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BACKBONE COLORINGS ALONG STARS AND MATCHINGS IN SPLIT GRAPHS: THEIR SPAN IS CLOSE TO THE CHROMATIC NUMBER*

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Abstract

We continue the study on backbone colorings, a variation on classical vertex colorings that was introduced at WG2003. Given a graph $G = (V, E)$ and a spanning subgraph H of G (the backbone of G), a λ-backbone coloring for G and H is a proper vertex coloring $V \to \{1, 2, \ldots\}$ of G in which the colors assigned to adjacent vertices in H differ by at least λ. The algorithmic and combinatorial properties of backbone colorings have been studied for various types of backbones in a number of papers. The main outcome of earlier studies is that the minimum number ℓ of colors, for which such colorings $V \to \{1, 2, \ldots, \ell\}$ exist, in the worst case is a factor times the chromatic number (for

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path, tree, matching and star backbones). We show here that for split graphs and matching or star backbones, ℓ is at most a small additive constant (depending on λ) higher than the chromatic number. Our proofs combine algorithmic and combinatorial arguments. We also indicate other graph classes for which our results imply better upper bounds on ℓ than the previously known bounds.

Keywords: backbone coloring, split graph, matching, star.

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1. Introduction and Related Research

Coloring has been a central area in Graph Theory for more than 150 years. In [3] backbone colorings are introduced, motivated and put into the following general framework of coloring problems related to frequency assignment.

Given two graphs G_1 and G_2 with the property that G_1 is a spanning subgraph of G_2, one considers the following type of coloring problems: Determine a coloring of (G_1 and) G_2 that satisfies certain restrictions of type 1 in G_1, and restrictions of type 2 in G_2.

Many known coloring problems fit into this general framework, such as distance-2 coloring, radio coloring, radio labeling, and so on, see [2] for an overview.

In the WG2003 paper [3], a situation is modeled in which the transmitters form a network in which a certain substructure of adjacent transmitters (called the backbone) is more crucial for the communication than the rest of the network. This means more restrictions are put on the assignment of frequency channels along the backbone than on the assignment of frequency channels to other adjacent transmitters. The backbone could, e.g., model hot spots in a (sensor) network where a very busy pattern of communications takes place (the sensors with the highest computational power and energy), whereas the other adjacent transmitters supply a more moderate service.

Postponing the relevant definitions to the next subsections, we consider the problem of coloring the graph G_2 (that models the whole network) with a proper vertex coloring such that the colors on adjacent vertices in G_1 (that models the backbone) differ by at least $\lambda \geq 2$. This is a continuation of the study in [3] and [5]. Just as in [5] we consider two types of backbones in this paper: matchings and disjoint unions of stars. We are interested in split graphs for the following reasons.

1. In all worst cases the matching and star backbone coloring numbers grow proportionally to a multiplicative factor times the chromatic number [5]. Although these upper bounds are tight, they are probably only reached for very special graphs. To analyze this further, we turn to study the special case of split graphs. Split graphs have nice structural properties, which may lead to substantially better upper bounds on the number of colors in this context of backbone colorings. This was motivated by a similar study in [3]. There, the authors showed that for split graphs and tree (path) backbones the 2-backbone coloring number differs at most 2 (1) from the chromatic number.

2. Every graph can be turned into a split graph by considering any (e.g., a maximum or maximal) independent set and turning the remaining vertices into a clique. The number of colors needed to color the resulting split graph is an upper bound for the number of colors one needs to color the original graph. This way we find classes of non-split graphs for which our results also imply better upper bounds.

3. Although split graphs have a very special structure, they are not completely artificial in the context of, e.g., sensor networks. As an example, consider a sensor network within a restricted area (like a lab) with two distinct types of nodes: weak sensors with a very low battery capacity, like heat sensors, smoke sensors, body tags, etc., and PCs, laptops, etc., with much stronger power properties. The weak sensors are very unlikely to interfere with one another (especially if they are put with a certain purpose on fixed locations), while the other equipment is likely to interfere (within this restricted area). Weak sensors interfere with pieces of the other equipment within their vicinity. In such cases, the situation can be modeled as a split graph.

2. Terminology

For undefined terminology we refer to [1]. Let $G = (V, E)$ be a graph, where $V = V_G$ is a finite set of vertices and $E = E_G$ is a set of unordered pairs of two different vertices, called edges. A function $f : V \to \{1, 2, 3, \ldots\}$ is a *vertex coloring* of V if $|f(u) - f(v)| \geq 1$ holds for all edges $uv \in E$. A vertex coloring $f : V \to \{1, \ldots, k\}$ is called a *k-coloring*, and the *chromatic number* $\chi(G)$ is the smallest integer k for which there exists a k-coloring. A set $V' \subseteq V$ is *independent* if its vertices are mutually nonadjacent; it is a *clique* if its vertices are mutually adjacent. By definition, a k-coloring partitions V into k independent sets $V_1, \ldots, V_k$.

Let H be a *spanning subgraph* of G, i.e., $H = (V_G, E_H)$ with $E_H \subseteq E_G$. Given an integer $\lambda \geq 1$, a vertex coloring f is a λ-*backbone coloring* of (G, H), if $|f(u) - f(v)| \geq \lambda$ holds for all edges $uv \in E_H$. A λ-backbone coloring $f : V \to \{1, \ldots, \ell\}$ is called a λ-*backbone ℓ-coloring*. The λ-*backbone coloring number* $\text{BBC}_\lambda(G, H)$ of (G, H) is the smallest integer ℓ for which there exists a λ-backbone ℓ-coloring. Since a 1-backbone coloring is equivalent to a vertex coloring, we assume from now on that $\lambda \geq 2$.

For $q \geq 1$, a *star* S_q is a complete 2-partite graph with independent sets $V_1 = \{r\}$ and V_2 with $|V_2| = q$; the vertex r is called the *root* and the vertices in V_2 are called the *leaves* of the star S_q. For the star S_1, we arbitrarily choose one of its two vertices to be the root. In our context a *matching* M is a collection of pairwise vertex-disjoint stars that are all copies of S_1. A matching M of a graph G is called *perfect* if it is a spanning subgraph of G. We call a spanning subgraph H of a graph G a *star backbone* of G if H is a collection of pairwise disjoint stars, and a *matching backbone* if H is a perfect matching.

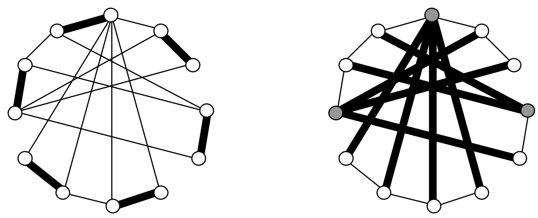

Figure 1. Matching and star backbones.

See Figure 1 for an example of a graph G with a matching backbone M (left) and a star backbone S (right). The thick edges are matching or star edges, respectively. The grey circles indicate root vertices of the stars in S.

Obviously, $\text{BBC}_\lambda(G, H) \geq \chi(G)$ holds for any backbone H of a graph G. We are interested in tight upper bounds for $\text{BBC}_\lambda(G, H)$ in terms of $\chi(G)$.

3. New Results

For convenience we give the definition of a split graph. A *split graph* is a graph whose vertex set can be partitioned into a clique and an independent set, with possibly edges in between. The size of a largest clique in G is denoted by $\omega(G)$. The size of a largest independent set in G is denoted by $\alpha(G)$. Split graphs were introduced by Hammer & Földes [8]; see also the book [7] by Golumbic. They form an interesting subclass of the class of perfect graphs. Hence, split graphs satisfy $\chi(G) = \omega(G)$, and many NP-hard problems are polynomially solvable when restricted to split graphs. The combinatorics of most graph problems becomes easier when the problem is restricted to split graphs. In this paper we study the special case of λ-backbone colorings of split graphs where their backbones are matchings or stars.

3.1. Matching backbones

In this section we present sharp upper bounds on the λ-backbone coloring numbers of split graphs along matching backbones. Our result on matching backbones is summarized in the next theorem which will be proven in Section 4.. We would like to mention here that this proof turned out to be far more involved than for the other studied backbones of split graphs. In particular, we use two coloring heuristics and elements of extremal graph theory to complete the proof.

Theorem 3.1. *Let $\lambda \geq 2$ and let $G = (V, E)$ be a split graph with $\chi(G) = k \geq 2$. For every matching backbone $M = (V, E_M)$ of G,*

$$\mathrm{BBC}_\lambda(G, M) \leq \begin{cases} \lambda + 1 & \text{if } k = 2, & \text{(i)} \\ k + 1 & \text{if } k \geq 4 \text{ and } \lambda \leq \min\{\frac{k}{2}, \frac{k+5}{3}\}, & \text{(ii)} \\ k + 2 & \text{if } k = 9 \text{ or } k \geq 11 \text{ and } \frac{k+6}{3} \leq \lambda \leq \lceil \frac{k}{2} \rceil, & \text{(iii)} \\ \lceil \frac{k}{2} \rceil + \lambda & \text{if } k = 3, 5, 7 \text{ and } \lambda \geq \lceil \frac{k}{2} \rceil, & \text{(iv)} \\ \lceil \frac{k}{2} \rceil + \lambda + 1 & \text{if } k = 4, 6 \text{ or } k \geq 8 \text{ and } \lambda \geq \lceil \frac{k}{2} \rceil + 1. & \text{(v)} \end{cases}$$

All the bounds are tight.

We will now show how these results can yield upper bounds for non-split graphs. For this purpose we first implicitly define a function f by the upper bounds $\mathrm{BBC}_\lambda(G, M) \leq f(\lambda, \chi(G))$ from the above theorem. Note that f is a nondecreasing function in λ and $\chi(G)$. Let $G = (V, E)$ be a graph and $V_1 \subseteq V$ be an independent set with $|V_1| = \alpha(G)$, and let $V_2 = V \setminus V_1$. Let W be the subset of V_1 consisting of vertices that are adjacent to all vertices in V_2. If W is non-empty, then we choose one $v \in W$ and move it to V_2, i.e., $V_2 := V_2 \cup \{v\}$. The meaning of this choice will become clear after the next sentence. Let $S(G)$ be the split graph with clique V_2

and independent set V_1. Since we moved one vertex from W to V_2 in case $W \neq \emptyset$, we guarantee that no vertex of V_1 is adjacent to all vertices of V_2. So $\chi(S(G)) = \omega(S(G)) = |V(G)| - \alpha(G)$ or $\chi(S(G)) = |V(G)| - \alpha(G) + 1$. Let the edges between V_1 and V_2 be defined according to E. Then we obtain: $\text{BBC}_\lambda(G, M) \leq \text{BBC}_\lambda(S(G), M) \leq f(\lambda, \chi(S(G))) \leq f(\lambda, |V(G)| - \alpha(G) + 1)$.

When can these bounds be useful for other (non-split) graphs? To answer this question, we should compare the new bound $f(\lambda, |V(G)| - \alpha(G) + 1)$ with the bound $(2 - \frac{2}{\lambda+1})\chi(G)$ from [5].

To get some insight into situations for which this gives an improvement, we apply a very rough calculation in which we use that the first bound is roughly of order $|V(G)| - \alpha(G)$ (disregarding some additive constant depending on λ), and the second one is roughly of order $2\chi(G)$ (disregarding the factor $\frac{2}{\lambda+1}$). Adopting these rough estimates, the first bound is better than the second one whenever $|V(G)| - \alpha(G) \leq 2\chi(G)$. This is, of course, the case when G is a split graph, since then $|V(G)| - \alpha(G) \leq \omega(G) = \chi(G)$. Now suppose we have a graph G with the following structure: An independent set I of G with cardinality $\alpha(G)$ shares at most one vertex with a clique C of G with cardinality $\omega(G)$, and $r = |V(G) \setminus (I \cup C)| \leq \omega(G)$. Then clearly $|V(G)| - \alpha(G) \leq 2\omega(G) \leq 2\chi(G)$. This gives large classes of non-split graphs for which the new bounds are better than the old bounds. Also if we apply a more careful analysis: If r is small compared to $(1 - \frac{2}{\lambda+1})\omega(G) + \lambda$, we get an improvement. We omit the details.

3.2. Star backbones

For split graphs with star backbones we obtained the following theorem. We have chosen to leave the proof of this result out of this paper, since its case analysis goes along the same lines as the proofs of previous results.

Theorem 3.2. *Let $\lambda \geq 2$ and let $G = (V, E)$ be a split graph with $\chi(G) = k \geq 2$. For every star backbone $S = (V, E_S)$ of G,*

$$\text{BBC}_\lambda(G, S) \leq \begin{cases} k + \lambda & \text{if either } k = 3 \text{ and } \lambda \geq 2 \text{ or } k \geq 4 \text{ and } \lambda = 2, \\ k + \lambda - 1 & \text{in the other cases.} \end{cases}$$

All the bounds are tight.

We can apply the results to obtain upper bounds for certain non-split graphs that improve bounds in [5], in a similar way as we did in the case of matching backbones, using a function $g(\lambda, \chi(G))$ which is implicitly defined by the upper bounds from Theorem 3.2. We omit the details.

4. Proof of Theorem 3.1

Let $G = (V, E)$ be a split graph with a perfect matching backbone $M = (V, E_M)$. A vertex $u \in V$ is called a *matching neighbor* of $v \in V$ if $uv \in E_M$, denoted by $u = mn(v)$.

Let C, I be two vertex-disjoint subsets of V such that C is a clique and I is an independent set of G with $mn(v) \in C$ for all $v \in I$. We note that, if C is a largest clique of V, then the condition $mn(v) \in C$ for all $v \in I$ is immediately satisfied. A *set of nonneighbors of an element* $u \in C$ is defined as the set of vertices $v \in I$ for which $uv \notin E$. Similarly, a *set of nonneighbors of an element* $v \in I$ is defined as the set of vertices $u \in C$ for which $vu \notin E$. The set of nonneighbors of a vertex u will be denoted by $NN(u)$. Note that in G, every vertex of I has at least one nonneighbor in C, if C is a largest clique of G. However, for a vertex $u \in C$, the set $NN(u)$ may be empty. Given C, I as above, a *splitting set of cardinality* p, named an s-set for short, is a subset $\{v_1, \ldots, v_p\} \subseteq I$ such that

$$\left\{ \bigcup_{i=1 \ldots p} NN(v_i) \right\} \bigcap \left\{ \bigcup_{i=1 \ldots p} \{mn(v_i)\} \right\} = \emptyset.$$

Note that if (G, M) has an s-set of cardinality p, then it also has an s-set of cardinality q, for all $q \leq p$. See Figure 2 for an example of a split graph with a matching backbone that has an s-set: the thick edges form the matching backbone and the grey vertices form the s-set.

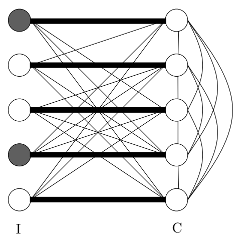

Figure 2. A graph with a matching backbone that has an s-set.

We need the following technical lemmas on the existence of certain s-sets for our proof.

Lemma 4.1. *Let $G = (V, E)$ be a split graph with a perfect matching backbone $M = (V, E_M)$. Let C, I be two vertex-disjoint subsets of V such that C is a clique of G with $|C| = k$, and I is an independent set of G with $|I| = i$. If $i = k$ and every vertex in I has at most one nonneighbor in C and every vertex in I has exactly one matching neighbor in C and $\lceil \frac{k}{3} \rceil \geq p$, then (G, M) has an s-set of cardinality p.*

Proof. Below we partition the disjoint sets C and I in the sets C_1, C_2, I_1 and I_2 with cardinalities c_1, c_2, i_1 and i_2, respectively. Then we show that one can pick at least $\lceil \frac{i_1}{3} \rceil$ vertices from I_1 and at least $\lceil \frac{i_2}{3} \rceil$ vertices from I_2 to form an s-set with cardinality $q \geq \lceil \frac{i_1}{3} \rceil + \lceil \frac{i_2}{3} \rceil \geq \lceil \frac{k}{3} \rceil$, which will prove the lemma.

C and I are split up in the following way: C_1 consists of all the vertices in C that either have zero nonneighbors in I or have at least two nonneighbors in I or have exactly one nonneighbor in I, whose matching neighbor in C has no nonneighbors in I; C_2 consists of all other vertices in C. Obviously, they all have exactly one nonneighbor in I; I_1 consists of the matching neighbors of the vertices in C_1; I_2 consists of the matching neighbors of the vertices in C_2.

Clearly, $i_1 = c_1$ and $i_2 = c_2$. Now assume that there are ℓ_1 vertices in C_1 that have no nonneighbors in I and put them in L_1. Also assume that there are ℓ_2 vertices in C_1 that have at least two nonneighbors in I and put them in L_2. Finally, assume that there are ℓ_3 vertices in C_1 that have exactly one nonneighbor in I, whose matching neighbor has no nonneighbors in I and put them in L_3. Then $\ell_1 \geq \ell_2$ and $\ell_1 \geq \ell_3$ and $c_1 = \ell_1 + \ell_2 + \ell_3$, so $c_1 \leq 3\ell_1$.

Let L_1', L_2' and L_3' be the sets of matching neighbors of the vertices in L_1, L_2 and L_3, respectively. Now we pick from I_1 the ℓ_1 vertices in L_1' and put them in the s-set. Notice that these vertices do not violate the definition of an s-set, because the set of their nonneighbors and the set of their matching neighbors are two disjoint sets. The matching neighbors of the nonneighbors of the ℓ_1 vertices in the s-set are either in L_2' or in L_3', so we exclude the vertices in these two sets for use in the s-set. On the other hand, the matching neighbors of the ℓ_1 vertices in the s-set do not have nonneighbors, so we do not have to worry about that. From the observations above it is clear that we can pick $\ell_1 \geq \lceil \frac{c_1}{3} \rceil = \lceil \frac{i_1}{3} \rceil$ vertices from I_1 that can be used in the s-set. Moreover, any vertices from I_2 that we will put in the s-set do not conflict with the vertices from L_1' that are in the s-set already. So the only thing we have to do now is to pick at least $\lceil \frac{i_2}{3} \rceil$ vertices from I_2 that can be used in the s-set. Simply pick an arbitrary vertex from I_2 and put it in the s-set. Now delete from I_2 the matching neighbor of its nonneighbor and the unique nonneighbor of its matching neighbor if they happen to be in I_2. Continuing this way, we 'throw away' at most two vertices of I_2 for every vertex of I_2 that we put in the s-set. It is easy to see that we can pick at least $\lceil \frac{i_2}{3} \rceil$ vertices from I_2 that we can put in the s-set. Therefore, the cardinality of the s-set will be at least $\lceil \frac{i_1}{3} \rceil + \lceil \frac{i_2}{3} \rceil \geq \lceil \frac{i}{3} \rceil = \lceil \frac{k}{3} \rceil$, which proves the lemma. ■

Lemma 4.2. *Let $G = (V, E)$ be a split graph with a perfect matching backbone $M = (V, E_M)$. Let C, I be a partition of V such that C is a* maximum *clique with $|C| = k$, and I is an independent set with $|I| = i$. If $i \leq k$ and every vertex in I has exactly one nonneighbor in C and $\lceil \frac{k}{3} \rceil \geq p$, then (G, M) has an s-set S with $|S| = p - \frac{k-i}{2}$ such that there are no matching edges between elements of the set of nonneighbors of vertices of S.*

Proof. To prove this lemma, we first define three disjoint subsets of C: C_1 consists of the i vertices of C that have a matching neighbor in I; C_2 contains, for each matching edge in C for which both end vertices have at least one nonneighbor in I, the end vertex with the fewest nonneighbors in I. If both end vertices have the same number of nonneighbors in I, then one arbitrary end vertex will be in C_2; C_3 contains, for each matching edge in C for which both end vertices have at least one nonneighbor in I, the end vertex that is not in C_2.

Let m be the sum of the number of nonneighbors of the vertices in C_2 and let n be the sum of the number of nonneighbors of vertices in C_3. Then clearly, $n \geq m$ and there are at least $m + n$ vertices in C_1 that have zero nonneighbors in I.

We now give a partition of I into four sets, $I_1, \ldots, I_4$ with $|I_1| = i_1$, $|I_2| = i_2$, $|I_3| = i_3$, and $|I_4| = i_4$. We show that one can pick n vertices from I_2 and at least $p - \frac{k-i}{2} - n$ vertices from I_4 that together will form an s-set of (G, M) with cardinality at least $p - \frac{k-i}{2}$. It will turn out that this will be sufficient to prove the lemma.

I_1 consists of all the nonneighbors of the vertices in C_2; I_2 consists of the matching neighbors of n vertices in C_1 that have no nonneighbors in I and whose matching neighbors are not already in I_1; I_3 consists of the matching neighbors of the nonneighbors of the elements of I_2 that are in I but not in I_1; I_4 consists of the other vertices of I.

It is easily verified that $i_1 = m$, $i_2 = n$, $i_3 \leq n$ and $i_4 \geq i - (2n + m)$. By construction of I_2, the matching neighbor of each vertex in I_2 does not have any nonneighbors. Hence, I_2 is an s-set of (G, M). Furthermore, the matching neighbors of the nonneighbors of the vertices in I_2 are in $(I_1 \cup I_3)$ or not in I at all. So, if we add vertices from I_4 to I_2, we only have to check whether the condition of being an s-set is satisfied for any pair of vertices of I_4.

Let $C_4 \subset C$ with $|C_4| = k_4$ be the set of the matching neighbors of I_4 in C. Every vertex in I_4 has at most one nonneighbor in C_4 and exactly one matching neighbor in C_4. Moreover, $i_4 = k_4$ and, since $n \geq m$ and $i_4 \geq i - (2n + m)$, we find that

$$\left\lceil \frac{k_4}{3} \right\rceil = \left\lceil \frac{i_4}{3} \right\rceil \geq \left\lceil \frac{k - (k - i) - (2n + m)}{3} \right\rceil \geq \left\lceil \frac{k}{3} \right\rceil - \left\lceil \frac{k - i}{3} \right\rceil - \left\lceil \frac{2n + m}{3} \right\rceil$$
$$\geq p - \left\lceil \frac{k - i}{2} \right\rceil - n = p - \frac{k - i}{2} - n.$$

Thus, according to Lemma 4.1, (G, M) has an s-set $I_4' \subseteq I_4$ of cardinality $p - \frac{k-i}{2} - n$. As observed before, then $S = I_2 \cup I_4'$ is also an s-set of (G, M). The cardinality of this set is $|S| = n + p - \frac{k-i}{2} - n = p - \frac{k-i}{2}$. We finish our proof of this lemma by observing that there is no matching edge between the two nonneighbors of any two vertices $x, y \in S$. If such a matching edge would exist, then one of the two vertices x, y would be in I_1. This is a contradiction. ■

4.1. Proof of the bounds in Theorem 3.1

First of all, note that for technical reasons we split up the proof in more and different subcases than there appear in the formulation of the theorem. The exact relation between the subcases in the theorem and those in the following proof is given as follows: Subcase **i** of the theorem is proven in **a**. The proof of subcase **ii** can be found in **b**. For even k the proof of subcase **iii** is given in **c**, for odd k in **d**. The three cases with $k = 3$ and $\lambda = 2$, $k = 5$ and $\lambda = 3$ and $k = 7$ and $\lambda = 4$ from subcase **iv** are treated in **b**, the others in **e**. Finally, subcase **v** is proven in **f** for even k and in **g** for odd k.

In all subcases, $G = (V, E)$ is a split graph with a perfect matching backbone $M = (V, E_M)$, and we let C, I be a partition of V such that C with $|C| = k$ is a clique of maximum size, and I is an independent set with

$|I| = i$. Without loss of generality, we assume that every vertex in I has exactly one nonneighbor in C.

Subcase **a.** If $k = 2$ then G is bipartite, and we use colors 1 and $\lambda + 1$.

Subcase **b.** Here we consider the cases with $k \geq 4$ and $\lambda \leq \min\{\frac{k}{2}, \frac{k+5}{3}\}$ together with the three separate cases with $k = 3$ and $\lambda = 2$, $k = 5$ and $\lambda = 3$ and $k = 7$ and $\lambda = 4$. The reason for this is that these are exactly the cases for which we obtain $k \geq 2\lambda - 1$ and $\lceil \frac{k}{3} \rceil \geq \lambda - 1$ and for which we need show the existence of a λ-backbone coloring using at most $k + 1$ colors. By Lemma 4.2, we find that (G, M) has an s-set of cardinality $y = \lambda - 1 - \frac{k-i}{2}$ such that there are no matching edges between the nonneighbors of vertices in the s-set. We make a partition of C into six disjoint sets $C_1, \ldots, C_6$, with cardinalities $c_1, \ldots, c_6$, respectively, as follows: C_1 consists of those vertices in C that have a matching neighbor in C and a nonneighbor in the s-set. Notice that by definition of the s-set, there are no matching edges between vertices in C_1; C_2 consists of those vertices in C that have a matching neighbor in I and a nonneighbor in the s-set; C_3 contains one end vertex of each matching edge in C that has no end vertex in C_1; C_4 consists of those vertices in C whose matching neighbor is in I and that are neither matching neighbor nor nonneighbor of any vertex in the s-set; C_5 consists of those vertices in C that have a matching neighbor in the s-set; C_6 consists of those vertices in C that have a matching neighbor in C and that are not already in C_1 or C_3. It is easily verified that

$$c_1 + c_2 \leq y, \qquad c_3 = \frac{k-i}{2} - c_1, \qquad c_4 = i - y - c_2,$$
$$c_5 = y, \qquad c_6 = \frac{k-i}{2}, \qquad \sum_{i=1}^{6} c_i = k.$$

An algorithm that constructs a feasible λ-backbone coloring of (G, M) with at most $k + 1$ colors is given below. In this algorithm I' denotes the set of vertices of I that are not in the s-set.

Coloring Algorithm 1

1 Color the vertices in C_1 with colors from the set $\{1, \ldots, c_1\}$.

2 Color the vertices in C_2 with colors from the set $\{c_1 + 1, \ldots, c_1 + c_2\}$.

3 Color the vertices in the s-set by assigning to them the same colors as their nonneighbors in C_1 or C_2. Note that different vertices in the s-set can have the same nonneighbor in C_1 or C_2, so a color may occur more than once in the s-set.

4 Color the vertices in C_3 with colors from the set $\{c_1 + c_2 + 1, \ldots, c_1 + c_2 + c_3\}$.

5 Color the vertices in C_4 with colors from the set $\{c_1 + c_2 + c_3 + 1, \ldots, c_1 + c_2 + c_3 + c_4\}$.

6 Color the vertices in C_5 with colors from the set $\{c_1 + c_2 + c_3 + c_4 + 1, \ldots, c_1 + c_2 + c_3 + c_4 + c_5\}$; start with assigning the lowest color from this set to the matching neighbor of the vertex in the s-set with the lowest color and continue this way.

7 Color the vertices in C_6 with colors from the set $\{c_1 + c_2 + c_3 + c_4 + c_5 + 1, \ldots, c_1 + c_2 + c_3 + c_4 + c_5 + c_6\}$; start with assigning the lowest color from this set to the matching neighbor with the lowest color in $C_1 \cup C_3$ and continue this way.

8 Finally, color the vertices of I' with color $k + 1$.

We prove the correctness of this algorithm as follows. First, it is immediately clear that vertices in C all get different colors and that vertices in I either get a color that does not occur in C or get the same color as their nonneighbor in C. There are now three types of matching edges for which we have to verify that the distance between the colors of their end vertices is at least λ.

1. Matching edges in the clique. They have one end vertex in $C_1 \cup C_3$ and one end vertex in C_6. It is easy to see that the smallest distance between two colors here occurs in the matching edges that have one end vertex in C_3 and one end vertex in C_6. This distance is $c_4 + c_5 + c_6$. However, $c_4 + c_5 + c_6 = i - c_2 + \frac{k-i}{2} \geq i - y + \frac{k-i}{2} = i - \lambda + 1 + \frac{k-i}{2} + \frac{k-i}{2} = k - \lambda + 1 \geq 2\lambda - 1 - \lambda + 1 = \lambda$, so the coloring causes no problems here.
2. Matching edges between the s-set and C. These are exactly y matching edges. They have one end vertex in the s-set and one end vertex in C_5, so one end vertex gets a color from the set $\{1, \ldots, c_1 + c_2\}$ and the other end vertex gets a color from the set $\{c_1+c_2+c_3+c_4+1, \ldots, c_1+c_2+c_3+c_4+c_5\}$. This last set contains exactly y colors, but the first set may contain less than y colors, because some of the colors of the first set may be used more than once in the s-set. However, it is not too hard to see that the smallest distance between colors here occurs in the matching edge with colors 1 and $c_1+c_2+c_3+c_4+1$. This distance is equal to $c_1+c_2+c_3+c_4 = k-c_5-c_6 = k-y-\frac{k-i}{2} = k-\lambda+1+\frac{k-i}{2}-\frac{k-i}{2} = k-\lambda+1 \geq 2\lambda-1-\lambda+1 = \lambda$,

so the coloring is also feasible for these matching edges.

3. Matching edges between I' and C. They have one end vertex I' and one end vertex in $C_2 \cup C_4$. It is clear that the smallest distance between two colors on a matching edge of this type is equal to $k+1-c_1-c_2-c_3-c_4$. This is equal to $c_5+c_6+1 = p+\frac{k-i}{2}+1 = \lambda-1-\frac{k-i}{2}+\frac{k-i}{2}+1 = \lambda$, so the coloring is feasible here as well.

These three checks show that the coloring provided by the algorithm indeed is a proper λ-backbone coloring of (G, M) with $k+1$ colors, which finishes the proof of this case.

Subcase **c.** Here we consider the case $k = 2m$, $m \geq 6$ and $\frac{k+6}{3} \leq \lambda \leq \frac{k}{2}$. We obtain $k \geq 2\lambda$. We color the k vertices in C with colors from the sets $\{2, \ldots, \frac{k}{2}+1\}$ and $\{\frac{k}{2}+2, \ldots, k+1\}$. If there are matching edges in C, then we color them such that the first colors from both sets are assigned to the end vertices of one matching edge, the second colors from both sets are assigned to the end vertices of another matching edge, and so on. For later reference we call this a greedy coloring. We can color up the two end vertices of $\frac{k}{2}$ matching edges in C this way, which suffices. Vertices in I get color $k+2$ if their matching neighbor in C is colored by a color from the first set, and vertices in I get color 1 if their matching neighbor in C is colored by a color from the second set. This yields a λ-backbone coloring of (G, M) with at most $k+2$ colors.

Subcase **d.** We now consider the case $k = 2m+1$, $m \geq 4$ and $\frac{k+6}{3} \leq \lambda \leq \frac{k+1}{2}$. We obtain $k \geq 2\lambda - 1$. For this case i is odd, otherwise there is no perfect matching in G. If $i = 1$, then there are $\frac{k-1}{2}$ matching edges in C. We can color their end vertices with colors from the two sets $\{1, \ldots, \frac{k-1}{2}\}$ and $\{\frac{k-1}{2}+3, \ldots, k+1\}$ by a greedy coloring. The distance between the colors of the end vertices of a matching edge in C is then $\frac{k-1}{2}+2 \geq \frac{2\lambda-2}{2}+2 = \lambda+1$. For the other vertex in C we use color $\frac{k-1}{2}+1$ and its matching neighbor in I gets color $k+2$. Note that $k+2-\frac{k-1}{2}-1 = \frac{k+3}{2} \geq \frac{2\lambda+2}{2} = \lambda+1$. If $3 \leq i \leq k$, there are $\frac{k-i}{2}$ matching edges in C. We color their end vertices with colors from the two sets $\{2, \ldots, \frac{k-i}{2}+1\}$ and $\{\frac{k+i}{2}+2, \ldots, k+1\}$ by a greedy coloring. The distance between the colors of the end vertices in a matching edge in C is then $\frac{k+i}{2} \geq \frac{2\lambda-1+i}{2} \geq \frac{2\lambda+2}{2} = \lambda+1$. The other i vertices in C are colored with colors from the sets $\{\frac{k-i}{2}+2, \ldots, \frac{k+3}{2}\}$ and $\{\frac{k+3}{2}+1, \ldots, \frac{k+i}{2}+1\}$. The cardinality of the first set is $\frac{i+1}{2}$ and of the second set $\frac{i-1}{2}$, adding up to exactly i. Vertices in I get color $k+2$ if their matching neighbor in C is colored by a color from the first set, or get color 1 if their matching neighbor in C is colored by a color from the second set. Notice that $k+2-\frac{k+3}{2}=\frac{2k+4-k-3}{2}=\frac{k+1}{2}\geq\frac{2\lambda}{2}=\lambda$ and $\frac{k+3}{2}+1-1=\frac{k+3}{2}\geq\frac{2\lambda+2}{2}=\lambda+1$, so this yields a λ-backbone coloring of (G,M) with at most $k+2$ colors.

Subcase **e.** Next, we consider the case $k=3,5,7$ and $\lambda\geq\frac{k+6}{3}$. We obtain $\lambda>\frac{k+1}{2}$ and $\lceil\frac{k}{3}\rceil=\frac{k-1}{2}$. By Lemma 4.2, we find that (G,M) has an s-set of cardinality $z=\frac{k-1}{2}-\frac{k-i}{2}=\frac{i-1}{2}$ such that there are no matching edges between the nonneighbors of vertices in the s-set. We have to construct a λ-backbone coloring of (G,M) using at most $\frac{k+1}{2}+\lambda$ colors. Obviously, colors from the set $\{\frac{k+1}{2}+1,\dots,\lambda\}$ can not be used at all, so we must find a λ-backbone coloring with colors from the sets $\{1,\dots,\frac{k+1}{2}\}$ and $\{\lambda+1,\dots,\frac{k+1}{2}+\lambda\}$. We partition C in six disjoint sets exactly like we did in (b). For the cardinalities of the sets, we now find the following relations:

$$c_1+c_2\leq\frac{i-1}{2},\qquad c_3=\frac{k-i}{2}-c_1,\qquad c_4=i-z-c_2,$$
$$c_5=z,\qquad c_6=\frac{k-i}{2},\qquad \sum_{i=1}^{6}c_i=k.$$

The following variation on Coloring Algorithm 1 constructs a feasible λ-backbone coloring of (G,M).

Coloring Algorithm 2

1 - 5 are the same as in Coloring Algorithm 1.

6 Color the vertices in C_5 with colors from the set $\{\lambda+1,\dots,\lambda+c_5\}$; start with assigning the lowest color from this set to the matching neighbor of the vertex in the s-set with the lowest color and continue this way.

7 Color the vertices in C_6 with colors from the set $\{\lambda+c_5+1,\dots,\lambda+c_5+c_6\}$; start with assigning the lowest color from this set to the matching neighbor with the lowest color in $C_1\cup C_3$ and continue this way.

8 Finally, color the vertices in I' with color $\frac{k+1}{2}+\lambda$.

We prove the correctness of this algorithm as follows. Since $\sum_{i=1}^{4}c_i=\frac{k+1}{2}$, it is clear that vertices in C all get different colors and that vertices in I either get a color that does not occur in C or get the same color as their nonneighbor in C. Again, there are three types of matching edges for which we have to verify that the distance the colors of their end vertices is at least λ.

1. Matching edges in the clique. They have one end vertex in $C_1 \cup C_3$ and one end vertex in C_6. It is easy to see that the smallest distance occurs in the matching edges that have one end vertex in C_3 and one end vertex in C_6. This distance is $\lambda + c_5 + c_6 - c_1 - c_2 - c_3 = \lambda + \frac{i-1}{2} + \frac{k-i}{2} - \frac{k-i}{2} - c_2 \geq \lambda + \frac{i-1}{2} - \frac{i-1}{2} = \lambda$.
2. Matching edges between the s-set and C. These are exactly $z = \frac{i-1}{2}$ matching edges. They have one end vertex in the s-set and one end vertex in C_5, so one end vertex gets a color from the set $\{1, \ldots, c_1 + c_2\}$ and the other gets a color from the set $\lambda + 1, \ldots, \lambda + c_5\}$. This last set contains exactly z colors, but the first set may contain less than z colors, because some of the colors of the first set may be used more than once in the s-set. However, it can be verified that the smallest distance here occurs in the matching edge with colors 1 and $\lambda + 1$ and this distance is equal to λ.
3. Matching edges between I' and C. They have one end vertex in I' and one end vertex in $C_2 \cup C_4$. It is clear that the smallest distance in a matching edge of this type is equal to $\frac{k+1}{2} + \lambda - c_1 - c_2 - c_3 - c_4$. This is equal to $\frac{k+1}{2} + \lambda - \frac{k-i}{2} - i + \frac{i-1}{2} = \lambda + \frac{k+1-k+i-2i+i-1}{2} = \lambda$.

These observations show that the coloring induced by Coloring Algorithm 2 indeed is a proper λ-backbone coloring of (G, M) using only colors from $\{1, \ldots, \frac{k+1}{2} + \lambda\}$.

Subcase **f.** We consider the case $k = 2m$, $m \geq 2$ and $\lambda \geq \frac{k}{2} + 1$. For this case we find that i is even, otherwise there is no perfect matching of G. If $i = 0$, then there are $\frac{k}{2}$ matching edges in C. We can use color pairs $\{1, \lambda + 1\}, \{2, \lambda + 2\}, \ldots, \{\frac{k}{2}, \frac{k}{2} + \lambda\}$ for their end vertices, because $\lambda + 1 > \frac{k}{2}$. If $i \geq 2$, then there are $\frac{k-i}{2}$ matching edges in C. We can color their end vertices with colors from the two sets $\{2, \ldots, \frac{k-i}{2} + 1\}$ and $\{\frac{i}{2} + \lambda + 1, \ldots, \frac{k}{2} + \lambda\}$, using greedy coloring. The distance between the two colors on every matching edge in C is then $\frac{i}{2} + \lambda - 1 \geq \lambda$. The other i vertices in C are colored with colors from the sets $\{\frac{k-i}{2} + 2, \ldots, \frac{k}{2} + 1\}$ and $\{\lambda + 1, \ldots, \frac{i}{2} + \lambda\}$, which are exactly i colors. The colors in the first set have distance at least λ to color $\frac{k}{2} + \lambda + 1$, so we color the matching neighbors in I of the vertices in C that are colored with colors from this set with color $\frac{k}{2} + \lambda + 1$. The colors in the second set have distance at least

λ to color 1, so we color the matching neighbors in I of the vertices in C that are colored with colors from this set with color 1. This yields a feasible λ-backbone coloring of (G, M) with at most $\frac{k}{2} + \lambda + 1$ colors.

Subcase **g.** Finally, we consider the case $k = 2m + 1$, $m \geq 4$ and $\lambda \geq \frac{k+1}{2} + 1$. For this case we find that i is odd, otherwise there is no perfect matching of G. There are $\frac{k-i}{2}$ matching edges in C. We can color their end vertices with colors from the two sets $\{2, \ldots, \frac{k-i}{2} + 1\}$ and $\{\frac{i+3}{2} + \lambda, \ldots, \frac{k+1}{2} + \lambda\}$ by a greedy coloring. Notice that $\frac{i+3}{2} + \lambda - \frac{k-i}{2} - 1 = \frac{i+3+2\lambda-k+i-2}{2} = \frac{2i+1-k+2\lambda}{2} \geq \frac{2i+1-k+k+2}{2} > 0$, so that these sets are disjoint. The distance between the two colors on every matching edge in C is $\frac{i-1}{2} + \lambda \geq \lambda$. The other i vertices in C are colored with colors from the sets $\{\frac{k-i}{2} + 2, \ldots, \frac{k+1}{2}\}$ and $\{\lambda + 1, \ldots, \frac{i+1}{2} + \lambda\}$, which are exactly i colors that have not been used so far. Vertices in I get color $\frac{k+1}{2} + \lambda + 1$ if their matching neighbor in C is colored by a color from the first set, and get color 1 otherwise. This yields a λ-backbone coloring of (G, M) with at most $\frac{k+1}{2} + \lambda + 1$ colors.

4.2. Proof of the tightness of the bounds in Theorem 3.1

Again, different subcases will be used in the proof than there appear in the formulation of the theorem. The case $k = 2$ is trivial.

We first consider the subcases **ii**, **iv** and the cases with even k in subcase **v** from Theorem 3.1: subcase **ii** together with the three cases $k = 3$ and $\lambda = 2$, $k = 5$ and $\lambda = 3$ and $k = 7$ and $\lambda = 4$ from subcase **iv** are proven in **a**. The other cases from subcase **iv** are treated in **b**, whereas subcase **v** for even k can be found in **c**.

For all three cases **a**, **b**, **c**, we consider a split graph G with matching backbone M that is defined as follows. G is partitioned in a clique of k vertices $v_1, \ldots, v_k$ and an independent set of k vertices $u_1, \ldots, u_k$. Every vertex u_i for $i = 1, \ldots, k-1$ is adjacent to all vertices v_j for $i = 1, \ldots, k-1$. The vertex u_k is adjacent to all vertices v_j for $j = 2, \ldots, k$. The perfect matching M contains the k edges $u_i v_i$ for $i = 1, \ldots, k$.

Subcase **a.** For these cases, we must show that there is no feasible λ-backbone coloring of (G, M) using less than $k + 1$ colors. Suppose to the contrary that $\text{BBC}_\lambda(G, M) \leq k$. Then all k colors are used in the clique and the vertices u_i with $i = 1, \ldots, k-1$ must get the same color as the color of v_k. However, one color can be used at most $k - \lambda \leq k - 2$ times in the independent set, since else the corresponding matching neighbors in the clique can not be colored, so we find a contradiction.

Subcase **b.** Here $k = 3, 5$ or 7 and $\lambda \geq \frac{k+6}{3}$. Suppose to the contrary that $\text{BBC}_\lambda(G, M) \leq \frac{k-1}{2} + \lambda$. Then colors from the set $\{\frac{k-1}{2} + 1, \ldots, \lambda\}$ can not be used at all, since for these colors there are no colors at distance of at least λ within the color set $\{1, \ldots, \frac{k-1}{2} + \lambda\}$. Since there are only $k - 1$ colors left to use and there is no way to color a clique of size k with only $k - 1$ colors, we find a contradiction

Subcase **c.** Suppose to the contrary that $\text{BBC}_\lambda(G, M) \leq \frac{k}{2} + \lambda$. Then colors from the set $\{\frac{k}{2} + 1, \ldots, \lambda\}$ can not be used at all, since for these colors there are no colors at a distance of at least λ within the color set $\{1, \ldots, \frac{k}{2} + \lambda\}$. Therefore, only the other k colors can be used and they all appear in the clique. The vertices u_i for $i = 1, \ldots, k - 1$ must then get the same color as v_k, but then we find a contradiction, since one color can be used at most $\frac{k}{2} \leq k - 2$ times in the independent set.

We are now ready to prove the remaining subcases of Theorem 3.1. Subcase **iii** of the theorem will shortly be proven in **d**, whereas the proof of subcase **v** for odd values of k can be found in **e**.

We first need the following definition. Let G be a split graph on $2k$ vertices with $k = \omega(G) = \alpha(G)$. Let C, I be a partition of V such that C is a largest clique, and I is an independent set of G. Let G have a matching backbone M that contains k edges between C and I. We let every vertex in I have exactly one nonneighbor in C, and we let the matching edges together with the nonneighbor relations (see these nonneighbor relations as some imaginary edges) form a cycle of length $2k$. By $C_{k,k}$, we then mean the representation of G only by its vertices, its matching edges and the nonneighbor relations between C and I, i.e., $C_{k,k}$ is the graph obtained from G after deleting all edges between two vertices in C together with all nonmatching edges between vertices from I and C, and after adding an edge uv for each $u \in C, v \in I$ that are nonadjacent in G.

Now, in **d** and **e**, we consider a split graph G with matching backbone M that is defined by the following three characteristics (see Figure 3 for an example).

1. $\omega(G) = \alpha(G) = k$,
2. $|NN(v) \cap C| = 1, \ \forall v \in I$,
3. The representation by its vertices, matching edges and nonneighbor relations between C and I consists of exactly $\lceil \frac{k}{3} \rceil$ copies of $C_{3,3}$ or $C_{2,2}$.

More specifically, there are q copies of $C_{3,3}$ for $k = 3q$, there are $q-1$ copies of $C_{3,3}$ and two copies of $C_{2,2}$ for $k = 3q+1$, and there are q copies of $C_{3,3}$ and one copy of $C_{2,2}$ for $k = 3q+2$.

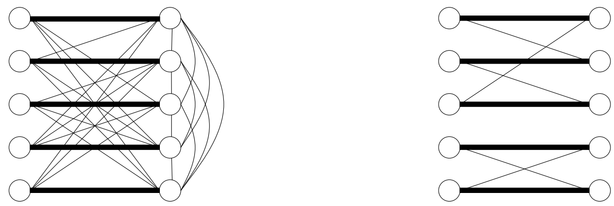

Figure 3. A split graph satisfying 1-3, together with its representation in condition 3.

Subcase **d.** Suppose to the contrary that $\text{BBC}_\lambda(G, M) \leq k+1$. Then the following observations can be made.

Observation 4.3. *There is exactly one color that is not used in C, which we will call the* independent *color. Without loss of generality, we may assume that the independent color is in the set $\{\lambda+1, \ldots, k+1\}$. The independent color may be used p times in I, where $p \leq k+1-\lambda$. All vertices in I that are not colored with the independent color must get the same color as their unique nonneighbor in C, hence all these other colors can only occur once in I.*

Observation 4.4. *Assume that the independent color is in the set $\{\lambda+1, \ldots, k+1\}$ and that this color is used p times in I. Then the other colors from $\{\lambda+1, \ldots, k+1\}$ can be used on at most $k+1-\lambda-p$ vertices of I.*

Indeed, if the independent color is used $k+1-\lambda$ times, then all the possible colors for matching neighbors in C of the vertices in I with the other colors from $\{\lambda+1, \ldots, k+1\}$ are already in use by matching neighbors of the vertices that are colored with the independent color.

Observation 4.5. *Assuming that the independent color is in the set $\{\lambda+1, \ldots, k+1\}$, the colors from $\{1, \ldots, \lambda\}$ can be used at most once in I. Even stronger, from the set $\{1, \ldots, \lambda\}$ we can choose only $\lceil \frac{k}{3} \rceil$ colors that can be used in I.*

Indeed, if we choose more, there would be at least two colors from $\{1,\ldots,\lambda\}$ in one $C_{2,2}$ or $C_{3,3}$. This means that there would be a matching edge violating the minimally required distance of λ between the two colors of its end vertices.

By these three observations, we derived the following. Firstly, we can use the independent color at most p times in I. Secondly, we can use the other colors from $\{\lambda+1,\ldots,k+1\}$ on at most $k+1-\lambda-p$ vertices of I. Thirdly, we can use colors from $\{1,\ldots,\lambda\}$ for at most $\lceil\frac{k}{3}\rceil$ vertices of I. Since $\lceil\frac{k}{3}\rceil < \lambda-1$, we can only color at most $k+1-\lambda+\lceil\frac{k}{3}\rceil < k$ vertices of I. We find a contradiction.

Subcase **e.** Suppose to the contrary that $\mathrm{BBC}_\lambda(G,M) \le \frac{k+1}{2}+\lambda$ for the case $k=2m+1$, $m\ge 4$ and $\lambda \ge \frac{k+1}{2}+1$. It is clear that colors from the set $\{\frac{k+1}{2}+1,\ldots,\lambda\}$ can not be used at all. So, we can only use the $k+1$ colors from the two sets $\{1,\ldots,\frac{k+1}{2}\}$ and $\{\lambda+1,\ldots,\frac{k+1}{2}+\lambda\}$. Hence, we have one independent color. Without loss of generality, we may assume that this independent color is in $\{\lambda+1,\ldots,\frac{k+1}{2}+\lambda\}$. By Observation 4.3, we can use the independent color at most p times in I, where $p \le \frac{k+1}{2}$. By Observation 4.4, we can use the other colors from $\{\lambda+1,\ldots,\frac{k+1}{2}+\lambda\}$ on at most $\frac{k+1}{2}-p$ vertices of I. Since $\frac{k+1}{2} < \lambda$, by Observation 4.5, we can use colors from $\{1,\ldots,\frac{k+1}{2}\}$ for at most $\lceil\frac{k}{3}\rceil$ vertices of I. So we can only color up to $\frac{k+1}{2}+\lceil\frac{k}{3}\rceil$ vertices of I. Since for $k\ge 9$, it holds that $\frac{k+1}{2}+\lceil\frac{k}{3}\rceil < \frac{k+1}{2}+\frac{k}{3}+1 = \frac{5k+9}{6} \le k$, we find a contradiction.

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