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Strong robustness in multi-phase adaptive control: the basic scheme

Maria Cadic[†] and Jan Willem Polderman[*,‡]

Faculty of Electrical Engineering, Mathematics and Computer Sciences, University of Twente, P.O. Box 217, 7500-AE Enschede, The Netherlands

SUMMARY

The general structure of adaptive control systems based on strong robustness is introduced. This approach splits into two phases. In the first phase, emphasis is put on identification until enough information is obtained in order to design a controller that stabilizes the actual system, and even under adaptation. This is achieved if the input sequence is computed in such a way that the uncertainty on the parameters of the system to be controlled becomes sufficiently small. Then, in the second phase, effort is shifted to control via a traditional certainty equivalence type of strategy. Copyright © 2004 John Wiley & Sons, Ltd.

KEY WORDS: linear SISO systems; strong robustness; adaptive control

1. INTRODUCTION

The majority of adaptive control algorithms are based on the certainty equivalence principle [1, 2]. In each iteration of these algorithms, a new model of the unknown true system is computed from the previous model and measured data, and a new controller is designed on the basis of this model as if it were the true system. However, because of the model uncertainty, there is little reason to believe that the controller based on this model stabilizes the true system. Hence, undesired transients may arise in the closed-loop system behaviour. In addition, even in the case where closed-loop stability is obtained at each frozen time instant, time-variations induced by the model adaptation might destroy the closed-loop stability of the time-varying system and completely disrupt the performance.

To avoid bad transients, we design a test checking on-line if the model-based controller stabilizes the true system, even when updated at each time instant. We would apply the model-based controller to the system only when we know that the closed-loop stability is guaranteed, this for any choice of the model in the model set, and at any (present and future) time of the design. We construct this test on the basis of the set of all candidate models, the *uncertainty set*.

*Correspondence to: J. W. Polderman, Department of Applied Mathematics, University of Twente, P.O. Box 217, 7500-AE Enschede, The Netherlands.
† E-mail: m.a.cadic@math.utwente.nl
‡ E-mail: j.w.polderman@math.utwente.nl

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At each time the uncertainty set is updated, we check whether any time-varying model taken from this set yields a (time-varying) controller that stabilizes any fixed system in the set. If the uncertainty set has this property, it is said to be *strongly robust*. In particular, since the uncertainty set contains the true system to be controlled, this would guarantee that any model taken from this set stabilizes the true system, even under adaptation. Moreover, since in our framework the uncertainty set cannot expand with time, this would guarantee that stability will be maintained at each future time of the design, independently of the model update law, provided that the sequence of models stays within the uncertainty set.

Inspired by this idea, we construct an adaptive control system based on two phases. In the first phase, emphasis is put on identification until the identified uncertainty set is strongly robust. When strong robustness is achieved, the adaptive system is then allowed to switch to the second phase where the effort is shifted to control, using a certainty equivalence routine similar to classical adaptive control.

Our purpose in this paper is to describe at a very general level the structure of the adaptive control scheme based on strong robustness and to refer to previous work on related issues. In the next section, we define the problem set-up. In Section 3, we introduce the general structure of the adaptive control system based on strong robustness. Next, in Section 4, we address two fundamental issues: the existence of strongly robust sets of systems and their characterization. In addition, we show how the identification input should be computed to guarantee that the presented scheme switches to the second phase in finite time. Further, in Section 5, the analysis of the proposed adaptive control scheme is discussed. As a suggestion for future research, we then address in Section 6.2 two issues that would potentially lead to an improvement of the presented scheme in the sense that the time at which the control phase can start is decreased. Finally, we conclude in Section 7.

2. GENERAL SET-UP OF THE PROBLEM

We first introduce some notation.

Definition 2.1 (Models)
We denote by $\mathscr{P}_n$ the set of linear time-invariant systems of order n described in discrete time by

$$y(k+1) = \theta^{\mathrm{T}} \phi(k), \quad \forall k \tag{1}$$

where $\phi(k)$ represents the regressor vector given by

$$\phi(k) = (-y(k), \ldots, -y(k-n+1), u(k), \ldots, u(k-n+1))^{\mathrm{T}} \in \mathbb{R}^{2n} \tag{2}$$

denoting by u, y the input and output sequences, respectively, and

$$\theta = (a_{n-1}, \ldots, a_0, b_{n-1}, \ldots, b_0)^{\mathrm{T}} \in \mathbb{R}^{2n} \tag{3}$$

denotes the parameter vector.

To keep notation simple, we will associate $\mathscr{P}_n$ to $\mathbb{R}^{2n}$. In the sequel, '$\theta \in \mathscr{P}_n$' should be read as 'the system parameterized by θ and described by (1)'.

Definition 2.2 (Asymptotically stable systems and controllable systems in $\mathscr{P}_n$)
We denote by $\mathscr{S}_n$ the set of asymptotically stable systems in $\mathscr{P}_n$ and by $\mathscr{C}_n$ the set of controllable systems in $\mathscr{P}_n$. Controllability here refers to the case where no pole-zero cancellation occurs.

Assumption 2.3 (System to be controlled)
The system to be controlled is described by

$$y(k+1) = (\theta^0)^{\mathrm{T}}\phi(k) + \delta(k), \quad k \geqslant 0 \tag{4}$$

where $\theta^0 = (a^0_{n-1}, \ldots, a^0_0, b^0_{n-1}, \ldots, b^0_0)^{\mathrm{T}} \in \mathscr{S}_n \cap \mathscr{C}_n$ denotes the unknown parameter vector and $\delta(k)$ represents the modelling error which is unknown but bounded, with a *known* bound d, i.e., $|\delta(k)| \leqslant d$, $\forall k \geqslant 0$.

Remark 2.4

The assumption that system (4) is controllable is motivated by our control purpose. The assumption that the system to be controlled is open-loop asymptotically stable is for the sake of open-loop identification in the first phase of the algorithm. As stated below, the control objective is not only to stabilize the real plant but also to improve its performance. The assumption that a bound d on the modelling error is known is chosen as the simplest case of approximate modelling. No stochastic assumptions on the modelling error is made, since for small data sets they may not be justified.

The control objective is left unspecified. Instead, we make the following assumption.

Assumption 2.5 (Controllers)
There exists a single-valued continuous map f assigning to any system $\theta \in \mathscr{C}_n$ its controller $f(\theta)$ leading to the control law

$$u(k) = f(\theta)x(k) \quad \forall k \tag{5}$$

where x denotes the non-minimal state-vector

$$x(k) = (-y(k), \ldots, -y(k-n+1), u(k-1), \ldots, u(k-n+1))^{\mathrm{T}} \in \mathbb{R}^{2n-1} \tag{6}$$

and such that the resulting closed-loop system (1), (5) is asymptotically stable.

We now define strongly robust sets of systems in $\mathscr{C}_n$.

Definition 2.6 (Strongly robust sets)
A set $\Omega \subset \mathscr{C}_n$ is strongly robust if for any system $\theta \in \Omega$ and for any sequence of systems $\{\theta(k)\}_{k\in\mathbb{N}} \subset \Omega$, the sequence of controllers $\{f(\theta(k))\}_{k\in\mathbb{N}}$ according to (5) is such that the time-varying closed-loop system defined by

$$\begin{aligned} y(k+1) &= \theta^{\mathrm{T}}\phi(k) \\ u(k) &= f(\theta(k))x(k) \end{aligned} \tag{7}$$

where $\phi(k)$ and $x(k)$ are defined in (2) and (6), respectively, is asymptotically stable. More precisely, $\Omega \subset \mathscr{C}_n$ is strongly robust if for any system $\theta \in \Omega$, for any sequence of systems $\{\theta(k)\}_{k\in\mathbb{N}} \subset \Omega$, for any initial state $x(0) \in \mathbb{R}^{2n-1}$, if $u(k) = 0$, $\forall k \geqslant 0$, then $\lim_{k\to\infty} x(k) = 0$.

Remark 2.7

Continuity in Assumption 2.5 is essential for the existence of strongly robust sets of systems [3]. Pole placement and LQ control satisfy Assumption 2.5.

Remark 2.8

Whether a set of systems is strongly robust or not depends on the control objective, i.e. a given uncertainty set can be strongly robust with respect to some control objectives and not strongly robust with respect to some others.

3. MULTI-PHASE ADAPTIVE CONTROL SYSTEMS BASED ON STRONG ROBUSTNESS: THE GENERAL STRUCTURE

Figure 1 describes the general scheme of the adaptive control systems based on strong robustness. Two phases are distinguished. In the first phase, the *identification phase*, data measurements are used to compute the set of all candidate models that are consistent with these measurements, and at each time, it is tested whether this set is strongly robust or not. If this test is positive, the system switches to the second phase, the *control phase*, where a certainty

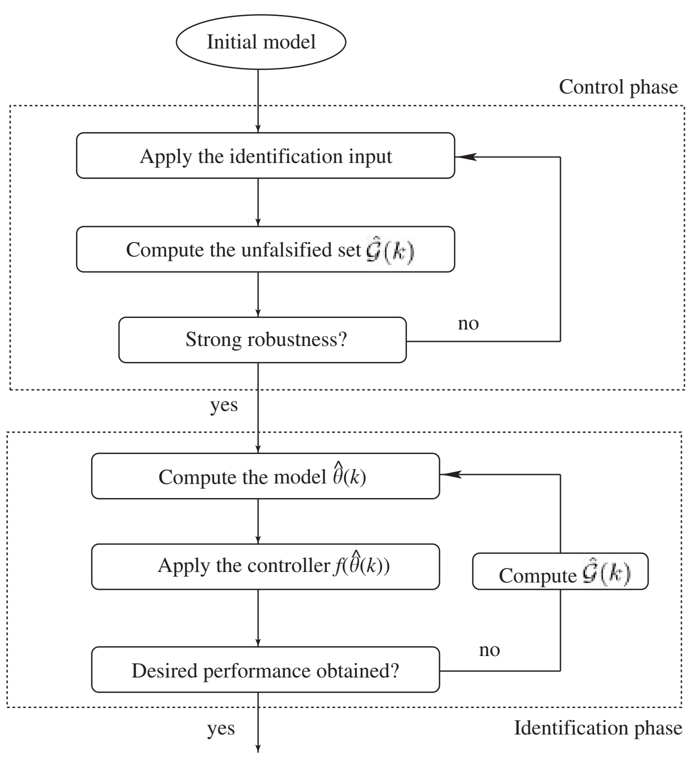

Figure 1. Iterative scheme.

equivalence type of control strategy is applied: at each measurement the model is updated within the strongly robust model set and the controller is designed on this model. Applying this controller on the actual plant leads to new input–output measurements, and subsequently, the uncertainty set, the model and the controller can be tuned more accurately so as to improve the closed-loop performance.

3.1. Identification phase

During the identification phase, two tasks are completed at each measurement: the *unfalsified set* is computed, and a test is applied on this set to check whether it is strongly robust or not.

Compute the unfalsified set $\hat{\mathcal{G}}(k)$: Model set estimation in the case of unknown but bounded modelling error has received some constant attention [4–10]. It consists in computing the unfalsified set $\hat{\mathcal{G}}(k)$, defined as the set of parameter vectors θ consistent with the i/o data measurements up to time k and the known bound on the modelling error. Formally, $\hat{\mathcal{G}}(k)$ is the polytope defined by

$$\hat{\mathcal{G}}(k) = \{\theta \in \mathbb{R}^{2n} : |y(i) - \theta^{\mathrm{T}}\phi(i-1)| \leqslant d, \quad \forall i \leqslant k\} \tag{8}$$

and is computed as the intersection of $2k$ half spaces in the parameter space defined by equations of the form

$$\begin{aligned} y(i) - \theta^{\mathrm{T}}\phi(i-1) \leqslant d, \quad \forall i \leqslant k \\ y(i) - \theta^{\mathrm{T}}\phi(i-1) \geqslant -d, \quad \forall i \leqslant k \end{aligned} \tag{9}$$

Denoting by $\mathcal{G}(k)$ the set of parameter vectors consistent with the i/o data at time k:

$$\mathcal{G}(k) = \{\theta : |y(k) - \theta^{\mathrm{T}}\phi(k-1)| \leqslant d\} \tag{10}$$

$\hat{\mathcal{G}}(k)$ can be computed as

$$\hat{\mathcal{G}}(k) = \mathcal{G}(k) \cap \hat{\mathcal{G}}(k-1) = \bigcap_{i \leqslant k} \mathcal{G}(i) \tag{11}$$

Remark 3.1

The set defined in (11) is convex.

Check Strong Robustness: This test indicates when to switch to the second phase of the algorithm. If the model set $\hat{\mathcal{G}}(k)$ is strongly robust, the adaptive system switches to the second phase.

3.2. Control phase

Let us suppose that the set $\hat{\mathcal{G}}(k)$ defined in (11) becomes strongly robust at the time $T > 0$. The control phase is then started. During this phase, the model is updated, and the controller is designed on the basis of this estimate.

Compute the model $\hat{\theta}(k)$: At each measurement, the model $\hat{\theta}(k)$ of the true parameter vector θ^0 is updated, leading to the new model $\hat{\theta}(k+1)$. Since $\theta^0 \in \hat{\mathcal{G}}(k)$, we naturally choose $\hat{\theta}(T)$ as a member of $\hat{\mathcal{G}}(T)$, and $\forall k \geqslant T$, the model $\hat{\theta}(k+1)$ is computed as the orthogonal projection of the previous estimate $\hat{\theta}(k)$ on the set $\hat{\mathscr{G}}(k+1)$. This leads to the following update procedure:

$$\hat{\theta}(T) \text{ is arbitrarily chosen in } \hat{\mathscr{G}}(T)$$

$$\hat{\theta}(k+1) = \arg \min_{\theta \in \hat{\mathscr{G}}(k+1)} \{(\theta - \hat{\theta}(k))^{\mathrm{T}}(\theta - \hat{\theta}(k))\} \quad \forall k \geqslant T \tag{12}$$

Remark 3.2

From our prior knowledge in Assumption 2.3, the true system θ_0 is open-loop asymptotically stable and controllable. Hence it seems that a 'good' model should also be open-loop asymptotically stable and controllable, in which case we should have $\hat{\theta}(k) \in \hat{\mathscr{G}}(k) \cap \mathscr{S}_n \cap \mathscr{C}_n$. Now, note that if the model set is strongly robust, then all its members are controllable. Hence, in the control phase, the update law (12) guarantees that $\hat{\theta}(k) \in \hat{\mathscr{G}}(k) \cap \mathscr{C}_n$. On the other hand, open-loop asymptotic stability of the estimate is not guaranteed by the update law (12), but we emphasize that stability of the model is not required. Moreover, $\mathscr{S}_n$ is neither convex nor closed, hence $\hat{\mathscr{G}}(k) \cap \mathscr{S}_n$ may be neither convex nor closed. Hence, generating the new estimate in $\hat{\mathscr{G}}(k) \cap \mathscr{S}_n$ using orthogonal projection as in (12) might be computationally expensive or infeasible.

Design a controller: Since $\hat{\mathscr{G}}(k)$ is strongly robust $\forall k \geqslant T$, it follows that $\hat{\mathscr{G}}(k) \subset \mathscr{C}_n$ $\forall k \geqslant T$. Following Assumption 2.5, at each time $k \geqslant T$, we compute the controller $f(\hat{\theta}(k))$, leading to the updated control input law:

$$u(k) = f(\hat{\theta}(k))x(k) \quad \forall k \geqslant T \tag{13}$$

From Assumption 2.5, and because of strong robustness, the time-varying controller $f(\hat{\theta}(k))$ stabilizes the true plant θ_0 $\forall k \geqslant T$.

4. STRONGLY ROBUST MODEL SET: EXISTENCE AND CHARACTERIZATION

Clearly, since the objective is to perform control of the unknown plant, we must guarantee that the criterion that decides if the control phase starts in finite time, i.e. we must ensure that the model set $\hat{\mathscr{G}}(k)$ is strongly robust in finite time. This remark raises two key issues: first, it is crucial that the identification input yields a strongly robust model set. Then, we must be able to test practically at each measurement whether the model set is strongly robust or not. We now address these two issues.

4.1. Strong robustness of the model set: input design

In Reference [3] the following theorem is proved.

Theorem 4.1 (Existence of strongly robust sets)
For any system θ^0 in $\mathscr{C}_n$, there exists an open neighbourhood of θ^0 contained in $\mathscr{C}_n$ which is strongly robust.

In the sequel, if the uncertainty set $\hat{\mathscr{G}}(k)$ is sufficiently small, i.e. if the radius of the smallest sphere containing $\hat{\mathscr{G}}(k)$ is sufficiently small, then it is strongly robust. From Theorem 4.1 the following result follows.

Theorem 4.2

If the identification input sequence $\{u(k)\}$ is such that $\lim_{k\to\infty} \rho(\hat{\mathscr{G}}(k)) = 0$, where $\rho(\hat{\mathscr{G}}(k))$ denotes the radius of the smallest sphere containing $\hat{\mathscr{G}}(k)$, then there exists k_0 such that $\hat{\mathscr{G}}(k)$ is strongly robust $\forall k \geqslant k_0$.

Therefore, in the case where the input sequence is such that $\hat{\mathscr{G}}(k)$ is bounded and shrinks with time, a strongly robust uncertainty set is identified in finite time. This guarantees that the adaptive system described in Figure 1 switches to the control phase in finite time. The design of an input sequence leading to a strongly robust set in finite time is proposed in Reference [11] where the authors show that any input of the form

$$u(k) = \gamma_k \cdot \tilde{u}(k) \tag{14}$$

where $\{\tilde{u}(k)\}$ is the $2n$ periodic sequence defined by

$$\tilde{u}(k) = u_{t(k)}, \qquad t(k) = k \bmod 2n \quad \forall k \tag{15}$$

where the $2n$ values $u_0, \ldots, u_{2n-1}$ are chosen so that

$$\gcd\left(\sum_{i=0}^{2n-1} u_i \xi^i, \xi^{2n} - 1\right) = 1 \tag{16}$$

and where the gain sequence $\{\gamma_k\}$ satisfies

$$\lim_{k\to\infty} \gamma_k = \infty \quad \text{and} \quad \lim_{k\to\infty} (\gamma_{k+1} - \gamma_k) = 0 \tag{17}$$

yields an uncertainty set $\hat{\mathscr{G}}(k)$ which is bounded in finite time and such that its radius uniformly decreases with time. More precisely, it has been shown in Reference [11] that the identification of the system (4) with the input sequence given in (14)–(17) is such that there exists $K \geqslant 0$ such that $\hat{\mathscr{G}}(k)$ is bounded $\forall k \geqslant K$ and

$$\forall k \geqslant K, \quad \exists k' \geqslant k : \rho(\hat{\mathscr{G}}(k')) < \rho(\hat{\mathscr{G}}(k)) \tag{18}$$

where $\rho(\hat{\mathscr{G}}(k))$ denotes the radius of the smallest sphere containing $\hat{\mathscr{G}}(k)$. Hence, it follows from Theorem 4.1 that $\hat{\mathscr{G}}(k)$ is strongly robust in finite time.

Remark 4.3

Because of the increase of the input gain γ, our approach (14)–(17) might not look appealing at first sight. If indeed the designer apply input (14)–(17) with the only objective of securing strong robustness and without concern for system performance, choosing for the gain sequence a sequence γ that increases very fast, then poor transients may still occur. Therefore, it seems that the increase rate of γ should be chosen in an adaptive way rather than arbitrarily, so as not to exceed the minimum input level needed to reach strong robustness. However, since the system is unknown, no information tells us *a priori* what value this gain should take so that the uncertainty set becomes small enough to allow control to be started. Hence to increase γ is somehow unavoidable. Having observed that increasing the gain is necessary for control purposes, it should also be emphasized that within a complete adaptive control scheme the input gain will never be increased more than it is necessary to exhibit a strongly robust uncertainty set. Otherwise stated, the idea is to no longer step γ_k as soon as the uncertainty set is small enough to be useful for control. We show later (see Theorem 4.6) that this happens after a finite number of iterations. Thus the input sequence stays bounded during identification.

4.2. *Characterizing strong robustness*

At each time, it must be tested whether the updated unfalsified set is strongly robust or not, so as to know when the adaptive system can switch to the control phase. Thus it is crucial to have an explicit test to check whether a given set of systems is strongly robust or not.

First it should be noted that if a set of systems in $\mathscr{P}_n$ is strongly robust, then all its elements are controllable. Therefore, if the uncertainty set has uncontrollable systems, it is not strongly robust and the adaptive scheme stays in the identification phase. Hence, we should first check if the uncertainty set is a subset of the controllable systems in $\mathscr{P}_n$, although to check this property is not clear in the general case. Then, the construction of a necessary and sufficient test for characterizing strongly robust sets in $\mathscr{C}_n$ is not trivial, and is still under investigation. However, note that what we essentially need is a sufficient test for strong robustness. In this respect, we have the following theorem [12].

Theorem 4.4

Let Ω be a subset of $\mathscr{C}_n$. Given any system $\theta \in \Omega$, we denote by $(A, b) \in \mathbb{R}^{N\times N} \times \mathbb{R}^{N\times 1}$ a state space representation of dimension N of θ, rewriting (4) by

$$x(k+1) = Ax(k) + bu(k) \tag{19}$$

and (5) by

$$u(k) = \tilde{f}(A, b)x(k) \tag{20}$$

Ω is strongly robust if the following inequality holds:

$$\forall \theta_1, \theta_2 \in \Omega, \|f(\theta_2) - f(\theta_1)\| \leqslant r^{\mathbb{C}}_{A_1+b_1\tilde{f}(A_1,b_1)} \tag{21}$$

where $\|.\|$ denotes the Euclidean norm in $\mathbb{R}^{1\times N}$ and $\forall \theta \in \Omega$, $r^{\mathbb{C}}_{A+b\tilde{f}(A,b)}$ denotes the complex stability radius of the Schur matrix $A + b\tilde{f}(A, b)$ with respect to the perturbation structure (b, I_N) as it is defined in Reference [13].

Remark 4.5

Condition (21) is only a sufficient condition for strong robustness and may hence be conservative. However, the achievement of Theorem 4.4 is that testing if a set $\Omega \subset \mathscr{C}_n$ is strongly robust, which *a priori* involves time-varying systems, is reduced into a test involving time-invariant quantities only.

Back to the adaptive control scheme presented in Section 3, Theorem 4.4 provides us with a sufficient condition on the uncertainty set $\hat{\mathscr{G}}(k)$ to guarantee its strong robustness. As seen in Section 3, in order to switch to the control phase, the overall adaptive scheme needs to know explicitly when the uncertainty set satisfies condition (21). In this respect we have the following result.

Theorem 4.6

If the identification input sequence $\{u(k)\}$ is such that

$$\lim_{k\to\infty} \rho(\hat{\mathscr{G}}(k)) = 0 \tag{22}$$

where $\rho(\hat{\mathscr{G}}(k))$ denotes the radius of the smallest sphere containing $\hat{\mathscr{G}}(k)$, then there exists k_1 such that (21) is satisfied, taking $\Omega = \hat{\mathscr{G}}(k)\ \forall k \geqslant k_1$.

Proof

Suppose the identification input is such that $\lim_{k\to\infty} \rho(\hat{\mathscr{G}}(k)) = 0$. Since $\theta^0 \in \hat{\mathscr{G}}(k)$, $\forall k$, and $\theta^0 \in \mathscr{C}_n$, then there exists a time K such that $\forall k \geqslant K$, $\hat{\mathscr{G}}(k) \subset \mathscr{C}_n$. Then by continuity of the map f (Assumption 2.5), we have that $\lim_{k\to\infty} \rho(f(\hat{\mathscr{G}}(k))) = 0$, where $\rho(f(\hat{\mathscr{G}}(k)))$ denotes the radius of the smallest sphere containing $f(\hat{\mathscr{G}}(k))$ and where for any set $S \subset \mathscr{P}_n$, $f(S)$ denotes the set defined by $f(S) = \{f(\theta) : \theta \in S\}$. Therefore, $\forall \varepsilon > 0$, $\exists k_1$ such that $\forall f, f' \in \hat{\mathscr{G}}(k_1)$ then $\|f - f'\| \leqslant \varepsilon$. Now, choose any $\theta_1 \in \hat{\mathscr{G}}(k_1)$ and $\varepsilon = r^{\mathbb{C}}_{A_1 + b_1\tilde{f}(A_1,b_1)}$, where $r^{\mathbb{C}}_{A_1 + b_1\tilde{f}(A_1,b_1)}$ denotes the complex stability radius of the Schur matrix $A_1 + b_1\tilde{f}(A_1, b_1)$ with respect to the perturbation structure (b_1, I_N) [13]. We then have: $\forall f, f' \in \hat{\mathscr{G}}(k_1)$ then $\|f - f'\| \leqslant r^{\mathbb{C}}_{A_1 + b_1\tilde{f}(A_1,b_1)}$. Hence, $\forall \theta_2 \in \hat{\mathscr{G}}(k_1)$ we have: $\|f(\theta_1) - f(\theta_2)\| \leqslant r^{\mathbb{C}}_{A_1 + b_1\tilde{f}(A_1,b_1)}$, this for any $\theta_1 \in \hat{\mathscr{G}}(k_1)$. Thus (21) is satisfied taking $\Omega = \hat{\mathscr{G}}(k_1)$. Clearly, this implies that (21) is satisfied, taking $\Omega = \hat{\mathscr{G}}(k)\ \forall k \geqslant k_1$. □

The identification input design (14)–(17) provides us with an uncertainty set that satisfies (22). Hence the sufficient test for strong robustness (21) is satisfied in finite time, allowing the overall scheme to switch to the control phase in finite time.

5. ANALYSIS OF THE ADAPTIVE SYSTEM

In this section we are concerned with the analysis of the proposed adaptive control scheme.

5.1. Finite switching time

The identification input is constructed so that the uncertainty set becomes strongly robust in finite time. Therefore, the control phase is guaranteed to start in finite time. Obviously, this switching time depends on the characteristics of the system to be controlled (initial conditions and value of the unknown parameter vector θ^0), on the characteristics of the modelling error signal δ and the assumed value of its bound d, and on the chosen identification input. Intuitively, the switch from the identification phase to the control phase is expected to occur faster in the case of a small modelling error and a small bound d. On the contrary, for large modelling error and very conservative bound d, we expect that more measurements will be necessary before the system switches to the control phase. But it seems quite natural that small prior knowledge on the system to be controlled requires a longer learning phase.

5.2. Convergence of the model to the real system

The convergence of $\hat{\theta}(k)$ to the true parameter vector is not *a priori* guaranteed and is dependent on the input–output data that specify the uncertainty set $\hat{\mathscr{G}}(k)$. Note that the proposed scheme has the property of *neutrality*, i.e. in the case where the present uncertainty set is not falsified by the new input–output measurement, the model and hence the model-based controller are not updated. Hence, the adaptation process might stop after a finite time during the control phase, leading to a frozen adaptive system. Of course, such a case does by no means imply that the model error is zero, but simply means that the newly observed data do not bring any useful information with respect to the identification process. The model update law (12) provides the following properties [14].

Property 5.1

The model error sequence $\{\hat{\theta}(k) - \theta^0\}$ is bounded and non-increasing:

$$\|\hat{\theta}(k) - \theta^0\| \leqslant \|\hat{\theta}(k-1) - \theta^0\| \quad \forall k \tag{23}$$

and is asymptotically slow, i.e

$$\lim_{k\to\infty} (\|\hat{\theta}(k) - \theta^0\| - \|\hat{\theta}(k-1) - \theta^0\|) = 0 \tag{24}$$

5.3. Transient analysis

The proposed adaptive control approach mainly differs from classical approaches in the first phase, therefore the transient analysis is key in its analysis. Intuitively, since at no time the adaptive control system based on strong robustness does involve any destabilizing controller, the transient behaviour is expected to be superior to classical certainty equivalence-based schemes where, on the contrary, the model-based controller may be temporarily destabilizing. A rigorous proof of this intuitive result in the case of pole placement can be found in Reference [15], followed by a simulation example in the case of first-order systems and pole placement. It is shown that because of insufficient prior knowledge on the system to be controlled, classical pole placement might generate poor model-based controllers for an arbitrarily large number of iterations, yielding bad transients in the input–output response of the closed-loop system.

In addition it is worth noting that, even in the case where at each frozen time instant the closed-loop system would be obtained, stability of the time-varying system is not necessarily maintained in classical approaches if adaptation is too fast. In comparison, at no time in our strategy a destabilizing controller is applied to the real system, and closed-loop stability is guaranteed, regardless how fast the adaptation goes. Still, identification inputs generated as in Section 4.1 may induce poor transients, but this appears to be the inevitable price to be paid due to insufficient prior knowledge, whereas in classical adaptive control approaches, the bad transients serve no purpose.

5.4. Asymptotic analysis

Once the adaptive system has switched to the control phase, which is guaranteed to occur in finite time, a classical adaptive control approach is used. Hence the asymptotic analysis is fairly standard. For instance, if the control objective is pole placement, the interested reader can refer to Reference [14] for the asymptotic analysis. In addition, at no time in our design the time-variations induced by the adaptation process can destroy stability of the closed-loop system. Hence, bad asymptotic behaviour caused by fast adaptation cannot occur.

5.5. *Bounded input*

For the sake of the identification of a strongly robust uncertainty set, it is shown in Reference [11] the input energy level must be increased so that the criterion for strong robustness described in Section 4.2 is satisfied. By construction of this identification input depicted in Section 4.1, strong robustness is satisfied in finite time, hence after such time the identification phase stops, i.e. the identification input does not have to be increased any longer. This implies that the input sequence stays bounded in the first phase. Later, in the second phase, the input is designed according to (5), which also stays bounded since the estimate on which the controller is designed is guaranteed to be controllable.

6. REVISITING THE ADAPTIVE SCHEME: FUTURE RESEARCH

The adaptive control scheme presented in the previous sections can switch to the second phase only when condition (21) is satisfied, which may take a long time. Now, rather than switching from identification to control as it has been presented in Section 3, a suggestion would be to gradually start control before strong robustness is reached, under some precautions so that global stability of the adaptive scheme is not endangered. Also, we remark that the time at which strong robustness is achieved, and hence the time at which control is actually started, depend on the desired control objective. Next, based on these two remarks, we discuss how the introduced adaptive control scheme can be revisited.

6.1. *Dwell time*

The adaptive control scheme exposed in the previous sections requires that a strongly robust uncertainty set is identified before the control phase can be started. However, the certification of the sufficient condition for strong robustness (21) may be obtained only when the uncertainty set is very small. As a result, it may take long before the adaptive system actually switches to the control phase. Now, one can perhaps re-consider the presented scheme and proceed in the following way. Strong robustness requires that the controller based on any sequence of systems in the uncertainty set stabilizes any system in the set. Hence, a first condition to be satisfied is that the controller based on any system in the uncertainty set stabilizes any other system in the set. This condition on the uncertainty set is already stringent but is satisfied in finite and reasonable time when using input sequence (14)–(17). To this respect, it has been proven in a slightly different context that stabilizing the identified class of models automatically leads to stabilization of the true unknown system [16, 17]. Once it has been checked that the controller based on any system in the uncertainty set stabilizes any other system in this set, time-variations of the controller should also be taken into account so that strong robustness is achieved: it should be checked that the time-varying controller based on any sequence of systems in the uncertainty set stabilizes any fixed system in the set. This could be done by checking if the uncertainty set satisfies condition (21) at any time. Or, one may already start control using adaptation of a model and certainty equivalence, while checking at any time that the stability of the time-varying closed-loop system is not disrupted. More precisely, at each time we would estimate a model according to (12), compute the controller on the basis of this estimate according to (13), but at the same time force the time-variations of the controller to be mild enough so that global stability of the overall scheme is preserved. Such an idea suggests to introduce a so-called *dwell time* [16] between consecutive instants at which the model is updated, in such a way that it would be adaptively selected on the basis of collected data measurements. However, how to compute adaptively such a dwelling time in our framework is not clear yet and requires further investigation.

6.2. Weak strong robustness

In Remark 2.8, we note that the outcome of the strong robustness test depends on the control objective. Otherwise stated, a set of systems which is not strongly robust with respect to a control objective might be strongly robust if we update the control objective. Hence, the performance of our algorithm depends on the desired control objective. This might be of interest if the control objective is defined as a set of control objectives rather than being defined uniquely. Within these control objectives, there exists one for which the algorithm is completed in a minimal time. We define a weakly strongly robust set of systems as follows.

Definition 6.1

A set $\Omega \subset \mathscr{C}_n$ is weakly strongly robust if there exists a control objective such that Ω is strongly robust with respect to this control objective. Any control objective solution of this problem is then said to be weakly achievable.

Now, remark that the 'distance' between the set of weakly achievable control objectives and the desired control objective is likely to decrease iteratively if the uncertainty set uniformly shrinks with time, where the notion of distance between two control objectives should be specified. Based on this discussion, we could thus revisit the proposed adaptive control scheme based on strong robustness so that, at each time of the design, the control objective is optimized amongst the class of control objectives that are weakly achievable with respect to the actual uncertainty set. If some degree of freedom exists on the desired control objective, one would pick the optimal weakly achievable control objective, so that a strongly robust set is found in a minimal time. Moreover, even in the case where the desired control objective is fixed, the distance between the set of weakly achievable control objectives and this desired control objective might give some insight on how the input sequence may be generated so that the uncertainty set becomes strongly robust. The study of multi-phase adaptive control based on weak strong robustness is still in progress.

7. CONCLUSION

This paper introduces the structure of a two-step adaptive control system based on a criterion deciding when to put most of the effort on identification or on control. This criterion consists in checking whether the parameter uncertainty set is such that the controller based on any model taken in this set leads to a stable system when applied to any other model in this set, regardless how fast the adaptation is done. Our motivation is to avoid the case where a destabilizing controller would be applied to the plant to be controlled, what may occur in usual classical adaptive control, due to two factors: lack of prior knowledge on the system to be controlled and fast time-variations. Our design guarantees the existence of a time at which strong robustness is reached, however, this time cannot be predicted since it depends on the unknown system to be controlled. Hence, we have to assume that a large observation set of i/o data is available, although this is a quite restricting assumption. Moreover, securing strong robustness along the presented sufficient test for strong robustness may lead to the strong robustness certification only when the uncertainty set is very small. In this case, the identification phase should be prolonged for quite some time before control can start, although the actual uncertainty set is strongly robust before the time at which the scheme switch to the second phase. To avoid this, a necessary and sufficient test for strong robustness would be preferred; the construction of such a test is still under investigation.

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