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Stability estimates for h-p spectral element methods for general elliptic problems on curvilinear domains

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Abstract. In this paper we show that the h-p spectral element method developed in [3,8,9] applies to elliptic problems in curvilinear polygons with mixed Neumann and Dirichlet boundary conditions provided that the Babuska–Brezzi inf–sup conditions are satisfied. We establish basic stability estimates for a non-conforming h-p spectral element method which allows for simultaneous mesh refinement and variable polynomial degree. The spectral element functions are non-conforming if the boundary conditions are Dirichlet. For problems with mixed boundary conditions they are continuous only at the vertices of the elements. We obtain a stability estimate when the spectral element functions vanish at the vertices of the elements, which is needed for parallelizing the numerical scheme. Finally, we indicate how the mesh refinement strategy and choice of polynomial degree depends on the regularity of the coefficients of the differential operator, smoothness of the sides of the polygon and the regularity of the data to obtain the maximum accuracy achievable.

Keywords. Corner singularities; geometrical mesh; mixed Neumann and Dirichlet boundary conditions; curvilinear polygons; inf–sup conditions; stability estimates; fractional Sobolev norms.

1. Introduction

In this paper we generalize all the results we have obtained in [3] and seek a numerical solution to an elliptic boundary value problem where the differential operator satisfies the Babuska–Brezzi inf–sup conditions. We solve the boundary value problem on a curvilinear polygon whose sides are piecewise analytic (smooth) and we assume the boundary conditions are of mixed Neumann and Dirichlet type as in [1,2,5].

We now briefly describe the contents of this paper. In §2 we discuss function spaces and obtain differentiability estimates for the solution in modified polar coordinates in a sectoral neighbourhood of the vertices. Here we examine two cases viz. when the coefficients of the differential operator, sides of the polygon and the data are analytic and when they have finite regularity.

In §3 we obtain a stability theorem for a non-conforming spectral element representation of the solution for problems with mixed boundary conditions. We let the spectral element functions to be polynomials of variable degree, where the degree of all these polynomials is bounded by W, and let M denote the number of elements or layers in a sectoral neighbourhood of each of the vertices in the radial direction as shown in figure 1. We then define a quadratic form $\mathcal{V}^{M,W}$ which measures the sum of squares of a weighted squared norm of the partial differential equation and fractional Sobolev norms of the boundary conditions and a term which measures the jumps in the function and its derivatives at inter-element boundaries in appropriate Sobolev norms. In each of the sectoral neighbourhoods of the corners we use modified polar coordinates and a global coordinate system in the remaining part of the domain. We prove that the sum of the squares of the H^2 norms of the spectral element functions is bounded by the quadratic form $\mathcal{V}^{M,W}$ multiplied by a factor which grows logarithmically in W for problems with Dirichlet boundary conditions. For problems with mixed boundary conditions this factor can grow as M^4, provided W is not too large, and thus the method displays algebraic instability.

We choose as our approximate solution the unique spectral element function which minimizes a functional $r^{M,W}$ closely related to the quadratic form $\mathcal{V}^{M,W}$ as defined in [3,8,9]. In case the solution is analytic, we choose M proportional to W, and show that $r^{M,W}$ decays exponentially in M. Now the error is bounded by $r^{M,W}$ multiplied by a factor which grows at most algebraically in M. Hence the order of convergence remains exponential. If the solution has finite regularity then we choose M proportional to $\ln W$ and show that $r^{M,W}$ decays algebraically in W. Now the error is bounded by $r^{M,W}$ multiplied by a factor which grows polylogarithmically in W and hence the error decays algebraically in W.

We now come to the aspect of parallelization of the numerical scheme. For problems with Dirichlet boundary conditions the spectral element functions are non-conforming and we can use the stability theorem to parallelize the scheme in an optimal manner. It should be noted that the method is assymptotically faster then the h-p finite element method. For problems with mixed boundary conditions we cannot use this stability theorem to parallelize our method since the factor in the stability estimate can grow as M^4. To get around this problem we make the spectral element functions continuous at the vertices of the elements only. We then prove a stability theorem for mixed problems when the spectral element functions vanish at the vertices of their elements. The values of the spectral element functions at the vertices of their elements constitute the set of common boundary values we have to solve for. It should be noted that the cardinality of the set of common boundary values is much smaller than for finite element methods where the functions have to be continuous along the edges of the elements. Since the cardinality of the set of common boundary values is small we can construct an accurate approximation to the Schur complement matrix from its definition. As a result the method is faster than the standard h-p finite element method [8].

2. Function spaces and differentiability estimates

Let Ω be a curvilinear polygon with vertices $A_1, A_2, \ldots, A_p$ and corresponding sides $\Gamma_1, \Gamma_2, \ldots, \Gamma_p$ where Γ_i joins the points A_{i-1} and A_i . We shall assume that the sides $\overline{\Gamma}_i$ are analytic (smooth) arcs, i.e.

$$\overline{\Gamma}_i = \{(\varphi_i(\xi),\ \psi_i(\xi))|\xi \in \overline{I} = [-1, 1].\}$$

with $\varphi_i(\xi)$ and $\psi_i(\xi)$ being analytic (smooth) functions on $\overline{I}$ and $|\varphi_i'(\xi)|^2 + |\psi_i'(\xi)|^2 \geq \alpha > 0$. By Γ_i we mean the open arc, i.e. the image of $I = (-1, 1)$. Let the angle subtended at A_j be ω_j. We shall denote the boundary $\partial\Omega$ of Ω by Γ. Further let $\Gamma = \Gamma^{[0]} \bigcup \Gamma^{[1]}$, $\Gamma^{[0]} = \bigcup_{i\in\mathcal{D}} \overline{\Gamma}_i$, $\Gamma^{[1]} = \bigcup_{i\in\mathcal{N}} \overline{\Gamma}_i$ where $\mathcal{D}$ is a subset of the set $\{i \mid i = 1, \ldots, p\}$ and $\mathcal{N} = \{i \mid i = 1, \ldots, p\} \setminus \mathcal{D}$. Let x denote the vector $x = (x_1, x_2)$.

Let $\mathfrak{L}$ be a strongly elliptic operator

$$\mathfrak{L}(u) = -\sum_{r,s=1}^{2}(a_{r,s}(x)u_{x_s})_{x_r} + \sum_{r=1}^{2} b_r(x)u_{x_r} + c(x)u, \tag{2.1}$$

where $a_{s,r}(x) = a_{r,s}(x)$, $b_r(x)$, $c_r(x)$ are analytic (smooth) functions on $\overline{\Omega}$ and for any $(\xi_1, \xi_2) \in \mathbb{R}$ and any $x \in \overline{\Omega}$,

$$\sum_{r,s=1}^{2} a_{r,s}\xi_r\xi_s \geq \mu_0(\xi_1^2 + \xi_2^2) \tag{2.2}$$

with $\mu_0 > 0$. Moreover let the bilinear form induced by the operator $\mathfrak{L}$ satisfy the inf–sup conditions.

In this paper we shall consider the boundary value problem

$$\begin{aligned} &\mathfrak{L}u = f && \text{on } \Omega, \\ &u = g^{[0]} && \text{on } \Gamma^{[0]}, \\ &\left(\frac{\partial u}{\partial N}\right)_A = g^{[1]} && \text{on } \Gamma^{[1]}, \end{aligned} \tag{2.3}$$

where $\left(\frac{\partial u}{\partial N}\right)_A$ denotes the usual conormal derivative which we shall now define. Let A denote the 2×2 matrix whose entries are given by

$$A_{r,s}(x) = a_{r,s}(x)$$

for $r, s = 1, 2$. Let $N = (N_1, N_2)$ denote the outward normal to the curve Γ_i for $i \in \mathcal{N}$. Then $\left(\frac{\partial u}{\partial N}\right)_A$ is defined as follows:

$$\left(\frac{\partial u}{\partial N}\right)_A (x) = \sum_{r,s=1}^{2} N_r a_{r,s}\frac{\partial u}{\partial x_s}. \tag{2.4}$$

We shall assume that the given data f is analytic (smooth) on $\overline{\Omega}$ and $g^{[l]}$ is analytic (smooth) on every closed arc $\overline{\Gamma}_i$ and $g^{[0]}$ is continuous on $\Gamma^{[0]}$.

We need to state our regularity estimates in terms of local variables which are defined on a geometrical mesh imposed on Ω as in §5 of [2]. We first divide Ω into subdomains. Thus we divide Ω into p subdomains $S^1, \ldots, S^p$, where S^i denotes a domain which contains the vertex A^i and no other, and on each S^i we define a geometrical mesh. Let $\mathfrak{S}^k = \{\Omega^k_{i,j}, j = 1, \ldots, J_k, i = 1, \ldots, I_{k,j}\}$ be a partition of S^k and let $\mathfrak{S} = \bigcup_{k=1}^{p} \mathfrak{S}^k$. Here $J_k = M + O(1)$ and $I_{k,j} \leq I$ for all k and j, where I is a constant. As has been stated earlier M denotes the number of elements or layers in a sectoral neighbourhood of each of the vertices in the radial direction.

We now put some restrictions on $\mathfrak{S}$. Let (r_k, θ_k) denote polar coordinates with center at A_k. Let $\tau_k = \ln r_k$. We choose ρ so that the curvilinear sector Ω^k with sides Γ_k and Γ_{k+1}, center at A_k and radius ρ satisfies

$$\Omega^k \subseteq \bigcup_{\Omega^k_{i,j} \in \mathfrak{S}_k} \overline{\Omega}^k_{i,j}.$$

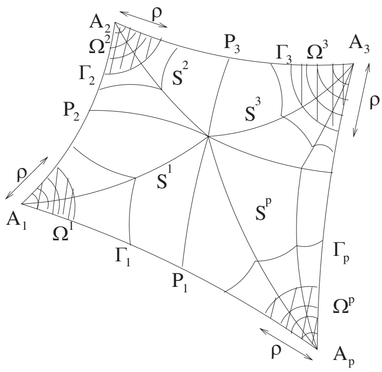

Figure 1. Geometric mesh with M layers in the radial direction in the curvilinear domain.

Ω^k may be represented as

$$\Omega^k = \{(x, y) \in \Omega : 0 < r_k < \rho\}. \tag{2.5}$$

The geometrical mesh we have imposed on Ω is as shown in figure 1.

Let $\gamma^k_{i,j,l}$, $1 \le l \le 4$ be the side of the quadrilateral $\Omega^k_{i,j} \in \mathfrak{S}$. Then we assume that

$$\gamma^k_{i,j,l} : \begin{cases} x = h^k_{i,j}\varphi^k_{i,j,l}(\xi), \\ y = h^k_{i,j}\psi^k_{i,j,l}(\xi), \end{cases} \quad 0 \le \xi \le 1, l = 1, 3 \tag{2.6a}$$

$$\gamma^k_{i,j,l} : \begin{cases} x = h^k_{i,j}\varphi^k_{i,j,l}(\eta), \\ y = h^k_{i,j}\psi^k_{i,j,l}(\eta), \end{cases} \quad 0 \le \eta \le 1, l = 2, 4 \tag{2.6b}$$

and that for some $C \ge 1$ and $L \ge 1$ independent of i, j, k and l

$$\left|\frac{\mathrm{d}^t}{\mathrm{d}s^t}\varphi^k_{i,j,l}(s)\right|, \left|\frac{\mathrm{d}^t}{\mathrm{d}s^t}\psi^k_{i,j,l}(s)\right| \le CL^t t!, t = 1, 2, \ldots . \tag{2.7}$$

We shall also examine the case when they are smooth. Some of the elements may be triangles too [7]. We shall place further restrictions on the geometric mesh we impose on Ω^k later.

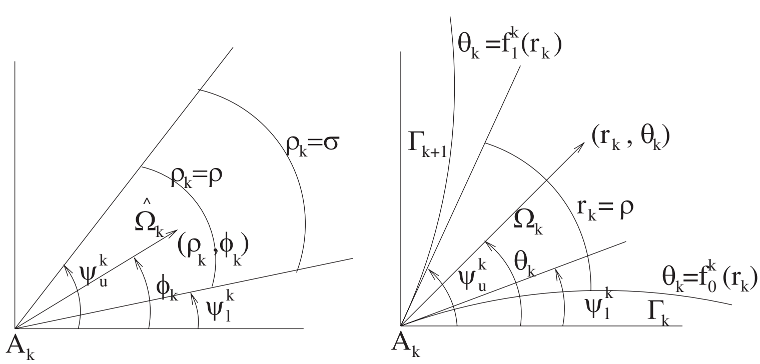

Figure 2. Curvilinear sectors.

Let (r_k, θ_k) be polar coordinates with center at A_k. Then Ω^k is the open set bounded by the curvilinear arcs Γ_k, Γ_{k+1} and a portion of the circle $r_k = \rho$. We subdivide Ω^k into curvilinear rectangles by drawing M circular arcs $r_k = \sigma_j^k = \rho\mu_k^{M+1-j}$, $j = 2, \ldots, M+1$, where $\mu_k < 1$ and $I_k - 1$ analytic curves $C_2, \ldots, C_{I_k}$ whose exact form we shall prescribe in what follows. We define $\sigma_1^k = 0$. Thus $I_{k,j} = I_k$ for $j \leq M$; in fact, we shall let $I_{k,j} = I_k$ for $j \leq M+1$. Moreover $I_{k,j} \leq I$ for all k, j where I is a fixed constant. Let

$$\Gamma_{k+j} = \{(r_k, \theta_k) | \theta_k = f_j^k(r_k), 0 < r_k < \rho\},$$

$j = 0, 1$ in a neighbourhood A_k of Ω^k. Then the mapping

$$r_k = \rho_k, \quad \theta_k = \frac{1}{(\psi_u^k - \psi_l^k)}[(\phi_k - \psi_l^k) f_1^k(\rho_k) - (\phi_k - \psi_u^k) f_0^k(\rho_k)], \quad (2.8)$$

where f_j^k is analytic in r_k for $j = 0, 1$, maps locally the cone

$$\{(\rho_k, \phi_k) : 0 < \rho_k < \sigma, \psi_l^k < \phi_k < \psi_u^k\}$$

onto a set containing Ω^k as in §3 of [2]. The functions f_j^k satisfy $f_0^k(0) = \psi_l^k$, $f_1^k(0) = \psi_u^k$ and $(f_j^k)'(0) = 0$ for $j = 0, 1$. It is easy to see that the mapping defined in (2.8) has two bounded derivatives in a neighbourhood of the origin which contains the closure of the open set

$$\widehat{\Omega}^k = \{(\rho_k, \phi_k) : 0 < \rho_k < \rho, \psi_l^k < \phi_k < \psi_u^k\}.$$

We choose the I_{k-1} curves $C_2, \ldots, C_{I_k}$ as

$$C_i : \phi_k(r_k, \theta_k) = \psi_i^k$$

for $i = 2, \ldots, I_k$. Here $\psi_l^k = \psi_1^k < \psi_2^k < \cdots < \psi_{I_k+1}^k = \psi_u^k$. Let $\Delta\psi_i^k = \psi_{i+1}^k - \psi_i^k$. Then we choose $\{\psi_i^k\}_{i,k}$ so that

$$\max_{i,k}(\Delta\psi_i^k) < \lambda(\min_{i,k}(\Delta\psi_i^k)) \quad (2.9)$$

for some constant λ. We need another set of local variables (τ_k, θ_k) in a neighbourhood of Ω^k where $\tau_k = \ln r_k$. In addition we need one final set of local variables (ν_k, ϕ_k) in the cone

$$\{(\rho_k, \phi_k) : 0 \leq \rho_k \leq \rho, \psi_l^k \leq \phi_k \leq \psi_u^k\},$$

where $\nu_k = \ln \rho_k$. Let $S_\mu^k = \{(r_k, \theta_k) : 0 \leq r_k \leq \mu\} \cap \Omega$. Then the image $\widehat{S}_\mu^k$ in (ν_k, ϕ_k) variables of S_μ^k is given by

$$\widehat{S}_\mu^k = \{(\nu_k, \phi_k) : -\infty \leq \nu_k \leq \ln \mu, \psi_l^k \leq \phi^k \leq \psi_u^k\}.$$

Now the relationship between the variables (τ_k, θ_k) and (ν_k, ϕ_k) is given by $(\tau_k, \theta_k) = M^k(\nu_k, \phi_k)$, viz.

$$\tau_k = \nu_k,$$
$$\theta_k = \frac{1}{(\psi_u^k - \psi_l^k)}[(\phi_k - \psi_l^k) f_1^k(e^{\nu_k}) - (\phi_k - \psi_u^k) f_0^k(e^{\nu_k})]. \quad (2.10)$$

Hence it is easy to see that $J_{M^k}(\nu_k, \phi_k)$, the Jacobian of the above transformation, satisfies $C_1 \leq |J_{M^k}(\nu_k, \phi_k)| \leq C_2$ for all $(\nu_k, \phi_k) \in \widehat{S}^k_\mu$, for all $0 < \mu \leq \rho$.

We should mention here that it is not necessary to choose the system of curves we have chosen to impose a geometric mesh on S^k_μ. However it is necessary to choose the curve $r_k = \rho$ as the boundary of Ω^k and no other, as will become apparent in what follows. Any other additional set of analytic curves which imposes a geometrical mesh on S^k_μ would do equally well. However the set of curves we have chosen is, in some sense, the most natural as the image $\widehat{\Omega}^k_{i,j}$ of a curvilinear rectangle $\Omega^k_{i,j}$ for $j \geq 2$ in (ν_k, ϕ_k) variables is given by a rectangle with straight lines for sides and for $j = 1$ is a semi-infinite strip with straight lines for sides.

We now state the differentiability estimates for the solution u of (2.3) which will be needed in this paper.

PROPOSITION 2.1

Consider the case when the coefficients of the differential operator are analytic on $\overline{\Omega}$ *and the sides of the curvilinear polygon are analytic. Moreover let the geometric mesh satisfy* (2.7). *Let the data* f *be analytic on* $\overline{\Omega}$ *and let* $g^{[l]}$ *be analytic on every closed arc* $\overline{\Gamma_i}^{[l]}$, *for* $l = 0, 1$, *and let* $g^{[0]}$ *be continuous on* $\Gamma^{[0]}$. *Let* $U^k_{i,j}(\nu_k, \phi_k) = u(\nu_k, \phi_k)$ *for* $(\nu_k, \phi_k) \in \widehat{\Omega}^k_{i,j}$ *for* $j \leq M$ *and* $a_k = u(A_k)$. *Now there is an analytic mapping* $M^k_{i,j} : S \to \Omega^k_{i,j}$ *for* $j > M$ *given by* $M^k_{i,j}(\xi, \eta) = (X^k_{i,j}(\xi, \eta), Y^k_{i,j}(\xi, \eta))$. *Here* S *is the unit square. Let* $U^k_{i,j}(\xi, \eta) = u(X^k_{i,j}(\xi, \eta), Y^k_{i,j}(\xi, \eta))$. *Then we can show as in* [3,8] *that*

$$\|U^k_{i,j}(\nu_k, \phi_k) - a_k\|^2_{m,\widehat{\Omega}^k_{i,j}} \leq (Cm!d^m \mu_k^{(1-\beta_k)(M-j+2)})^2 \tag{2.11a}$$

for $1 \leq j \leq M, k = 1, \ldots, p, 1 \leq i \leq I_k$ *and*

$$\|U^k_{i,j}(\xi, \eta)\|^2_{m,S} \leq (Cm!d^m)^2 \tag{2.11b}$$

for $M < j \leq J_k$, $1 \leq i \leq I_{k,j}$, $1 \leq k \leq p$. *Here* C, d *and* β_k *are constants and* $0 < \beta_k < 1$ *for* $1 \leq k \leq p$. ■

We next consider the case when the data has finite regularity. To state the differentiability results in this case we shall need to use the space $H^{k,l}_\beta(\Omega)$ with $k \geq l$ defined in [1]. We now cite Remark 3 after Theorem 2.1 of [1]. Let $\overline{\Gamma_j} \in C^{m+2}(\bar{I})$ for $j = 1, \ldots, p$ and let the coefficients of the differential operator $\in C^m(\overline{\Omega})$. Let $g^{[0]} \in H^{m+\frac{3}{2},\frac{3}{2}}_\beta(\Gamma^{[0]})$, $g^{[1]} \in H^{m+\frac{1}{2},\frac{1}{2}}_\beta(\Gamma^{[1]})$ and $f \in H^{m,0}_\beta(\Omega)$. Then there exists a constant K_m such that

$$\|u\|_{H^{m+2,2}_\beta(\Omega)} \leq K_m \left(\|f\|_{H^{m,0}_\beta(\Omega)} + \sum_{j=0}^{1} \|g^{[j]}\|_{H^{m+\frac{3}{2}-j,\frac{3}{2}-j}_\beta(\Gamma^{[j]})} \right). \tag{2.12}$$

PROPOSITION 2.2

Consider the case when the differential operator and data satisfy the conditions stated above. We assume moreover that the curves $\phi^k_{i,j,l}$ *and* $\psi^k_{i,j,l}$ *defined in* (2.6a), (2.6b) *satisfy*

$$\|\phi^k_{i,j,l}\|_{m+2,\infty,\bar{I}}, \|\psi^k_{i,j,l}\|_{m+2,\infty,\bar{I}} \leq E_{m+2}$$

where E_{m+2} is a constant independent of i, j, k and l. Let $U^k_{i,j}(\nu_k, \phi_k) = u(\nu_k, \phi_k)$ for $(\nu_k, \phi_k) \in \hat{\Omega}^k_{i,j}$ for $j \leq M$ and $a_k = u(A_k)$. Now there is a smooth mapping $M^k_{i,j} : S \to \Omega^k_{i,j}$ for $j > M$ given by $M^k_{i,j}(\xi, \eta) = (X^k_{i,j}(\xi, \eta), Y^k_{i,j}(\xi, \eta))$. Here S is the unit square. Let $U^k_{i,j}(\xi, \eta) = U(X^k_{i,j}(\xi, \eta), Y^k_{i,j}(\xi, \eta))$. Then using (2.12) *we can show that*

$$\|U^k_{i,j}(\nu_k, \phi_k) - a_k\|^2_{m+2,\hat{\Omega}^k_{i,j}} \leq K_{m+2}(\mu_k^{(1-\beta_k)(M-j+2)})^2 \tag{2.13a}$$

for $1 \leq j \leq M, k = 1, \ldots, p, 1 \leq i \leq I_k$ *and*

$$\|U^k_{i,j}(\xi, \eta)\|^2_{m+2,S} \leq K_{m+2} \tag{2.13b}$$

for $M < j \leq J_k, 1 \leq i \leq I_{k,j}, 1 \leq k \leq p$. *Here K_{m+2} denotes a constant.* ■

3. Stability estimates

3.1 *Preliminaries*

Let

$$\mathfrak{L}u = -\sum_{r,s=1}^{2}(a_{r,s}(x)u_{x_s})_{x_r} + \sum_{r=1}^{2} b_r(x)u_{x_r} + c(x)u \tag{3.1}$$

be a strongly elliptic operator which satisfies the inf–sup conditions. Hence there exists a positive constant $\mu_0 > 0$ such that

$$\sum_{r,s=1}^{2} a_{r,s}(x)\xi_r\xi_s \geq \mu_0(\xi_1^2 + \xi_2^2),$$

for all $x \in \overline{\Omega}$.

Let $H = H^1_0(\Omega)$ where $w \in H^1_0(\Omega)$ if $w \in H^1(\Omega)$ and $\text{trace}(w)|_{\Gamma^{[0]}} = 0$. Consider the bilinear form $B(u, v)$ defined on $H \times H$ as follows:

$$B(u, v) = \int_\Omega \left(\sum_{r,s=1}^{2} a_{r,s}(x)u_{x_s}v_{x_r} + \sum_{r=1}^{2} b_r(x)u_{x_r}v + cuv\right) \mathrm{d}x. \tag{3.2}$$

Then $B(u, v)$ is a continuous mapping from $H \times H \to \mathbb{R}$ and there exists a constant C_1 such that

$$|B(u, v)| \leq C_1\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)} \tag{3.3}$$

for all $u, v \in H^1_0(\Omega)$. Moreover we assume that the inf–sup conditions [7]

$$\inf_{0\neq u\in H} \sup_{0\neq v\in H} \frac{B(u, v)}{\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}} \geq C_2 > 0, \tag{3.4a}$$

and

$$\sup_{u\in H} B(u, v) > 0 \quad \text{for every } 0 \neq v \in H \tag{3.4b}$$

hold. Then for every continuous linear functional $F(v)$ defined on $H_0^1(\Omega)$ there exists unique $u_0 \in H_0^1(\Omega)$ such that $B(u_0, v) = F(v)$ for all $v \in H_0^1(\Omega)$. Moreover, the *a priori* estimate

$$\|u_0\|_{H_0^1(\Omega)} \le \frac{1}{C_2} \sup_{0 \ne v \in H_0^1(\Omega)} \frac{|F(v)|}{\|v\|_{H^1(\Omega)}} \tag{3.5}$$

holds.

Now consider the following mixed boundary value problem

$$\mathfrak{L}u = f \quad \text{in } \Omega, \tag{3.6a}$$

$$\overline{\gamma}_0 u = u|_{\Gamma^{[0]}} = g^{[0]}, \tag{3.6b}$$

and

$$\overline{\gamma}_1 u = \left(\frac{\partial u}{\partial N}\right)_A \Bigg|_{\Gamma^{[1]}} = g^{[1]}. \tag{3.6c}$$

Here the conormal derivative $\overline{\gamma}_1 u$ is defined as follows. Let $\Gamma_i \subseteq \Gamma^{[1]}$ and let T and N denote the unit tangent vector and unit outward normal at a point P on Γ_i which we traverse in the clockwise direction. Let $T = (T_1, T_2)^t$ and $N = (N_1, N_2)^t$. Then

$$\overline{\gamma}_1 u\big|_{\Gamma_i} = \left(\frac{\partial u}{\partial N}\right)_A \Bigg|_{\Gamma_i} = \sum_{r,s=1}^{2} N_r a_{r,s} \frac{\partial u}{\partial x_s} = N^t A \nabla_x u. \tag{3.7a}$$

In the same way we define the cotangential derivative

$$\left(\frac{\partial u}{\partial T}\right)_A \Bigg|_{\Gamma_i} = \sum_{r,s=1}^{2} T_r a_{r,s} \frac{\partial u}{\partial x_s} = T^t A \nabla_x u, \tag{3.7b}$$

and the tangential vector

$$\left(\frac{\partial u}{\partial T}\right) \Bigg|_{\Gamma_i} = T^t \nabla_x u. \tag{3.7c}$$

We now consider the spectral elements which are not contained in the sectoral neighbourhoods of the vertices Ω^k for $k = 1, \dots, p$. Now $\Omega_{i,j}^k \subseteq \Omega^k$ for $1 \le i \le I_{k,j}$ and $1 \le j \le M$. Let

$$O^{p+1} = \{\Omega_{i,j}^k, 1 \le k \le p, M < j \le J_k, 1 \le i \le I_{k,j}\}.$$

Once more $J_k = M + O(1)$. We shall relabel the elements of O^{p+1} and write

$$O^{p+1} = \{\Omega_l^{p+1}, 1 \le l \le L\}.$$

We shall now introduce some notation so that the reader may proceed directly to the stability theorem 3.2 and examine the proof later as it is quite involved.

Consider the domain Ω_l^{p+1}. Then there is a mapping M_l^{p+1} from the master square $S = (0, 1) \times (0, 1)$ to Ω_l^{p+1}. Let $J_l^{p+1}(\xi, \eta)$ denote the Jacobian of the transformation M_l^{p+1}. We let

$$u_l^{p+1}(\xi, \eta) = \sum_{j=0}^{W} \sum_{i=0}^{W} h_{i,j} \xi^i \eta^j .$$

We choose the spectral element functions $\{u_{i,j}^k(\nu_k, \phi_k)\}_{i,j,k}$ for $1 \leq i \leq I_k$, $1 \leq j \leq M$ and $1 \leq k \leq p$ to be polynomials of the form

$$u_{i,j}^k(\nu_k, \phi_k) = \sum_{s=0}^{W_j} \sum_{r=0}^{W_j} a_{r,s} \nu_k^r \phi_k^s$$

for $j \neq 1$. Here $1 \leq W_j \leq W$. If $j = 1$ we choose $u_{i,1}^k(\nu_k, \phi_k) = g_k$ where g_k is a constant for $1 \leq i \leq I_k$. Let $\pi^{M,W}$ denote the space of polynomials $\{\{u_{i,j}^k(\nu_k, \phi_k)\}_{i,j,k}, \{u_l^{p+1}(\xi, \eta)\}_l\}$.

Remark 1. We shall always choose $M = O(W)$. In case the conditions of Proposition 2.1 are satisfied so that u is analytic we choose $W = M$. Once we have obtained the numerical solution we can define a correction to it so that the corrected solution is conforming and converges to the actual solution exponentially in M in the $H^1(\Omega)$ norm [8,9]. Thus the error in the $H^1(\Omega)$ norm is bounded by $C\mathrm{e}^{-bM}$ where C and b are constants. In case $u \in H_\beta^{m+2,2}(\Omega)$ we would choose M proportional to $m \ln W$. Once more we can define a corrected version of the solution so that it is conforming and converges to the actual solution in the $H^1(\Omega)$ norm and the error is bounded by $C(\ln W)^3 W^{-m+1}$. Hence for the method to converge we must have $m \geq 2$.

The stability theorem 3.2 holds provided the coefficients of the differential operator $\in C^3(\bar{\Omega})$ and the curves $\phi_{i,j,l}^k$, $\psi_{i,j,l}^k$ defined by (2.6a), (2.6b) satisfy

$$\|\phi_{i,j,l}^k\|_{3,\infty,\bar{I}}, \|\psi_{i,j,l}^k\|_{3,\infty,\bar{I}} \leq K_3,$$

where K_3 is a constant independent of i, j, k and l. In this paper however we prove Theorem 3.2 assuming that the coefficients of the differential operator are analytic on $\overline{\Omega}$ and the curves $\phi_{i,j,l}^k$, $\psi_{i,j,l}^k$ defined in (2.6a), (2.6b) are analytic and satisfy the condition (2.7). Now

$$\int_{\Omega_l^{p+1}} \int |\mathfrak{L} u_l^{p+1}(x, y)|^2 \mathrm{d}x\mathrm{d}y = \int_S \int |\mathfrak{L}_l^{p+1} u_l^{p+1}(\xi, \eta)| \mathrm{d}\xi \mathrm{d}\eta.$$

Here

$$\mathfrak{L}_l^{p+1} u_l^{p+1}(\xi, \eta) = (\mathfrak{L} u_l^{p+1})(x, y)\sqrt{J_l^{p+1}}.$$

Now

$$\mathfrak{L}_l^{p+1} w = A_l^{p+1} w_{\xi\xi} + 2B_l^{p+1} w_{\xi\eta} + C_l^{p+1} w_{\eta\eta} + D_l^{p+1} w_\xi + E_l^{p+1} w_\eta + F_l^{p+1} w,$$

where the coefficients of the differential operator are analytic (smooth) functions of ξ and η. Let $\widehat{A}_l^{p+1}$ be the unique polynomial which is the orthogonal projection of A_l^{p+1} into the space of polynomials of degree W in ξ and η with respect to the usual inner product in $H^2(S)$. We define $\widehat{B}_l^{p+1}$, $\widehat{C}_l^{p+1}$, $\widehat{D}_l^{p+1}$, $\widehat{E}_l^{p+1}$ and $\widehat{F}_l^{p+1}$ in the same way. We then define

$$(\mathfrak{L}_l^{p+1})^a w = \widehat{A}_l^{p+1} w_{\xi\xi} + 2\widehat{B}_l^{p+1} w_{\xi\eta} + \widehat{C}_l^{p+1} w_{\eta\eta} + \widehat{D}_l^{p+1} w_{\xi} + \widehat{E}_l^{p+1} w_{\eta} + \widehat{F}_l^{p+1} w.$$

Now let γ_l be a side of the element Ω_m^{p+1} and let it be the image of the side $\xi = 0$ under the mapping M_m^{p+1}. Clearly

$$\frac{\partial u_m^{p+1}}{\partial x} = (u_m^{p+1})_\xi \xi_x + (u_m^{p+1})_\eta \eta_x.$$

We now define

$$\left(\frac{\partial u_m^{p+1}}{\partial x}\right)^a \Bigg|_{\gamma_l} = ((u_m^{p+1})_\xi \widehat{\xi}_x + (u_m^{p+1})_\eta \widehat{\eta}_x)(0, \eta).$$

Here $\widehat{\xi}_x(0,\eta)$ and $\widehat{\eta}_x(0,\eta)$ are the unique polynomials which are the orthogonal projections of $\xi_x(0,\eta)$ and $\eta_x(0,\eta)$ into the space of polynomials of degree W in ξ and η with respect to the usual inner product in $H^2(I)$. In the same way we can define $(\partial u_m^{p+1}/\partial y)^a$ on γ_l. Now let γ_l be a side common to Ω_m^{p+1} and Ω_n^{p+1} and let it be the image of $\xi = 0$ under the mapping M_m^{p+1} and the image of $\xi = 1$ under the mapping M_n^{p+1}.

Let $[w]$ denote the jump in w across γ_l, where w is a smooth function on $\overline{\Omega}_m^{p+1}$ and $\overline{\Omega}_n^{p+1}$. We now define

$$\left\|\left[\left(\frac{\partial u}{\partial x}\right)^a\right]\right\|_{1/2,\gamma_l}^2 = \left\|\left(\frac{\partial u_m^{p+1}}{\partial x}\right)^a (0,\eta) - \left(\frac{\partial u_n^{p+1}}{\partial x}\right)^a (1,\eta)\right\|_{1/2,(0,1)}^2$$

and

$$\left\|\left[\left(\frac{\partial u}{\partial y}\right)^a\right]\right\|_{1/2,\gamma_l}^2 = \left\|\left(\frac{\partial u_m^{p+1}}{\partial y}\right)^a (0,\eta) - \left(\frac{\partial u_n^{p+1}}{\partial y}\right)^a (1,\eta)\right\|_{1/2,(0,1)}^2.$$

Finally we consider a side Γ_k of the polygonal domain Ω as shown in figure 1. Let γ_l be a side of Ω_m^{p+1} such that $\gamma_l \subseteq \Gamma_k$ and such that γ_l is the image of $\xi = 0$ under the mapping M_m^{p+1} and which maps the master square S to Ω_m^{p+1}. Then we can define $(\partial u_m^{p+1}/\partial T)^a$ and $(\partial u_m^{p+1}/\partial N)_A^a$ in the same way. Finally we define

$$\left\|\left(\frac{\partial u}{\partial T}\right)^a\right\|_{1/2,\gamma_l}^2 = \left\|\left(\frac{\partial u_m^{p+1}}{\partial T}\right)^a (0,\eta)\right\|_{1/2,(0,1)}^2$$

and

$$\left\|\left(\frac{\partial u}{\partial N}\right)_A^a\right\|_{1/2,\gamma_l}^2 = \left\|\left(\frac{\partial u_m^{p+1}}{\partial N}\right)_A^a (0,\eta)\right\|_{1/2,(0,1)}^2.$$

Now consider the sectoral domain Ω_k. Let us define the differential operator

$$\widetilde{\mathfrak{L}}^k w(\tau_k, \theta_k) = \mathrm{e}^{2\tau_k} \mathfrak{L} w(x, y)$$

as in [3]. Then

$$\widetilde{\mathfrak{L}}^k w(\tau_k, \theta_k) = \alpha^k w_{\tau_k \tau_k} + 2\beta^k w_{\tau_k \theta_k} + \gamma^k w_{\theta_k \theta_k} + \delta^k w_{\tau_k} + \epsilon^k w_{\theta_k} + \mu^k w,$$

where the coefficients of $\widetilde{\mathfrak{L}}^k$ are analytic functions of their arguments. Consider the element $\Omega^{k}_{i,j}$ with $1 < j \leq M$. Now the image of $\Omega^k_{i,j}$ in (ν_k, ϕ_k) coordinates is the rectangle $\widehat{\Omega}^k_{i,j}$. Clearly

$$\int_{\Omega^k_{i,j}} \int (\widetilde{\mathfrak{L}}^k w(\tau_k, \theta_k))^2 \mathrm{d}\tau_k \mathrm{d}\theta_k = \int_{\widehat{\Omega}^k_{i,j}} \int (\mathfrak{L}^k_{i,j} w(\nu_k, \phi_k))^2 \mathrm{d}\nu_k \mathrm{d}\phi_k.$$

Here

$$\mathfrak{L}^k_{i,j} w(\nu_k, \phi_k) = \widetilde{\mathfrak{L}}^k w(\tau_k, \theta_k) \sqrt{J_{M^k}(\nu_k, \phi_k)},$$

where J_{M^k} denotes the Jacobian of the transformation M^k defined in (2.10). Once more we can define a differential operator $(\mathfrak{L}^k_{i,j})^a$ by replacing the coefficients of $\mathfrak{L}^k_{i,j}$ by polynomials of degree W in ν_k and ϕ_k which are exponentially close approximation to them.

Now the highest order terms of the differential operator $\widetilde{\mathfrak{L}}^k$ are given by $\widetilde{\mathfrak{M}}^k$, where

$$\widetilde{\mathfrak{M}}^k w = \sum_{i,j=1}^{2} \frac{\partial}{\partial y_i} \left(\widetilde{a}^k_{i,j} \frac{\partial w}{\partial y_j} \right).$$

Here $y_1 = \tau_k$ and $y_2 = \theta_k$. Let $\widetilde{A}^k$ denote the 2×2 matrix such that $\widetilde{A}^k_{i,j} = \widetilde{a}^k_{i,j}$. Let γ_l be a side of the element $\Omega^k_{i,j}$ such that $\gamma_l \subseteq \Gamma_k$, where Γ_k is a side of the polygon Ω. Let $\widetilde{\gamma}_l$ be the image of γ_l in (y_1, y_2) coordinates given by $y_1 = y_1(\sigma)$, and $y_2 = y_2(\sigma)$. Let t and n denote the unit tangent and normal vector at a point P on $\widetilde{\gamma}_l$. We now define the conormal derivative

$$\left(\frac{\partial w}{\partial n} \right)_{\widetilde{A}^k} = n^t \widetilde{A}^k \nabla_y w.$$

Now the transformation M^k defined in (2.10) maps the rectangle $\widehat{\Omega}^k_{i,j}$ to $\widetilde{\Omega}^k_{i,j}$. Once more we can define $(\partial w/\partial n)^a_{\widetilde{A}^k}|_{\widehat{\gamma}_l}$ by replacing the coefficients of the first order differential operator $(\partial w/\partial n)_{\widetilde{A}^k}$ by polynomials of degree W in ν_k which are exponentially close approximations to them. We can now define $\|(\partial w/\partial n)^a_{\widetilde{A}^k}\|^2_{1/2,\widehat{\gamma}_l}$ as we have done before.

The reader can now proceed directly to the stability theorem 3.2 stated in §3.3 and examine the proof later.

3.2 *Technical results*

Consider some $\Omega^{p+1}_l \in O^{p+1}$, as shown in figure 3. Then Ω^{p+1}_l is a curvilinear quadrilateral whose sides are analytic arcs and the boundary $\partial\Omega^{p+1}_l$ is traversed in the clockwise direction.

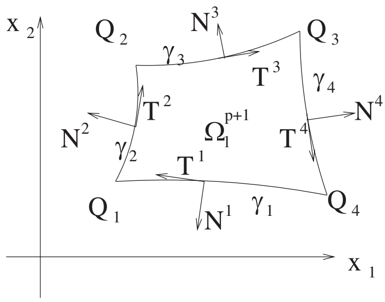

Figure 3. Element Ω_l^{p+1}.

Let γ be a smooth curve and let N and T denote the unit outward normal and tangent vectors to γ at a point P on γ. Let s be the arc length measured from a point on the curve in the clockwise direction. Then the second fundamental form is given by

$$\mathfrak{B}(\xi, \eta) = -\frac{\partial N}{\partial s} \cdot T\xi\eta = \frac{\partial T}{\partial s} \cdot N\xi\eta = \kappa\xi\eta, \tag{3.8}$$

where

$$\kappa = \pm\frac{\mathrm{d}T}{\mathrm{d}s}$$

is the curvature of γ at P. Clearly Trace$(\mathfrak{B}) = \kappa$.

Now we need to use Theorem 3.1.1.2 of [4]. Let v be a smooth vector field defined on $\overline{\Omega}_l^{p+1}$ where $v = (v_1, v_2)^t$. Consider the restriction of v to the boundary $\partial\Omega_l^{p+1}$. Now $\partial\Omega_l^{p+1} = (\bigcup_{i=1}^4 \gamma_i) \bigcup (\bigcup_{i=1}^4 Q_i)$, where γ_i are the sides of $\partial\Omega_l^{p+1}$ with end points deleted and Q_i are the vertices of Ω_l^{p+1}. We shall denote by v_T the projection of v on the tangent vector T to $\partial\Omega_l^{p+1}$ except at the vertices where this cannot be defined. Similarly by v_N we shall denote the component of v in the direction of N. Thus we have

$$v_N = v \cdot N$$

and

$$v_T = v \cdot T.$$

Lemma 3.1. *Let $u \in H^3(\Omega_l^{p+1})$. Then*

$$\frac{\mu_0^2}{2} \sum_{r,s=1}^{2} \int_{\Omega_l^{p+1}} \left|\frac{\partial^2 u}{\partial x_r \partial x_s}\right|^2 \mathrm{d}x$$

$$\leq \int_{\Omega_l^{p+1}} |\mathfrak{M}u|^2 \mathrm{d}x + \sum_{j=1}^{4} \int_{\gamma_j} |\kappa| \left(\left(\frac{\partial u}{\partial N}\right)_A^2 + \left(\frac{\partial u}{\partial T}\right)_A^2\right) \mathrm{d}s$$

$$+\frac{512R^4}{\mu_0^2}\sum_{r=1}^{2}\int_{\Omega_l^{p+1}}\left|\frac{\partial u}{\partial x_r}\right|^2 \mathrm{d}x + 2\sum_{j=1}^{4}\int_{\gamma_j}\left(\frac{\partial u}{\partial T}\right)_A \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial u}{\partial N}\right)_A \mathrm{d}s$$

$$+\sum_{j=1}^{4}\left\{\left(\frac{\partial u}{\partial N^{j+1}}\right)_A\left(\frac{\partial u}{\partial T^{j+1}}\right)_A - \left(\frac{\partial u}{\partial N^{j}}\right)_A\left(\frac{\partial u}{\partial T^{j}}\right)_A\right\}(Q_j). \tag{3.9}$$

We shall say that a bounded open subset of $\mathbb{R}^2$ with Lipschitz boundary Γ has a piecewise C^2 boundary if $\Gamma = \Gamma_0 \bigcup \Gamma_1$, where

(a) Γ_0 has zero measure (for the arc length measure ds)
(b) Γ_1 is open in Γ and each point $x \in \Gamma_1$ has a C^2 boundary as defined in 1.2.1.1 of [4].
 Then Theorem 3.1.1.2 of [4] may be stated as follows:

Let O be a bounded open subset of $\mathbb{R}^2$ with Lipschitz boundary Γ. Assume in addition that Γ is piecewise C^2. Then for all $v \in (H^2(\Omega))^2$ we have

$$\int_O |\mathrm{div}(v)|^2 \mathrm{d}x - \int_O \sum_{r,s=1}^{2}\frac{\partial v_r}{\partial x_s}\frac{\partial v_s}{\partial x_r}\mathrm{d}x$$
$$= \int_{\Gamma_1}\left\{\frac{\mathrm{d}}{\mathrm{d}s}(v_N v_T) - 2v_T\frac{\mathrm{d}}{\mathrm{d}s}v_N\right\}\mathrm{d}s - \int_{\Gamma_1}\{(\mathrm{tr}\mathfrak{B})v_N^2 + \mathfrak{B}(v_T, v_T)\}\mathrm{d}s. \tag{3.10}$$

To apply (3.10) we define the vector field

$$v = A\nabla_x u,$$

where A is the matrix

$$(A)_{r,s} = a_{r,s}.$$

We then observe that

$$\mathfrak{M}u = \sum_{r,s=1}^{2}\frac{\partial}{\partial x_r}\left(a_{r,s}\frac{\partial u}{\partial x_s}\right) = \mathrm{div}(v), \tag{3.11a}$$

$$\left(\frac{\partial u}{\partial N}\right)_A = \sum_{r,s=1}^{2} N_r a_{r,s}\frac{\partial u}{\partial x_s} = (\overline{\gamma}_0 v)\cdot N \tag{3.11b}$$

and

$$\left(\frac{\partial u}{\partial T}\right)_A = \sum_{r,s=1}^{2} T_r a_{r,s}\frac{\partial u}{\partial x_s} = (\overline{\gamma}_0 v)\cdot T. \tag{3.11c}$$

Hence (3.10) takes the form

$$
\begin{aligned}
&\int_{\Omega_l^{p+1}} |\mathfrak{M}u|^2 \mathrm{d}x - \sum_{r,s=1}^{2} \int_{\Omega_l^{p+1}} \frac{\partial v_r}{\partial x_s}\frac{\partial v_s}{\partial x_r}\mathrm{d}x \\
&= \sum_{j=1}^{4}\int_{\gamma_j} \frac{\mathrm{d}}{\mathrm{d}s}(v_N v_T)\mathrm{d}s - \sum_{j=1}^{4} 2\int_{\gamma_j} \left(\frac{\partial u}{\partial T}\right)_A \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial u}{\partial N}\right)_A \mathrm{d}s \\
&\quad - \sum_{j=1}^{4}\int_{\gamma_j} \kappa \left(\left(\frac{\partial u}{\partial N}\right)_A^2 + \left(\frac{\partial u}{\partial T}\right)_A^2\right)\mathrm{d}s. \qquad (3.12)
\end{aligned}
$$

Now by Lemma 3.1.3.4 of [4] the following inequality holds for all $u \in H^2(\Omega)$:

$$
\mu_0^2 \sum_{r,s=1}^{2} \left|\frac{\partial^2 u}{\partial x_r \partial x_s}\right|^2 \leq \sum_{r,s,k,l=1}^{2} a_{r,k} a_{s,l} \frac{\partial^2 u}{\partial x_s \partial x_k}\frac{\partial^2 u}{\partial x_r \partial x_l},
$$

a.e. in Ω. Thus it follows that

$$
\mu_0^2 \sum_{r,s=1}^{2} \left|\frac{\partial^2 u}{\partial x_r \partial x_s}\right|^2 \leq \sum_{r,s=1}^{2} \frac{\partial v_r}{\partial x_s}\frac{\partial v_s}{\partial x_r} + 2\sum_{r,s,k,l=1}^{2} \left| a_{r,k} \frac{\partial^2 u}{\partial x_s \partial x_k}\frac{\partial a_{s,l}}{\partial x_r}\frac{\partial u}{\partial x_l}\right|,
$$

a.e. in Ω. Integrating, we have

$$
\begin{aligned}
\mu_0^2 \sum_{r,s=1}^{2} \int \left|\frac{\partial^2 u}{\partial x_r \partial x_s}\right|^2 \mathrm{d}x &\leq \sum_{r,s=1}^{2}\int \frac{\partial v_r}{\partial x_s}\frac{\partial v_s}{\partial x_r}\mathrm{d}x \\
&\quad + 32R^2 \int_\Omega \sum_{r=1}^{2}\left|\frac{\partial u}{\partial x_r}\right| \sum_{r,s=1}^{2}\left|\frac{\partial^2 u}{\partial x_r \partial x_s}\right|\mathrm{d}x
\end{aligned}
$$

where R is a common bound for all the C^1 norms of all the $a_{r,s}$. Hence

$$
\frac{\mu_0^2}{2}\sum_{r,s=1}^{2}\int \left|\frac{\partial^2 u}{\partial x_r \partial x_s}\right|^2 \mathrm{d}x \leq \sum_{r,s=1}^{2}\int \frac{\partial v_r}{\partial x_s}\frac{\partial v_s}{\partial x_r}\mathrm{d}x + \frac{512R^4}{\mu_0^2}\sum_{r=1}^{2}\int \left|\frac{\partial u}{\partial x_r}\right|^2 \mathrm{d}x. \qquad (3.13)
$$

Next

$$
\begin{aligned}
\sum_{j=1}^{4}\int_{\gamma_j} \frac{\mathrm{d}}{\mathrm{d}s}(v_N v_T)\mathrm{d}s &= \sum_{j=1}^{4}\left\{-\left(\frac{\partial u}{\partial N^{j+1}}\right)_A \left(\frac{\partial u}{\partial T^{j+1}}\right)_A\right. \\
&\quad \left. + \left(\frac{\partial u}{\partial N^{j}}\right)_A \left(\frac{\partial u}{\partial T^{j}}\right)_A\right\}(Q_j). \qquad (3.14)
\end{aligned}
$$

Then combining (3.12)–(3.14) we obtain the result. ■

In a neighbourhood of the vertex A_k we move to polar coordinates. We take a curvilinear rectangle $\Omega_{i,j}^k$ which comprises part of the sectoral neighbourhood Ω^k of the vertex A_k and consider its image $\widetilde{\Omega}_{i,j}^k$ in (τ_k, θ_k) variables as shown in figure 4.

As in [3] we write the differential operator $\mathfrak{M}$ in modified polar coordinates, where

$$\mathfrak{M}u = \sum_{r,s=1}^{2} \frac{\partial}{\partial x_r}\left(a_{r,s}\frac{\partial u}{\partial x_s}\right).$$

Now

$$x_1 = x_1^k + \mathrm{e}^{\tau_k}\cos\theta_k$$

and

$$x_2 = x_2^k + \mathrm{e}^{\tau_k}\sin\theta_k.$$

Here $A_k = (x_1^k, x_2^k)$. We would like to obtain an estimate for

$$\int_{\Omega_{i,j}^k} r_k^2 |\mathfrak{M}u|^2 \mathrm{d}x = \int_{\widetilde{\Omega}_{i,j}^k} |\widetilde{\mathfrak{M}}^k u|^2 \mathrm{d}\tau_k \mathrm{d}\theta_k.$$

Let us define the new differential operator

$$\widetilde{\mathfrak{M}}^k u = \mathrm{e}^{2\tau_k} \sum_{r,s=1}^{2} \frac{\partial}{\partial x_r}\left(a_{r,s}\frac{\partial u}{\partial x_s}\right) = \sum_{r,s=1}^{2} \frac{\partial}{\partial y_r}\left(\tilde{a}_{r,s}\frac{\partial u}{\partial y_s}\right). \tag{3.15}$$

Here $y_1 = \tau_k$ and $y_2 = \theta_k$. Let O^k denote the matrix

$$O^k = \begin{bmatrix} \cos\theta_k & -\sin\theta_k \\ \sin\theta_k & \cos\theta_k \end{bmatrix} \tag{3.16a}$$

and $\widetilde{A}^k$ denote the matrix

$$\widetilde{A}^k = \begin{bmatrix} \widetilde{a}_{1,1}^k & \widetilde{a}_{1,2}^k \\ \widetilde{a}_{2,1}^k & \widetilde{a}_{2,2}^k \end{bmatrix}.$$

Then it can be easily shown that

$$\widetilde{A}^k = (O^k)^t A O^k. \tag{3.16b}$$

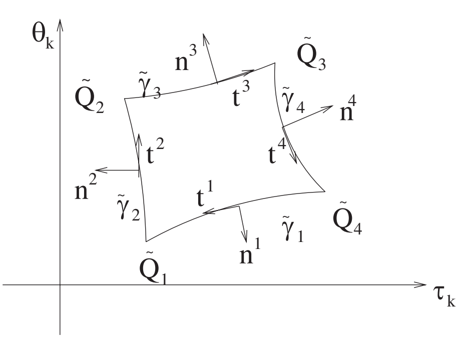

Figure 4. Element $\widetilde{\Omega}_{i,j}^k$.

Hence, since O^k is an orthogonal matrix, we have that

$$\sum_{r,s=1}^{2} \widetilde{a}^k_{r,s}\eta_r\eta_s \geq \mu_0(\eta_1^2+\eta_2^2). \tag{3.17}$$

Moreover the following relations hold:

$$(\widetilde{a}^k_{1,1})_{\theta_k} = 2\widetilde{a}^k_{1,2} + O(\mathrm{e}^{\tau_k}), \tag{3.18a}$$

$$(\widetilde{a}^k_{1,2})_{\theta_k} = \widetilde{a}^k_{2,2} - \widetilde{a}^k_{1,1} + O(\mathrm{e}^{\tau_k}), \tag{3.18b}$$

$$(\widetilde{a}^k_{2,2})_{\theta_k} = -2\widetilde{a}^k_{1,2} + O(\mathrm{e}^{\tau_k}), \tag{3.18c}$$

$$(\widetilde{a}^k_{1,1})_{\tau_k}, (\widetilde{a}^k_{1,2})_{\tau_k} \quad \text{and} \quad (\widetilde{a}^k_{2,2})_{\tau_k} = O(\mathrm{e}^{\tau_k}), \tag{3.18d}$$

as $\tau_k \to -\infty$. Next let γ be a curve given by

$$x_1 = x_1(s),$$
$$x_2 = x_2(s),$$

where s is the arc length along the curve γ. Then the curvature κ at a point P on the curve is given by

$$\kappa = \frac{\mathrm{d}x_1}{\mathrm{d}s}\frac{\mathrm{d}^2x_2}{\mathrm{d}s^2} - \frac{\mathrm{d}x_2}{\mathrm{d}s}\frac{\mathrm{d}^2x_1}{\mathrm{d}s^2}.$$

Let $\widetilde{\gamma}$ be the image of the curve in (y_1, y_2) coordinate given by

$$y_1 = y_1(\sigma),$$
$$y_2 = y_2(\sigma),$$

where σ is the arc length along the curve $\widetilde{\gamma}$. Then it is easy to verify that

$$\frac{\mathrm{d}s}{\mathrm{d}\sigma} = \mathrm{e}^{y_1}. \tag{3.19}$$

Now we can show that the curvature $\widetilde{\kappa}$ of the curve $\widetilde{\gamma}$ is given by

$$\widetilde{\kappa} = \kappa\mathrm{e}^{y_1} + \frac{\mathrm{d}y_2}{\mathrm{d}\sigma}.$$

Hence

$$|\widetilde{\kappa}| < |\kappa|\mathrm{e}^{\tau_k} + 1 \leq K, \tag{3.20}$$

where K is a uniform constant, for all the curves $\widetilde{\gamma}_s \subseteq \widetilde{\Omega}^k$.

We shall denote by t and n the unit tangent and outward normal vector at a point P on $\widetilde{\gamma}$, the boundary of $\widetilde{\Omega}^k_{i,j}$ except at its vertices where these are not defined.

Lemma 3.2. *Let* $u(y) \in H^3(\widetilde{\Omega}^k_{i,j})$. *Then*

$$\frac{\mu_0^2}{2}\sum_{r,s=1}^{2}\int_{\widetilde{\Omega}^k_{i,j}}\left|\frac{\partial^2 u}{\partial y_r \partial y_s}\right|^2 \mathrm{d}y$$

$$\leq \int_{\widetilde{\Omega}^k_{i,j}} |\widetilde{\mathfrak{M}}^k u|^2 \mathrm{d}y + 2\sum_{j=1}^{4}\int_{\widetilde{\gamma}_j}\left(\frac{\partial u}{\partial t}\right)_{\widetilde{A}^k}\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\left(\frac{\partial u}{\partial n}\right)_{\widetilde{A}^k}\right)\mathrm{d}\sigma$$

$$+\sum_{j=1}^{4}\left\{\left(\frac{\partial u}{\partial t^{j+1}}\right)_{\widetilde{A}^k}\left(\frac{\partial u}{\partial n^{j+1}}\right)_{\widetilde{A}^k} - \left(\frac{\partial u}{\partial t^{j}}\right)_{\widetilde{A}^k}\left(\frac{\partial u}{\partial n^{j}}\right)_{\widetilde{A}^k}\right\}(\widetilde{Q}_j)$$

$$+\sum_{j=1}^{4}\int_{\widetilde{\gamma}_j}|\widetilde{\kappa}|\left(\left(\frac{\partial u}{\partial t}\right)^2_{\widetilde{A}^k}+\left(\frac{\partial u}{\partial n}\right)^2_{\widetilde{A}^k}\right)\mathrm{d}\sigma + \frac{512}{\mu_0^2}R^4\sum_{r=1}^{2}\int_{\widetilde{\Omega}^k_{i,j}}\left|\frac{\partial u}{\partial y_r}\right|^2 \mathrm{d}y. \tag{3.21}$$

Now once more we use Theorem 3.1.1.2 of [4]. Clearly $\widetilde{\Omega}^k_{i,j}$ for $j \geq 2$ is a bounded open subset of $\mathbb{R}^2$ with Lipschitz boundary $\widetilde{\Gamma}$ that is a piecewise C^2. Thus $\widetilde{\Gamma} = \left(\bigcup_{i=1}^4 \widetilde{\gamma}_i\right) \bigcup \left(\bigcup_{i=1}^4 \widetilde{Q}_i\right)$ where $\widetilde{\gamma}_i$ are the sides of the open rectangle $\widetilde{\Omega}^k_{i,j}$ with the end points removed and $\widetilde{Q}_i$ are its vertices.

Now

$$\int_{\Omega^k_{i,j}} r_k^2 |\mathfrak{M}u|^2 \mathrm{d}x = \int_{\widetilde{\Omega}^k_{i,j}} \mathrm{e}^{4\tau_k}|\mathfrak{M}u|^2 \mathrm{d}\tau_k \mathrm{d}\theta_k = \int_{\widetilde{\Omega}^k_{i,j}} |\widetilde{\mathfrak{M}}^k u|^2 \mathrm{d}y.$$

Here

$$\widetilde{\mathfrak{M}}^k u = \sum_{r,s=1}^{2}\frac{\partial}{\partial y_r}\left(\widetilde{a}^k_{r,s}\frac{\partial u}{\partial y_s}\right)$$

as defined in (3.15). Then for all $w \in (H^2(\widetilde{\Omega}^k_{i,j}))^2$ we have

$$\int_{\widetilde{\Omega}^k_{i,j}} |\mathrm{div}(w)|^2 \mathrm{d}y - \sum_{r,s=1}^{2}\int_{\widetilde{\Omega}^k_{i,j}}\frac{\partial w_r}{\partial y_s}\frac{\partial w_s}{\partial y_r}\mathrm{d}y$$

$$= \sum_{j=1}^{4}\left\{\int_{\widetilde{\gamma}_j}\frac{\mathrm{d}}{\mathrm{d}\sigma}(w_n w_t) - 2w_t\frac{\mathrm{d}}{\mathrm{d}\sigma}w_n\right\}\mathrm{d}\sigma - \sum_{j=1}^{4}\int_{\widetilde{\gamma}_j}\widetilde{\kappa}(w_n^2+w_t^2)\mathrm{d}\sigma. \tag{3.22}$$

Here w_n and w_t are the projections of w on the normal and tangent vectors n and t respectively. We define

$$w = \widetilde{A}^k \nabla_y u.$$

Then

$$\widetilde{\mathfrak{M}}^k u = \sum_{r,s=1}^{2}\frac{\partial}{\partial y_r}\left(\widetilde{a}^k_{r,s}\frac{\partial u}{\partial y_s}\right) = \mathrm{div}(w), \tag{3.23a}$$

$$\left(\frac{\partial u}{\partial n}\right)_{\widetilde{A}^k} = \sum_{r,s=1}^{2} n_r \widetilde{a}^k_{r,s}\frac{\partial u}{\partial y_s} = w_n, \tag{3.23b}$$

and

$$\left(\frac{\partial u}{\partial t}\right)_{\widetilde{A}^k} = \sum_{r,s=1}^{2} t_r \widetilde{a}_{r,s}^k \frac{\partial u}{\partial y_s} = w_t. \tag{3.23c}$$

So (3.22) takes the form

$$\begin{aligned}
&\int_{\widetilde{\Omega}_{i,j}^k} |\widetilde{\mathfrak{M}}^k u|^2 \mathrm{d}y - \sum_{r,s=1}^{2} \frac{\partial w_r}{\partial y_s}\frac{\partial w_s}{\partial y_r}\mathrm{d}y \\
&= -2\sum_{j=1}^{4}\int_{\widetilde{\gamma}_j}\left(\frac{\partial u}{\partial t}\right)_{\widetilde{A}^k}\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\left(\frac{\partial u}{\partial n}\right)_{\widetilde{A}^k}\right)\mathrm{d}\sigma \\
&\quad - \sum_{j=1}^{4}\int \widetilde{\kappa}\left(\left(\frac{\partial u}{\partial t}\right)_{\widetilde{A}^k}^2 + \left(\frac{\partial u}{\partial n}\right)_{\widetilde{A}^k}^2\right)\mathrm{d}\sigma \\
&\quad - \sum_{j=1}^{4}\left\{\left(\frac{\partial u}{\partial t^{j+1}}\right)_{\widetilde{A}^k}\left(\frac{\partial u}{\partial n^{j+1}}\right)_{\widetilde{A}^k} - \left(\frac{\partial u}{\partial t^{j}}\right)_{\widetilde{A}^k}\left(\frac{\partial u}{\partial n^{j}}\right)_{\widetilde{A}^k}\right\}(\widetilde{Q}_j). \quad (3.24)
\end{aligned}$$

Now using Lemma 3.1.3.4 of [4] we obtain

$$\mu_0^2 \sum_{r,s=1}^{2}\left|\frac{\partial^2 u}{\partial y_r \partial y_s}\right|^2 \le \sum_{i,j=1}^{2}\frac{\partial w_r}{\partial y_s}\frac{\partial w_s}{\partial y_r} + 2\sum_{r,s,t,l=1}^{2}\left|\widetilde{a}_{r,t}^k \frac{\partial^2 u}{\partial y_s \partial y_t}\frac{\partial \widetilde{a}_{s,l}^k}{\partial y_r}\frac{\partial u}{\partial y_l}\right|$$

and by (3.18a)–(3.18d) there exists a constant R such that R is a common bound for the C^1 norms of all $\tilde{a}_{i,j}^k$. Hence

$$\begin{aligned}
\frac{\mu_0^2}{2}\sum_{r,s=1}^{2}\int_{\widetilde{\Omega}_{i,j}^k}\left|\frac{\partial^2 u}{\partial y_r \partial y_s}\right|^2 \mathrm{d}y &\le \sum_{r,s=1}^{2}\int_{\widetilde{\Omega}_{i,j}^k}\frac{\partial w_r}{\partial y_s}\frac{\partial w_s}{\partial y_r}\mathrm{d}y \\
&\quad + \frac{512}{\mu_0^2}R^4\sum_{r=1}^{2}\int_{\widetilde{\Omega}_{i,j}^k}\left|\frac{\partial u}{\partial y_r}\right|^2 \mathrm{d}y. \quad (3.25)
\end{aligned}$$

Thus combining (3.22), (3.24) and (3.25) we get the result. ■

We now need to write terms such as

$$2\rho^2\int_{\gamma_j}\left(\frac{\partial u}{\partial T}\right)_A \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial u}{\partial N}\right)_A \mathrm{d}s$$

in (3.21) where $\gamma_j \subseteq B_\rho^k = \{(x_1, x_2) : \rho_k = \rho\}$ in terms of (y_1, y_2) coordinates. Let γ be a smooth curve in $\Omega_\mu^k = \{(x_1, x_2) : (x_1, x_2) \in \Omega \text{ and } \rho_k < \mu\}$, where $\rho < \mu$, and let P be a point on γ such that P in polar coordinates has the representation (ρ_k, θ_k) with $\rho_k = \rho$.

Now

$$\mathrm{e}^{y_1}\nabla_x u = O^k \nabla_y u, \tag{3.26}$$

where O^k is the matrix defined in (3.16a), and

$$T = O^k t, \qquad N = O^k n. \tag{3.27}$$

Hence

$$e^{y_1}\left(\frac{\partial u}{\partial T}\right)_A(P)=t^t(O^k)^tAO^k\nabla_y u(\widetilde{P})=t^t\widetilde{A}^k\nabla_y u(\widetilde{P})=\left(\frac{\partial u}{\partial t}\right)_{\widetilde{A}^k}(\widetilde{P}) \tag{3.28a}$$

using (3.16a), (3.26) and (3.27). Here $\widetilde{P}$ is the image of the point P in (y_1, y_2) coordinates. Similarly, we have

$$e^{y_1}\left(\frac{\partial u}{\partial N}\right)_A(P)=\left(\frac{\partial u}{\partial n}\right)_{\widetilde{A}^k}(\widetilde{P}). \tag{3.28b}$$

PROPOSITION 3.1

Thus we can conclude that

$$2\rho^2\int_{\gamma_j}\left(\frac{\partial u}{\partial T}\right)_A\frac{d}{ds}\left(\frac{\partial u}{\partial N}\right)_A ds=2\int_{\widetilde{\gamma}_j}\left(\frac{\partial u}{\partial t}\right)_{\widetilde{A}^k}\frac{d}{d\sigma}\left(\frac{\partial u}{\partial n}\right)_{\widetilde{A}^k}d\sigma \tag{3.29a}$$

and

$$\left\{\rho^2\left(\frac{\partial u}{\partial T}\right)_A\left(\frac{\partial u}{\partial N}\right)_A\right\}(P)=\left\{\left(\frac{\partial u}{\partial t}\right)_{\widetilde{A}^k}\left(\frac{\partial u}{\partial n}\right)_{\widetilde{A}^k}\right\}(\widetilde{P}). \tag{3.29b}$$

■

In the same way we obtain the following results.

PROPOSITION 3.2

Consider the boundary γ common to $\Omega^k_{i,M+1}$ and $\Omega^k_{i,M}$. Then the following relations hold (*figure* 5):

$$\left\{\rho^2\left(\frac{\partial u}{\partial T^3}\right)_A\left(\frac{\partial u}{\partial N^3}\right)_A\right\}(Q_1)=\left\{\left(\frac{\partial u}{\partial t^3}\right)_{\widetilde{A}^k}\left(\frac{\partial u}{\partial n^3}\right)_{\widetilde{A}^k}\right\}(\widetilde{Q}_1), \tag{3.30a}$$

$$\left\{\rho^2\left(\frac{\partial u}{\partial T^2}\right)_A\left(\frac{\partial u}{\partial N^2}\right)_A\right\}(Q_1)=\left\{\left(\frac{\partial u}{\partial t^4}\right)_{\widetilde{A}^k}\left(\frac{\partial u}{\partial n^4}\right)_{\widetilde{A}^k}\right\}(\widetilde{Q}_1), \tag{3.30b}$$

$$\left\{\rho^2\left(\frac{\partial u}{\partial T^2}\right)_A\left(\frac{\partial u}{\partial N^2}\right)_A\right\}(Q_2)=\left\{\left(\frac{\partial u}{\partial t^4}\right)_{\widetilde{A}^k}\left(\frac{\partial u}{\partial n^4}\right)_{\widetilde{A}^k}\right\}(\widetilde{Q}_2), \tag{3.30c}$$

and

$$\left\{\rho^2\left(\frac{\partial u}{\partial T^1}\right)_A\left(\frac{\partial u}{\partial N^1}\right)_A\right\}(Q_2)=\left\{\left(\frac{\partial u}{\partial t^1}\right)_{\widetilde{A}^k}\left(\frac{\partial u}{\partial n^1}\right)_{\widetilde{A}^k}\right\}(\widetilde{Q}_2). \tag{3.30d}$$

■

Now let $\widetilde{\gamma}_l \subseteq \partial\widetilde{\Omega}^k_{i,j}$ for some $j \le M$ and further suppose $\widetilde{\gamma}_l \subseteq \widetilde{\Gamma}_j$ where $j \in \mathcal{D}$. Let n and t be the unit outward normal and tangent vectors, respectively, defined at every point of $\widetilde{\gamma}_l$. Then

$$\left(\frac{\partial u}{\partial t}\right)_{\widetilde{A}^k}(\sigma)=\widetilde{g}^k(\sigma)\left(\frac{\partial u}{\partial t}\right)(\sigma)+\widetilde{h}^k(\sigma)\left(\frac{\partial u}{\partial n}\right)_{\widetilde{A}^k}(\sigma). \tag{3.31a}$$

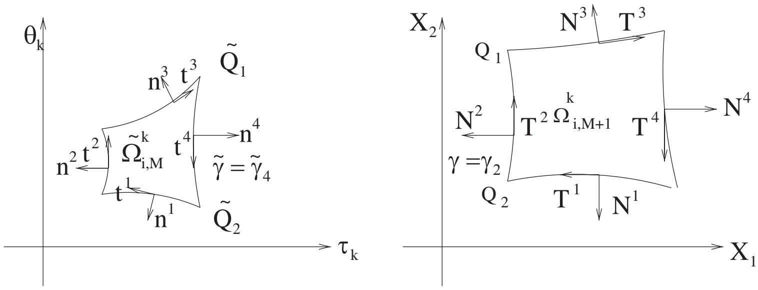

Figure 5. Elements $\widetilde{\Omega}^k_{i,M}$ and $\Omega^k_{i,M+1}$.

Here σ is the arc length measured from the point $\widetilde{G}$ (figure 6) where

$$\widetilde{g}^k(\sigma) = t^t \widetilde{A}^k t(\sigma) - \frac{(t^t \widetilde{A}^k n(\sigma))^2}{n^t \widetilde{A}^k n(\sigma)}, \tag{3.31b}$$

and

$$\widetilde{h}^k(\sigma) = \frac{t^t \widetilde{A}^k n(\sigma)}{n^t \widetilde{A}^k n(\sigma)}. \tag{3.31c}$$

Hence

$$\int_{\widetilde{\gamma}_l} \left(\frac{\partial u}{\partial t}\right)_{\widetilde{A}^k} \frac{\mathrm{d}}{\mathrm{d}\sigma} \left(\frac{\partial u}{\partial n}\right)_{\widetilde{A}^k} \mathrm{d}\sigma = \int_{\widetilde{\gamma}_l} \widetilde{g}^k(\sigma) \frac{\partial u}{\partial t} \frac{\mathrm{d}}{\mathrm{d}\sigma} \left(\frac{\partial u}{\partial n}\right)_{\widetilde{A}^k} \mathrm{d}\sigma + \int_{\widetilde{\gamma}_l} \frac{\widetilde{h}^k(\sigma)}{2} \frac{\mathrm{d}}{\mathrm{d}\sigma} \left(\left(\frac{\partial u}{\partial n}\right)^2_{\widetilde{A}^k}\right) \mathrm{d}\sigma.$$

And so we can conclude that the following holds.

PROPOSITION 3.3

$$\int_{\widetilde{\gamma}_l} \left(\frac{\partial u}{\partial t}\right)_{\widetilde{A}^k} \frac{\mathrm{d}}{\mathrm{d}\sigma} \left(\frac{\partial u}{\partial n}\right)_{\widetilde{A}^k} \mathrm{d}\sigma = \int_{\widetilde{\gamma}_l} \widetilde{g}^k(\sigma) \frac{\partial u}{\partial t} \frac{\mathrm{d}}{\mathrm{d}\sigma} \left(\frac{\partial u}{\partial n}\right)_{\widetilde{A}^k} \mathrm{d}\sigma - \frac{1}{2} \int_{\widetilde{\gamma}_l} \frac{\mathrm{d}\widetilde{h}^k}{\mathrm{d}\sigma} \left(\frac{\partial u}{\partial n}\right)^2_{\widetilde{A}^k} \mathrm{d}\sigma + \left.\frac{\widetilde{h}^k(\sigma)}{2} \left(\frac{\partial u}{\partial n}\right)^2_{\widetilde{A}^k}\right|_{\partial\widetilde{\gamma}_l}. \tag{3.32}$$

Here $\widetilde{g}^k(\sigma)$ and $\widetilde{h}^k(\sigma)$ are defined in (3.31b) *and* (3.31c). ■

Next let $\gamma_m \subseteq \partial\Omega^k_{i,j}$ for some $j > M$ such that $\gamma_m \subseteq \Gamma_j$ where $j \in \mathcal{D}$. Let N and T be the unit normal and tangent vectors, respectively, defined at every point of γ_m. Then

$$\left(\frac{\partial u}{\partial T}\right)_A (s) = g(s) \left(\frac{\partial u}{\partial T}\right)(s) + h(s) \left(\frac{\partial u}{\partial N}\right)_A (s), \tag{3.33a}$$

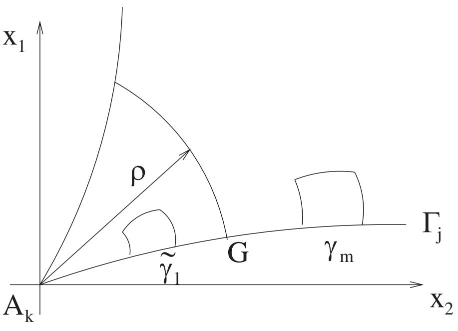

Figure 6. Arc length measured from the point G.

where s is the arc length measured from the point G as shown in figure 6. Here

$$g(s) = T^t AT - \frac{(T^t AN)^2}{N^t AN}, \tag{3.33b}$$

and

$$h(s) = \frac{T^t AN}{N^t AN}. \tag{3.33c}$$

So we obtain the following result.

PROPOSITION 3.4

$$\begin{aligned}
&\rho^2 \int_{\gamma_m} \left(\frac{\partial u}{\partial T}\right)_A \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial u}{\partial N}\right)_A \mathrm{d}s \\
&= \rho^2 \int_{\gamma_m} g(s) \frac{\partial u}{\partial T}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial u}{\partial N}\right)_A \mathrm{d}s \\
&\quad - \frac{\rho^2}{2}\int_{\gamma_m} \frac{\mathrm{d}h}{\mathrm{d}s}\left(\frac{\partial u}{\partial N}\right)_A^2 \mathrm{d}s + \frac{\rho^2 h}{2}\left(\frac{\partial u}{\partial N}\right)_A^2 \Bigg|_{\partial\gamma_m}.
\end{aligned} \tag{3.34}$$

■

Now by (3.28b) we have that

$$\rho^2 \left(\frac{\partial u}{\partial N}\right)_A^2 (G) = \left(\frac{\partial u}{\partial n}\right)_{\widetilde{A}^k}^2 (\widetilde{G}).$$

And moreover by (3.16a) and (3.27)

$$g(G) = \widetilde{g}^k(\widetilde{G}), \tag{3.35a}$$

and

$$h(G) = \widetilde{h}^k(\widetilde{G}). \tag{3.35b}$$

We can now prove the following estimate.

Lemma 3.3. *Let* $u_l^{p+1} \in H^3(\Omega_l^{p+1})$. *Then*

$$\sum_{|\alpha|=2} \int_S \int |D_\xi^{\alpha_1} D_\eta^{\alpha_2} u_l^{p+1}(\xi,\eta)|^2 \mathrm{d}\xi \mathrm{d}\eta$$
$$- C \left(\sum_{|\alpha|\leq 1} \int_S \int |D_\xi^{\alpha_1} D_\eta^{\alpha_2} u_l^{p+1}|^2 \mathrm{d}\xi \mathrm{d}\eta \right)$$
$$\leq K \int_S \int |\mathfrak{L}_l^{p+1} u_l^{p+1}|^2 \mathrm{d}\xi \mathrm{d}\eta + 2\rho^2 \sum_{r=1}^{4} \int \left(\frac{\partial u_l^{p+1}}{\partial T}\right)_A \frac{\mathrm{d}}{\mathrm{d}s} \left(\frac{\partial u_l^{p+1}}{\partial N}\right)_A \mathrm{d}s$$
$$+ \sum_{r=1}^{4} \rho^2 \left\{ \left(\frac{\partial u_l^{p+1}}{\partial N^{r+1}}\right)_A \left(\frac{\partial u_l^{p+1}}{\partial T^{r+1}}\right)_A - \left(\frac{\partial u_l^{p+1}}{\partial N^{r}}\right)_A \left(\frac{\partial u_l^{p+1}}{\partial T^{r}}\right)_A \right\} (Q_r)$$
$$+ \sum_{r=1}^{4} \int_{\gamma_r} |\kappa| \rho^2 \left(\left(\frac{\partial u_l^{p+1}}{\partial N}\right)_A^2 + \left(\frac{\partial u_l^{p+1}}{\partial T}\right)_A^2 \right) \mathrm{d}s. \tag{3.36}$$

Here S is the unit square and $\mathfrak{L}_l^{p+1}$ is the differential operator $\mathfrak{L}$ written in (ξ, η) coordinates. Here K and C are positive constants.

Recall that

$$\mathfrak{L}u = -\sum_{r,s=1}^{2} (a_{r,s}(x) u_{x_s})_{x_r} + \sum_{r=1}^{2} b_r(x) u_{x_r} + c(x)u$$
$$= \mathfrak{M}u + \mathfrak{N}u, \tag{3.37}$$

where

$$\mathfrak{N}u = \sum_{r=1}^{2} b_r(x) u_{x_r} + c(x)u.$$

Hence

$$\rho^2 \int_{\Omega_l^{p+1}} |\mathfrak{M}u|^2 \mathrm{d}x \leq 2\rho^2 \int_{\Omega_l^{p+1}} |\mathfrak{L}u|^2 \mathrm{d}x + 2\rho^2 \int_{\Omega_l^{p+1}} |\mathfrak{N}u|^2 \mathrm{d}x.$$

Using Lemma 3.1 we can conclude that there is a constant C such that the following estimate holds.

$$\frac{\rho^2 \mu_0^2}{2} \sum_{r,s=1}^{2} \int_{\Omega_l^{p+1}} \left| \frac{\partial^2 u_l^{p+1}}{\partial x_r \partial x_s} \right|^2 \mathrm{d}x$$
$$- C\rho^2 \left(\sum_{r=1}^{2} \left(\int_{\Omega_l^{p+1}} \left| \frac{\partial u_l^{p+1}}{\partial x_r} \right|^2 \mathrm{d}x \right) + \int_{\Omega_l^{p+1}} |u_l^{p+1}|^2 \mathrm{d}x \right)$$

$$
\begin{aligned}
&\leq 2\rho^2 \int_{\Omega_l^{p+1}} |\mathfrak{L} u_l^{p+1}|^2 \mathrm{d}x + 2\rho^2 \sum_{j=1}^{4} \int_{\gamma_j} \left(\frac{\partial u_l^{p+1}}{\partial T}\right)_A \frac{\mathrm{d}}{\mathrm{d}s} \left(\frac{\partial u_l^{p+1}}{\partial N}\right)_A \mathrm{d}s \\
&\quad + \sum_{j=1}^{4} \rho^2 \left\{ \left(\frac{\partial u_l^{p+1}}{\partial N^{j+1}}\right)_A \left(\frac{\partial u_l^{p+1}}{\partial T^{j+1}}\right)_A - \left(\frac{\partial u_l^{p+1}}{\partial N^{j}}\right)_A \left(\frac{\partial u_l^{p+1}}{\partial T^{j}}\right)_A \right\} (Q_j) \\
&\quad + \rho^2 \sum_{j=1}^{4} \int_{\gamma_j} |\kappa| \left(\left(\frac{\partial u_l^{p+1}}{\partial N}\right)_A^2 + \left(\frac{\partial u_l^{p+1}}{\partial T}\right)_A^2 \right) \mathrm{d}s. \qquad (3.38)
\end{aligned}
$$

Writing the above in (ξ, η) coordinates we obtain the result. ■

In the same way we can prove the following estimate.

Lemma 3.4. *Let $u_{i,j}^k \in H^3(\Omega_{i,j}^k)$. Then*

$$
\begin{aligned}
&\beta \sum_{|\alpha|=2} \int_{\widehat{\Omega}_{i,j}^k}\!\!\int |D_{\nu_k}^{\alpha_1} D_{\phi_k}^{\alpha_2} u_{i,j}^k|^2 \mathrm{d}\nu_k \mathrm{d}\phi_k \\
&\quad - C \left(\sum_{|\alpha|=1} \int_{\widehat{\Omega}_{i,j}^k}\!\!\int |D_{\nu_k}^{\alpha_1} D_{\phi_k}^{\alpha_2} u_{i,j}^k|^2 \mathrm{d}\nu_k \mathrm{d}\phi_k + \int_{\widehat{\Omega}_{i,j}^k}\!\!\int |u_{i,j}^k|^2 \mathrm{e}^{4\nu_k} \mathrm{d}\nu_k \mathrm{d}\phi_k \right) \\
&\leq K \int_{\widehat{\Omega}_{i,j}^k}\!\!\int |\mathfrak{L}_{i,j}^k u_{i,j}^k|^2 \mathrm{d}\nu_k \mathrm{d}\phi_k + 2 \sum_{r=1}^{4} \int_{\widetilde{\gamma}_r} \left(\frac{\partial u_{i,j}^k}{\partial t}\right)_{\widetilde{A}^k} \frac{\mathrm{d}}{\mathrm{d}\sigma} \left(\frac{\partial u_{i,j}^k}{\partial n}\right)_{\widetilde{A}^k} \mathrm{d}\sigma \\
&\quad + \sum_{r=1}^{4} \left\{ \left(\frac{\partial u_{i,j}^k}{\partial n^{r+1}}\right)_{\widetilde{A}^k} \left(\frac{\partial u_{i,j}^k}{\partial t^{r+1}}\right)_{\widetilde{A}^k} - \left(\frac{\partial u_{i,j}^k}{\partial n^{r}}\right)_{\widetilde{A}^k} \left(\frac{\partial u_{i,j}^k}{\partial t^{r}}\right)_{\widetilde{A}^k} \right\} (\widetilde{Q}_r) \\
&\quad + \sum_{r=1}^{4} \int_{\widetilde{\gamma}_r} |\widetilde{\kappa}| \left(\left(\frac{\partial u_{i,j}^k}{\partial n}\right)_{\widetilde{A}^k}^2 + \left(\frac{\partial u_{i,j}^k}{\partial t}\right)_{\widetilde{A}^k}^2 \right) \mathrm{d}\sigma. \qquad (3.39)
\end{aligned}
$$

Here $\widehat{\Omega}_{i,j}^k = (\psi_i^k, \psi_{i+1}^k) \times (\alpha_j^k, \alpha_{j+1}^k)$ and β, C and K are positive constants.

For

$$
\begin{aligned}
\widetilde{\mathfrak{L}}^k u &= \mathrm{e}^{2y_1} \left(-\sum_{r,s=1}^{2} (a_{r,s}(x) u_{x_s})_{x_r} + \sum_{r=1}^{2} b_r(x) u_{x_r} + c(x) u \right) \\
&= \left(\sum_{r,s=1}^{2} -(\widetilde{a}_{r,s}^k(y) u_{y_s})_{y_r} \right) + \left(\sum_{r=1}^{2} \widetilde{b}_r^k(y) u_{y_r} + \widetilde{c}^k(y) u \right) \\
&= \widetilde{\mathfrak{M}}^k u + \widetilde{\mathfrak{N}}^k u.
\end{aligned}
$$

Here

$$
\widetilde{\mathfrak{N}}^k u = \sum_{r=1}^{2} \widetilde{b}_r^k(y) u_{y_r} + \widetilde{c}^k(y) u \qquad (3.40)
$$

and $y = (y_1, y_2) = (\tau_k, \theta_k)$ for some k. Moreover the coefficients of $\widetilde{\mathfrak{N}}^k$ satisfy

$$\widetilde{b}_r^k = O(\mathrm{e}^{\tau_k}) \text{ for } r = 1, 2$$

and

$$\widetilde{c}^k = O(\mathrm{e}^{2\tau_k})$$

as $\tau_k \to -\infty$.

Once more

$$\int_{\widetilde{\Omega}_{i,j}^k} |\widetilde{\mathfrak{M}}^k u|^2 \mathrm{d}y \le 2 \left(\int_{\widetilde{\Omega}_{i,j}^k} |\widetilde{\mathfrak{L}}^k u|^2 \mathrm{d}y + \int_{\widetilde{\Omega}_{i,j}^k} |\widetilde{\mathfrak{N}}^k u|^2 \mathrm{d}y \right).$$

Using Lemma 3.2 we can conclude that there exists a constant C such that the following estimate holds.

$$\begin{aligned}
&\frac{\mu_0^2}{2} \sum_{r,s=1}^{2} \int_{\widetilde{\Omega}_{i,j}^k} \left| \frac{\partial^2 u}{\partial y_r \partial y_s} \right|^2 \mathrm{d}y - C \left(\sum_{r=1}^{2} \int_{\widetilde{\Omega}_{i,j}^k} \left| \frac{\partial u}{\partial y_r} \right|^2 \mathrm{d}y + \int_{\widetilde{\Omega}_{i,j}^k} |u|^2 \mathrm{e}^{4y_1} \mathrm{d}y \right) \\
&\le 2 \int_{\widetilde{\Omega}_{i,j}^k} |\widetilde{\mathfrak{L}}^k u|^2 \mathrm{d}y + 2 \sum_{j=1}^{4} \int_{\widetilde{\gamma}_j} \left(\frac{\partial u}{\partial t} \right)_{\widetilde{A}^k} \frac{\mathrm{d}}{\mathrm{d}\sigma} \left(\frac{\partial u}{\partial n} \right)_{\widetilde{A}^k} \mathrm{d}\sigma \\
&\quad + \sum_{j=1}^{4} \left\{ \left(\frac{\partial u}{\partial n^{j+1}} \right)_{\widetilde{A}^k} \left(\frac{\partial u}{\partial t^{j+1}} \right)_{\widetilde{A}^k} - \left(\frac{\partial u}{\partial n^{j}} \right)_{\widetilde{A}^k} \left(\frac{\partial u}{\partial t^{j}} \right)_{\widetilde{A}^k} \right\} (\widetilde{Q}_j) \\
&\quad + \sum_{j=1}^{4} \int_{\widetilde{\gamma}_j} |\widetilde{\kappa}| \left(\left(\frac{\partial u}{\partial n} \right)_{\widetilde{A}^k}^2 + \left(\frac{\partial u}{\partial t} \right)_{\widetilde{A}^k}^2 \right) \mathrm{d}\sigma. \qquad (3.41)
\end{aligned}$$

Rewriting (3.41) in (ν_k, ϕ_k) coordinates (3.39) follows. ∎

We now need to obtain estimates for the spectral element functions in the H^1 norm which we do in the following theorem.

Theorem 3.1. *The following estimate holds*:

$$\begin{aligned}
&\sum_{k=1}^{p} \sum_{i=1}^{I_k} |u_{i,1}^k|^2 + \sum_{k=1}^{p} \sum_{j=2}^{M} \sum_{i=1}^{I_k} \| u_{i,j}^k(\nu_k, \phi_k) \|_{1,\widehat{\Omega}_{i,j}^k}^2 + \sum_{l=1}^{L} \| u_l^{p+1}(\xi, \eta) \|_{1,S}^2 \\
&\le C_M \left\{ \sum_{k=1}^{p} \sum_{j=2}^{M} \sum_{i=1}^{I_k} \| \mathfrak{L}_{i,j}^k u_{i,j}^k(\nu_k, \phi_k) \|_{0,\widehat{\Omega}_{i,j}^k}^2 \right. \\
&\quad + \sum_{k=1}^{p} \sum_{\gamma_s \subseteq \Omega^k} (\| [u] \|_{0,\widehat{\gamma}_s}^2 + \| [u_{\nu_k}] \|_{0,\widehat{\gamma}_s}^2 + \| [u_{\phi_k}] \|_{0,\widehat{\gamma}_s}^2) \\
&\quad + \sum_{l \in \mathcal{D}} \sum_{k=l-1}^{l} \sum_{\gamma_s \subseteq \partial\Omega^k \bigcap \Gamma_l, \mu(\widehat{\gamma}_s) < \infty} (\| u \|_{0,\widehat{\gamma}_s}^2 + \| u_{\nu_k} \|_{0,\widehat{\gamma}_s}^2)
\end{aligned}$$

$$+\sum_{k=1}^{p}\sum_{\gamma_s\subseteq B_\rho^k}(\|[u]\|_{0,\widehat{\gamma}_s}^2+\|[u_{\nu_k}]\|_{0,\widehat{\gamma}_s}^2+\|[u_{\phi_k}]\|_{0,\widehat{\gamma}_s}^2)$$

$$+\sum_{l\in\mathcal{N}}\sum_{k=l-1}^{l}\sum_{\gamma_s\subseteq\partial\Omega^k\bigcap\Gamma_l}\left\|\left(\frac{\partial u}{\partial n}\right)_{\widetilde{A}^k}\right\|_{0,\widehat{\gamma}_s}^2$$

$$+\sum_{l=1}^{L}\int_S\int|\mathfrak{L}_l^{p+1}u_l^{p+1}(\xi,\eta)|^2\mathrm{d}\xi\mathrm{d}\eta$$

$$+\sum_{\gamma_s\subseteq\Omega^{p+1}}(\|[u]\|_{0,\gamma_s}^2+\|[u_{x_1}]\|_{0,\gamma_s}^2+\|[u_{x_2}]\|_{0,\gamma_s}^2)$$

$$+\sum_{l\in\mathcal{D}}\sum_{\gamma_s\subseteq\partial\Omega^{p+1}\bigcap\Gamma_l}\left(\|u\|_{0,\gamma_s}^2+\left\|\frac{\partial u}{\partial T}\right\|_{0,\gamma_s}^2\right)$$

$$+\sum_{l\in\mathcal{N}}\sum_{\gamma_s\subseteq\partial\Omega^{p+1}\bigcap\Gamma_l}\left\|\left(\frac{\partial u}{\partial N}\right)_A\right\|_{0,\gamma_s}^2\Bigg\}. \tag{3.42}$$

Here $C_M = CM^4$ if there exists a vertex A_j such that Neumann boundary conditions are imposed on the adjoining sides Γ_j and Γ_{j+1} and $C_M = C$ otherwise. C denotes a constant and $\mu(\hat{\gamma}_s)$ the length of $\widehat{\gamma}_s$.

To prove the estimate (3.42) we shall use (3.5). To do so we have to define a corrected version of the spectral element functions so that it is conforming.

Let $\{\{u_{i,j}^k(\nu_k,\phi_k)\}_{i,j\le M,k},\{u_{i,j}^k(\xi,\eta)\}_{i,j>M,k}\}_k$ be a set of spectral element functions $\in\pi^{M,W}$. Here $\pi^{M,W}$ is the set of spectral element functions such that $u_{i,1}^k=g_k$, a constant for all i, and $u_{i,j}^k$ is a polynomial of degree W in each variable for $j\ge 2$. Then there is a set of spectral element functions

$$\{\lambda_{i,j}^k(\nu_k,\phi_k)\}_{i,j\le M,k},\{\lambda_{i,j}^k(\xi,\eta)\}_{i,j>M,k}\in\pi^{M,W}$$

such that the function $\varphi(x_1,x_2)$ defined as

$$\varphi(x_1,x_2)=\begin{cases}(u_{i,j}^k+\lambda_{i,j}^k)(\nu_k(x_1,x_2),\phi_k(x_1,x_2)) & \text{if } (x_1,x_2)\in\Omega_{i,j}^k \text{ for } j\le M\\ (u_{i,j}^k+\lambda_{i,j}^k)(\xi(x_1,x_2),\eta(x_1,x_2)) & \text{if } (x_1,x_2)\in\Omega_{i,j}^k \text{ for } j>M\end{cases}$$

is a differentiable function of its arguments and $\varphi\in H_0^1(\Omega)$. This can be shown as in Lemma 4.57 of [7].

Moreover the estimate

$$\sum_{k=1}^{p}\sum_{i=1}^{I_k}|\lambda_{i,1}^k|^2+\sum_{k=1}^{p}\sum_{j=2}^{M}\sum_{i=1}^{I_k}\|\lambda_{i,j}^k(\nu_k,\phi_k)\|_{1,\widehat{\Omega}_{i,j}^k}^2$$

$$+\sum_{k=1}^{p}\sum_{j=M+1}^{J_k}\sum_{i=1}^{I_{k,j}}\|\lambda_{i,j}^k(\xi,\eta)\|_{1,S}^2$$

$$
\begin{aligned}
&\leq C\left\{\left(\sum_{l\in\mathcal{D}}\sum_{k=l-1}^{l}\sum_{\gamma_s\subseteq\Gamma_l\cap\partial\Omega^k,\mu(\widehat{\gamma}_s)<\infty}(\|u\|^2_{0,\widehat{\gamma}_s}+\|u_{\nu_k}\|^2_{0,\widehat{\gamma}_s})\right)\right.\\
&\quad+\sum_{k=1}^{p}\sum_{\gamma_s\subseteq\Omega^k,\mu(\widehat{\gamma}_s)<\infty}(\|[u]\|^2_{0,\widehat{\gamma}_s}+\|[u_{\nu_k}]\|^2_{0,\widehat{\gamma}_s}+\|[u_{\phi_k}]\|^2_{0,\widehat{\gamma}_s})\\
&\quad+\sum_{k=1}^{p}\sum_{\gamma_s\subseteq B^k_\rho}(\|[u]\|^2_{0,\widehat{\gamma}_s}+\|[u_{\nu_k}]\|^2_{0,\widehat{\gamma}_s}+\|[u_{\phi_k}]\|^2_{0,\widehat{\gamma}_s})\\
&\quad+\sum_{\gamma_s\subseteq\Omega^{p+1}}\left(\|[u]\|^2_{0,\gamma_s}+\left\|\left[\frac{\partial u}{\partial T}\right]\right\|^2_{0,\gamma_s}\right)\\
&\quad\left.+\sum_{l\in\mathcal{D}}\sum_{\gamma_s\subseteq\partial\Omega^{p+1}\cap\Gamma_l}\left(\|u\|^2_{0,\gamma_s}+\left\|\frac{\partial u}{\partial T}\right\|^2_{0,\gamma_s}\right)\right\} \qquad (3.43)
\end{aligned}
$$

holds.

We now explain the notation we have used in (3.43). Let $d\widehat{\sigma}$ denote an element of arc length in (ν_k,ϕ_k) coordinates. Then

$$\|w\|^2_{0,\widehat{\gamma}_s}=\int_{\widehat{\gamma}_s}|w(\nu_k,\phi_k)|^2 d\widehat{\sigma}.$$

Moreover if γ_s is given by $\gamma_s=\partial\Omega^{p+1}_m\bigcap\partial\Omega^{p+1}_n$ then

$$\left\|\left[\frac{\partial u}{\partial T}\right]\right\|^2_{0,\gamma_s}=\int_{\gamma_s}\left(\frac{\partial u^{p+1}_m}{\partial T}-\frac{\partial u^{p+1}_n}{\partial T}\right)^2 ds.$$

Here $\partial/\partial T$ denotes the tangential derivative in (x_1,x_2) variables, i.e.

$$\frac{\partial u}{\partial T}=T^t\nabla_x u.$$

The other terms in the right-hand side of (3.43) are similarly defined.

Now consider the bilinear form

$$
\begin{aligned}
B(\varphi,v)&=\int_\Omega\left(\sum_{r,s=1}^{2}a_{r,s}(x)\varphi_{x_s}v_{x_r}+\sum_{r=1}^{2}b_r(x)\varphi_{x_r}v+c\varphi v\right)dx\\
&=\sum_{k=1}^{p}\sum_{j=1}^{M}\sum_{i=1}^{I_k}B(\varphi,v)_{\Omega^k_{i,j}}+\sum_{l=1}^{L}B(\varphi,v)_{\Omega^{p+1}_l}.
\end{aligned}
$$

Here

$$B(\varphi,v)_\Delta=\int_\Delta\left(\sum_{r,s=1}^{2}a_{r,s}(x)\varphi_{x_s}v_{x_r}+\sum_{r=1}^{2}b_r(x)\varphi_{x_r}v+c\varphi v\right)dx,$$

where Δ is a domain contained in Ω and $v\in H^1_0(\Omega)$.

Now

$$B(\varphi, v)_{\Omega_l^{p+1}} = \int_{\Omega_l^{p+1}} \left(\sum_{r,s=1}^{2} a_{r,s}(x)\varphi_{x_s} v_{x_r} + \sum_{r=1}^{2} b_r(x)\varphi_{x_r} v + c\varphi v \right) \mathrm{d}x$$

$$= \int_{\Omega_l^{p+1}} \mathfrak{L}\varphi v \mathrm{d}x + \int_{\partial\Omega_l^{p+1}} \left(\frac{\partial \varphi}{\partial N}\right)_A v \mathrm{d}s.$$

Similarly if $1 \leq j \leq M$ we have

$$B(\varphi, v)_{\Omega_{i,j}^k} = \int_{\widetilde{\Omega}_{i,j}^k} \widetilde{\mathfrak{L}}^k \varphi v \mathrm{d}\tau_k \mathrm{d}\theta_k + \int_{\partial\widetilde{\Omega}_{i,j}^k} \left(\frac{\partial \varphi}{\partial n}\right)_{\widetilde{A}^k} v \mathrm{d}\sigma.$$

Moreover if $j = 1$,

$$B(\varphi, v)_{\Omega_{i,1}^k} = \int_{\widetilde{\Omega}_{i,1}^k} c\varphi v \mathrm{e}^{2\tau_k} \mathrm{d}\tau_k \mathrm{d}\theta_k + \int_{\partial\widetilde{\Omega}_{i,1}^k} \left(\frac{\partial \varphi}{\partial n}\right)_{\widetilde{A}^k} v \mathrm{d}\sigma$$

since φ is a constant on $\widetilde{\Omega}_{i,1}^k$.

Finally if $j = M+1$ we obtain

$$B(\varphi, v)_{\Omega_{i,M+1}^k} = \int_{\Omega_{i,M+1}^k} \mathfrak{L}\varphi v \mathrm{d}x + \int_{\widetilde{B}_\rho^k} \left(\frac{\partial \varphi}{\partial n}\right)_{\widetilde{A}^k} v \mathrm{d}\sigma$$

$$+ \int_{\partial\Omega_{i,M+1}^k \setminus B_\rho^k} \left(\frac{\partial \varphi}{\partial N}\right)_A v \mathrm{d}s.$$

For by (3.28b)

$$\rho \left(\frac{\partial \varphi}{\partial N}\right)_A (P) = \left(\frac{\partial \varphi}{\partial n}\right)_{\widetilde{A}^k} (\widetilde{P})$$

and $\mathrm{d}s = \rho \mathrm{d}\sigma$. Here P is any point on the circular arc B_ρ^k and $\widetilde{P}$ is its image in (τ_k, θ_k) coordinates. Now

$$B(\varphi, v) = \sum_{k=1}^{p} \sum_{j=1}^{M} \sum_{i=1}^{I_k} B(\varphi, v)_{\Omega_{i,j}^k} + \sum_{l=1}^{L} B(\varphi, v)_{\Omega_l^{p+1}}$$

$$= \sum_{k=1}^{p} \sum_{j=1}^{M} \sum_{i=1}^{I_k} B(u_{i,j}^k, v)_{\Omega_{i,j}^k} + \sum_{l=1}^{L} B(u_l^{p+1}, v)_{\Omega_l^{p+1}}$$

$$+ \left(\sum_{k=1}^{p} \sum_{j=1}^{M} \sum_{i=1}^{I_k} B(\lambda_{i,j}^k, v)_{\Omega_{i,j}^k} + \sum_{l=1}^{L} B(\lambda_l^{p+1}, v)_{\Omega_l^{p+1}} \right)$$

$$= \sum_{k=1}^{p} \sum_{j=1}^{M} \sum_{i=1}^{I_k} \int_{\widetilde{\Omega}_{i,j}^k} \widetilde{\mathfrak{L}}^k u_{i,j}^k v \mathrm{d}\tau_k \mathrm{d}\theta_k + \sum_{l=1}^{L} \int_{\Omega_l^{p+1}} \mathfrak{L} u_l^{p+1} v \mathrm{d}x_1 \mathrm{d}x_2$$

$$+ \sum_{k=1}^{p} \sum_{\gamma_s \subseteq \Omega^k, \mu(\widetilde{\gamma}_s) < \infty} \int_{\widetilde{\gamma}_s} \left[\left(\frac{\partial u}{\partial n}\right)_{\widetilde{A}^k} \right] v \mathrm{d}\sigma$$

$$+ \sum_{\gamma_s \subseteq \Omega^{p+1}} \int_{\gamma_s} \left[\left(\frac{\partial u}{\partial N} \right)_A \right] v \mathrm{d}s + \sum_{k=1}^{p} \sum_{\gamma_s \subseteq B_\rho^k} \int_{\widetilde{\gamma}_s} \left[\left(\frac{\partial u}{\partial n} \right)_{\widetilde{A}^k} \right] v \mathrm{d}\sigma$$

$$+ \sum_{l \in \mathcal{N}} \sum_{k=l-1}^{l} \sum_{\gamma_s \subseteq \Gamma_l \cap \partial\Omega^k, \mu(\widetilde{\gamma}_s) < \infty} \int_{\widetilde{\gamma}_s} \left(\frac{\partial u}{\partial n} \right)_{\widetilde{A}^k} v \mathrm{d}\sigma$$

$$+ \sum_{l \in \mathcal{N}} \sum_{r=1}^{L} \sum_{\gamma_s \subseteq \partial\Omega_r^{p+1} \cap \Gamma_l} \int_{\gamma_s} \left(\frac{\partial u}{\partial N} \right)_A v \mathrm{d}s$$

$$+ \left(\sum_{k=1}^{p} \sum_{i=1}^{I_k} B(\lambda_{i,1}^k, v)_{\Omega_{i,1}^k} + \sum_{k=1}^{p} \sum_{j=2}^{M} \sum_{i=1}^{I_k} B(\lambda_{i,j}^k, v)_{\Omega_{i,j}^k} \right.$$

$$\left. + \sum_{l=1}^{L} B(\lambda_l^{p+1}, v)_{\Omega_l^{p+1}} \right). \tag{3.44}$$

Now

$$\int_{\widetilde{\Omega}_{i,1}^k} \widetilde{\mathfrak{L}}^k \lambda_{i,1}^k v \mathrm{d}\tau_k \mathrm{d}\theta_k = \int_{\widetilde{\Omega}_{i,1}^k} c \lambda_{i,1}^k v \mathrm{e}^{2\tau_k} \mathrm{d}\tau_k \mathrm{d}\theta_k.$$

Here

$$\lambda_{i,1}^k = \begin{Bmatrix} -u_{i,1}^k, & \text{if } \Gamma_k \text{ or } \Gamma_{k+1} \subseteq \Gamma^{[0]} \\ 0, & \text{otherwise} \end{Bmatrix}.$$

Now $c_k = c(A_k)$, a constant, and $c(x_1, x_2)$ is an analytic function of x_1 and x_2. Hence

$$\left| \int_{\widetilde{\Omega}_{i,1}^k} \widetilde{\mathfrak{L}}^k \lambda_{i,1}^k v \mathrm{d}\tau_k \mathrm{d}\theta_k \right| \leq 2c_k \left(\int |\lambda_{i,1}^k|^2 \mathrm{e}^{2\tau_k} \mathrm{d}\tau_k \mathrm{d}\theta_k \right)^{1/2}$$
$$\times \left(\int v^2 \mathrm{e}^{2\tau_k} \mathrm{d}\tau_k \mathrm{d}\theta_k \right)^{1/2}$$

for M large enough. And so we obtain

$$\left| \int_{\widetilde{\Omega}_{i,1}^k} \widetilde{\mathfrak{L}}^k \lambda_{i,1}^k v \mathrm{d}\tau_k \mathrm{d}\theta_k \right| \leq \varepsilon |\lambda_{i,1}^k| \| v(x_1, x_2) \|_{0, \Omega_{i,1}^k},$$

where ε is exponentially small in M. Now, let $2 \leq j \leq M$. Then

$$\left| \int_{\widetilde{\Omega}_{i,j}^k} \widetilde{\mathfrak{L}}^k u_{i,j}^k v \mathrm{d}\tau_k \mathrm{d}\theta_k \right| \leq \| \widetilde{\mathfrak{L}}^k u_{i,j}^k (\tau_k, \theta_k) \|_{0, \widetilde{\Omega}_{i,j}^k} \| v(\tau_k, \theta_k) \|_{0, \widetilde{\Omega}_{i,j}^k}.$$

Finally

$$\left| \int_{\Omega_l^{p+1}} (\mathfrak{L} u_l^{p+1}) v \mathrm{d}x \right| \leq \| \mathfrak{L} u_l^{p+1} (x_1, x_2) \|_{0, \Omega_l^{p+1}} \| v(x_1, x_2) \|_{0, \Omega_l^{p+1}}.$$

Now

$$\sum_{k=1}^{p}\sum_{j=2}^{M}\sum_{i=1}^{I_k}\|v(\nu_k,\phi_k)\|^2_{0,\widehat{\Omega}^k_{i,j}} \le K_M\|v(x_1,x_2)\|^2_{1,\Omega}.$$

Here $K_M = KM^2$ if there is a vertex A_j such that Neumann boundary conditions are imposed on the adjoining sides Γ_j and Γ_{j+1} and $K_M = K$, otherwise. K denotes a constant. Hence

$$\sum_{k=1}^{p}\sum_{j=2}^{M}\sum_{i=1}^{I_k}\|v(\nu_k,\phi_k)\|^2_{1,\widehat{\Omega}^k_{i,j}} \le K_M\|v(x_1,x_2)\|^2_{1,\Omega}. \tag{3.45}$$

Now using the trace theorem for Sobolev spaces we obtain

$$\sum_{k=1}^{p}\sum_{j=2}^{M}\sum_{i=1}^{I_k}\|v\|^2_{0,\partial\widehat{\Omega}^k_{i,j}} \le K_M\|v(x_1,x_2)\|^2_{1,\Omega}.$$

And so we can conclude that

$$\sum_{k=1}^{p}\sum_{j=2}^{M}\sum_{i=1}^{I_k}\int_{\partial\widetilde{\Omega}^k_{i,j}} v^2\,\mathrm{d}\sigma \le K_M\|v(x_1,x_2)\|^2_{1,\Omega}. \tag{3.46}$$

Using the Cauchy–Schwartz inequality in (3.44) and using (3.45) and (3.46) we can conclude that

$$\begin{aligned}
|B(\varphi,v)|^2 \le K\Bigg\{&\sum_{k=1}^{p}\sum_{j=2}^{M}\sum_{i=1}^{I_k}\|\widetilde{\mathfrak{L}}^k u^k_{i,j}(\tau_k,\theta_k)\|^2_{0,\widetilde{\Omega}^k_{i,j}} + \sum_{k=1}^{p}\sum_{i=1}^{I_k}\varepsilon|u^k_{i,1}|^2\\
&+\sum_{k=1}^{p}\left(\sum_{\gamma_s\subseteq\Omega^k}\int_{\widetilde{\gamma}_s}\left[\left(\frac{\partial u}{\partial n}\right)_{\widetilde{A}^k}\right]^2\mathrm{d}\sigma + \sum_{\gamma_s\subseteq B^k_\rho}\int_{\widetilde{\gamma}_s}\left[\left(\frac{\partial u}{\partial n}\right)_{\widetilde{A}^k}\right]^2\mathrm{d}\sigma\right)\\
&+\sum_{l\in\mathcal{N}}\sum_{k=l-1}^{l}\sum_{\gamma_s\subseteq\Gamma_l\bigcap\partial\Omega^k}\int_{\widetilde{\gamma}_s}\left(\frac{\partial u}{\partial n}\right)^2_{\widetilde{A}^k}\mathrm{d}\sigma\\
&+\sum_{l=1}^{L}\int_{\Omega^{p+1}_l}\int|\mathfrak{L}u^{p+1}_l(x_1,x_2)|^2\mathrm{d}x_1\mathrm{d}x_2\\
&+\sum_{\gamma_s\subseteq\Omega^{p+1}}\int_{\gamma_s}\left[\left(\frac{\partial u}{\partial N}\right)_A\right]^2\mathrm{d}s\\
&+\sum_{l\in\mathcal{N}}\sum_{\gamma_s\subseteq\Gamma_l\bigcap\partial\Omega^{p+1}}\int_{\gamma_s}\left(\frac{\partial u}{\partial N}\right)^2_A\mathrm{d}s\\
&+\varepsilon\sum_{k=1}^{p}\sum_{i=1}^{I_k}|\lambda^k_{i,1}|^2 + \sum_{k=1}^{p}\sum_{j=2}^{M}\sum_{i=1}^{I_k}\|\lambda^k_{i,j}(\tau_k,\theta_k)\|^2_{1,\widetilde{\Omega}^k_{i,j}}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{l=1}^{L}\|\lambda_l^{p+1}(x,y)\|^2_{1,\Omega_l^{p+1}}\Bigg\}\cdot\Bigg\{\sum_{k=1}^{p}\sum_{i=1}^{I_k}\|v(x_1,x_2)\|^2_{0,\Omega^k_{i,1}}\\
&+\sum_{k=1}^{p}\sum_{j=2}^{M}\sum_{i=1}^{I_k}\|v(\tau_k,\theta_k)\|^2_{1,\widetilde{\Omega}^k_{i,j}}\\
&+\sum_{l=1}^{L}\|v_l^{p+1}(x,y)\|^2_{1,\Omega_l^{p+1}}+\sum_{k=1}^{p}\sum_{j=2}^{M}\sum_{i=1}^{I_k}\int_{\partial\widetilde{\Omega}^k_{i,j}}v^2\mathrm{d}\sigma\\
&+\sum_{\gamma_s\subseteq\Omega^{p+1}}\int_{\gamma_s}v^2\mathrm{d}s+\sum_{l\in\mathcal{N}}\sum_{\gamma_s\subseteq\partial\Omega^{p+1}\bigcap\Gamma_l}\int_{\gamma_s}v^2\mathrm{d}s\Bigg\}.
\end{aligned}
$$

Now $v\in H_0^1(\Omega)$ and $\mathfrak{L}$ satisfies the inf–sup conditions (3.4). Hence using (3.5), (3.43) and (3.46) we obtain

$$
\begin{aligned}
\|\varphi\|^2_{1,\Omega}\le K_M&\Bigg\{\sum_{k=1}^{p}\sum_{j=2}^{M}\sum_{i=1}^{I_k}\|\widetilde{\mathfrak{L}}^k u^k_{i,j}(\tau_k,\theta_k)\|^2_{0,\widetilde{\Omega}^k_{i,j}}\\
&+\sum_{k=1}^{p}\Bigg(\sum_{\gamma_s\subseteq\Omega^k}(\|[u]\|^2_{0,\widehat{\gamma}_s}+\|[u_{\nu_k}]\|^2_{0,\widehat{\gamma}_s}+\|[u_{\phi_k}]\|^2_{0,\widehat{\gamma}_s})\\
&+\sum_{l\in\mathcal{N}}\sum_{k=l-1}^{l}\sum_{\gamma_s\subseteq\partial\Omega^k\bigcap\Gamma_l}\int_{\widetilde{\gamma}_s}\left(\frac{\partial u}{\partial n}\right)^2_{\widetilde{A}^k}\mathrm{d}\sigma\\
&+\sum_{l\in\mathcal{D}}\sum_{k=l-1}^{l}\sum_{\gamma_s\subseteq\Gamma_l\bigcap\partial\Omega^k,\mu(\widehat{\gamma}_s)<\infty}(\|u\|^2_{0,\widehat{\gamma}_s}+\|u_{\nu_k}\|^2_{0,\widehat{\gamma}_s})\Bigg)\\
&+\sum_{k=1}^{p}\sum_{\gamma_s\subseteq B^k_\rho}(\|[u]\|^2_{0,\widehat{\gamma}_s}+\|[u_{\nu_k}]\|^2_{0,\widehat{\gamma}_s}+\|[u_{\phi_k}]\|^2_{0,\widehat{\gamma}_s})\\
&+\sum_{l=1}^{L}\int_{\Omega_l^{p+1}}\int|\mathfrak{L}u_l^{p+1}(x_1,x_2)|^2\mathrm{d}x_1\mathrm{d}x_2\\
&+\sum_{\gamma_s\subseteq\Omega^{p+1}}(\|[u]\|^2_{0,\gamma_s}+\|[u_{x_1}]\|^2_{0,\gamma_s}+\|[u_{x_2}]\|^2_{0,\gamma_s})\\
&+\sum_{l\in\mathcal{D}}\sum_{\gamma_s\subseteq\partial\Omega^{p+1}\bigcap\Gamma_l}\left(\|u\|^2_{0,\gamma_s}+\left\|\frac{\partial u}{\partial T}\right\|^2_{0,\gamma_s}\right)\\
&+\sum_{l\in\mathcal{N}}\sum_{\gamma_s\subseteq\partial\Omega^{p+1}\bigcap\Gamma_l}\int\left(\frac{\partial u}{\partial N}\right)^2_{A}\mathrm{d}s+\varepsilon\sum_{k=1}^{p}\sum_{i=1}^{I_k}(|u^k_{i,1}|^2+|\lambda^k_{i,1}|^2)\Bigg\}.
\end{aligned}
$$

Here ε is exponentially small in M.

Using (3.43) and (3.45) once more we obtain the result. ■

We now define differential operators $(\mathfrak{L}^k_{i,j})^a$ which are second order differential operators with polynomial coefficients in ν_k and ϕ_k of degree W such that these coefficients are exponentially close approximation to the coefficients of $(\mathfrak{L}^k_{i,j})$ as has been described in the beginning of this section. In the same way we define the differential operator $(\partial u/\partial n)^a_{\widetilde{A}^k}$ to be a first order differential operator with polynomial coefficients in ν_k and ϕ_k such that these coefficients are exponentially close approximations to the coefficients of $(\partial u/\partial n)_{\widetilde{A}^k}$. The other approximations are similarly defined.

From the above, it is easy to conclude that

$$\sum_{k=1}^{p}\sum_{i=1}^{I_k}\left(|u^k_{i,1}|^2+\sum_{j=2}^{M}\|u^k_{i,j}(\nu_k,\phi_k)\|^2_{1,\widehat{\Omega}^k_{i,j}}\right) + \sum_{l=1}^{L}\|u^{p+1}_l(\xi,\eta)\|^2_{1,S}\le C_M(\mathcal{I}), \tag{3.47}$$

where

$$\begin{aligned}\mathcal{I}=\Bigg\{&\sum_{k=1}^{p}\sum_{j=2}^{M}\sum_{i=1}^{I_k}\|(\mathfrak{L}^k_{i,j})^a u^k_{i,j}(\nu_k,\phi_k)\|^2_{0,\widehat{\Omega}^k_{i,j}}\\
&+\sum_{k=1}^{p}\sum_{\gamma_s\subseteq\Omega^k}(\|[u]\|^2_{0,\widehat{\gamma}_s}+\|[u_{\nu_k}]\|^2_{0,\widehat{\gamma}_s}+\|[u_{\phi_k}]\|^2_{0,\widehat{\gamma}_s})\\
&+\sum_{l\in\mathcal{D}}\sum_{k=l-1}^{l}\sum_{\gamma_s\subseteq\partial\Omega^k\bigcap\Gamma_l,\mu(\widehat{\gamma}_s)<\infty}(\|u\|^2_{0,\widehat{\gamma}_s}+\|u_{\nu_k}\|^2_{0,\widehat{\gamma}_s})\\
&+\sum_{l\in\mathcal{N}}\sum_{k=l-1}^{l}\sum_{\gamma_s\subseteq\partial\Omega^k\bigcap\Gamma_l,\mu(\widehat{\gamma}_s)<\infty}\left\|\left(\frac{\partial u}{\partial n}\right)^a_{\widetilde{A}^k}\right\|^2_{0,\widehat{\gamma}_s}\\
&+\sum_{k=1}^{p}\sum_{\gamma_s\subseteq B^k_\rho}(\|[u]\|^2_{0,\widehat{\gamma}_s}+\|[(u)^a_{\nu_k}]\|^2_{0,\widehat{\gamma}_s}+\|[(u)^a_{\phi_k}]\|^2_{0,\widehat{\gamma}_s})\\
&+\sum_{l=1}^{L}\|(\mathfrak{L}^{p+1}_l)^a u^{p+1}_l(\xi,\eta)\|^2_{0,S}\\
&+\sum_{\gamma_s\subseteq\Omega^{p+1}}(\|[u]\|^2_{0,\gamma_s}+\|[u_{x_1}]^a\|^2_{0,\gamma_s}+\|[u_{x_2}]^a\|^2_{0,\gamma_s})\\
&+\sum_{l\in\mathcal{D}}\sum_{\gamma_s\subseteq\partial\Omega^{p+1}\bigcap\Gamma_l}\left(\|u\|^2_{0,\gamma_s}+\left\|\left(\frac{\partial u}{\partial T}\right)^a\right\|^2_{0,\gamma_s}\right)\\
&+\sum_{l\in\mathcal{N}}\sum_{\gamma_s\subseteq\partial\Omega^{p+1}\bigcap\Gamma_l}\left\|\left(\frac{\partial u}{\partial N}\right)^a_A\right\|^2_{0,\gamma_s}\Bigg\}.\end{aligned}$$

Here C_M is as defined in Theorem 3.1.

3.3 *The estimates*

We now define the quadratic form

$$
\begin{aligned}
&\mathcal{V}^{M,W}(\{u_{i,j}^k(\nu_k,\phi_k)\}_{i,j,k},\{u_l^{p+1}(\xi,\eta)\}_l) \\
&= \Bigg\{ \sum_{k=1}^{p}\sum_{j=2}^{M} \|(\mathfrak{L}_{i,j}^k)^a u_{i,j}^k(\nu_k,\phi_k)\|^2_{0,\widehat{\Omega}_{i,j}^k} \\
&\quad + \sum_{k=1}^{p}\sum_{\gamma_s\subseteq\Omega^k} (\|[u]\|^2_{0,\widehat{\gamma}_s} + \|[u_{\nu_k}]\|^2_{1/2,\widehat{\gamma}_s} + \|[u_{\phi_k}]\|^2_{1/2,\widehat{\gamma}_s}) \\
&\quad + \sum_{l\in\mathcal{D}}\sum_{k=l-1}^{l}\sum_{\gamma_s\subseteq\partial\Omega^k\bigcap\Gamma_l,\mu(\widehat{\gamma}_s)<\infty} (\|u\|^2_{0,\widehat{\gamma}_s} + \|u_{\nu_k}\|^2_{1/2,\widehat{\gamma}_s}) \\
&\quad + \sum_{l\in\mathcal{N}}\sum_{k=l-1}^{l}\sum_{\gamma_s\subseteq\partial\Omega^k\bigcap\Gamma_l,\mu(\widehat{\gamma}_s)<\infty} \left\| \left(\frac{\partial u}{\partial n}\right)^a_{\widetilde{A}^k} \right\|^2_{1/2,\widehat{\gamma}_s} \\
&\quad + \sum_{k=1}^{p}\sum_{\gamma_s\subseteq B_\rho^k} (\|[u]\|^2_{0,\widehat{\gamma}_s} + \|[(u)^a_{\nu_k}]\|^2_{1/2,\widehat{\gamma}_s} + \|[(u)^a_{\phi_k}]\|^2_{1/2,\widehat{\gamma}_s}) \\
&\quad + \sum_{l=1}^{L} \|(\mathfrak{L}_l^{p+1})^a u_l^{p+1}(\xi,\eta)\|^2_{0,S} \\
&\quad + \sum_{\gamma_s\subseteq\Omega^{p+1}} (\|[u]\|^2_{0,\gamma_s} + \|[u_{x_1}]^a\|^2_{1/2,\gamma_s} + \|[u_{x_2}]^a\|^2_{1/2,\gamma_s}) \\
&\quad + \sum_{l\in\mathcal{D}}\sum_{\gamma_s\subseteq\partial\Omega^{p+1}\bigcap\Gamma_l} \left(\|u\|^2_{0,\gamma_s} + \left\| \left(\frac{\partial u}{\partial T}\right)^a \right\|^2_{1/2,\gamma_s} \right) \\
&\quad + \sum_{l\in\mathcal{N}}\sum_{\gamma_s\subseteq\partial\Omega^{p+1}\bigcap\Gamma_l} \left\| \left(\frac{\partial u}{\partial N}\right)^a_A \right\|^2_{1/2,\gamma_s} \Bigg\}
\end{aligned}
\tag{3.48}
$$

We can now state the main result of this section.

Theorem 3.2. *Let* $\mathcal{V}^{M,W}(\{u_{i,j}^k(\nu_k,\phi_k)\}_{i,j,k},\{u_l^{p+1}(\xi,\eta)\}_l)$ *be as defined in* (3.48). *Then for M and W large enough the estimate*

$$
\begin{aligned}
&\sum_{k=1}^{p}\sum_{i=1}^{I_k}\left(|u_{i,1}^k|^2 + \sum_{j=2}^{M}\|u_{i,j}^k(\nu_k,\phi_k)\|^2_{2,\widehat{\Omega}_{i,j}^k}\right) + \sum_{l=1}^{L}\|u_l^{p+1}(\xi,\eta)\|^2_{2,S} \\
&\le C_{M,W}\mathcal{V}^{M,W}(\{u_{i,j}^k(\nu_k,\phi_k)\}_{i,j,k},\{u_l^{p+1}(\xi,\eta)\}_l)
\end{aligned}
\tag{3.49}
$$

holds for all $\{\{u_{i,j}^k(\nu_k,\phi_k)\}_{i,j,k},\{u_l^{p+1}(\xi,\eta)\}_l\}\in\pi^{M,W}$.

Here $C_{M,W} = C$ *maximum* $(M^4,(\ln W)^2)$ *if there is a vertex* A_j *such that Neumann boundary conditions are imposed on the adjoining sides* Γ_j *and* Γ_{j+1} *and* $C_{M,W} = C(\ln W)^2$ *otherwise. C is a constant, independent of M and W.*

Adding a weighted combination of (3.36), (3.39) and (3.47) and using the techniques and results of [3] the result follows. ■

Remark 2. The stability theorem 3.2 holds provided the coefficients of the differential operator $\in C^3(\overline{\Omega})$ and the curves $\phi^k_{i,j,l}$ and $\psi^k_{i,j,l}$ defined in (2.6a), (2.6b) satisfy (2.7) for $t = 1, \ldots, 3$.

For problems with mixed boundary conditions the factor multiplying the right-hand side of (3.49) grows rapidly with M. This creates difficulties in parallelizing the numerical scheme. To overcome this we make the spectral element functions continuous at the vertices of the elements. Let $\pi_V^{M,W}$ denote the space of spectral element functions which are continuous at the vertices of their elements. We define $\pi_0^{M,W}$ to be the space of spectral element functions which vanish at the vertices of their element. We now need to state a version of Theorem 3.2 when the spectral element functions vanish at the vertices of their elements.

To do so, we have to prove the following result.

Lemma 3.5. *Let $u^k_{i,j}(\xi,\eta)$ be a polynomial of degree W in ξ and η separately, defined on the unit square $S = (0,1)\times(0,1)$, and which is zero at all the vertices of the square. Then there exists a positive constant C such that*

$$|u^k_{i,j}(\xi,\eta)|^2_{0,S} \le C(|u^k_{i,j}(\xi,\eta)|^2_{1,S} + |u^k_{i,j}(\xi,\eta)|^2_{2,S}). \tag{3.50}$$

Consider $u^k_{i,j}(\xi,\eta)$ defined on $(0,1)\times(0,1)$. Now $u^k_{i,j}(0,0) = 0$. Hence

$$u^k_{i,j}(\xi,0) = \int_0^\xi \frac{\partial u^k_{i,j}}{\partial \xi'}(\xi',0)\mathrm{d}\xi'.$$

And so we can conclude that

$$|u^k_{i,j}(\xi,0)|^2 \le \xi \int_0^1 \left|\frac{\partial u^k_{i,j}}{\partial \xi}(\xi,0)\right|^2 \mathrm{d}\xi.$$

Integrating the above with respect to ξ we obtain

$$\begin{aligned}\int_0^1 |u^k_{i,j}(\xi,0)|^2\mathrm{d}\xi &\le \frac{1}{2}\int_0^1 \left|\frac{\partial u^k_{i,j}}{\partial \xi}(\xi,0)\right|^2 \mathrm{d}\xi \\ &\le K(|u^k_{i,j}(\xi,\eta)|^2_{1,S} + |u^k_{i,j}|^2_{2,S})\end{aligned} \tag{3.51}$$

by the trace theorem for Sobolev spaces. Again

$$u^k_{i,j}(\xi,\eta) = u^k_{i,j}(\xi,0) + \int_0^\eta \frac{\partial u^k_{i,j}}{\partial \eta'}(\xi,\eta')\mathrm{d}\eta'.$$

Therefore

$$|u^k_{i,j}(\xi,\eta)|^2 \le 2|u^k_{i,j}(\xi,0)|^2 + 2\eta\int_0^1 \left|\frac{\partial u^k_{i,j}}{\partial \eta}(\xi,\eta)\right|^2 \mathrm{d}\eta.$$

Integrating the above with respect to ξ and η we get

$$\int_S\int |u^k_{i,j}(\xi,\eta)|^2 \mathrm{d}\xi\mathrm{d}\eta \leq 2\int_0^1 |u^k_{i,j}(\xi,0)|^2\mathrm{d}\xi + \int_S\int \left|\frac{\partial u^k_{i,j}}{\partial\eta}(\xi,\eta)\right|^2 \mathrm{d}\xi\mathrm{d}\eta.$$

Combining the above with (3.51) we obtain the required result. ■

Clearly Lemma 3.5 applies equally well to any of the function elements $u^k_{i,j}(\nu_k,\phi_k)$ for $2\leq j\leq M$, $1\leq i\leq I_k$, $1\leq k\leq p$, although with a constant C_k which depends on k. Taking the supremum over the constant C_k (as given in (3.50)) we conclude that

$$|u^k_{i,j}(\nu_k,\phi_k)|^2_{0,\widehat{\Omega}^k_{i,j}} \leq C(|u^k_{i,j}(\nu_k,\phi_k)|^2_{1,\widehat{\Omega}^k_{i,j}} + |u^k_{i,j}(\nu_k,\phi_k)|^2_{2,\widehat{\Omega}^k_{i,j}}), \tag{3.52}$$

for all function elements with $1\leq k\leq p$, $1\leq i\leq I_k$, $2\leq j\leq M$. Here C, of course, denotes a generic constant. We can now state the final result of this section.

Theorem 3.3. *Let* $\{\{u^k_{i,j}(\nu_k,\phi_k)\}_{i,j,k},\{u^{p+1}_l(\xi,\eta)\}_l\}$ *belong to the space of functions* $\pi_0^{M,W}$ *which are zero at the vertices of the elements on which they are defined. Then the following estimate holds*:

$$\sum_{k=1}^{p}\sum_{j=2}^{M}\sum_{i=1}^{I_k} \|u^k_{i,j}(\nu_k,\phi_k)\|^2_{2,\widehat{\Omega}^k_{i,j}} + \|u^{p+1}_l(\xi,\eta)\|^2_{2,S} \leq C(\ln W)^2\mathcal{V}^{M,W}(\{u^k_{i,j}(\nu_k,\phi_k)\}_{i,j,k},\{u^{p+1}_l(\xi,\eta)\}_l) \tag{3.53}$$

for M and W large enough.

In the above $u^k_{i,1}(\nu_k,\phi_k)$ is taken to be identically zero for $1\leq k\leq p$ and $1\leq i\leq I_k$. Combining the estimates (3.50) and (3.52) with the earlier results (3.53) follows. ■

4. Conclusion

We can use the stability theorem 3.2 to formulate a numerical scheme to obtain an approximate solution to the elliptic boundary value problem (2.1) as has been described in [8,9]. For problems with Dirichlet boundary conditions we choose our solution to be a non-conforming spectral element representation which minimizes a functional which is the sum of the squares of weighted squared norms of the residuals in the partial differential equation and fractional Sobolev norms of the residuals in the boundary conditions and a term which measures the sum of the jumps in the function and its derivatives in appropriate Sobolev norms at inter-element boundaries. In a sectoral neighbourhood of the corners these quantities are computed using modified polar coordinates and in the remaining part of the domain we use a global coordinate system. This method is faster than the h-p finite element method as there are no common boundary values to solve for [8,9].

For problems with mixed boundary conditions we have to make the spectral element functions continuous only at the vertices of the elements. As a result the Schur complement matrix has a small dimension and an accurate inverse can be computed. Hence the numerical scheme has a computational complexity which is less for finite element methods.

Moreover, the construction of a pre-conditioner for the Schur complement matrix is very simple unlike the case for finite element methods. In fact, for problems in three dimensions the construction of pre-conditioners for the Schur complement matrix becomes quite complex for finite element methods [6].

Though the ideas in these papers deal with problems in two dimensions, they generalize to three dimensions. We intend to study these problems both theoretically and computationally in future work.

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