Prolongation structure of the Landau–Lifshitz equation

G. H. M. Roelofs and R. Martini
University of Twente, Department of Applied Mathematics, P.O. Box 217, 7500 AE Enschede, The Netherlands

(Received 24 June 1992; accepted for publication 4 August 1992)

The prolongation method of Wahlquist and Estabrook is applied to the Landau–Lifshitz equation. The resulting prolongation algebra is shown to be isomorphic to a subalgebra of the tensor product of the Lie algebra so(3) with the elliptic curve $v_\alpha^2 - v_\beta^2 = j_\beta - j_\alpha$ $(\alpha,\beta=1,2,3)$, which is essentially a subalgebra of the Lie algebra applied by Date *et al.* in a different context. Taking a matrix representation of so(3) gives rise to a Lax pair of the Landau–Lifshitz equation in agreement with the results found by Sklyanin. A system of related equations is deduced which can be used for the computation of auto-Bäcklund transformations of the Landau–Lifshitz equation.

I. INTRODUCTION

In this paper we shall apply the prolongation method of Wahlquist and Estabrook to the Landau–Lifshitz equation

$$\mathbf{S}_t = \mathbf{S} \times \mathbf{S}_{xx} + \mathbf{S} \times J\mathbf{S}, \quad |\mathbf{S}| = 1 \tag{1}$$

with $\mathbf{S} = (S_1, S_2, S_3)$ and $J = \mathrm{diag}(j_1, j_2, j_3)$. $\mathbf{S}(x,t)$ is called the magnetization vector and the matrix J characterizes the interaction anisotropy. Throughout this paper we assume $j_1 \neq j_2$, $j_2 \neq j_3$, and $j_3 \neq j_1$, which corresponds to full anisotropy.

Many authors studied the isotropic case $(J=0)$, cf. for instance, Lee *et al.*,[1] or special anisotropies ($j_1 = j_2$ or $j_2 = j_3$). However, Date *et al.*[2] and Mikhailov[3] also paid attention to the case of full anisotropy. Date *et al.* described Eq. (1) in terms of a free fermion on an elliptic curve and applied an infinite-dimensional Lie algebra of quadratic forms, which acts on solutions as infinitesimal Bäcklund transformations.

Sklyanin found a Lax pair of the fully anisotropic Landau–Lifshitz equation, in which the elliptic curve

$$v_\alpha^2 - v_\beta^2 = j_\beta - j_\alpha \quad (\alpha,\beta = 1,2,3) \tag{2}$$

played an important role.

In this paper we show that applying the prolongation method of Wahlquist and Estabrook to Eq. (1) gives rise to a prolongation algebra in which the elliptic curve (2) again plays an essential role. In fact, we prove that the prolongation algebra is isomorphic to a subalgebra of $\mathrm{so}(3) \otimes A$, where A is the quotient algebra of $\mathbb{C}[v_1, v_2, v_3]$ with respect to the ideal generated by relations (2). Essentially this is a subalgebra of the algebra found by Date *et al.*

Realizing this algebra by means of an appropriate matrix representation leads to a Lax pair which agrees with the results found by Sklyanin. If we take a vector field representation of so(3) instead, we obtain a system of related equations which can be used for the computation of auto-Bäcklund transformations for the Landau–Lifshitz equation.

Moreover, the results obtained in this paper, especially the explicit structure of the prolongation algebra, may clarify the relationship with the algebraic approach to the Landau–Lifshitz equation by Date *et al.*[2]

Related work on Heisenberg models has been performed recently by De *et al.*[4] and Zhang *et al.*[5]

0022-2488/93/062394-06$06.00

II. PROLONGATION OF THE LANDAU–LIFSHITZ EQUATION

If $\mathscr{Y}=\{\mathbf{S}_t-\mathbf{S}\times\mathbf{S}_2-\mathbf{S}\times J\mathbf{S}=0,|\mathbf{S}|-1=0\}\subset J^2(x,t;\mathbf{S})$ is the Landau–Lifshitz equation and $\mathscr{Y}_\infty\subset J^\infty(x,t;\mathbf{S})$ its infinite prolongation in the infinite jet bundle, we can consider on $\mathscr{Y}_\infty$ the set of total differential operators D_x and D_t given by

$$D_x=\partial_x+\sum_{i=0}^{\infty}\mathbf{S}_{i+1}\cdot\partial_{\mathbf{S}_i},\quad D_t=\partial_t+\sum_{i=0}^{\infty}\mathbf{S}_{i,t}\cdot\partial_{\mathbf{S}_i}$$

with $\mathbf{S}_{i,t}=D_x^i\mathbf{S}_t$ and $\partial_{\mathbf{S}_i}=(\partial_{S_{1,i}},\partial_{S_{2,i}},\partial_{S_{3,i}})$. Obviously the integrability condition $[D_x,D_t]=0$ is satisfied.

In this setting prolongation can be described as follows (cf. Ref. 6): find a covering $(\widetilde{\mathscr{Y}}_\infty,\widetilde{D}_x,\widetilde{D}_t)$ such that there exists a projection $\tau:\widetilde{\mathscr{Y}}_\infty\to\mathscr{Y}_\infty$ with $\tau^*(\widetilde{D}_x)=D_x$, $\tau^*(\widetilde{D}_t)=D_t$ and $\widetilde{D}_x$ and $\widetilde{D}_t$ satisfy the integrability condition $[\widetilde{D}_x,\widetilde{D}_t]=0$.

If we consider, in local coordinates, coverings of the form $\widetilde{\mathscr{Y}}_\infty=\mathscr{Y}_\infty\times W$, where W is some finite dimensional manifold, the total differential operators $\widetilde{D}_x$ and $\widetilde{D}_t$ take the form $\widetilde{D}_x=D_x+X$, $\widetilde{D}_t=D_t+T$, where X and T are differential operators on W with coefficients in $\widetilde{\mathscr{Y}}_\infty$. If W has local coordinates $(w_1,...,w_n)$, X and T can be written as

$$\begin{aligned}X&=\sum_{i=1}^{n}w_{i,x}\partial_{w_i},\quad \widetilde{D}_x(w_i)=w_{i,x},\\ T&=\sum_{i=1}^{n}w_{i,t}\partial_{w_i},\quad \widetilde{D}_t(w_i)=w_{i,t}\end{aligned}\tag{3}$$

with $w_{i,x}$ and $w_{i,t}$ differential functions on $\widetilde{\mathscr{Y}}_\infty$.

For the actual computation of the prolongation structure of the Landau–Lifshitz equation we temporarily use spherical coordinates $\mathbf{S}=(\cos v\sin u,\sin v\sin u,\cos u)$, in which case Eq. (1) can be rewritten to

$$\begin{aligned}u_t&=-\sin u\, v_{xx}-2\cos u\, u_xv_x+(j_1-j_2)\sin u\cos v\sin v,\\ v_t&=u_{xx}/\sin u-\cos u\, v_x^2+\cos u(j_1\cos^2 v+j_2\sin^2 v-j_3).\end{aligned}\tag{4}$$

Wahlquist and Estabrook prolongation is equivalent to taking $X=X(u,v,u_x,v_x)$ and $T=T(u,v,u_x,v_x)$. In this case the integrability condition

$$[\widetilde{D}_x,\widetilde{D}_t]=D_x(T)-D_t(X)+[X,T]=0$$

gives rise to an overdetermined system of differential equations which can be solved to give

$$\begin{aligned}X&=\mathbf{P}\cdot\mathbf{S}+p_4,\\ T&=(\mathbf{P}\times\mathbf{S})\cdot\mathbf{S}_x+(\mathbf{P}\times\mathbf{P})\cdot\mathbf{S}+p_5,\end{aligned}\tag{5}$$

where $\mathbf{P}=(p_1,p_2,p_3)$ and $\mathbf{P}\times\mathbf{P}=([p_2,p_3],[p_3,p_1],[p_1,p_2])$. In addition $p_1,...,p_5$ have to satisfy the relations

$$[p_i,p_4]=[p_i,p_5]=0\quad (i=1,...,5)\tag{6}$$

and

$$[p_1,[p_2,p_3]]=[p_2,[p_3,p_1]]=[p_3,[p_1,p_2]]=0,$$
$$[p_2,[p_2,p_3]]-[p_1,[p_1,p_3]]+(j_1-j_2)p_3=0,$$
$$[p_3,[p_3,p_1]]-[p_2,[p_2,p_1]]+(j_2-j_3)p_1=0, \tag{7}$$
$$[p_1,[p_1,p_2]]-[p_3,[p_3,p_2]]+(j_3-j_1)p_2=0.$$

III. IDENTIFICATION OF THE PROLONGATION ALGEBRA

Since p_4 and p_5 are elements of the center of the prolongation algebra, we can restrict ourselves to identifying the Lie algebra P defined as the algebra generated by p_1, p_2, and p_3 and subjected to relations (7). In order to identify P we first notice that P admits a $\mathbb{Z}_2^3$- grading given by

$$\deg(p_1)=(0,1,1),\quad \deg(p_2)=(1,0,1),\quad \deg(p_3)=(1,1,0) \tag{8}$$

such that relations (7) are homogeneous. Since the Lie algebra $\mathrm{so}(3)=\{x,y,z\}$ with standard relations

$$[x,y]=z,\quad [y,z]=x,\quad [z,x]=y$$

admits the same grading, with

$$\deg(x)=(0,1,1),\quad \deg(y)=(1,0,1),\quad \deg(z)=(1,1,0),$$

we propose that p_1, p_2, and p_3 be of the form

$$p_1=x\otimes v_1,\quad p_2=y\otimes v_2,\quad p_3=z\otimes v_3. \tag{9}$$

Substituting Eq. (9) into Eq. (7) we find that v_1, v_2, and v_3 satisfy the set of relations (2).

Therefore, let A be the elliptic curve defined by

$$A=\mathbb{C}[v_1,v_2,v_3]/I(v_\alpha^2-v_\beta^2+j_\alpha-j_\beta,\alpha,\beta=1,2,3),$$

and let R be the subalgebra of $\mathrm{so}(3)\otimes A$ generated by $x\otimes v_1$, $y\otimes v_2$, and $z\otimes v_3$. We shall prove that P is isomorphic to R. In order to prove this we need the following lemma.

Lemma 3.1: If we define a filtration $(R^n)_{n\in\mathbb{N}^*}$ on R by

$$R^1=\langle x\otimes v_1,y\otimes v_2,z\otimes v_3\rangle,\quad R^n=\sum_{i+j\leq n}[R^i,R^j]\quad (n>1),$$

then $\dim(R^n/R^{n-1})=3$ for every $n>1$.

Proof: From the definition of the filtration and relations (2) it is clear that R^n/R^{n-1} is generated as a linear space by elements of the form

$$x\otimes T_1^n(v_1,v_2,v_3),\quad y\otimes T_2^n(v_1,v_2,v_3),\quad z\otimes T_3^n(v_1,v_2,v_3)$$

with T_1^n, T_2^n, and T_3^n monomials in v_1, v_2, and v_3 of total degree n. Furthermore, using an induction argument and the relations in so(3) one can easily prove that the power of v_i in T_i^n has to be odd for n odd and even for n even, and the opposite for the powers of both $v_{(i+1)\bmod 3}$ and $v_{(i+2)\bmod 3}$ in T_i^n.

Using relations (2), it is easily shown that modulo R^{n-1}, for n is odd or even, we can always find a representant for which the power of v_i in T_i^n is 1 or 0 and the power of

$v_{(i+1)\bmod 3}$ is 0 or 1, respectively. Thus we see that T_i^n has only one degree of freedom and hence $\dim(R^n/R^{n-1})=3$. □

Using lemma 3.1 we can directly prove

Theorem 3.2: The Lie algebra P is isomorphic to the algebra R, the isomorphism being given by

$$p_1 \mapsto x\otimes v_1,\quad p_2 \mapsto y\otimes v_2,\quad p_3\mapsto z\otimes v_3.$$

Proof: Since R satisfies relations (7) with $p_1=x\otimes v_1$, $p_2=y\otimes v_2$, and $p_3=z\otimes v_3$ it is clear that R is a quotient algebra of P. On P we define a filtration $(P^n)_{n\in\mathbb{N}}$ by

$$P^1=\langle p_1,p_2,p_3\rangle,\quad P^n=\sum_{i+j<n}[P^i,P^j]\quad (n>1).$$

Since R^n/R^{n-1} is a quotient space of P^n/P^{n-1} we have $\dim(P^n/P^{n-1})\geqslant 3$. Due to lemma 3.1 we are done if we can prove that $\dim(P^n/P^{n-1})=3$ for all n. Therefore, we define

$$x_1=p_1,\quad y_1=p_2,\quad z_1=p_3$$

and inductively define for $n>1$

$$x_n=[y_1,z_{n-1}],\quad y_n=[z_1,x_{n-1}],\quad z_n=[x_1,y_{n-1}].$$

Notice that the grading on P defined by Eq. (8) can be extended to x_n, y_n, and z_n by

$$\deg(x_n)=(0,1,1),\quad \deg(y_n)=(1,0,1),\quad \deg(z_n)=(1,1,0)\quad (n\geqslant 1).$$

Using the Jacobi identity, we shall prove by induction that for every $n>1$

(1) $[x_i,x_j]=[y_i,y_j]=[z_i,z_j]=0$ for $i+j=n$.
(2) $[x_i,y_j]=z_n \bmod P^{n-1}$ for $i+j=n$, and similarly for cyclic permutations of x, y, and z.

Hence, $P^n/P^{n-1}=\langle x_n,y_n,z_n\rangle$, the desired result.

For $n=2$, 3 this is obvious, due to relations (7). For $n\geqslant 4$ we have

(1) $[y_j,z_k] = x_{j+k} + \Sigma_{l<j+k}c_l x_l$ for $j+k<n$ due to the grading on P and the induction hypothesis. Hence the Jacobi identity for x_i, y_j, and z_k with $i+j+k=n$ and the induction hypothesis yield

$$[x_i,x_{j+k}]+[y_j,y_{i+k}]+[z_k,z_{i+j}]=0.$$

From these equations we shall prove that

$$[x_i,x_{n-i}]=[y_i,y_{n-i}]=[z_i,z_{n-i}]=0$$

for $i=1,\ldots,[(n-1)/2]$ inductively. Namely, if we fix $j+k=n-i$, we get for $j=1,\ldots,\ n-i-1$

$$[x_i,x_{n-i}]+[y_j,y_{n-j}]+[z_{n-i-j},z_{i+j}]=0.$$

Using the antisymmetry of the bracket and the fact that we have already proved that $[x_l,x_{n-l}] =[y_l,y_{n-l}]=[z_l,z_{n-l}]=0$ for $l=1,\ldots,i-1$, the sum of these equations is

$$(n-i-1)[x_i,x_{n-i}]+[y_i,y_{n-i}]+[z_i,z_{n-i}]=0.$$

From this equation, together with its cyclic permutations, it is simple linear algebra to show that $[x_i,x_{n-i}]=[y_i,y_{n-i}]=[z_i,z_{n-i}]=0$.

(2) The Jacobi identity for x_i, x_j, and z_k with $i+j+k=n$ and the induction hypothesis give

$$[x_i,y_{j+k}]=[x_j,y_{i+k}] \bmod P^{n-1}.$$

Hence $[x_i,y_j]=z_n \bmod P^{n-1}$ for $i+j=n$. □

IV. REALIZATIONS OF so(3)

A. Lax pair of the Landau–Lifshitz equation

Once we have found the structure of the prolongation algebra of the Landau–Lifshitz equation we can easily reconstruct the Lax pair found by Sklyanin (cf. Ref. 2),

$$\frac{\partial W}{\partial x_1}=LW, \quad \frac{\partial W}{\partial x_2}=MW, \tag{10a}$$

with $x_1=x$, $x_2=-it$ and

$$L=\sum_{\alpha=1}^{3} z_\alpha S_\alpha \sigma_\alpha,$$

$$M=i\sum_{\alpha,\beta,\gamma=1}^{3} z_\alpha \sigma_\alpha S_\beta S_{\gamma x}\epsilon^{\alpha\beta\gamma}+2z_1z_2z_3\sum_{\alpha=1}^{3} z_\alpha^{-1} S_\alpha\sigma_\alpha, \tag{10b}$$

$$\sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \sigma_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \quad \sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

In Eq. (10b) $\epsilon^{\alpha\beta\gamma}$ is the complete antisymmetric tensor and the spectral parameters (z_1,z_2,z_3) satisfy

$$z_\alpha^2-z_\beta^2=\tfrac{1}{4}(j_\alpha-j_\beta) \quad (\alpha,\beta=1,2,3). \tag{10c}$$

In order to reconstruct this Lax pair from the prolongation structure we notice that a matrix representation of so(3) can be obtained by taking $x=-\frac{1}{2}i\sigma_1$, $y=-\frac{1}{2}i\sigma_2$, $z=-\frac{1}{2}i\sigma_3$. Choosing $p_4=p_5=0$ and p_1, p_2, p_3 as in Eq. (9), formula (5) exactly yields the Lax pair (10), if we take $x_1=-x$, $x_2=it$, $z_1=\frac{1}{2}iv_1$, $z_2=\frac{1}{2}iv_2$ and $z_3=\frac{1}{2}iv_3$. The spectral condition (10c) directly follows from condition (2) in this case.

B. Related systems of equations

If we consider the vector fields

$$x=w_3\partial_{w_2}-w_2\partial_{w_3}, \quad y=w_1\partial_{w_3}-w_3\partial_{w_1}, \quad z=w_2\partial_{w_1}-w_1\partial_{w_2}, \tag{11}$$

then it is obvious that x, y, and z satisfy the relations of so(3). Thus taking p_1, p_2, p_3 as in Eq. (9), $p_4=p_5=0$, combining Eqs. (5) and (3) yield the following system of related equations:

$$\mathbf{W}_x=\mathbf{W}\times V\mathbf{S},$$

$$\mathbf{W}_t=\mathbf{W}\times V(\mathbf{S}\times\mathbf{S}_x)+V(V\mathbf{W}\times\mathbf{S}),$$

where $\mathbf{W}=(w_1,w_2,w_3)$ and $V=\mathrm{diag}(v_1,v_2,v_3)$. Notice that the integral manifolds of Eq. (11) are spheres $|\mathbf{W}|=c>0$. Thus we may require that $\mathbf{W}$ satisfy the condition $|\mathbf{W}|=1$.

This system can be used to compute auto-Bäcklund transformations of the Landau–Lifshitz equation as follows. Suppose that $\mathbf{S}$ is a solution of the Landau–Lifshitz equation (1), then we are looking for a new solution $\widetilde{\mathbf{S}}=\widetilde{\mathbf{S}}(\mathbf{S},\mathbf{W})$ of Eq. (1) where $\widetilde{\mathbf{S}}_i=\widetilde{D}_x^i\widetilde{\mathbf{S}}$. Substituting $\widetilde{\mathbf{S}}$ into Eq. (1) will give rise to an overdetermined system, which may be solved to give the desired Bäcklund transformation. In this case, unfortunately, solving the overdetermined system is extremely hard, especially due to the sphere conditions $|\mathbf{S}|=|\mathbf{W}|=1$, and we do not know if it admits a nontrivial solution.

[1] Y. Q. Lee and J. F. Lu, Phys. Lett. A **135**, 117 (1989).
[2] E. Date, M. Jimbo, M. Kashiwara, and T. Miwa, J. Phys. A **16**, 221 (1983).
[3] A. V. Mikhailov, Phys. Lett. A **92**, 51 (1982).
[4] A. K. De and A. R. Chowdhury, J. Math. Phys. **28**, 319 (1987).
[5] D.-G. Zhang, G.-X. Yang, and J. Liu, J. Phys. A **24**, L219 (1991).
[6] I. S. Krasil'shchick and A. M. Vinogradov, Acta Appl. Math **15**, 161 (1989).