) \parallel \text{Inductance}_5(m_p) \parallel \text{Effort-Source}_2(F) \\
&\parallel \quad \text{Capacitance}_6\!\left(\frac{1}{k}\right) \parallel \text{Resistance}_7(b)
\end{aligned}
$$

Process algebraic calculations as described in [20, 21], allow the elimination of the parallel compositions in this description. The result is the following bisimilar constitutive hybrid process for the impact module as a whole. Indeed, it can be proved that: Module $\Leftrightarrow$ Module$'$. Note, that the equations $\dot{E}_i = e_i \cdot f_i$, $\dot{p}_i = e_i$ and $\dot{q}_i = f_i$, as well as $p_i^+ - p_i^- = \hat{e}$ and $q_i^+ - q_i^- = \hat{f}$, remain implicit.

Module$'$:

$$
\left(\begin{bmatrix}
(E_1^+ - E_1^-) = -(E_3^+ - E_3^-) = \frac{(p_1^+)^2 - (p_1^-)^2}{2 \cdot m_s} \\
(E_4^+ - E_4^-) = (E_5^+ - E_5^-) = \frac{(p_5^+)^2 - (p_5^-)^2}{2 \cdot m_p} \\
(E_1^+ - E_1^-) + (E_5^+ - E_5^-) = (E_2^+ - E_2^-) \\
= (E_6^+ - E_6^-) = (E_7^+ - E_7^-) = 0 \\
\hat{e}_1 = -\hat{e}_3 \\
\hat{e}_4 = \hat{e}_5 \\
\hat{e}_1 + \hat{e}_5 = \hat{e}_2 = \hat{e}_6 = \hat{e}_7 = 0 \\
\hat{f}_1 = \hat{f}_2 = \hat{f}_3 = \hat{f}_4 = \hat{f}_5 = \hat{f}_6 = \hat{f}_7 = 0 \\
\text{Act}^-
\end{bmatrix}
\gg
\begin{Bmatrix}
p_1 = m_s \cdot f_1 \\
p_5 = m_p \cdot f_5 \\
q_6 = \frac{1}{k} \cdot e_6 \\
e_2 = F \\
e_7 = b \cdot f_7 \\
e_2 - e_1 = e_3 = e_4 = e_5 + e_6 + e_7 \\
f_1 = f_2 = f_3 = f_4 = f_5 = f_6 = f_7 \\
\text{Act}
\end{Bmatrix}
\right) \oplus
$$

$$
\begin{bmatrix}
(E_1^+ - E_1^-) = (E_2^+ - E_2^-) = (E_3^+ - E_3^-) \\
= (E_4^+ - E_4^-) = (E_5^+ - E_5^-) = (E_6^+ - E_6^-) = (E_7^+ - E_7^-) = 0 \\
\hat{e}_1 = \hat{e}_2 = \hat{e}_3 = \hat{e}_4 = \hat{e}_5 = \hat{e}_6 = \hat{e}_7 = 0 \\
\hat{f}_1 = \hat{f}_2 = \hat{f}_3 = \hat{f}_4 = \hat{f}_5 = \hat{f}_6 = \hat{f}_7 = 0 \\
\text{Inact}^-
\end{bmatrix}
\gg
\begin{Bmatrix}
p_1 = m_s \cdot f_1 \\
p_5 = m_p \cdot f_5 \\
q_6 = \frac{1}{k} \cdot e_6 \\
e_2 = F \\
e_7 = b \cdot f_7 \\
e_2 - e_1 = e_3 = 0 \\
e_4 = e_5 + e_6 + e_7 = 0 \\
f_1 = f_2 = f_3 \\
f_4 = f_5 = f_6 = f_7 \\
\text{Inact}
\end{Bmatrix}
\oplus
$$

$$\left[\begin{array}{l} (E_1^+ - E_1^-) = -(E_3^+ - E_3^-) = \frac{(p_1^+)^2 - (p_1^-)^2}{2 \cdot m_s} \\ (E_4^+ - E_4^-) = (E_5^+ - E_5^-) = \frac{(p_5^+)^2 - (p_5^-)^2}{2 \cdot m_p} \\ (E_1^+ - E_1^-) + (E_5^+ - E_5^-) \le (E_2^+ - E_2^-) \\ = (E_6^+ - E_6^-) = (E_7^+ - E_7^-) = 0 \\ \hat{e}_1 = -\hat{e}_3 \\ \hat{e}_4 = \hat{e}_5 \\ \hat{e}_1 + \hat{e}_5 = \hat{e}_2 = \hat{e}_6 = \hat{e}_7 = 0 \\ \hat{f}_1 = \hat{f}_2 = \hat{f}_3 = \hat{f}_4 = \hat{f}_5 = \hat{f}_6 = \hat{f}_7 = 0 \\ \text{Inact}^- \end{array}\right] \gg \left\{\begin{array}{l} p_1 = m_c \cdot f_1 \\ p_5 = m_p \cdot f_5 \\ q_6 = \frac{1}{k} \cdot e_6 \\ e_2 = F \\ e_7 = b \cdot f_7 \\ e_2 - e_1 = e_3 = e_4 = e_5 + e_6 + e_7 \\ f_1 = f_2 = f_3 = f_4 = f_5 = f_6 = f_7 \\ \text{Act} \end{array}\right\} \oplus$$

$$\left[\begin{array}{l} (E_1^+ - E_1^-) = -(E_3^+ - E_3^-) = \frac{(p_1^+)^2 - (p_1^-)^2}{2 \cdot m_s} \\ (E_4^+ - E_4^-) = (E_5^+ - E_5^-) = \frac{(p_5^+)^2 - (p_5^-)^2}{2 \cdot m_p} \\ (E_1^+ - E_1^-) + (E_5^+ - E_5^-) \le (E_2^+ - E_2^-) \\ = (E_6^+ - E_6^-) = (E_7^+ - E_7^-) = 0 \\ \hat{e}_1 = -\hat{e}_3 \\ \hat{e}_4 = \hat{e}_5 \\ \hat{e}_1 + \hat{e}_5 = \hat{e}_2 = \hat{e}_6 = \hat{e}_7 = 0 \\ \hat{f}_1 = \hat{f}_2 = \hat{f}_3 = \hat{f}_4 = \hat{f}_5 = \hat{f}_6 = \hat{f}_7 = 0 \\ \text{Act}^- \end{array}\right] \gg \left\{\begin{array}{l} p_1 = m_c \cdot f_1 \\ p_5 = m_p \cdot f_5 \\ q_6 = \frac{1}{k} \cdot e_6 \\ e_2 = F \\ e_7 = b \cdot f_7 \\ e_2 - e_1 = e_3 = 0 \\ e_4 = e_5 + e_6 + e_7 = 0 \\ f_1 = f_2 = f_3 \\ f_4 = f_5 = f_6 = f_7 \\ \text{Inact} \end{array}\right\}\Biggr) \blacktriangleright \text{Module}'.$$

In other work [31], this constitutive hybrid process is used as a starting point for the design and analysis of a safe control strategy for the pick-and-place module, intended to limit the forces and energy exchange that may occur during impact.

6.2 An Electrical Circuit

In this section, the electrical circuit depicted in Fig. 11 is studied. The bond graph associated with this circuit, is depicted in Fig. 12. This model is taken from [39], and uses controlled junctions to model the electrical switch and the diode. The state of the diode depends on the flow and effort in the bond, or bonds, connected to it. For the electrical switch, we use the predicate *Closed* to represent a closed switch, and the predicate *Open* to represent an open switch.
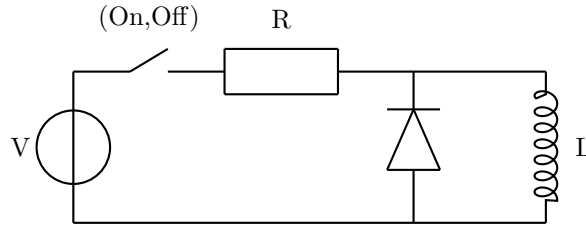
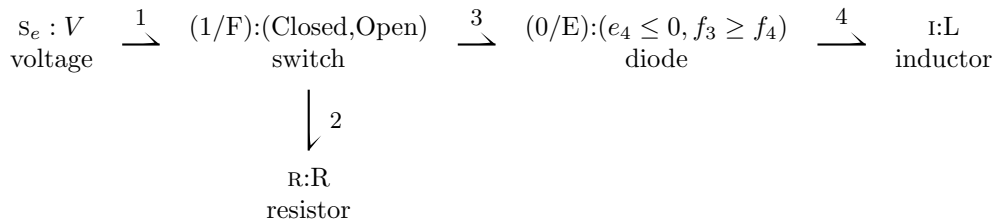Figure 11: Electrical circuit with switch and diode

Figure 12: Bond graph model using controlled junctions

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Calculation on the parallel composition of constitutive hybrid processes of the bond graph depicted in Fig. 12 show that there are six possible re-initializations and four possible flow clauses. Combinations of these re-initializations and flow clauses are restricted by the switching predicates. The following constitutive process is obtained for the circuit:

$$
\begin{aligned}
\text{Circuit} \quad \approx \quad & \big((d_1 \wedge [\ f_3^- \geq f_4^-\ ] \wedge [\ \text{Closed}^-\ ]) \gg c_1 \\
\oplus \quad & (d_1 \wedge [\ f_3^- \geq f_4^-\ ] \wedge [\ \text{Closed}^-\ ]) \gg c_2 \\
\oplus \quad & (d_3 \wedge [\ f_3^- \geq f_4^-\ ] \wedge [\ \text{Closed}^-\ ]) \gg c_3 \\
\oplus \quad & (d_4 \wedge [\ f_3^- \geq f_4^-\ ] \wedge [\ \text{Closed}^-\ ]) \gg c_4 \\
\oplus \quad & (d_1 \wedge [\ f_3^- \geq f_4^-\ ] \wedge [\ \text{Open}^-\ ]) \gg c_1 \\
\oplus \quad & (d_1 \wedge [\ f_3^- \geq f_4^-\ ] \wedge [\ \text{Open}^-\ ]) \gg c_2 \\
\oplus \quad & (d_4 \wedge [\ f_3^- \geq f_4^-\ ] \wedge [\ \text{Open}^-\ ]) \gg c_3 \\
\oplus \quad & (d_1 \wedge [\ f_3^- \geq f_4^-\ ] \wedge [\ \text{Open}^-\ ]) \gg c_4 \\
\oplus \quad & (d_3 \wedge [\ e_4^- \leq 0\ ] \wedge [\ \text{Closed}^-\ ]) \gg c_1 \\
\oplus \quad & (d_4 \wedge [\ e_4^- \leq 0\ ] \wedge [\ \text{Closed}^-\ ]) \gg c_2 \\
\oplus \quad & (d_2 \wedge [\ e_4^- \leq 0\ ] \wedge [\ \text{Closed}^-\ ]) \gg c_3 \\
\oplus \quad & (d_5 \wedge [\ e_4^- \leq 0\ ] \wedge [\ \text{Closed}^-\ ]) \gg c_4 \\
\oplus \quad & (d_4 \wedge [\ e_4^- \leq 0\ ] \wedge [\ \text{Open}^-\ ]) \gg c_1 \\
\oplus \quad & (d_1 \wedge [\ e_4^- \leq 0\ ] \wedge [\ \text{Open}^-\ ]) \gg c_2 \\
\oplus \quad & (d_5 \wedge [\ e_4^- \leq 0\ ] \wedge [\ \text{Open}^-\ ]) \gg c_3 \\
\oplus \quad & (d_1 \wedge [\ e_4^- \leq 0\ ] \wedge [\ \text{Open}^-\ ]) \gg c_4 \big) \\
\blacktriangleright \quad & \text{Circuit}
\end{aligned}
$$

The following abbreviations were used:

$$
d_1 = \left[\begin{array}{l}
(E_1^+ - E_1^-) = (E_2^+ - E_2^-) = (E_3^+ - E_3^-) = (E_4^+ - E_4^-) = 0 \\
\hat{e}_1 = \hat{e}_2 = \hat{e}_3 = \hat{e}_4 = 0 \\
\hat{f}_1 = \hat{f}_2 = \hat{f}_3 = \hat{f}_4 = 0
\end{array} \right]
$$

$$
d_2 = \left[\begin{array}{l}
(E_1^+ - E_1^-) = (E_2^+ - E_2^-) = (E_3^+ - E_3^-) = (E_4^+ - E_4^-) = 0 \\
\hat{e}_1 = \hat{e}_2 = 0 \\
\hat{e}_3 = \hat{e}_4 \\
\hat{f}_1 = \hat{f}_2 = \hat{f}_3 = \hat{f}_4 = 0 \\
(p_4^+)^2 = (p_4^-)^2
\end{array} \right]
$$

$$
d_3 = \left[\begin{array}{l}
(E_1^+ - E_1^-) = (E_2^+ - E_2^-) = 0 \\
(E_3^+ - E_3^-) = (E_4^+ - E_4^-) = \frac{(p_4^+)^2 - (p_4^-)^2}{2 \cdot L} \leq 0 \\
\hat{e}_1 = \hat{e}_2 = 0 \\
\hat{e}_3 = \hat{e}_4 \\
\hat{f}_1 = \hat{f}_2 = \hat{f}_3 = \hat{f}_4 = 0
\end{array} \right]
$$

$$
d_4 = \left[\begin{array}{l}
(E_1^+ - E_1^-) = (E_2^+ - E_2^-) = 0 \\
(E_4^+ - E_4^-) = \frac{(p_4^+)^2 - (p_4^-)^2}{2 \cdot L} \leq (E_3^+ - E_3^-) \leq 0 \\
\hat{e}_1 = \hat{e}_2 = 0 \\
\hat{e}_3 = \hat{e}_4 \\
\hat{f}_1 = \hat{f}_2 = \hat{f}_3 = \hat{f}_4 = 0
\end{array} \right]
$$

$$
d_5 = \left[\begin{array}{l}
(E_1^+ - E_1^-) = (E_2^+ - E_2^-) = (E_3^+ - E_3^-) = 0 \\
(E_4^+ - E_4^-) = \frac{(p_4^+)^2 - (p_4^-)^2}{2 \cdot L} \leq 0 \\
\hat{e}_1 = \hat{e}_2 = 0 \\
\hat{e}_3 = \hat{e}_4 \\
\hat{f}_1 = \hat{f}_2 = \hat{f}_3 = \hat{f}_4 = 0
\end{array} \right]
$$

$$
c_1 = \begin{Bmatrix} e_1 - e_2 = e_3 = e_4 \\ f_1 = f_2 = f_3 = f_4 \\ e_1 = V \\ e_2 = R \cdot f_2 \\ p_4 = \frac{1}{L} \cdot f_4 \\ e_4 \leq 0 \ \wedge \ \text{Closed} \end{Bmatrix}
\qquad
c_2 = \begin{Bmatrix} e_2 = e_3 = e_4 \\ f_1 = f_2 = f_3 = f_4 = 0 \\ e_1 = V \\ p_4 = 0 \\ e_4 \leq 0 \ \wedge \ \text{Open} \end{Bmatrix}
$$

$$
c_3 = \begin{Bmatrix} e_1 - e_2 = e_3 = e_4 = 0 \\ f_1 = f_2 = f_3 \\ e_1 = V \\ e_2 = R \cdot f_2 \\ p_4 = \frac{1}{L} \cdot f_4 \\ f_3 \geq f_4 \ \wedge \ \text{Closed} \end{Bmatrix}
\qquad
c_4 = \begin{Bmatrix} e_2 = e_3 = e_4 = 0 \\ f_1 = f_2 = f_3 = 0 \\ e_1 = V \\ p_4 = \frac{1}{L} \cdot f_4 \\ f_4 \leq 0 \ \wedge \ \text{Open} \end{Bmatrix}
$$

Again, the equations $\dot{E}_i = e_i \cdot f_i$, $\dot{p}_i = e_i$, $\dot{q}_i = f_i$, $p^+ - p^- = \hat{e}$, and $q^+ - q^- = \hat{f}$ have been left implicit.

Further study on the flow clauses shows a peculiarity in the behavior of c_2. Using calculation on derivatives, $e_4 = \dot{p}_4 = 0$ is obtained. As it turns out, the only case where time progresses in c_2, is when both the voltage over the diode, and the current through the diode are zero. Interestingly, the set of solutions of c_2 is a subset of the set of solutions of c_4, indicating that the diode can be interpreted as both conducting and blocking.

Process algebraic manipulation of the constitutive hybrid process allows combining the flow clauses c_2 and c_4, showing the character of implicit switching more clearly. The complete derivation is outside the scope of this paper, and can be found in [21]. The central observation in the derivation is that all solutions from c_4 that start with $p_4 = 0$, are also solutions from c_2. This allows combining certain process terms and the following alternative description of the circuit is obtained:

$$
\begin{aligned}
\text{Circuit}' \quad : \quad & \big(\big(d_1 \wedge [\ f_3^- \geq f_4^-\ ] \wedge [\ \text{Closed}^-\ ] \big) \gg c_1 \\
\oplus \quad & \big(d_3 \wedge [\ f_3^- \geq f_4^-\ ] \wedge [\ \text{Closed}^-\ ] \big) \gg c_3 \\
\oplus \quad & \big(((d_1 \wedge p_4^+ = 0) \vee d_4) \wedge [\ f_3^- \geq f_4^-\ ] \wedge [\ \text{Closed}^-\ ] \big) \gg c_4 \\
\oplus \quad & \big(d_1 \wedge [\ f_3^- \geq f_4^-\ ] \wedge [\ \text{Open}^-\ ] \big) \gg c_1 \\
\oplus \quad & \big(d_4 \wedge [\ f_3^- \geq f_4^-\ ] \wedge [\ \text{Open}^-\ ] \big) \gg c_3 \\
\oplus \quad & \big(d_1 \wedge [\ f_3^- \geq f_4^-\ ] \wedge [\ \text{Open}^-\ ] \big) \gg c_4 \\
\oplus \quad & \big(d_3 \wedge [\ e_4^- \leq 0\ ] \wedge [\ \text{Closed}^-\ ] \big) \gg c_1 \\
\oplus \quad & \big(d_2 \wedge [\ e_4^- \leq 0\ ] \wedge [\ \text{Closed}^-\ ] \big) \gg c_3 \\
\oplus \quad & \big(((d_4 \wedge p_4^+ = 0) \vee d_5) \wedge [\ f_3^- \geq f_4^-\ ] \wedge [\ \text{Closed}^-\ ] \big) \gg c_4 \\
\oplus \quad & \big(d_4 \wedge [\ e_4^- \leq 0\ ] \wedge [\ \text{Open}^-\ ] \big) \gg c_1 \\
\oplus \quad & \big(d_5 \wedge [\ e_4^- \leq 0\ ] \wedge [\ \text{Open}^-\ ] \big) \gg c_3 \\
\oplus \quad & \big(d_1 \wedge [\ e_4^- \leq 0\ ] \wedge [\ \text{Open}^-\ ] \big) \gg c_4 \big) \\
\blacktriangleright \quad & \text{Circuit}'
\end{aligned}
$$

Indeed, it can be proved that $\text{Circuit} \Leftrightarrow \text{Circuit}'$.

From this description, it becomes clear that, when the switch is opened, this enforces that the diode starts conducting. If the current through the inductance is negative, the implicit switch from blocking to conducting, leads to an energy loss, such that after the discontinuity, the generalized momentum of the inductance, and the current through the inductance, are zero. This phenomenon of implicit switching has been observed in other work [40, 41] and is consistent with the results computed from the hybrid process algebra description.

7   Conclusions

To remain competitive, computational models are becoming essential in the design of engineered systems. Such systems typicallly consist of a significant physical part. Modeling physics at a macroscopic level is often well done by means of algebraic and differential equations. The inherent continuity constraints of physics are inherently reflected by such a system of equations. To provide additional insight and support analysis, bond graphs have shown to be a useful language that still is independent of the physical domain, has a rigorous mapping onto a system of equations, while enforcing energy conservation constraints.

In many cases, the abstraction of reality to arrive at useful models results in with piecewise continuous behavior interspersed with discrete state changes. For example, an electrical switch may be modeled to operate in one of two modes, open and closed, with disctinctly different behavior in each of these modes.

To support such mode changes, bond graphs can be extended by a *controlled junction* that allows capturing dynamic changes to the model structure. The resulting behavior then is of a *hybrid* nature, mixing continuous-time and discrete state transition evolution. The combined behavior also requires the discrete model part to detect events in the continuous model part as well as re-initialization of the continuous states when discrete changes occur.

Previous work has developed *hybrid bond graphs* as a formal framework to express such models. In this work, a formal semantics is given to hybrid bond graphs in a unifying underlying representation based on a hybrid process algebra. This facilitates formal reasoning about the comprehensive system behavior. Furthermore, because it is not necessary to transform a hybrid bond graph in a set of global representations for the piecewise continuous behavior, the analysis is directly linked to the hybrid bond-graph topology. Maintaining the model structure is essential for analysis purposes. This also preserves the hierarchy properties of the hybrid bond-graph model, allowing upscaling towards industrial-sized models.

An additional benefit of using a process algebra is the support for nondeterministic behavior. When fast continuous behavior is abstracted into a discrete mode change, the details of the underlying behavior may be difficult to capture quantitatively. Nondeterminism allows analysis of a set of possible behaviors, within constraints that are specified. In the broadest sense, for example, this constraint is not to generate energy when a discrete state change occurs. Hence, in early phases of the design process, proper models can be made, which can be detailed out in a later phase, when necessary.

Case studies illustrate the applicability and power of this approach. Analysis of complicated phenomena that would result in dynamic changes of the continuous-time state space was effectively conducted using the approach documented in this work.

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