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Note

A note on equivalence of consistency and bilateral consistency through converse consistency

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Abstract

In the framework of (set-valued or single-valued) solutions for coalitional games with transferable utility, the three notions of consistency, bilateral consistency, and converse consistency are frequently used to provide axiomatic characterizations of a particular solution (like the core, prekernel, prenucleolus, Shapley value). Our main equivalence theorem claims that a solution satisfies consistency (with respect to an arbitrary reduced game) if and only if the solution satisfies both bilateral consistency and converse consistency (with respect to the same reduced game). The equivalence theorem presumes transitivity of the reduced game technique as well as difference independence on payoff vectors for two-person reduced games.

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1. Definitions and conventions

Since the present paper deals with a continuation of Chang and Hu's recent result about the converse consistency property for transferable utility games, we shall follow the mathematical notation of their paper (Chang and Hu, 2007) to a large extent.

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Let $\mathbb{N}$ be the set of potential *players*. A *coalition* is a nonempty finite subset of $\mathbb{N}$ and let $\mathcal{N} := \{N \mid N \subseteq \mathbb{N}\}$ denote the set of all coalitions of $\mathbb{N}$. A *coalitional game with transferable utility* (a so-called TU-game) is a pair $\langle N, v\rangle$, where the player set $N \in \mathcal{N}$ is a coalition with at least two players and $v : 2^N \to \mathbb{R}$ is the so-called *characteristic function* that assigns to each subcoalition S of N a real number $v(S)$, that is $v(S) \in \mathbb{R}$ for all $S \in \mathcal{N}$ with $S \subseteq N$. The number $v(S)$ is called the *worth* of S in the game v, and it is always assumed $v(\emptyset) := 0$. Further, let $|N|$ denote the cardinality of the coalition $N \in \mathcal{N}$. This concludes the modeling part.

For the purpose of the solution part, recall that $\mathbb{R}^N$ denotes the set of all functions from the coalition $N \in \mathcal{N}$ to the set $\mathbb{R}$ of real numbers. In fact, we will consider functions $x \in \mathbb{R}^N$ as $|N|$-dimensional vectors whose coordinates are indexed by the members of the coalition N, that is, we write x_i for $x(i)$ and $\vec{x} = (x_i)_{i\in N}$ is called a *payoff vector*.

With every TU-game $\langle N, v\rangle$, there is associated the set $X^*(N, v)$ of *feasible payoff vectors* and the set $X(N, v)$ of *Pareto optimal feasible payoff vectors* respectively by

$$X^*(N,v) = \left\{\vec{x} \in \mathbb{R}^N \;\Big|\; \sum_{k\in N} x_k \leqslant v(N)\right\} \quad \text{and} \quad X(N,v) = \left\{\vec{x} \in \mathbb{R}^N \;\Big|\; \sum_{k\in N} x_k = v(N)\right\}.$$

A *solution* on the class $\mathcal{G}$ of all TU-games with at least two players is a function σ on $\mathcal{G}$ which associates with each TU-game $\langle N, v\rangle$ a subset $\sigma(N, v) \subseteq X^*(N, v)$ of feasible payoff vectors. Throughout this paper we do not deal with set-valued solutions because the forthcoming fundamental Theorem 2.4 concerns single-valued solutions. Single-valued solutions are called *values* and their corresponding solution set $\sigma(N, v)$ consists of a singleton denoted by $(\sigma_i(N, v))_{i\in N} \in \mathbb{R}^N$.

With any TU-game $\langle N, v\rangle$, any subcoalition $N' \subseteq N$, and any payoff vector $\vec{x} = (x_k)_{k\in N} \in \mathbb{R}^N$, there is associated a *reduced game* $\langle N', r^{\vec{x}}_{N'}(v)\rangle$ of any type, provided the worth for the empty set and its grand coalition are determined by $(r^{\vec{x}}_{N'}(v))(\emptyset) = 0$ and $(r^{\vec{x}}_{N'}(v))(N') = v(N) - \sum_{k\in N\setminus N'} x_k$.

In this framework, non-members of the subcoalition N' are supposed to leave the initial game and it is assumed that all members of the initial player set N agree that the non-members of N' are paid according to the payoff vector $\vec{x}$. The members of N' form the player set of the reduced game and in order to describe the worth of a non-trivial coalition $T \subseteq N'$ in the reduced game, the members of T may consider various options to cooperate with possible subsets consisting of non-members of N' (subject to the foregoing agreement about payments). Well-known examples of this type of a reduced game are Davis and Maschler's maximum reduced game and Moulin's complement reduced game (Chang and Hu, 2007).

Definition 1.1. A value σ on $\mathcal{G}$ is said to possess the *consistency property with respect to a specific type of reduced game* $\langle N', r^{\vec{x}}_{N'}(v)\rangle$ if for all TU-games $\langle N, v\rangle$ with $|N| \geqslant 3$, any subcoalition $N' \subseteq N$, the following holds:

$$\sigma_k\big(N', r^{\vec{y}}_{N'}(v)\big) = \sigma_k(N, v) \quad \text{for all } k \in N', \text{ where } \vec{y} := \sigma(N, v). \tag{1.1}$$

The consistency property for values states that the solution of the initial TU-game is reconfirmed as the solution of the reduced game (subject to the foregoing agreement that payments to non-members of the reduced game are settled according to the solution of the initial game). Sobolev (1975) used the maximum reduced game with reference to values in order to axiomatize the prenucleolus (on the set of all TU-games) by means of single-valuedness, covariance, anonymity, and the consistency property with respect to the maximum reduced game. Sobolev's complicated proof can be found in Peleg and Sudhölter's introductory book (Peleg and Sudhölter, 2003) about cooperative game theory (cf. Theorem 6.3.1 and its proof on pp. 149–153). The set-valued extension of the consistency property (with respect to the maximum reduced game) is due to Peleg (1986) who axiomatized the set-valued solutions of the core and prekernel.

A value is said to possess the *bilateral consistency property* if the consistency property (1.1) applies in the framework of *all pairs of players*, i.e., the consistency property (1.1) holds for all two-person subcoalitions $N' \subseteq N$. Clearly, consistency implies bilateral consistency, but the converse is not true in general.

For future purposes, recall the common knowledge that, for values satisfying Pareto optimality for two-person games, (bilateral) consistency implies Pareto optimality for all games. Indeed, let $\langle N, v\rangle$ be a TU-game with $|N| \geqslant 3$ and consider any two-person subcoalition $N' = \{i, j\}$. Put $\vec{y} = \sigma(N, v)$. From the (bilateral) consistency property (1.1) of σ and the Pareto optimality of σ applied to the two-person reduced game $\langle N', r^{\vec{y}}_{N'}(v)\rangle$, it follows

$$y_i + y_j = \sigma_i\big(N', r^{\vec{y}}_{N'}(v)\big) + \sigma_j\big(N', r^{\vec{y}}_{N'}(v)\big) = \big(r^{\vec{y}}_{N'}(v)\big)(N') = v(N) - \sum_{k\in N\setminus N'} y_k.$$

Thus, $\vec{y} = \sigma(N, v) \in X(N, v)$ for all games $\langle N, v\rangle$ whenever the (bilaterally) consistent value σ verifies Pareto optimality for two-person games. It remains to add one more fundamental property due to Peleg (1986).

Definition 1.2. A value σ on $\mathcal{G}$ is said to possess the *converse consistency property with respect to a specific type of reduced game* $\langle N', r^{\vec{x}}_{N'}(v)\rangle$ if for all TU-games $\langle N, v\rangle$ with $|N| \geqslant 3$, all Pareto optimal feasible payoff vectors $\vec{x} \in X(N, v)$, the following holds:

$$\text{If } \vec{x}_{N'} = \sigma\big(N', r^{\vec{x}}_{N'}(v)\big) \text{ for all two-person subcoalitions } N' \subseteq N,$$
$$\text{then } \vec{x} = \sigma(N, v). \tag{1.2}$$

2. Equivalence theorem

This section is devoted to interrelationships between the consistency, bilateral consistency, and converse consistency properties for a value. Obviously, consistency implies bilateral consistency. Roughly speaking, we claim that consistency and bilateral consistency are equivalent through converse consistency. In other words, we claim that, under certain circumstances, consistency holds if and only if both bilateral consistency and converse consistency hold.

The "if" statement is due to a recent result for set-valued solutions by Chang and Hu (2007). The validity of their conclusion is strongly based on a natural assumption, called *transitivity*, of the underlying reduced game technique. The reduced game $\langle N', r^{\vec{x}}_{N'}(v)\rangle$ is said to behave *transitive* if the repeated use of the reduced game does not depend on the consecutive order of removal of players, i.e., $r^{\vec{x}_{N'}}_{N''}(r^{\vec{x}}_{N'}(v)) = r^{\vec{x}}_{N''}(v)$ for all TU-games $\langle N, v\rangle$, all subcoalitions $\emptyset \neq N'' \subseteq N' \subseteq N$ and all payoff vectors $\vec{x} \in \mathbb{R}^N$.

Theorem 2.1. *(Cf. "Equivalence" Lemma 18 in Chang and Hu, 2007.) Let a value σ on $\mathcal{G}$ satisfy Pareto optimality for two-person games. If σ satisfies both bilateral consistency and converse consistency with respect to a transitive reduced game technique, then σ satisfies consistency* (*with respect to the same reduced game*) *as well.*

The main goal of this paper is to establish the converse of Chang and Hu's recent result. We claim that consistency implies, besides bilateral consistency, converse consistency as well. As a counterpart of Chang and Hu's requirement for reduced games to behave transitive, we introduce a new requirement for two-person reduced games as follows.

Definition 2.2. The *two-person reduced game* $\langle N', r^{\vec{x}}_{N'}(v)\rangle$ is said to fulfill the *difference independence on payoff vectors* if the difference of the worth of the singletons in the two-person reduced game does not depend on the payoff vector $\vec{x} \in \mathbb{R}^N$, i.e., for all $i \in N$, $j \in N$, $i \neq j$,

$$\text{the difference} \quad \big(r^{\vec{x}}_{ij}(v)\big)\big(\{i\}\big) - \big(r^{\vec{x}}_{ij}(v)\big)\big(\{j\}\big)$$
$$\text{is the same for all payoff vectors } \vec{x} \in \mathbb{R}^N. \tag{2.1}$$

As a minor requirement, the value is supposed to behave standard for two-person games. The common notion of 1-standardness for two-person games is extended to the general notion of λ-standardness for two-person games. In words, given that each of the two players receives the same fraction (denoted by $\lambda \in \mathbb{R}$) of the individual worth, the surplus is divided equally among them.

Definition 2.3. Let $\lambda \in \mathbb{R}$. A value σ on $\mathcal{G}$ is said to satisfy *λ-standardness for two-person games* if the following holds for any two-person game $\langle\{i,j\}, v\rangle$ $(i \neq j)$:

$$\sigma_k\big(\{i,j\}, v\big) = \lambda \cdot v\big(\{k\}\big) + \frac{1}{2} \cdot \big[v\big(\{i,j\}\big) - \lambda \cdot v\big(\{i\}\big) - \lambda \cdot v\big(\{j\}\big)\big] \quad \text{for } k \in \{i,j\}.$$

Theorem 2.4. *Let $\lambda \in \mathbb{R}$. Suppose a value σ on $\mathcal{G}$ satisfies both λ-standardness for two-person games and consistency with respect to a specific type of reduced game $\langle N', r^{\vec{x}}_{N'}(v)\rangle$ such that the two-person reduced game fulfills the difference independence* (2.1) *on payoff vectors. Then σ satisfies converse consistency* (*with respect to the same reduced game*) *as well.*

Proof. Let $\langle N, v\rangle$ be a TU-game with $|N| \geqslant 3$. In order to establish the converse consistency of σ, suppose $\vec{x} \in X(N,v)$ satisfies $\vec{x}_{N'} = \sigma(N', r^{\vec{x}}_{N'}(v))$ for all two-person subcoalitions $N' \subseteq N$. We aim to prove $\vec{x} = \sigma(N,v)$.

Fix the two-person subcoalition N' by writing $N' = \{i,j\}$. To shorten the proof, put $\vec{z} = \vec{x}$ or $\vec{z} = \sigma(N,v)$. Our first claim is that $\vec{z} \in X(N,v)$ due to the assumption $\vec{x} \in X(N,v)$ or due to Pareto optimality of $\sigma(N,v)$ resulting from the consistency property of σ together with Pareto optimality of σ for two-person games. Our second claim is that $z_i = \sigma_i(\{i,j\}, r^{\vec{z}}_{ij}(v))$ due to the basic assumption on $\vec{x}$ or due to the consistency property of σ. Recall that the worth of the grand coalition of a reduced game is determined by $(r^{\vec{z}}_{ij}(v))(\{i,j\}) = v(N) - \sum_{k \in N\setminus\{i,j\}} z_k = z_i + z_j$ where the last equality holds because of the first claim $\vec{z} \in X(N,v)$. From this, jointly with the λ-standardness of σ applied to the two-person (reduced) game $\langle\{i,j\}, r^{\vec{z}}_{ij}(v)\rangle$, we deduce the following:

$$\begin{aligned} z_i = \sigma_i\big(\{i,j\}, r^{\vec{z}}_{ij}(v)\big) &= \lambda \cdot \alpha^v_i(\vec{z}) + \frac{1}{2} \cdot \big[\big(r^{\vec{z}}_{ij}(v)\big)\big(\{i,j\}\big) - \lambda \cdot \alpha^v_i(\vec{z}) - \lambda \cdot \alpha^v_j(\vec{z})\big] \\ &= \lambda \cdot \alpha^v_i(\vec{z}) + \frac{1}{2} \cdot \big[z_i + z_j - \lambda \cdot \alpha^v_i(\vec{z}) - \lambda \cdot \alpha^v_j(\vec{z})\big] \end{aligned} \tag{2.2}$$

where $\alpha^v_k(\vec{z}) := (r^{\vec{z}}_{ij}(v))(\{k\})$ for $k \in \{i,j\}$. Observe that (2.2) reduces to

$$z_i - \lambda \cdot \alpha^v_i(\vec{z}) = z_j - \lambda \cdot \alpha^v_j(\vec{z}), \quad \text{that is,} \quad z_i - z_j = \lambda \cdot \big[\alpha^v_i(\vec{z}) - \alpha^v_j(\vec{z})\big]. \tag{2.3}$$

From (2.3), together with the difference independence (2.1) of the two-person reduced game with respect to payoff vectors, we conclude immediately that $x_i - x_j = \sigma_i(N, v) - \sigma_j(N, v)$, or equivalently, $x_i - \sigma_i(N, v) = x_j - \sigma_j(N, v)$ for all $i \in N$, $j \in N$, $i \neq j$. From this, together with $\sum_{k \in N} x_k = v(N) = \sum_{k \in N} \sigma_k(N, v)$, it follows that $\vec{x} = \sigma(N, v)$, as was to be shown. □

Corollary 2.5 *(Equivalence theorem). Let a value σ on $\mathcal{G}$ satisfy λ-standardness for two-person games* (*for some $\lambda \in \mathbb{R}$*)*. With respect to any transitive reduced game technique that is difference independent on payoff vectors for two-person reduced games, the value σ satisfies consistency if and only if σ satisfies both bilateral consistency and converse consistency. In other words, under these two special circumstances, consistency and bilateral consistency are equivalent through converse consistency.*

Without going into details, we remark that Yanovskaya and Driessen's linear reduced game (Yanovskaya and Driessen, 2001) (inclusive of Moulin's complement reduced game as well as Sobolev's reduced game) satisfies difference independence on payoff vectors and moreover, the linear reduced game behaves transitive under certain circumstances (called path-independence (Yanovskaya and Driessen, 2001)). Therefore, the unique value σ on $\mathcal{G}$ that is λ-standard for two-person games (for some $\lambda \in \mathbb{R}$) and satisfies consistency with respect to the linear reduced game, can also be axiomatized through bilateral consistency or converse consistency instead of consistency. Its proof is based on Thomson's "Elevator Lemma" (cf. Lemma 16 in Chang and Hu, 2007). In general, Davis and Maschler's maximum reduced game fails to meet difference independence on payoff vectors.

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References

Chang, C., Hu, C.-C., 2007. Reduced games and converse consistency. Games Econ. Behav. 59, 260–278.

Peleg, B., 1986. On the reduced game property and its converse. Int. J. Game Theory 15, 187–200.

Peleg, B., Sudhölter, P., 2003. Introduction to the Theory of Cooperative Games. Kluwer Academic, Dordrecht, The Netherlands.

Sobolev, A.I., 1975. The characterization of optimality principles in cooperative games by functional equations. In: Vorob'ev, N.N. (Ed.), Mathematical Methods in the Social Sciences, vol. 6, Vilnius, pp. 94–151 (in Russian).

Yanovskaya, E., Driessen, T.S.H., 2001. On linear consistency of anonymous values for TU-games. Int. J. Game Theory 30, 601–609.