On Circuits and Pancyclic Line Graphs

A. Benhocine
UNIVERSITE DU MAINE
LE MANS, FRANCE

L. Clark
UNIVERSITY OF NEW MEXICO
ALBUQUERQUE, NM

N. Köhler
GERLINGERSTRASSE 6,
D-1000 BERLIN 47

H. J. Veldman
TWENTE UNIVERSITY OF TECHNOLOGY
ENSCHEDE, THE NETHERLANDS

ABSTRACT

Clark proved that $L(G)$ is hamiltonian if G is a connected graph of order $n \geq 6$ such that $\deg u + \deg v \geq n - 1 - p(n)$ for every edge uv of G, where $p(n) = 0$ if n is even and $p(n) = 1$ if n is odd. Here it is shown that the bound $n - 1 - p(n)$ can be decreased to $(2n + 1)/3$ if every bridge of G is incident with a vertex of degree 1, which is a necessary condition for hamiltonicity of $L(G)$. Moreover, the conclusion that $L(G)$ is hamiltonian can be strengthened to the conclusion that $L(G)$ is pancyclic. Lesniak-Foster and Williamson proved that G contains a spanning closed trail if $|V(G)| = n \geq 6$, $\delta(G) \geq 2$ and $\deg u + \deg v \geq n - 1$ for every pair of nonadjacent vertices u and v. The bound $n - 1$ can be decreased to $(2n + 3)/3$ if G is connected and bridgeless, which is necessary for G to have a spanning closed trail.

1. TERMINOLOGY

We use [4] for basic terminology and notation, but speak of vertices and edges instead of points and lines. Accordingly we denote the edge set of a graph G by $E(G)$. In [7] a circuit was defined as a nontrivial closed trail. Here the following subtle variation on this definition will be more convenient. A circuit C of a graph G is a nontrivial eulerian subgraph of G. Alternatively, C is a circuit if

Journal of Graph Theory, Vol. 10, No. 3, 411–425 (1986)
 CCC 0364-9024/86/030411-15$04.00

and only if C is a nontrivial connected subgraph such that every vertex of C has even degree in C. If C is a circuit of G, then $\beta(C)$ denotes the number of edges of G incident with at least one vertex of C. A spanning circuit, or briefly S-circuit, of a graph G is a circuit that contains all vertices of G. A dominating circuit or D-circuit of G is a circuit such that every edge of G is incident with at least one vertex of the circuit. If H is a subgraph of G, then vertices of $G - V(H)$ which are adjacent to at least one vertex of H are called neighbors of H. We denote the neighbors of $H = \{v\}$ by $N\{v\}$. A graph of order n is pancyclic if it contains a cycle of length i for each i with $3 \leq i \leq n$. A chord of a cycle C in G is an edge in $E(G) - E(C)$ whose ends are in C. A connected graph G is said to be almost bridgeless if every bridge of G is incident with a vertex of degree 1. If x is a real number, then $\lfloor x \rfloor$ and $\lceil x \rceil$ denote, respectively, the greatest integer smaller than or equal to x and the smallest integer greater than or equal to x.

2. DOMINATING CIRCUITS AND PANCYCLIC LINE GRAPHS

In [5] the following relation between D-circuits in graphs and hamiltonian cycles in line graphs is established.

Theorem 1. (Harary and Nash-Williams [5]). The line graph $L(G)$ of a graph G contains a hamiltonian cycle if and only if G has a D-circuit or G is isomorphic to $K_{1,s}$ for some $s \geq 3$.

In [3] Clark proved that the line graph $L(G)$ of a graph G is hamiltonian if G is connected, $|V(G)| = n \geq 6$ and $\deg u + \deg v \geq n - 1 - p(n)$ for every edge uv of G, where $p(n) = 0$ if n is even and $p(n) = 1$ if n is odd. The graphs showing that Clark's result is best possible all contain a bridge which is not incident with a vertex of degree 1. If a graph G contains a bridge uv with $\deg u \neq 1 \neq \deg v$, then the vertex of $L(G)$ corresponding to uv is a cut vertex of $L(G)$, so that $L(G)$ is nonhamiltonian. Hence a necessary condition for $L(G)$ to have a hamiltonian cycle, and for G to have a D-circuit, is that G is almost bridgeless. Using Theorem 1 we will show how Clark's bound $n - 1 - p(n)$ can be decreased if G is additionally required to be almost bridgeless. Before presenting our result we state two lemmas, the first of which is easily proved and frequently used in [2] and [3].

Lemma 2. Let G be a connected graph and C a circuit of G with maximum number of vertices. Then G contains no circuit C' satisfying $V(C') \cap V(C) \neq \emptyset \neq V(C') \cap V(G) - V(C)$ and $|E(C') \cap E(C)| \leq 1$.

Lemma 3. Let G be a connected graph, C a circuit of G with maximum number of vertices, K a component of $G - V(C)$ and u_1 and u_2 two neighbors of K on C. Then the following assertions hold.

a. u_1 and u_2 are nonadjacent.
b. If $w \in N(u_1) \cap N(u_2) - V(K)$, then none of the vertex pairs $\{u_1, w\}$ and $\{u_2, w\}$ has a common neighbor.
c. If $w_1 \in N(u_1) - V(K)$, $w_2 \in N(u_2) - V(K)$ and $w_1w_2 \in E(G)$, then at most one of the pairs $\{u_1, w_1\}$, $\{u_2, w_2\}$, and $\{w_1, w_2\}$ has a common neighbor.
d. If $v \in V(K)$ and $w \in N(u_1) \cap N(u_2) - V(K)$, then v and w are nonadjacent and have no common neighbor in $G - (V(K) \cup \{u_1, u_2\})$.
e. If $w_1, w_2 \in N(u_1) \cap N(u_2) - V(K)$, then w_1 and w_2 are nonadjacent and have no common neighbor in $G - \{u_1, u_2\}$.

Proof. Let G be a connected graph, C a circuit of G of maximum order, K a component of $G - V(C)$ and u_1 and u_2 two neighbors of K on C. Throughout the proof P will denote a u_1-u_2 path with $\emptyset \neq V(P) - \{u_1, u_2\} \subset V(K)$.

a. Suppose $u_1u_2 \in E(G)$. Then the cycle with edge set $E(P) \cup \{u_1u_2\}$ contradicts the assertion of Lemma 2. Hence u_1 and u_2 are nonadjacent.
b. Let w be a vertex of $N(u_1) \cap N(u_2) - V(K)$. If $u_1w \notin E(C)$ or $u_2w \notin E(C)$ then the cycle with edge set $E(P) \cup \{u_1w, u_2w\}$ contradicts Lemma 2. Hence $u_1w, u_2w \in E(C)$. Suppose, for example, u_1 and w have a common neighbor v. From Lemma 2 we deduce that $v \in V(C)$ and at least one of the edges u_1v and vw is in $E(C)$. Depending on whether or not each of the edges u_1v and vw is in $E(C)$ we now define a subgraph C' of G by specifying $E(C') - E(C)$ and $E(C) - E(C')$; $V(C')$ will be the set of vertices of G incident with at least one edge of $E(C')$. In the table below there is a column for each of the edges u_1v and vw; a one in such a column means that the relevant edge is in $E(C)$, while a zero means that it is in $E(G) - E(C)$.

u_1v	vw	$E(C') - E(C)$	$E(C) - E(C')$
1	1	$E(P)$	$\{u_1w, u_2w\}$
1	0	$E(P) \cup \{vw\}$	$\{u_1v, u_2w\}$
0	1	$E(P) \cup \{u_1v\}$	$\{vw, u_2w\}$

If, for example, $u_1v \in E(C)$ and $vw \notin E(C)$, then C' is defined as the subgraph of G with $V(C') = V(C) \cup V(P)$ and $E(C') = E(C) \cup E(P) \cup \{vw\} - \{u_1v, u_2w\}$, as indicated in the second row of the table. In all cases the fact that C is connected implies that C' is connected. Furthermore, since all vertices of C have even degree in C, all vertices of C' have even degree in C'. It follows that C' is a circuit with $|V(C')| = |V(C) \cup V(P)| > |V(C)|$, contradicting the choice of C and completing the proof of (b).
c. Let w_1 and w_2 be vertices of G such that $w_1 \in N(u_1) - V(K)$, $w_2 \in N(u_2) - V(K)$ and $w_1w_2 \in E(G)$. By Lemma 2 at least two of the edges u_1w_1, w_1w_2 and u_2w_2 are in $E(C)$. If one of the three edges is in $E(G) -$

$E(C)$, then a slight variation on the arguments used in (a) yields that the vertices incident with each of the remaining edges have no common neighbor. Hence assume $u_1w_1, w_1w_2, u_2w_2 \in E(C)$. Suppose that at least two of the pairs $\{u_1, w_1\}$, $\{w_1, w_2\}$ and $\{u_2, w_2\}$ have a common neighbor. We derive contradictions in two cases.

Case 1. There exists a vertex w of G which is adjacent to at least three of the vertices u_1, u_2, w_1, w_2.

From Lemma 2 and (b) we deduce that $w \in V(C) - \{u_1, u_2, w_1, w_2\}$ and w is adjacent to w_1, w_2 and exactly one of the vertices u_1 and u_2, u_1 say. Lemma 2 also implies that at least one of the edges wu_1 and ww_2 is in $E(C)$. In all possible cases we now specify, like in the proof of (b), a circuit C' of G with $|V(C')| > |V(C)|$, contradicting the choice of C.

wu_1	ww_1	ww_2	$E(C') - E(C)$	$E(C) - E(C')$
1	1	1	$E(P)$	$\{u_1w_1, u_2w_2, w_1w_2\}$
1	1	0	$E(P) \cup \{ww_2\}$	$\{wu_1, u_2w_2\}$
1	0	1	$E(P) \cup \{ww_1\}$	$\{wu_1, w_1w_2, u_2w_2\}$
0	1	1	$E(P) \cup \{wu_1\}$	$\{ww_2, u_2w_2\}$
1	0	0	$E(P) \cup \{ww_2\}$	$\{wu_1, u_2w_2\}$
0	0	1	$E(P) \cup \{wu_1\}$	$\{ww_2, u_2w_2\}$

Case 2. Each vertex of G is adjacent to at most two of the vertices u_1, u_2, w_1, w_2.

We assume that u_i and w_i have a common neighbor v_i ($i = 1, 2$); the remaining subcases are similar. From Lemma 2 we deduce that $v_1, v_2 \in V(C)$ and at least one of the edges u_1v_1, v_1w_1, u_2v_2 and v_2w_2 is in $E(C)$. Again a circuit C' of G with $|V(C')| > |V(C)|$ can be specified in all possible cases. We only treat two representative cases.

u_1v_1	v_1w_1	u_2v_2	v_2w_2	$E(C') - E(C)$	$E(C) - E(C')$
1	1	1	0	$E(P) \cup \{v_2w_2\}$	$\{u_1w_1, w_1w_2, u_2v_2\}$
0	0	0	1	$E(P) \cup \{u_1v_1, v_1w_1, u_2v_2\}$	$\{w_1w_2, v_2w_2\}$

d. Let v be a vertex of K and w a vertex in $N(u_1) \cap N(u_2) - V(K)$. For $i = 1, 2$, let P_i be a $v - u_i$ path with all internal vertices in K. From Lemma 2 it follows that $vw \notin E(G)$ and $u_1w, u_2w \in E(C)$. Suppose v and w have a common neighbor u in $G - (V(K) \cup \{u_1, u_2\})$. Then $uw \in E(C)$ by Lemma 2. If w is not a cut vertex of C or if u_1, u_2 and u are in the same component of $C - w$, then the subgraph C' of G with $V(C') = V(C) \cup V(P_1)$ and $E(C') = E(C) \cup E(P_1) \cup \{uv\} - \{uw, u_1w\}$ is connected, implying that C' is a circuit of G with $|V(C')| > |V(C)|$. Hence assume that w is a cut vertex of C and, for example, u and u_2 are in different components H_1 and H_2 of $C - w$, respectively. Let C_i be the subgraph of C

induced by $V(H_i) \cup \{w\}$ $(i = 1, 2)$. Then C_1 and C_2 are subcircuits of C. In particular, C_1 and C_2 are bridgeless, so $C_1 - uw$ and $C_2 - u_2w$ are connected subgraphs of C. It follows that $C - \{uw, u_2w\}$ is connected. But then the circuit C' with $V(C') = V(C) \cup V(P_2)$ and $E(C') = E(C) \cup E(P_2) \cup \{uv\} - \{uw, u_2w\}$ contradicts the choice of C.

e. Let w_1 and w_2 be two vertices in $N(u_1) \cup N(u_2) - V(K)$. Then $u_iw_j \in E(C)$ by Lemma 2 $(i = 1, 2; j = 1, 2)$. The table below shows that a circuit C' with $|V(C')| > |V(C)|$ can be constructed if $w_1w_2 \in E(G)$.

w_1w_2	$E(C') - E(C)$	$E(C) - E(C')$
1	$E(P)$	$\{u_1w_1, u_2w_1\}$
0	$E(P) \cup \{w_1w_2\}$	$[u_1w_1, u_2w_2\}$

Suppose w_1 and w_2 have a common neighbor v in $G - \{u_1, u_2\}$. Again a circuit C' with $|V(C')| > |V(C)|$ can be specified. Note that in the fourth row of the table below v may be a vertex of P.

vw_1	vw_2	$E(C') - E(C)$	$E(C) - E(C')$
1	1	$E(P)$	$\{u_1w_1, u_2w_1\}$
1	0	$E(P) \cup \{vw_2\}$	$\{u_1w_1, vw_1, u_2w_2\}$
0	1	$E(P) \cup \{vw_1\}$	$\{u_1w_1, vw_2, u_2w_2\}$
0	0	$E(P) \cup \{vw_1, vw_2\}$	$\{u_1w_1, u_2w_2\}$

∎

Theorem 4. Let G be a nontrivial connected, almost bridgeless graph of order n with $G \not\cong K_{1,n-1}$. If $\deg u + \deg v \geq (2n + 1)/3$ for every edge uv of G, then G contains a D-circuit.

Proof. Let G be a connected, almost bridgeless graph of order n with $G \not\cong K_{1,n-1}$. Assuming that G contains no D-circuit, we will exhibit two adjacent vertices with degree-sum at most $\frac{2}{3}n$. Since G is almost bridgeless and $G \not\cong K_{1,n-1}$, deletion of all vertices of degree 1 yields a nontrivial bridgeless graph, implying that G contains a circuit. Let C be a circuit of G such that $|V(C)|$ is maximum and $\beta(C) \geq \beta(C')$ for every circuit C' with $|V(C')| = |V(C)|$. Since C is not a D-circuit, $G - V(C)$ has a nontrivial component K. From Lemma 2 and the fact that G is almost bridgeless we conclude that K has at least two neighbors on C. We distinguish three cases.

Case 1. K has two neighbors on C which are joined by a path of length 2 contained in $G - V(K)$.

Let u_1 and u_2 be two neighbors of K on C which are joined by the path $u_1w_1u_2$, where $w_1 \notin V(K)$. Let P be a $u_1 - w_2$ path with $\emptyset \neq V(P) - \{u_1, u_2\} \subset V(K)$ such that $|V(P)|$ is minimum. Define v_1 as the immediate successor of u_1 on P. If $V(P) - \{u_1, u_2\} = \{v_1\}$, let v_2 be an arbitrary neighbor of v_1 in K, otherwise let v_2 be the successor of v_1 on P. Finally, let H be the induced subgraph $\langle V(P) \cup v_2, w_1\}\rangle$ of G. From Lemmas 2, 3(b) and 3(d) it follows that

$$N(u_1) \cap N(v_1) \cap (V(G) - V(H)) = N(u_1) \cap N(w_1) \cap (V(G) - V(H))$$
$$= N(v_1) \cap N(w_1) \cap (V(G) - V(H)) = \emptyset. \quad (1)$$

We next show that

$$V(G) - (V(H) \cup N(u_1) \cup N(v_1) \cup N(w_1)) \neq \emptyset. \quad (2)$$

Since each vertex of C has even degree in C, u_2 has a neighbor w_2 on C with $w_2 \neq w_1$. If $u_1w_2 \notin E(G)$, then, by Lemmas 2 and 3(b), w_2 is not adjacent to any of the vertices u_1, v_1 and w_1, implying (2). Now assume $u_1w_2 \in E(G)$. Then by Lemma 2 we have $u_1w_2, u_2w_2 \in E(C)$ and $v_2w_2 \notin E(G)$. There exists a vertex w in $G - V(H)$ which is adjacent to w_2, otherwise the circuit C' with $V(C') = V(C) \cup V(P) - \{w_2\}$ and $E(C') = E(C) \cup E(P) - \{u_1w_2, u_2w_2\}$ satisfies $|V(C')| \geq |V(C)|$ and $\beta(C') > \beta(C)$, contradicting the choice of C. By Lemma 2, $w \notin V(K)$. Application of Lemmas 3(b), 3(d) and 3(e) yields that w is adjacent to none of the vertices u_1, v_1 and w_1, implying (2).

Equation (1) expresses that each vertex of $G - V(H)$ is adjacent to at most one of the vertices u_1, v_1 and w_1. Together with (2) we obtain

$$\deg u_1 + \deg v_1 + \deg w_1 \leq n - |V(H)| - 1 + \deg_H u_1 + \deg_H v_1 + \deg_H w_1. \quad (3)$$

Similarly,

$$\deg u_1 + \deg v_2 + \deg w_1 \leq n - |V(H)| - 1 + \deg_H u_1 + \deg_H v_2 + \deg_H w_1. \quad (4)$$

Summation of the inequalities (3) and (4) yields

$$2(\deg u_1 + \deg w_1) + \deg v_1 + \deg v_2$$
$$\leq 2(n - |V(H)| - 1 + \deg_H u_1 + \deg_H w_1) + \deg_H v_1 + \deg_H v_2. \quad (5)$$

From Lemma 2, Lemma 3(a) and the minimality of $|V(P)|$ we conclude that every vertex of $H - \{v_1, v_2\}$ has degree 2 in H. Furthermore, $\deg_H v_1 = \deg_H v_2 = 2$ if $v_2 \in V(P)$, while $\deg_H v_1 = 3$ and $\deg_H v_2 = 1$ otherwise. Observing that $|V(H)| \geq 5$ we now deduce from (5) that

$$2(\deg u_1 + \deg w_1) + \deg v_1 + \deg v_2 \leq 2n.$$

It follows that either $\deg u_1 + \deg w_1 \leq \frac{2}{3}n$ or $\deg v_1 + \deg v_2 \leq \frac{2}{3}n$, settling Case 1.

Case 2. Case 1 does not apply and K has two neighbors on C which are joined by a path of length 3 contained in $G - V(K)$.

Let u_1 and u_2 be two neighbors of K on C which are joined by the path $u_1w_1w_2u_2$, where $w_1, w_2 \notin V(K)$. Define P, v_1 and v_2 as in Case 1 and put $H = \langle V(P) \cup \{v_2, w_1, w_2\}\rangle$. By Lemma 3(c) at least one of the pairs $\{u_1, w_1\}$ and $\{u_2, w_2\}$, $\{u_1, w_1\}$ say, has no common neighbor. In particular,

$$N(u_1) \cap N(w_1) \cap (V(G) - V(H)) = \emptyset . \tag{6}$$

By Lemma 2, v_1 and w_1 have no common neighbor outside C. Suppose v_1 and w_1 have a common neighbor u on C with $u \neq u_1$. Then Case 1 applies to the neighbors u and u_1 of K on C, contrary to assumption. We conclude that

$$N(v_1) \cap N(w_1) \cap (V(G) - V(H)) = \emptyset . \tag{7}$$

Another application of Lemma 2 gives us

$$N(u_1) \cap N(v_1) \cap (V(G) - V(H)) = \emptyset . \tag{8}$$

From (6), (7), and (8) we deduce that

$$\deg u_1 + \deg v_1 + \deg w_1 \leq n - |V(H)| + \deg_H u_1 + \deg_H v_1 + \deg_H w_1 . \tag{9}$$

Similarly,

$$\deg u_1 + \deg v_2 + \deg w_1 \leq n - |V(H)| + \deg_H u_1 + \deg_H v_2 + \deg_H w_1 . \tag{10}$$

Summation of (9) and (10) yields

$$\begin{aligned} &2(\deg u_1 + \deg w_1) + \deg v_1 + \deg v_2 \\ &\qquad \leq 2(n - |V(H)| + \deg_H u_1 + \deg_H w_1) + \deg_H v_1 + \deg_H v_2 . \end{aligned} \tag{11}$$

By Lemmas 2, 3(a), 3(b) and the minimality of $|V(P)|$, every vertex of $H - \{v_1, v_2\}$ has degree 2 in H, while $\deg_H v_1 + \deg_H v_2 = 4$. Observing that $|V(H)| \geq 6$, we deduce from (11) that

$$2(\deg u_1 + \deg w_1) + \deg v_1 + \deg v_2 \leq 2n ,$$

implying that either $\deg u_1 + \deg w_1 \leq \frac{2}{3}n$ or $\deg v_1 + \deg v_2 \leq \frac{2}{3}n$.

Case 3. Neither Case 1 nor Case 2 applies.

Let u_1 and u_2 be two arbitrary neighbors of K on C and w a vertex in $N(u_2) - V(K)$. Define P, v_1 and v_2 as in Case 1 and put $H = \langle V(P) \cup \{v_2, w\}\rangle$.

By Lemma 2 and by assumption we have

$$N(u_1) \cap N(v_1) \cap (V(G) - V(H)) = N(u_2) \cap N(v_1) \cap (V(G) - V(H)) = N(u_1) \cap N(u_2) \cap (V(G) - V(H)) = \emptyset, \tag{12}$$

implying that

$$\deg u_1 + \deg v_1 + \deg u_2 \le n - |V(H)| + \deg_H u_1 + \deg_H v_1 + \deg_H u_2 \le n - 5 + 1 + 3 + 2 = n + 1.$$

Suppose $\deg u_1 + \deg v_1 + \deg u_2 = n + 1$. Then, putting $U_1 = N(u_1) \cap V(C)$, $U_2 = N(u_2) \cap V(C)$ and $V_1 = N(v_1) \cap V(C) - \{u_1, u_2\}$, we have $U_1 \neq \emptyset \neq U_2$ and each vertex of $C - \{u_1, u_2\}$ is in exactly one of the sets U_1, U_2 and V_1. Since C is connected, there exists an edge uv of C with $u \in U_1$ and $v \in U_2 \cup V_1$. If $v \in V_1$, then Case 1 applies to the neighbors u_1 and v of K on C, contrary to assumption. If $v \in U_2$, then Case 2 applies to u_1 and u_2, again contrary to assumption. We conclude that

$$\deg u_1 + \deg v_1 + \deg u_2 \le n. \tag{13}$$

By Lemma 2, $N(v_1) \cap N(w) \cap (V(G) - V(C)) = N(u_1) \cap N(w) \cap (V(G) - V(C)) = \emptyset$. Assuming that $N(v_1) \cap N(w) \cap V(C) - \{u_2\} \neq \emptyset$ or $N(u_1) \cap N(w) \cap V(C) \neq \emptyset$, we reach the contradiction that Case 1 or Case 2 applies. Hence

$$N(v_1) \cap N(w) \cap (V(G) - V(H)) = N(u_1) \cap N(w) \cap (V(G) - V(H)) = \emptyset. \tag{14}$$

Together with (12) we obtain

$$\deg u_1 + \deg v_1 + \deg w \le n - |V(H)| + \deg_H u_1 + \deg_H v_1 + \deg_H w \le n - 5 + 1 + 3 + 1 = n. \tag{15}$$

Summation of (13) and (15) yields

$$2(\deg u_1 + \deg v_1) + \deg u_2 + \deg w \le 2n,$$

so that either $\deg u_1 + \deg v_1 \le \frac{2}{3}n$ or $\deg u_2 + \deg w \le \frac{2}{3}n$. ∎

Corollary 5. Let G be a connected, almost bridgeless graph of order $n \ge 4$ such that $\deg u + \deg v \ge (2n + 1)/3$ for every edge uv of G. Then $L(G)$ is hamiltonian. Moreover, if $G \not\cong C_4, C_5$, then $L(G)$ is pancyclic.

Proof. Let G be a connected, almost bridgeless graph of order $n \geq 4$ such that $\deg u + \deg v \geq (2n + 1)/3$ for every edge uv of G. The existence of a hamiltonian cycle in $L(G)$ immediately follows from the combination of Theorems 1 and 4. If $G \cong K_{1,n-1}$, then $L(G)$ is complete and hence pancyclic. Now assume $G \not\cong C_4, C_5, K_{1,n-1}$ and $L(G)$ is not pancyclic. Let $k = \max\{i \mid L(G)$ does not contain $C_i\}$.

We have $\Delta(G) \geq 3$, so $k \geq 4$. Let $D = u_1u_2 \ldots u_pu_1$ be a shortest cycle in G and suppose $p \geq 5$. Then every vertex of $G - V(D)$ is adjacent to at most one vertex of D, implying that

$$p(2n + 1)/6 \leq \sum_{i=1}^{p} \deg u_i \leq n - p + 2p,$$

so that $n \leq \lfloor 5p/(2p - 6) \rfloor \leq 6$. However, it is easily checked that every graph of order at most 6 satisfying our assumptions has a cycle of length at most 4. Hence, in fact, $p \leq 4$ and

$$\begin{aligned} \beta(D) &\geq p + \sum_{i=1}^{p} (\deg u_i - 2) \geq \lceil -p + p(2n + 1)/6 \rceil \\ &= \lceil p(2n - 5)/6 \rceil \geq \lceil (2n - 5)/2 \rceil = n - 2. \end{aligned} \tag{16}$$

Observing that, for any circuit C of G, $L(G)$ contains a cycle of length i for every i with $|E(C)| \leq i \leq \beta(C)$, we conclude that $k \geq n - 1$.

$L(G)$ is hamiltonian, so $k < |E(G)|$ and $L(G)$ contains C_{k+1}. Hence G contains a circuit C with $|E(C)| \leq k + 1 \leq \beta(C)$. In fact $|E(C)| = k + 1$, otherwise $L(G)$ contains C_k. Since C is a circuit, there exists edge-disjoint cycles $D_1, D_2, \ldots, D_r$ such that $C = \bigcup_{i=1}^{r} D_i$. We now derive contradictions in two cases.

Case 1. $r = 1$.

Since $|E(C)| = k + 1 \geq n$, C is a hamiltonian cycle of G and $k = n - 1$. Let D' be a shortest cycle among all cycles of G that contain exactly one chord of C. Let D' have length q. If $q = 3$, then G, and hence $L(G)$ too, contains C_{n-1}, a contradiction. If $q \geq 4$, then $n \geq 6$ and as in (16) we obtain

$$\beta(D') \geq \lceil q(2n - 5)/6 \rceil \geq \lceil 4(2n - 5)/6 \rceil \geq n - 1,$$

again implying the contradiction that $L(G)$ contains C_{n-1}.

Case 2. $r \geq 2$.

Let H be the graph with $V(H) = \{D_1, D_2, \ldots, D_r\}$ and $D_iD_j \in E(H)$ if and only if $V(D_i) \cap V(D_j) \neq \emptyset$. Since H is connected, at least two vertices of H are not cut vertices of H. Equivalently, there are at least two values of j for which $\bigcup_{1 \leq i \leq r, i \neq j} D_i$ is a connected subgraph of G and hence a circuit of G. Assume without loss of generality that $C' = \bigcup_{i=2}^{r} D_i$ and $C'' = D_1 \cup \bigcup_{i=3}^{r} D_i$ are circuits of G. If $E(D_2 - V(C'')) = \emptyset$, then $|E(C'')| < |E(C)| = k + 1 \leq \beta(C'')$, so that $L(G)$ contains C_k. Hence there exists an edge uv of D_2 with $u, v \notin V(C'')$. Let E_1 be the set of edges of D_1 incident with at least one vertex of C' and $E_2 = E(D_1) - E_1$. Then

$$\beta(C') \geq |E(C')| + |E_1| + \deg u - 2 + \deg v - 2 \geq |E(C)| - |E_2| + (2n + 1)/3 - 4 .$$

On the other hand, since $L(G)$ does not contain C_k,

$$\beta(C') \leq k - 1 = |E(C)| - 2 .$$

It follows that $|E_2| \geq (2n - 5)/3$. Hence $|V(D_1 - V(C'))| \geq (2n - 2)/3$ and similarily $|V(D_2 - V(C''))| \geq (2n - 2)/3$. But then

$$n = |V(G)| \geq |V(D_1 - V(C'))| + |V(D_2 - V(C''))| + 1 \geq 2(2n - 2)/3 + 1 > n ,$$

a contradiction. ∎

We do not know any connected, almost bridgeless graph G of order n without a D-circuit such that $G \not\cong K_{1,n-1}$ and $\deg u + \deg v \geq \frac{2}{3}n$ for every edge uv of G. We conjecture that, for n sufficiently large, the bound $(2n + 1)/3$ in Theorem 4 and Corollary 5 can be decreased to $(2n - 9)/5$. If true, this conjecture is best possible. To see this, construct for $i \geq 3$ a graph $G(i)$ as follows: take five disjoint copies of K_i, label them $G_1, \ldots, G_5$; choose three vertices u_1, u_2, u_3 in G_1, three vertices v_1, v_2, v_3 in G_2, two vertices x_1, x_2 in G_3, two vertices y_1, y_2 in G_4 and two vertices z_1, z_2 in G_5; obtain $G(i)$ as $\bigcup_{j=1}^{5} G_j + \{u_1x_1, u_2y_1, u_3z_1, v_1x_2, v_2y_2, v_3z_2\}$. Then $G(i)$ is 2-connected and $\deg u + \deg v \geq (2|V(G(i))| - 10)/5$ for every edge uv of $G(i)$, while $G(i)$ contains no D-circuit and hence $L(G(i))$ is nonhamiltonian.

Although Corollary 5 may not be best possible, it is strong enough to contain Clark's result.

Corollary 6. (Clark [3]). Let G be a connected graph of order $n \geq 6$. If $\deg u + \deg v \geq n - 1 - p(n)$ for every edge uv of G, where $p(n) = 0$ if n is even and $p(n) = 1$ if n is odd, then $L(G)$ is hamiltonian.

Proof. Let G be a connected graph of order $n \geq 6$ such that $\deg u + \deg v \geq n - 1 - p(n)$ for every edge uv of G. Since $n \geq 6$, $n - 1 - p(n) \geq (2n + 1)/3$. Hence we are done by Corollary 5 if G is shown to be almost bridgeless. Suppose G contains a bridge u_1u_2 with $\deg u_1 \neq 1 \neq$

deg u_2. Let H_i be the component of $G - u_1u_2$ containing u_i ($i = 1, 2$). Assume without loss of generality that $|V(H_1)| \leq |V(H_2)|$, so that $|V(H_1)| \leq (n - p(n))/2$. Since $|V(H_1)| \geq 2$, $H_1 - u_1$ contains a vertex u. If u has a neighbor v with $v \neq u_1$, then $\deg u + \deg v \leq 2(|V(H_1)| - 1) \leq n - p(n) - 2$, a contradiction. If u has no neighbor in $H - u_1$, then $uu_1 \in E(G)$ and $\deg u = 1$, so that $\deg u + \deg u_1 \leq 1 + |V(H_1)| \leq 1 + (n - p(n))/2$. For $n \geq 6$ we have $1 + (n - p(n))/2 \leq n - 2 - p(n)$. Thus $\deg u + \deg u_1 \leq n - 2 - p(n)$, again a contradiction. ∎

The bound $(2n + 1)/3$ in Corollary 5 can be decreased in case only hamiltonian graphs are considered.

Theorem 7. Let G be a hamiltonian graph of order $n \geq 13$. If $\deg u + \deg v \geq n/2$ for every edge uv of G, then $L(G)$ is pancyclic.

For the proof of Theorem 7 we refer to [1].

3. SPANNING CIRCUITS

In [6] Lesniak-Foster and Williamson proved that a graph G contains an S-circuit if $|V(G)| = n \geq 6$, $\delta(G) \geq 2$ and $\deg u + \deg v \geq n - 1$ for every pair of nonadjacent vertices u and v. All graphs showing that this result is best possible contain a bridge. For a graph G to have an S-circuit it is necessary that G is connected and contains no bridges. We now show how the above result can be improved by additionally imposing these necessary conditions.

Theorem 8. Let G be a connected bridgelsss graph of order $n \geq 3$. If $\deg u + \deg v \geq (2n + 3)/3$ for every pair of nonadjacent vertices u and v, then G contains an S-circuit.

Proof. Let G be a connected bridgeless graph of order $n \geq 3$. Assuming that G contains no S-circuit, we will exhibit two nonadjacent vertices with degree-sum smaller than $(2n + 3)/3$. Since G is bridgeless, G contains a circuit. Let C be a circuit of G of maximum order and K a component of $G - V(C)$. By Lemma 2 and the fact that G is bridgeless, K has at least two neighbors on C. We distinguish three cases.

Case 1. K has two neighbors on C which are joined by a path of length 2 contained in $G - V(K)$.

Let u_1 and u_2 be two neighbors of K on C which are joined by the path $u_1w_1u_2$, where $w_1 \notin V(K)$. Let P be a $u_1 - u_2$ path with $\emptyset \neq V(P) - \{u_1, u_2\} \subset V(K)$ such that $|V(P)|$ is minimum and let v be an arbitrary vertex in $V(P) \cap V(K)$. We distinguish two subcases.

Case 1.1. u_1 and u_2 have a common neighbor $w_2 \in V(G) - (V(K) \cup \{w_1\})$.

Put $H = \langle V(P) \cup \{w_1, w_2\}\rangle$. Lemmas 2, 3(d) and 3(e) imply that $\{v, w_1, w_2\}$ is an independent set and each vertex of $G - V(H)$ is adjacent to at most one of the vertices v, w_1, and w_2. Together with the minimality of $|V(P)|$ we obtain

$$\begin{aligned}\deg v + \deg w_1 + \deg w_2 &\le n - |V(H)| + \deg_H v + \deg_H w_1 + \deg_H w_2 \\ &\le n - 5 + 2 + 2 + 2 = n + 1 .\end{aligned}$$

It follows that at least one of the nonadjacent vertex pairs $\{v, w_1\}$, $\{v, w_2\}$ and $\{w_1, w_2\}$ has degree-sum at most $2(n + 1)/3$, settling Case 1.1.

Case 1.2. u_1 and u_2 have no common neighbor in $V(G) - (V(K) \cup \{w_1\})$.

Put $H = \langle V(P) \cup \{w_1\}\rangle$. By Lemmas 2, 3(b) and 3(d), each vertex of $G - V(H)$ is adjacent to at most one of the vertices u_1, u_2, v, w_1, so that

$$\begin{aligned}\deg u_1 + \deg u_2 + \deg v + \deg w_1 &\le n - |V(H)| + \deg_H u_1 + \deg_H u_2 \\ &+ \deg_H v + \deg_H w_1 \le n - 4 + 2 + 2 + 2 + 2 = n + 4 .\end{aligned}$$

It follows that at least one of the nonadjacent vertex pairs $\{u_1, u_2\}$ and $\{v, w_1\}$ has degree-sum at most $(n + 4)/2$. If $n > 6$, then $(n + 4)/2 < (2n + 3)/3$ and we are done. Now assume $n \le 6$. Since $\deg_C u_i \ge 2$, u_i has a neighbor v_i on C with $v_i \ne w_1$ $(i = 1, 2)$. By assumption v_1 and v_2 do not coincide, so that $n \ge 6$ and hence $n = 6$. By Lemmas 2 and 3(b), $(N(v) \cup N(w_1)) \cap \{v_1, v_2\} = \emptyset$. Thus $\deg v = \deg w_1 = 2$, so that $\deg v + \deg w_1 = 4 < 5 = (2n + 3)/3$.

Case 2. Case 1 does not apply and K has two neighbors on C which are joined by a path of length 3 contained in $G - V(K)$.

Let u_1 and u_2 be two neighbors of K on C which are joined by the path $u_1w_1w_2u_2$, where $w_1, w_2 \notin V(K)$. Define P and v as in Case 1 and put $H = \langle V(P) \cup \{w_1, w_2\}\rangle$. By Lemma 3(c) at least one of the following three subcases applies.

Case 2.1. $N(u_1) \cap N(w_1) = N(u_2) \cap N(w_2) = \emptyset$.

By Lemma 2 and the fact that Case 1 does not apply, each vertex of $G - V(H)$ is adjacent to at most one of the vertices u_1, v and w_1. Hence

$$\begin{aligned}\deg u_1 + \deg v + \deg w_1 &\le n - |V(H)| + \deg_H u_1 + \deg_H v + \deg_H w_1 \\ &\le n - 5 + 2 + 2 + 2 = n + 1 . \qquad (17)\end{aligned}$$

Similarly,

$$\deg u_2 + \deg v + \deg w_2 \le n + 1 . \qquad (18)$$

Assuming without loss of generality that $\deg w_1 \leq \deg w_2$ we deduce from (17) and (18) that

$$2(\deg v + \deg w_1) + \deg u_1 + \deg u_2 \leq 2 \deg v + \deg w_1 + \deg w_2 + \deg u_1 + \deg u_2 \leq 2n + 2 .$$

Hence one of the nonadjacent vertex pairs $\{v, w_1\}$ and $\{u_1, u_2\}$ has degree-sum at most $(2n + 2)/3$.

Case 2.2 $N(u_1) \cap N(w_1) = N(w_1) \cap N(w_2) = \emptyset$.
Similar arguments as used in Case 2.1 now yield

$$\deg u_1 + \deg v + \deg w_1 \leq n + 1$$

and

$$\deg v + \deg w_1 + \deg w_2 \leq n + 1 ,$$

implying that

$$2(\deg v + \deg w_1) + \deg u_1 + \deg w_2 \leq 2n + 2 .$$

Hence either $\deg v + \deg w_1 \leq (2n + 2)/3$ or $\deg u_1 + \deg w_2 \leq (2n + 2)/3$.

Case 2.3. $N(u_2) \cap N(w_2) = N(w_1) \cap N(w_2) = \emptyset$.
This case is symmetric to Case 2.2.

Case 3. Neither Case 1 nor Case 2 applies.
Let u_1 and u_2 be two neighbors of K on C and, for $i = 1, 2$, w_i a vertex in $N(u_i) - V(K)$. Define P and v as in Case 1 and put $H = \langle V(P) \cup \{w_1, w_2\} \rangle$. By Lemma 2 and the fact that neither Case 1 nor Case 2 applies, each vertex of $G - V(H)$ is adjacent to at most one of the vertices u_1, v and w_2. Hence

$$\deg u_1 + \deg v + \deg w_2 \leq n - |V(H)| + \deg_H u_1 + \deg_H v + \deg_H w_2 \leq n - 5 + 2 + 2 + 1 = n .$$

Similarly,

$$\deg u_2 + \deg v + \deg w_1 \leq n .$$

Assuming without loss of generality that $\deg w_1 \leq \deg w_2$, we obtain

$$2(\deg v + \deg w_1) + \deg u_1 + \deg u_2 \leq 2n .$$

Hence either $\deg v + \deg w_1 \leq \frac{2}{3}n$ or $\deg u_1 + \deg u_2 \leq \frac{2}{3}n$. ∎

The graph $K_{2,3}$ is the only known example of a connected bridgeless graph of order $n \geq 3$ without an S-circuit such that $\deg u + \deg v \geq (2n + 2)/3$ for every pair of nonadjacent vertices u and v. We conjecture that the bound in Theorem 8, too, can be decreased to $(2n - 9)/5$ if n is sufficiently large. Such an improvement would be best possible in view of the graphs $G(i)$ defined in Section 2.

Theorem 8 implies the result of Lesniak-Foster and Williamson mentioned above.

Corollary 9. (Lesniak-Foster and Williamson [6]). Let G be a graph with $|V(G)| = n \geq 6$ and $\delta(G) \geq 2$. If $\deg u + \deg v \geq n - 1$ for every pair of nonadjacent vertices u and v, then G contains an S-circuit.

Proof. Let G be a graph with $|V(G)| = n \geq 6$ and $\delta(G) \geq 2$ such that $\deg u + \deg v \geq n - 1$ for every pair of nonadjacent vertices u and v. It is easily seen that G must be connected. Since $n \geq 6$, $n - 1 \geq (2n + 3)/3$. In view of Theorem 8 it remains to be shown that G is bridgeless. Suppose G contains a bridge u_1u_2. Let H_i be the component of $G - u_1u_2$ containing u_i $(i = 1,2)$. Since $\delta(G) \geq 2$, H_i is nontrivial, say that $v_i \in V(H_i) - \{u_i\}$ $(i = 1,2)$. Then $v_1v_2 \notin E(G)$ and $\deg v_1 + \deg v_2 \leq |V(H_1)| - 1 + |V(H_2)| - 1 = n - 2$, a contradiction. ∎

4. DOMINATING CIRCUITS REVISITED

A slight variation on the proof of Theorem 8 gives us the following counterpart of Theorem 4.

Theorem 10. Let G be a connected, almost bridgeless graph of order $n \geq 3$. If $\deg u + \deg v \geq (2n + 1)/3$ for every pair of nonadjacent vertices u and v, then G contains a D-circuit.

Proof outline. Let G be a connected, almost bridgeless graph of order $n \geq 3$. We will exhibit a nonadjacent vertex pair with degree-sum smaller than $(2n + 1)/3$ under the assumption that G contains no D-circuit. Let C be a circuit of G of maximum order and K a nontrivial component of $G - V(C)$. K has at least two neighbors on C.

Distinguish the same cases as in the proof of Theorem 8. In each case define P as a shortest $u_1 - u_2$ path with $\emptyset \neq V(P) - \{u_1, u_2\} \subset V(K)$ and v_1 as the successor of u_1 on P. If $V(P) - \{u_1, u_2\} = \{v_1\}$, let v be an arbitrary neighbor of v_1 in K, otherwise let v be the successor of v_1 on P. Now all upper bounds on degree-sums in the proof of Theorem 8 can be decreased to obtain a vertex pair as desired. ∎

Without proof we mention that the corresponding counterpart of Corollary 5 also holds.

Corollary 11. Let G be a connected, almost bridgeless graph of order $n \geq 3$ such that $\deg u + \deg v \geq (2n + 1)/3$ for every pair of nonadjacent vertices u and v. Then $L(G)$ is hamiltonian. Moreover, if $G \neq C_4, C_5$, then $L(G)$ is pancyclic.

Again we conjecture, as a best possible improvement of Theorem 10 and Corollary 11, that the bound $(2n + 1)/3$ can be decreased to $(2n - 9)/5$ for n sufficiently large.

Note added in proof. A graph G is cyclically 2-edge-connected if no two cycles of G can be separated by the removal of at most one edge. Suppose G has order $n \geq 5$ with $\deg u + \deg v \geq (2n + 1)/3$ for every edge uv of G. Then G is connected and almost bridgeless if and only if G is cyclically 2-edge-connected and has no isolated vertices. Consequently, a corollary of Theorem 4 is the following: Let G be a nontrivial cyclically 2-edge-connected graph of order n with no isolated vertices. If $\deg u + \deg v \geq (2n + 1)/3$ for every edge uv of G, then G contains a D-circuit. Here the bound $(2n + 1)/3$ is best possible, as the following example shows. Let u be any vertex in $K_{(n/3)-1}$, v the center of the star $K_{1,(2n/3)-2}$ and $G = (K_{(n/3)-1} \cup K_{1,(2n/3)-2}) + uv$. Then G satisfies the above conditions with $(2n + 1)/3$ replaced with $2n/3$ but G has no D-circuit, since $L(G)$ is not hamiltonian.

References

[1] A. Benhocine, *Problèmes hamiltoniens et pancyclisme dans les graphes adjoints*. Thesis, Paris XI, France (1986).
[2] R. A. Brualdi and R. F. Shanny, Hamiltonian line graphs. *J. Graph Theory* 5 (1981) 307–314.
[3] L. Clark, On hamiltonian line graphs. *J. Graph Theory* 8 (1984) 303–307.
[4] F. Harary, Graph Theory. Addison-Wesley, Reading, Mass. (1969).
[5] F. Harary and C. St. J. A. Nash-Williams, On eulerian and hamiltonian graphs and line graphs. *Canad. Math. Bull.* 8 (1965) 701–710.
[6] L. Lesniak-Foster and J. E. Williamson, On spanning and dominating circuits in graphs. *Canad. Math. Bull.* 20 (1977) 215–220.
[7] H. J. Veldman, Existence of spanning and dominating trails and circuits. To appear in *J. Graph Theory*.