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Brief paper

H_2-optimal control of systems with multiple i/o delays: Time domain approach[☆]

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Abstract

In this paper the H_2-optimal control problem of systems with multiple i/o delays is presented. The problem is first converted to an equivalent H_2 regulator problem with multiple delays. The idea is to view the regulator problem in time-domain as a linear quadratic regulator problem with multiple input delays. It is shown that the rational part of the optimal controller has the same dimension as the plant and the non-rational part may be chosen to have finite impulse response. Furthermore, the regulator problem solution is also used to solve the H_2 filtering problem with multiple measurement delays.

Keywords: Time delay systems; H_2 control; LQG control

1. Introduction

The presence of delays in a control system poses quite a challenge to the design of optimal controllers. A lot of results have been put forward to solve the problem of controlling systems with delays optimally in some sense.

In the area of H_∞ control, Foias et al. (1996) treated a general class of infinite dimensional H_∞ control problems that includes i/o delay problems. Meinsma and Zwart (1998, 2000) solved the H_∞ suboptimal control problem for systems with a single delay using J-spectral factorization approach. At the same time Mirkin (2003) introduced the idea of extracting the dead-time compensator from the delay-free parametrization. This approach reduces the control problem to a special one-block problem. The ideas from Meinsma and Zwart (2000) and Mirkin (2003) were put together in Meinsma et al. (2002). The result was then extended to the multiple i/o delays case in Meinsma and Mirkin (2003).

Using semigroup approach, the papers (Delfour & Mitter, 1972a,b; Delfour et al., 1975) treat the LQ control of retarded differential equations. This approach is an application of general LQ theory for infinite dimensional systems (see Curtain and Zwart (1995, Chapter 6) and references therein). Extensions to general delay equations with delayed inputs and outputs may be found in Pritchard and Salamon (1985, 1987) and Delfour (1986). Earlier, the H_2 (LQG) problem for systems with a single i/o delay was solved in Kleinman (1969). Recently, it is shown in Mirkin and Raskin (2003) that the solution for this problem may be obtained from the Youla parametrization. A related problem is the H_2 preview control problem, in which the controller is fed with an advanced version of the external input signal. The problem is treated in Kojima (2004).

In Moelja et al. (2003), the multiple i/o delays case of the H_2 problem is treated. Here, the technique from Mirkin (2003) of reducing the problem to a one-block problem is also used. In Moelja et al. (2003), the one-block problem is further converted to an H_2 regulator problem, which in turn is solved using a frequency domain approach. However,

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the frequency domain method used in Moelja et al. (2003) works well only for open-loop stable plants. For unstable plants, pre-multiplication of the regulator equation by a certain all-pass transfer function, which makes the formulas unnecessarily complicated, is needed. In addition, the resulting optimal controller appears to have a considerably large state dimension. In this paper, a time-domain approach is proposed to circumvent the drawbacks of the frequency domain approach. The H_2 regulator problem is viewed as an LQR problem with multiple input delays. It is shown that the LQR problem with delays may be reduced to a number of standard LQR problems. It is done by splitting the optimization time interval into time regions whose boundaries are the delays and then apply dynamic programming ideas. Within each time region, the optimization becomes essentially delay-free and may be solved using standard methods. The idea of splitting the optimization interval according to the delay has been previously applied in Tadmor (1997) for the single input delay case in the context of robust control in the gap. Another solution of the LQR problem with multiple input delays, which is based on the infinite dimensional systems theory, may be found in Kojima and Ishijima (2003). The optimal input derived there shares a similar structure with the result from this paper. However, in Kojima and Ishijima (2003) the problem is treated as the limiting problem of an H_∞ problem, which requires an additional condition corresponding to a certain transcendental equation.

The controller structure resulting from the method in this paper is quite simple. The controller consists of a rational transfer matrix block, a finite impulse response block, and delay components. The rational part of the controller has the same state dimension as the plant and the method can handle unstable plants. The solution to the H_2-optimal control problem considered in this paper is not only interesting from a theoretical point of view, but also has potential for application. For example, the paper (Grimble and Hearns, 1998) considers the problem of steel-sheet profile control at a hot strip mill, which has different delays in its measurement channel. The problem is formulated as an LQG problem, which is equivalent to the H_2 problem. However the method developed in Grimble and Hearns (1998) assumes that noise and disturbance models are block-diagonal. Certain approximation has to be made to meet this assumptions, resulting in a controller that is not only sub-optimal but also of high order. Using the method derived in this paper, it is possible to compute and simulate the optimal H_2-controller for the same control problem without the approximation (see Moelja and Meinsma, 2004).

The paper is organized as follows. After an introduction and the problem formulation, the method for converting the standard problem to an equivalent H_2 regulator problem is briefly reviewed. In the next section, the time-domain solution of the regulator problem is presented followed by a section describing the state space realization of the optimal controller. The paper is concluded by a numerical example.

2. Preliminaries

The transfer matrix and the impulse response of a linear time-invariant system G are denoted by $G(s)$ and $G(t)$, respectively. The squared H_2-norm of G may be computed as

$$\|G\|_2^2 = \int_{-\infty}^{\infty} \operatorname{trace}[G(t)^{\mathrm{T}} G(t)]\,\mathrm{d}t. \tag{1}$$

A transfer matrix $G(s)$ is said to be stable if $G(s) \in H_\infty$. A lower linear fractional transformation (LFT) of two transfer matrices M and U of appropriate dimension is

$$F_\ell(M, U) := M_{11} + M_{12}U(I - M_{22}U)^{-1}M_{21}, \tag{2}$$

while the left homographic transformation of two transfer matrices N and V of appropriate dimensions is

$$C_\ell(N, V) := -(N_{11} - VN_{21})^{-1}(N_{12} - VN_{22}). \tag{3}$$

The matrices M and N are appropriately partitioned:

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \quad N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}.$$

As in Mirkin (2003), we define the "truncation" operator $\tau_h(\cdot)$. The operation sets the impulse response to zero after $t = h$. Given $G(s) = C(sI - A)^{-1}B$ we may express the truncation of $G(s)$ as

$$\tau_h(G) := C(sI - A)^{-1}B - \mathrm{e}^{-sh}C\mathrm{e}^{Ah}(sI - A)^{-1}B, \tag{4}$$

which has a finite impulse response with support on $[0, h]$. A column vector with elements $a_1, \ldots, a_n$ is denoted as $\operatorname{col}[a_1, \ldots, a_n]$ and $\mathbb{1}(t)$ is the step function.

3. Problem formulation

We consider standard control systems in which time delays are present in either the control input or the measurement output. Such control systems are depicted in Fig. 1(a) for the input delay case, and in Fig. 1(b) for the output delay case. Here the plant $P(s)$ is a rational transfer matrix which is assumed of having the following realization

$$P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A_P & B_{P1} & B_{P2} \\ \hline C_{P1} & 0 & D_{P12} \\ C_{P2} & D_{P21} & 0 \end{array}\right] \tag{5}$$

connected with a proper controller $K_s(s)$, and a multiple delay operator. The output and input multiple delay operator are of the form

$$\Lambda_y(s) = \operatorname{diag}(\mathrm{e}^{-sh_{y1}}, \mathrm{e}^{-sh_{y2}}, \ldots, \mathrm{e}^{-sh_{ym}}), \tag{6}$$

$$\Lambda_u(s) = \operatorname{diag}(\mathrm{e}^{-sh_{u1}}, \mathrm{e}^{-sh_{u2}}, \ldots, \mathrm{e}^{-sh_{up}}), \tag{7}$$

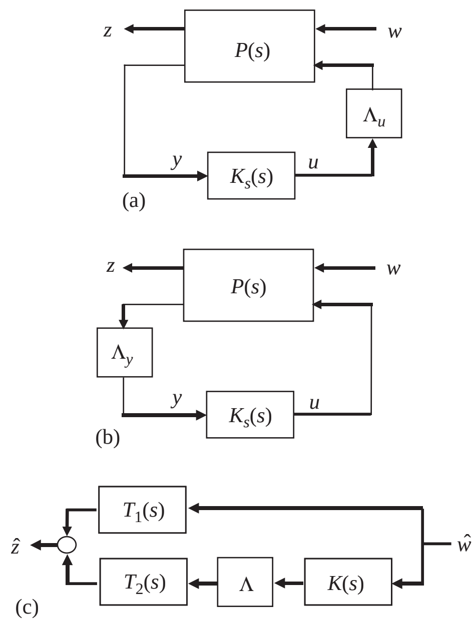

Fig. 1. (a) The input delay problem, (b) the output delay problem, and (c) the equivalent regulator problem.

where m and p are the dimensions of y and u, respectively. We assume that $P(s)$ satisfies the following standard assumptions:

A1. (C_{P2}, A_P, B_{P2}) is detectable and stabilizable;
A2. $R_1 = D_{P12}^{\mathrm{T}} D_{P12} > 0$ and $R_2 = D_{P21} D_{P21}^{\mathrm{T}} > 0$,
A3. $\begin{bmatrix} A_P - j\omega I & B_{P2} \\ C_{P1} & D_{P12} \end{bmatrix}$ and $\begin{bmatrix} A_P - j\omega I & B_{P1} \\ C_{P2} & D_{P21} \end{bmatrix}$ have full column rank and full row rank, respectively $\forall \omega \in \mathbb{R}$.

The H_2-optimal control problem is to find a stabilizing LTI causal controller $K_s(s)$ such that the H_2-norm of the transfer function from w to z is minimized.

4. From standard problem to regulator problem

In Moelja et al. (2003), using techniques developed in Mirkin (2003) and Mirkin and Raskin (2003), it is shown that solving the standard problem of Fig. 1(a) and (b) is equivalent to solving an H_2 regulator problem of Fig. 1(c). The subsequent lemmas, which are given without proof, summarize the formulas needed for the transformation. First, we need to define several quantities. Let X and Y be the stabilizing solutions of the two familiar Riccati equations:

$$A_X^{\mathrm{T}} X + X A_X - X B_{P2} R_1^{-1} B_{P2}^{\mathrm{T}} X + Q_X = 0, \tag{8}$$

$$A_Y Y + Y A_Y^{\mathrm{T}} - Y C_{P2}^{\mathrm{T}} R_2^{-1} C_{P2} Y + Q_Y = 0, \tag{9}$$

where $A_X = (A_P - B_{P2} R_1^{-1} D_{P21}^{\mathrm{T}} C_{P1})$, $Q_X = C_{P1}(I - D_{P12} R_1^{-1} D_{P12}^{\mathrm{T}}) C_{P1}$, $A_Y = (A_P - B_{P1} D_{P21}^{\mathrm{T}} R_2^{-1} C_{P2})$, and $Q_Y = B_{P1}(I - D_{P21}^{\mathrm{T}} R_2^{-1} D_{P21}) B_{P1}^{\mathrm{T}}$. Define the matrices F and L and the transfer function $G(s)$:

$$F := -R_1^{-1}(B_{P2}^{\mathrm{T}} X + D_{P12}^{\mathrm{T}} C_{P1}), \tag{10}$$

$$L := -(Y C_{P2}^{\mathrm{T}} + B_{P1} D_{P21}^{\mathrm{T}}) R_2^{-1}, \tag{11}$$

$$G(s) = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A_P & -L R_2^{\frac{1}{2}} & B_{P2} \\ \hline -R_1^{\frac{1}{2}} F & 0 & R_1^{\frac{1}{2}} \\ C_{P2} & R_2^{\frac{1}{2}} & 0 \end{array}\right] \tag{12}$$

Also define $B_{P2,i}$ and $C_{P2,j}$ as the ith column of B_{P2} and the jth row of C_{P2}. According to Theorem 13.7 of Zhou and Doyle (1998), the squared optimal H_2-norm for the delay-free case ($\Lambda_u = I$, $\Lambda_y = I$) is

$$J_{\mathrm{df}} = \min_{K_s} \|F_\ell(P, K_s)\|_2^2 = \mathrm{tr}(B_{P1}^{\mathrm{T}} X B_{P1}) + \mathrm{tr}(R_1 F Y F^{\mathrm{T}}).$$

Lemma 1 (*Input delays*). *Consider the problem of minimizing $\|F_\ell(P(s), \Lambda_u(s) K_s(s))\|_2$ over stabilizing, causal K_s, where $P(s)$ and $\Lambda_u(s)$ are given by* (5) *and* (7), *respectively. It is equivalent to minimizing $\|G_{11}(s) + G_{12}(s)\Lambda_u(s)K(s)\|_2$ over causal K, with $G(s)$ given by* (12), *where there is a causal bijection between K and K_s. The bijection is governed by*

$$K_s(s) = (I - K_{s,1}(s)\Phi_{22}(s))^{-1} K_{s,1}(s),$$

$$K_{s,1}(s) = K(s)(I + G_{21}^{-1}(s)\tilde{G}_{22}(s)K(s))G_{21}^{-1}(s),$$

where $\tilde{G}_{22}(s) = C_{P2}(sI - A_P)^{-1}\tilde{B}_{P2}$,

$$\Phi_{22}(s) = \tilde{G}_{22}(s) - G_{22}(s)\Lambda_u(s),$$

$$\tilde{B}_{P2} = [\mathrm{e}^{-A_P h_{u1}} B_{P2,1} \dots \mathrm{e}^{-A_P h_{up}} B_{P2,p}].$$

Moreover, the squared optimal H_2-norm is given by

$$\min_{K_s} \|F_\ell(P, \Lambda_u K_s)\|_2^2 = J_{\mathrm{df}} + \min_K \|G_{11} + G_{12}\Lambda_u K\|_2^2.$$

Proof. See Moelja et al. (2003). □

Lemma 2 (*Output delays*). *Consider the problem of minimizing $\|F_\ell(P(s), K_s(s)\Lambda_y(s))\|_2$ over stabilizing, causal K_s, where $P(s)$ and $\Lambda_y(s)$ are given by* (5) *and* (6), *respectively. It is equivalent to minimizing $\|G_{11}(s) + K(s)\Lambda_y(s)G_{21}(s)\|_2$ over causal K, with $G(s)$ given by* (12), *where there is a causal bijection between K and K_s. The bijection is governed by*

$$K_s(s) = (I - K_{s,1}(s)\Phi_{22}(s))^{-1} K_{s,1}(s),$$

$$K_{s,1}(s) = G_{12}^{-1}(s)K(s)(I + \tilde{G}_{22}(s)G_{12}^{-1}(s)K(s)),$$

where $\tilde{G}_{22}(s) = \tilde{C}_{P2}(sI - A_P)^{-1}B_{P2}$,

$$\Phi_{22}(s) = \tilde{G}_{22}(s) - \Lambda_y(s)G_{22}(s),$$

$$\tilde{C}_{P2} = \mathrm{col}[C_{P2,1}\mathrm{e}^{-A_P h_{y1}}, \dots, C_{P2,m}\mathrm{e}^{-A_P h_{ym}}].$$

Moreover, the squared optimal H_2-norm is given by

$$\min_{K_s} \|F_\ell(P, K_s \Lambda_y)\|_2^2 = J_{\mathrm{df}} + \min_K \|G_{11} + K\Lambda_y G_{21}\|_2^2.$$

Proof. See Moelja et al. (2003). □

5. Time-domain solution of the H_2 regulator problem

In the previous section we showed that the standard problem of Fig. 1(a), (b) may be transformed to the regulator problem of Fig. 1(c), which is addressed in this section. Consider the H_2 regulator problem of minimizing the H_2-norm of the transfer function from $\hat{w}$ to $\hat{z}$ in Fig. 1(c) over causal, LTI controller K. We assume that the LTI systems T_1 and T_2 have the joint realization

$$\begin{bmatrix} T_1(s) & T_2(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_2 \end{array}\right] \tag{13}$$

that satisfies the following standard assumptions:

A4. (C_1, A, B_2) is detectable and stabilizable;

A5. $R = D_2^{\mathrm{T}} D_2 > 0$ and $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_2 \end{bmatrix}$ has full column rank for all $\omega \in \mathbb{R}$.

Without loss of generality, we may also assume that the delays in the delay operator Λ are in ascending order according to their magnitude so that it may be written in the form

$$\Lambda(s) = \mathrm{diag}(\mathrm{e}^{-sh_0} I_0, \mathrm{e}^{-sh_1} I_1, \ldots, \mathrm{e}^{-sh_N} I_N),$$
$$0 = h_0 < h_1 < \cdots < h_N. \tag{14}$$

Note that I_0 may be empty which means that there are no undelayed channels. The approach is to convert the H_2 regulator problem to a time-domain optimal control problem with input delays. It is then solved by exploiting the fact that by confining ourselves to a time region between two delays, for instance on $t \in [h_1, h_2]$, the optimal control problem becomes essentially delay-free.

5.1. *Conversion to an LQR problem with multiple input delays*

In time domain, the H_2 regulator problem is to find the impulse response of the optimal controller $K(s)$:

$$K_{\mathrm{opt}}(t) = \arg\min_K \|T_1(t) + T_2(t) * \Lambda(t) * K(t)\|_2, \tag{15}$$

where $\Lambda(t) = \mathrm{diag}(\delta(t)I_0, \delta(t-h_1)I_1, \ldots, \delta(t-h_N)I_N)$ and the '$*$' operator denotes the convolution operator. It is straightforward to show that we may find any individual column of $K_{\mathrm{opt}}(t)$ independently and it is given by

$$K_{j,\mathrm{opt}}(t) = \arg\min_{K_j} \|T_{1,j}(t) + T_2(t) * \Lambda(t) * K_j(t)\|_2, \tag{16}$$

where $K_j(t)$ and $T_{1,j}(t)$ are the jth column of $K(t)$ and $T_1(t)$, respectively. Therefore without loss of generality we may assume that $T_1(s)$ and $K(s)$ are single-column transfer functions.

A6. $T_1(s)$ and $K(s)$ are single-column transfer functions.

The assumption A6 will be relaxed later. By assumption A6 $K(t)$ may be seen as a vector-valued signal, so that we may reformulate (16) into an optimal control problem with input delays. To this end, let us first partition T_2 and K according to the delays:

$$\begin{bmatrix} T_1(s) & T_2(s) \end{bmatrix} = \left[\begin{array}{c|c|c} A & B_1 & \begin{bmatrix} B_{2,0} & \cdots & B_{2,N} \end{bmatrix} \\ \hline C_1 & 0 & \begin{bmatrix} D_{2,0} & \cdots & D_{2,N} \end{bmatrix} \end{array}\right] \tag{17}$$

$$K(t) = \mathrm{col}[K_0(t), \ldots, K_N(t)]. \tag{18}$$

Here $B_{2,k}$, $D_{2,k}$ and $K_k(t)$ are the respective parts of B_2, D_2, and $K(t)$ that correspond to the delay h_k. In addition, the convolution of the delay operator and the controller is given by

$$\Lambda(t) * K(t) = \mathrm{col}[K_0(t), K_1(t-h_1), \ldots, K_N(t-h_N)].$$

From (16), it is apparent that we may view the quantity inside the norm brackets as the output of the system $[\, T_1(s) \quad T_2(s)\,]$ with $\mathrm{col}[\delta(t), \Lambda(t) * K(t)]$ as the input. Moreover, the realization (17) allows us to write down the state-space equation for this system:

$$\dot{x}(t) = Ax(t) + \sum_{k=0}^{N} B_{2,k} K_k(t-h_k), \quad x(0) = B_1 \tag{19}$$

and the objective (16) becomes minimizing

$$J(x_0, K) = \int_0^\infty \|C_1 x(t) + \sum_{k=0}^{N} D_{2,k} K_k(t-h_k)\|_2^2 \,\mathrm{d}t. \tag{20}$$

We see that the system description (19) and the associated criterion function (20) constitute an LQR problem with multiple input delays with $K(t)$ as the input.

5.2. *Solution of the LQR problem with multiple input delays*

In this section we show that we may reduce the optimal control problem (19), (20) to a series of standard LQR problems. To solve the problem we utilize a standard result in optimal control theory, namely the principle of optimality (see Anderson and Moore, 1989 for details). One implication of the principle is the following. Suppose we have an optimal control problem over the time region $t \in [t_0, t_e]$. Then we may solve the problem over $t \in [t_1, t_e]$, $t_1 > t_0$ independent of what happens during $t \in [t_0, t_1]$, provided that we start with the optimal state at $t=t_1$. Let us apply this principle to the problem (19), (20) for $t_0=0, t_1=h_N, t_e=\infty$. On $t\in[h_N,\infty]$, all inputs are active. Define the new input for $t\in[h_N,\infty]$:

$$\Phi_N(t)=\mathrm{col}[K_0(t),\ldots,K_N(t-h_N)]\mathbb{1}(t-h_N). \tag{21}$$

The state space Eq. (19) becomes

$$\dot{x}(t)=Ax(t)+B_2\Phi_N(t),$$
$$x(h_N)=x_N(x_0,K_0|_{t\leqslant h_N},\ldots,K_{N-1}|_{t\leqslant h_N}). \tag{22}$$

Assume that $x_{\mathrm{opt}}(t), t\leqslant h_N$ is known and apply the principle of optimality. We have that

$$\begin{aligned}&\min_{K_0,\ldots,K_N} J(x_0,K_0,\ldots,K_N)\\&\quad=\min_{K_0,\ldots,K_N}\int_0^\infty \|C_1x(t)+\sum_{k=0}^{N}D_{2,k}K_k(t-h_k)\|_2^2\,\mathrm{d}t\\&\quad=\int_0^{h_N}\|C_1x_{\mathrm{opt}}(t)+\sum_{k=0}^{N-1}D_{2,k}\,K_{k,\mathrm{opt}}(t-h_k)\|_2^2\,\mathrm{d}t\\&\qquad+\min_{\Phi_N} J_{[h_N,\infty]}(x_{\mathrm{opt}}(h_N),\Phi_N)\end{aligned} \tag{23}$$

where $J_{[h_N,\infty]}(x(h_N),\Phi_N)$

$$=\int_{h_N}^\infty \|C_1x(t;x(h_N))+D_2\Phi_N(t)\|_2^2\,\mathrm{d}t. \tag{24}$$

Eqs. (22) with $x(h_N)=x_{\mathrm{opt}}(h_N)$ and (24) constitute a standard infinite horizon LQR problem. By solving this problem, the optimal input from $t=h_N$ onwards is completely determined as a function of $x_{\mathrm{opt}}(h_N)$. Moreover, it is well known that the optimal cost $J_{[h_N,\infty],\mathrm{opt}}$ is quadratic in the initial state $x_{\mathrm{opt}}(h_N)$:

$$\min_{\Phi_N} J_{[h_N,\infty],\mathrm{opt}}=x_{\mathrm{opt}}(h_N)^{\mathrm{T}}M_Nx_{\mathrm{opt}}(h_N), \tag{25}$$

for a certain constant matrix $M_N\geqslant 0$. Thus substituting (25) in (23), we obtain

$$\begin{aligned}&\min_{K_0,\ldots,K_N} J(x_0,K_0,\ldots,K_N)\\&\quad=\int_0^{h_N}\|C_1x_{\mathrm{opt}}(t)+\sum_{k=0}^{N-1}D_{2,k}K_{k,\mathrm{opt}}(t-h_k)\|_2^2\,\mathrm{d}t\\&\qquad+x_{\mathrm{opt}}(h_N)^{\mathrm{T}}M_Nx_{\mathrm{opt}}(h_N)\\&\quad=\min_{K_0,\ldots,K_{N-1}} x(h_N)^{\mathrm{T}}M_Nx(h_N)\\&\qquad+\int_0^{h_N}\|C_1x(t)+\sum_{k=0}^{N-1}D_{2,k}K_k(t-h_k)\|_2^2\,\mathrm{d}t.\end{aligned} \tag{26}$$

This means that the infinite horizon problem (20) is reduced to a finite horizon problem (26). We may then continue to apply the principle of optimality for $t_1=h_{N-1}$ and $t_e=h_N$. By solving the resulting LQR problem over the time region $t\in[h_{N-1},h_N]$, we may express the cost contribution of the time region $t\in[h_{N-1},\infty]$ as a quadratic function of the state at $t=h_{N-1}$. By continuing in this fashion we may obtain the optimal input by solving the optimal control problem backward in time, time region by time region. The optimal input for each time region is expressed as a function of the optimal initial state of time region. Since we know the optimal state at $t=0$, we may then move forward in time to compute the optimal input for each time region. Let us define the input $\Phi_k(t)$ for the time region $t=[h_k,h_{k+1}]$ as

$$\begin{aligned}\Phi_k(t)=\mathrm{col}[K_0(t),\ldots,K_k(t-h_{h_k})][\mathbb{1}(t-h_k)\\-\mathbb{1}(t-h_{k+1})].\end{aligned} \tag{27}$$

Having computed the optimal new inputs Φ_k using the algorithm described earlier, we may recover the optimal original input $K(t)$, which is given by

$$K_{\mathrm{opt}}(t)=\Lambda^{\sim}(t)*\Phi(t), \tag{28}$$

where

$$\begin{aligned}\Lambda^{\sim}(t)&=\mathrm{diag}(\delta(t)I_0,\delta(t+h_1)I_1,\ldots,\delta(t+h_N)I_N),\\\Phi(t)&=\mathrm{col}[\Phi_{0,\mathrm{opt}}(t),0_1,\ldots,0_N]\\&\quad+\mathrm{col}[\Phi_{1,\mathrm{opt}}(t),0_2,\ldots,0_N]\\&\quad+\cdots+\mathrm{col}[\Phi_{N-1,\mathrm{opt}}(t),0_N]+\Phi_{N,\mathrm{opt}}(t).\end{aligned}$$

Here $0_k, k=1,\ldots,N$ is a zero column vector with dimension equal to the dimension of I_k in the delay operator (14). Note that $K_{\mathrm{opt}}(t)$ is causal since $\Phi_{k,\mathrm{opt}}$ is identically zero outside $t\in[h_k,h_{k+1}]$.

From the discussion, it is evident that the solution to the LQR problem with multiple input delays (19), (20) amounts to solving standard regional LQR problems for the time regions $t\in[h_N,\infty]$, $t\in[h_{N-1},h_N],\ldots,t\in[0,h_1]$. The solution to these regional LQR problems is reviewed in the appendix. Now we are in the position to formulate an algorithm that solves the optimal control problem (19), (20). The algorithm is formally stated in the following theorem. Note that this algorithm refers to formulas in Lemmas A.1 and A.2 of the appendix.

Theorem 3 (*Algorithm*). *Consider system* (19) *and the criterion function* (20). *Define the new input* Φ_k, $k=0,\ldots,N$ *as in* (27). *Also define*

$$\bar{B}_k:=[B_{2,0}\ \cdots\ B_{2,k}],\quad \bar{D}_k:=[D_{2,0}\ \cdots\ D_{2,k}]. \tag{29}$$

Then (19) *and* (20) *become*

$$\dot{x}(t)=Ax(t)+\sum_{k=0}^{N}\bar{B}_k\Phi_k(t),\quad x(0)=x_0:=B_1, \tag{30}$$

$$\begin{aligned}J(x_0,\Phi_0,\ldots,\Phi_N)=\int_0^\infty &\|C_1x(t)\\&+\sum_{k=0}^{N}\bar{D}_k\Phi_k(t)\|_2^2\,\mathrm{d}t.\end{aligned} \tag{31}$$

The optimal input may then be computed as followed.

(1) *Solve the infinite horizon LQR problem*

$$\dot{x}(t) = Ax(t) + B_2\Phi_N(t), \quad x(h_N) = x_N, \tag{32}$$

$$\min_{\Phi_N} \int_{h_N}^{\infty} \|C_1 x(t; x(h_N)) + D_2\Phi_N(t)\|_2^2 \,\mathrm{d}t. \tag{33}$$

Note that the optimal state at $t = h_N$, denoted by $x_N = x_{\mathrm{opt}}(h_N)$, is not yet known. By Lemma A.2 (*Appendix*), *the optimal cost is given by*

$$x_{\mathrm{opt}}(h_N)^{\mathrm{T}} M_N x_{\mathrm{opt}}(h_N), \tag{34}$$

where M_N is the stabilizing solution of the Riccati equation (A.18).[1] *The optimal input for this region is the impulse response of the transfer function*:

$$\Phi_{N,\mathrm{opt}}(s) = \mathrm{e}^{-sh_N} F_N(sI - S_N)^{-1} x_N \tag{35}$$

given by (A.20)[1], *where S_N and F_N are given by* (A.19)[1]. *Next, set $k := N - 1$.*

(2) *Solve the finite horizon LQR problem*

$$\dot{x}(t) = Ax(t) + \bar{B}_k\Phi_k(t), \quad x(h_k) = x_k, \tag{36}$$

$$\min_{\Phi_k} (x(h_{k+1})^{\mathrm{T}} M_{k+1} x(h_{k+1}) + \int_{h_k}^{h_{k+1}} \|C_1 x(t) + \bar{D}_k\Phi_k(t)\|^2 \,\mathrm{d}t). \tag{37}$$

Again $x_k = x_{\mathrm{opt}}(h_k)$ is not yet known. By Lemma A.1, *the optimal cost is given by*

$$x_{\mathrm{opt}}(h_k)^{\mathrm{T}} M_k x_{\mathrm{opt}}(h_k), \tag{38}$$

where M_k may be computed using (A.15).[2] *Then the optimal $\Phi_k(t)$ is the impulse response of the transfer function* (A.12)[2]:

$$\Phi_{k,\mathrm{opt}}(s) = \mathrm{e}^{-sh_k}\tau_{L_k}(F_k(sI - S_k)^{-1} P_k x_k), \tag{39}$$

where P_k, S_k, and F_k are given by (A.11)[2], (A.6)[2], *and* (A.4)[2], *respectively, while $L_k = h_{k+1} - h_k$. Next, set $k := k - 1$.*

(3) *If $k > 0$ then repeat step* 2, *otherwise go to the next step. At this stage $M_1, M_2, \ldots, M_N$ are known.*

(4) *Solve the LQR problem of the system* (36) *for $k=0$ with the criterion function* (37) *for the case $x(0) = x_0 = B_1$. By Lemma* A.1, *the optimal control $\Phi_{0,\mathrm{opt}}(t)$ is the impulse response of the transfer function* (39) *with $k=0$, which is a function of B_1. Furthermore, the optimal cost, which is also the optimal cost of the LQR problem with input delays* (19), (20), *is given by*

$$J_{[0,h_1],\mathrm{opt}}(x(0), \Phi_0) = B_1^{\mathrm{T}} M_0 B_1, \tag{40}$$

where M_0 may be computed using (A.15)[2] *for $k = 0$. Compute the optimal state at $t = h_1$, denoted by x_1, using* (A.14)[2] for $k = 0$.

(5) *Compute the optimal control $\Phi_{k+1,\mathrm{opt}}(t)$ by substituting $x(h_{k+1}) = x_{k+1}$ into* (39). *Next, compute the optimal state at $t = h_{k+2}$, denoted by x_{k+2}, using* (A.14)[2]. *Next, set $k := k + 1$.*

(6) *If $k < N$ repeat step* 5 *otherwise do the following. Substitute the optimal state at $t = h_N$, denoted by x_N to* (35) *to obtain $\Phi_{N,\mathrm{opt}}(t)$. Finally, substitute $\Phi_{k,\mathrm{opt}}$, $k = 0, \ldots, N$ to* (28) *to obtain K_{opt}.*

[1] See Lemma A.2 in the appendix.
[2] See Lemma A.1 in the appendix.

6. State-space realization of the optimal controller

In this section we relax assumption A6 and no longer assume that $T_1(s)$ and $K(s)$ are single-column transfer functions. We denote the jth column of $T_1(s)$, $K(s)$, and B_1 as $T_{1,j}(s)$, $K_j(s)$, and $B_{1,j}$ respectively.

In the previous section, it is shown that the impulse response of the optimal controller is given by (28). For $0 \leqslant k < N$, the jth column of $\Phi_{k,\mathrm{opt}}(t)$, denoted by $\Phi_{j,k,\mathrm{opt}}(t)$, is the impulse response of the transfer function (A.12)[2]:

$$\Phi_{j,k,\mathrm{opt}}(s) = \mathrm{e}^{-sh_k}\tau_{L_k}(F_k(sI - S_k)^{-1} P_k x_{j,k}), \tag{41}$$

where $L_k = h_{k+1} - h_k$ and the formulas for S_k, P_k, and F_k are given in Lemma A.1 in the appendix. For $k=N$, $\Phi_{j,N,\mathrm{opt}}$ is the impulse response of the transfer function (A.20)[1]

$$\Phi_{N,j,\mathrm{opt}}(s) = \mathrm{e}^{-sh_N}(F_N(sI - S_N)^{-1} x_{j,N}). \tag{42}$$

The formulas for S_N and F_N are given in Lemma A.2 in the appendix. The optimal state at $t = h_k$, denoted by $x_{j,k}$, $k=0, \ldots, N$, is the only thing that changes in (41), (42) as j changes. Moreover, the optimal state $x_{j,k}$ may be computed iteratively using (A.14)[2]: $x_{j,k+1} = \Gamma_k(L_k)x_{j,k}$, with the function $\Gamma_k(t)$ given by (A.9)[2]. Since $x_{j,k=0}$ is equal to the jth column of B_1, we have that

$$x_{j,k} = \left(\prod_{i=0}^{k-1} \Gamma_i(L_i)\right) B_{1,j}. \tag{43}$$

Therefore we may write for $k = 0, \ldots, N - 1$

$$\begin{aligned}\Phi_{k,\mathrm{opt}}(s) &= [\Phi_{1,k,\mathrm{opt}}(s) \quad \cdots \quad \Phi_{J,k,\mathrm{opt}}] \\ &= \mathrm{e}^{-sh_k}\tau_{L_k}(F_k(sI - S_k)^{-1} P_k [x_{1,k} \quad \cdots \quad x_{J,k}]) \\ &= \mathrm{e}^{-sh_k}\underbrace{\tau_{L_k}\left(F_k(sI - S_k)^{-1} P_k \left(\prod_{i=0}^{k-1} \Gamma_i(L_i)\right) B_1\right)}_{=:\tilde{\Phi}_{k,\mathrm{opt}}(s),}\end{aligned} \tag{44}$$

where J is the number of columns of the controller K. Similarly, for $k = N$ we have that

$$\begin{aligned}\Phi_{N,\mathrm{opt}}(s) &= [\Phi_{1,N,\mathrm{opt}}(s) \quad \cdots \quad \Phi_{J,N,\mathrm{opt}}] \\ &= \mathrm{e}^{-sh_N}(F_N(sI - S_N)^{-1}[x_{1,N} \quad \cdots \quad x_{J,N}]) \\ &= \mathrm{e}^{-sh_N}\underbrace{\left(F_N(sI - S_N)^{-1}\left(\prod_{i=0}^{N-1}\Gamma_i(L_i)\right)\right)}_{=:\tilde{\Phi}_{N,\mathrm{opt}}(s).}\end{aligned} \tag{45}$$

Hence, once the matrices S_k, F_k, P_k, $k = 1, \ldots, N-1$, S_N, and F_N have been computed, we may compute $\Phi_{k,\mathrm{opt}}$, $k = 0, \ldots, N$ using (44) and (45). Finally, the optimal controller $K_{\mathrm{opt}}(s)$ may be recovered using (28). It is straightforward to show that the optimal controller $K_{\mathrm{opt}}(s)$ may be written as the sum of a FIR block and a rational transfer matrix premultiplied by a multiple delay operator:

$$K_{\mathrm{opt}} = \tilde{\Phi}(s) + \mathrm{e}^{-sh_N}\mathrm{diag}(I_0, \mathrm{e}^{sh_1}, \ldots, \mathrm{e}^{sh_N} I_N)\tilde{\Phi}_N(s)) \tag{46}$$

where the rational $\tilde{\Phi}_N$ is defined in (45) and

$$\begin{aligned}\tilde{\Phi}(s) = \mathrm{col}[&\tilde{\Phi}^0_{0,\mathrm{opt}}(s), 0_{1,J}, \ldots, 0_{N-1,J}, 0_{N,J}] \\ &+ \mathrm{col}[\mathrm{e}^{-sh_1}\tilde{\Phi}^0_{1,\mathrm{opt}}(s), \tilde{\Phi}^1_{1,\mathrm{opt}}(s), \ldots, 0_{N-1,J}, 0_{N,J}] \\ &+ \cdots + \mathrm{col}[\mathrm{e}^{-s(h_{N-1})}\tilde{\Phi}^0_{N-1,\mathrm{opt}}(s), \\ &\quad \mathrm{e}^{-s(h_{N-1}-h_1)}\tilde{\Phi}^1_{N-1,\mathrm{opt}}(s), \ldots, \tilde{\Phi}^{N-1}_{N-1,\mathrm{opt}}(s), 0_{N,J}].\end{aligned}$$

Here $\tilde{\Phi}_k$ is defined in (44) and $\tilde{\Phi}^i_k$ with $k = 1, \ldots, N-1$ and $i = 0, \ldots, k$ are the rows of $\tilde{\Phi}_k$ corresponding to the delay h_k in the delay operator (14). The state dimension of the rational part of the optimal controller is the same as the state dimension of the plant. For the case where the controller is single-column, the optimal H_2-norm of the overall transfer function $T_1(s) + T_2(s)\Lambda(s)K(s)$ is given by (40). For the general case where $K(s)$ has multiple columns, the optimal H_2-norm may be obtained from the following expression:

$$\min_K \|T_1(s) + T_2(s)\Lambda(s)K(s)\|^2 = \mathrm{trace} B_1^{\mathrm{T}} M_0 B_1. \tag{47}$$

Remark 4. Since the H_2-norm of a transfer function is equal to the H_2-norm of its transpose, the technique developed in the last two sections may also be utilized to solve the filtering problem with multiple measurement delays, which is the transpose of the regulator problem.

7. Numerical example

Consider the regulator problem

$$\min_K \|T_1(s) + T_2(s)\Lambda(s)K(s)\|_2, \tag{48}$$

$$\left[T_1(s)\ T_2(s)\right] = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_2\end{array}\right] = \left[\begin{array}{c|c|c} 0 & 1 & \begin{bmatrix}1 & 1\end{bmatrix} \\ \hline \begin{bmatrix}1\\0\\0\end{bmatrix} & \begin{bmatrix}0\\0\\0\end{bmatrix} & \begin{bmatrix}0 & 0\\1 & 0\\0 & \frac{1}{\alpha}\end{bmatrix}\end{array}\right],$$

$$\Lambda(s) = \mathrm{diag}(1, \mathrm{e}^{-sh_1}), \quad h_1 > 0.$$

In this example $K(t)$ has a single column and thus the regulator problem may be converted directly to an LQR problem with input delays. In time domain the regulator problem (48) becomes

$$\min_K \|T_1(t) + T_2(t) * \Lambda(t) * K(t)\|_2. \tag{49}$$

By partitioning T_2 and K according to the delay:

$$T_2(s) = [T_{2,0}(s) \quad T_{2,1}(s)] = \begin{bmatrix} A & B_{2,0} & B_{2,1} \\ C_1 & D_{2,0} & D_{2,1}\end{bmatrix}, \tag{50}$$

$$K(s) = \mathrm{col}[K_0(s), K_1(s)], \tag{51}$$

where $B_{2,0} = B_{2,1} = 1$, $D_{2,0} = (0, 1, 0)^{\mathrm{T}}$, and $D_{2,1} = (0, 0, 1/\alpha)^{\mathrm{T}}$, the quantity inside the norm-bracket in (49) may be expressed as the output z of the state-equation

$$\dot{x}(t) = K_0(t) + K_1(t - h_1), \quad x(0) = B_1 = 1, \tag{52}$$

$$z(t) = \mathrm{col}\left[x(t), K_0(t), \frac{1}{\alpha}K_1(t)\right]. \tag{53}$$

In this framework, the objective (49) becomes

$$\begin{aligned}&\min_{K_0, K_1} J(x(0), K_0(t), K_1(t)) \\ &\quad = \min_{K_0, K_1}\int_0^\infty \left(x(t)^2 + K_0(t)^2 + \frac{1}{\alpha^2}K_1(t - h_1)^2\right)\mathrm{d}t.\end{aligned} \tag{54}$$

Eqs. (52) and (54) constitute the equivalent LQR problem, for which we define the new regional inputs:

$$\Phi_0(t) = K_0(t)[\mathbb{1}(t) - \mathbb{1}(t - h_1)], \tag{55}$$

$$\Phi_1(t) = \mathrm{col}[K_0(t), \quad K_1(t - h_1)]\mathbb{1}(t - h_1), \tag{56}$$

for the time regions $t \in [0, h_1]$ and $t \in [h_1, \infty]$, respectively. Now we may solve the problem backward in time using the algorithm of Theorem 3. For the time region $t \in [h_1, \infty]$, the state-space equation (52) becomes

$$\dot{x}(t) = [1 \quad 1]\Phi_1(t), \quad x(h_1) = x_{\mathrm{opt}}(h_1) \tag{57}$$

for a certain but not yet known $x_{\mathrm{opt}}(h_1)$ and the cost contribution over this period is

$$J_{[h_1,\infty]} = \int_{h_1}^\infty (x(t)^2 + \Phi_1(t)^{\mathrm{T}}\Phi_1(t))\,\mathrm{d}t. \tag{58}$$

Using Lemma A.1, it may be shown that the optimal input of this region is given by the state feedback

$$\Phi_{1,\mathrm{opt}}(t) = \frac{-1}{\sqrt{1+\alpha^2}}\begin{bmatrix}1\\ \alpha^2\end{bmatrix}x(t)\mathbb{1}(t - h_1) \tag{59}$$

resulting in the optimal state trajectory

$$x_{\text{opt}}(t) = \mathrm{e}^{-\sqrt{1+\alpha^2}(t-h_1)} x_{\text{opt}}(h_1), \quad t \in [h_1, \infty]. \tag{60}$$

Substituting (60) back into (59) we obtain the optimal input signal for $t \in [h_1, \infty]$:

$$\Phi_{1,\text{opt}}(t) = \frac{-1}{\sqrt{1+\alpha^2}} \begin{bmatrix} 1 \\ \alpha^2 \end{bmatrix} \times \mathrm{e}^{-\sqrt{1+\alpha^2}(t-h_1)} x_{\text{opt}}(h_1) \mathbb{1}(t-h_1). \tag{61}$$

Furthermore, the optimal cost over this region is

$$J_{[h_1,\infty],\text{opt}} = x_{\text{opt}}(h_1)^2/(\sqrt{1+\alpha^2}). \tag{62}$$

Applying the principle of optimality, we move to the time region $[0, h_1]$, in which the state equation is given by

$$\dot{x}(t) = \Phi_0(t), \quad x(0) = 1. \tag{63}$$

The infinite horizon criterion function (54) may be replaced by the cost contribution over the region $[0, h_1]$ plus the quadratic final state penalty (62):

$$\int_0^{h_1} (x(t)^2 + \Phi_0(t)^2)\, \mathrm{d}t + x(h_1)^2/(\sqrt{1+\alpha^2}). \tag{64}$$

By solving the finite horizon LQR problem (63), (64), it may be shown that the optimal input for the region $t \in [0, h_1]$ is given by

$$\Phi_{0,\text{opt}}(t) = [\sinh(t) - q(h_1)\cosh(t)][\mathbb{1}(t) - \mathbb{1}(t-h_1)], \tag{65}$$

where the function $q(h_1)$ is

$$q(h_1) = \frac{\sinh(h_1) + 1/\sqrt{1+\alpha^2}\cosh(h_1)}{\cosh(h_1) + 1/\sqrt{1+\alpha^2}\sinh(h_1)}, \tag{66}$$

resulting in the optimal state trajectory

$$x_{\text{opt}}(t) = [\cosh(t) - q(h_1)\sinh(t)], \quad t \in [0, h_1]. \tag{67}$$

We may then compute the optimal state at $t = h_1$ and the optimal cost over $t \in [0, \infty]$:

$$x_{\text{opt}}(h_1) = \frac{1}{\cosh(h_1) + \sinh(h_1)/\sqrt{1+\alpha^2}},$$
$$J_{[0,\infty],\text{opt}} = q(h_1).$$

The optimal state trajectory is obtained by combining the optimal state trajectories of the two time regions (60), (67). The optimal cost as a function of the parameter α for different values of the delay h_1 is shown in Fig. 2. As expected, larger values of the delay h_1 correspond to higher cost. It is also evident that the cost decreases as the parameter α increases. This observation may be explained as follows. From the criterion function (54), it is obvious that larger α makes the second input $K_1(t)$ cheaper, allowing the controller to inject large input while keeping the cost relatively small. Indeed, we can see from the optimal state trajectory depicted in Fig. 3 that for large α, as soon as the input $K_1(t)$ is active at $t = h_1$, the state is quickly driven to zero. The larger the parameter α, the quicker the state vanishes after $t = h_1$. By combining (61) and (65), we may retrieve the original input $K(t) = \text{col}(K_0(t), K_1(t))$ of the LQR problem (52), (54):

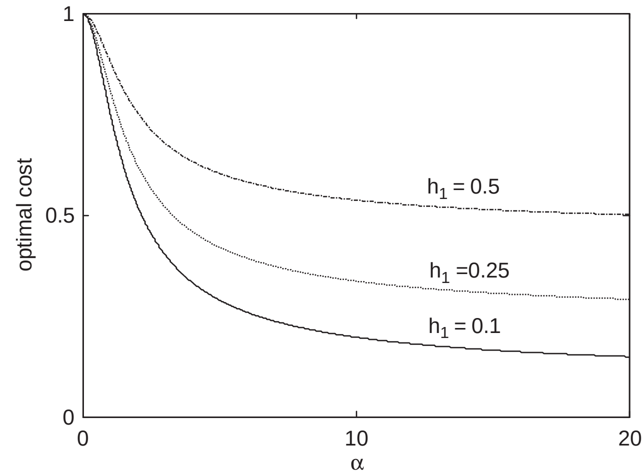

Fig. 2. The optimal cost as a function of α.

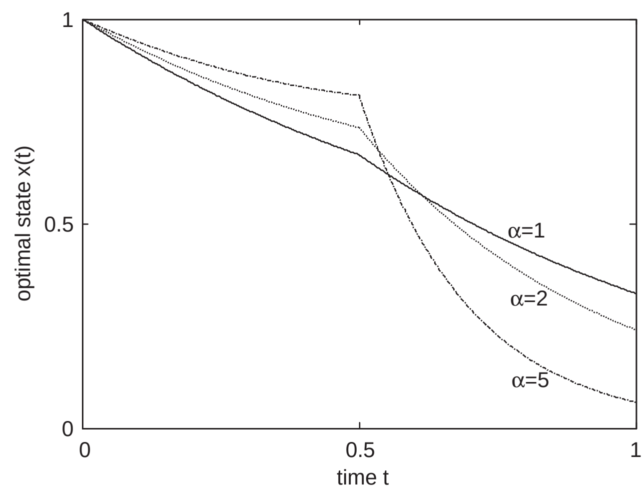

Fig. 3. The optimal state trajectory of the equivalent LQR problem for $h_1 = 0.5$

$$K_0(t) = \begin{cases} \sinh(t) - q(h_1)\cosh(t), & 0 \leqslant t \leqslant h_1, \\ \frac{-1}{\sqrt{1+\alpha^2}} x_{\text{opt}}(h_1) \mathrm{e}^{-\sqrt{1+\alpha^2}(t-h_1)}, & t > h_1. \end{cases}$$

$$K_1(t) = (-\alpha^2/(\sqrt{1+\alpha^2})) x_{\text{opt}}(h_1) \mathrm{e}^{-\sqrt{1+\alpha^2}t} \mathbb{1}(t).$$

Using formulas from Section 6, we may obtain the optimal controller for the regulator problem (48):

$$K_{\rm opt}(s) = \tau_{h_1}\begin{bmatrix} \dfrac{-q(h_1)}{s^2-1} \\ 0 \end{bmatrix} + \begin{bmatrix} e^{-sh_1} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \dfrac{-x_{\rm opt}(h_1)}{\sqrt{1+\alpha^2}(s+\sqrt{1+\alpha^2})} \\ \dfrac{-\alpha^2 x_{\rm opt}(h_1)}{\sqrt{1+\alpha^2}(s+\sqrt{1+\alpha^2})} \end{bmatrix}.$$

8. Concluding remarks

The standard H_2 problems of Fig. 1(a), (b) are equivalent to the H_2 regulator problem of Fig. 1(c). We showed that solving the H_2 regulator problem amounts to solving a series of LQR problems that results in the impulse response of the optimal controller. We also derived the state-space formulation of the optimal controller, which consists of a rational block, a FIR block, and delay components. Besides for solving the standard control problem, the theory may also be applied to solve the H_2 filtering problem with multiple delays.

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Appendix. Solution of the regional LQR problems

As discussed in Section 5.2, the solution of the LQR problem with multiple input delays amounts to solving an infinite horizon LQR problem over $t=[h_N,\infty]$ and N finite horizon LQR problems, each over $t=[h_k,h_{k+1}]$, $k=0,\dots,N$. The following two lemmas provide the solution of these problems. In this section we do not assume that $T_1(s)$ and $K(s)$ are single-column transfer functions. We denote the jth column of $T_1(s)$, $K(s)$, and B_1 as $T_{1,j}(s)$, $K_j(s)$, and $B_{1,j}$ respectively. Since the lemmas are standard results from LQR theory, they are provided without proofs. We refer to Anderson and Moore (1989) and Kwakernaak and Sivan (1972) for detailed presentation of the LQR theory.

Lemma A.1. *Consider LQR problem corresponding to the system*

$$\dot{x}(t) = Ax(t) + \bar{B}_k\Phi_{j,k}(t), \quad x(h_k) = x_{j,k} \tag{A.1}$$

with the criterion function

$$\min_{\Phi_{j,k}} x_{j,k}^{\rm T}M_{k+1}x_{j,k} + \int_{h_k}^{h_{k+1}} \|C_1x(t) + \bar{D}_k\Phi_{j,k}(t)\|^2\,{\rm d}t \tag{A.2}$$

where $x_{j,k}$ and $M_{k+1}\geqslant 0$ are known. Define

$$L_k := h_{k+1}-h_k, \quad R_k := \bar{D}_k^{\rm T}\bar{D}_k, \tag{A.3}$$

$$Q_k := C_1^{\rm T}(I-\bar{D}_kR_k^{-1}\bar{D}_k^{\rm T})C_1 \geqslant 0, \tag{A.4}$$

$$F_k := [-R_k^{-1}\bar{D}_k^{\rm T}C_1 \quad R_k^{-1}\bar{B}_k^{\rm T}], \tag{A.5}$$

$$S_k := \begin{bmatrix} (A-\bar{B}_kR_k^{-1}\bar{D}_k^{\rm T}C_1) & \bar{B}_kR_k^{-1}\bar{B}_k^{\rm T} \\ Q_k & -(A-\bar{B}_kR_k^{-1}\bar{D}_k^{\rm T}C_1)^{\rm T} \end{bmatrix} \tag{A.6}$$

$$\Sigma_k(t) = \begin{bmatrix} \Sigma_{k,11}(t) & \Sigma_{k,12}(t) \\ \Sigma_{k,21}(t) & \Sigma_{k,22}(t) \end{bmatrix} := e^{S_kt} \tag{A.7}$$

$$\tilde{\Sigma}_k(t) = \begin{bmatrix} \Sigma_{k,22}(t) & \Sigma_{k,21}(t) \\ -\Sigma_{k,12}(t) & -\Sigma_{k,11}(t) \end{bmatrix} \tag{A.8}$$

$$\Gamma_k(t) := \Sigma_{k,11}(t) + \Sigma_{k,12}(t)C_\ell(\tilde{\Sigma}_k(L_k), M_{k+1}) \tag{A.9}$$

$$\Xi_k(t) := \Sigma_{k,21}(t) + \Sigma_{k,22}(t)C_\ell(\tilde{\Sigma}_k(L_k), M_{k+1}) \tag{A.10}$$

$$P_k := {\rm col}[I, \quad C_\ell(\tilde{\Sigma}_k(L_k), M_{k+1})]. \tag{A.11}$$

Then the optimal input $\Phi_{j,k}(t)$ is the impulse response of the transfer function

$$\Phi_{j,k,{\rm opt}}(s) = e^{-sh_k}\tau_{L_k}(F_k(sI-S_k)^{-1}P_kx_{j,k}). \tag{A.12}$$

Moreover, the optimal cost and the optimal final state are respectively given by

$$x(h_k)^{\rm T}M_kx(h_k) \quad \textit{and} \tag{A.13}$$

$$x_{\rm opt}(h_{k+1}) = \Gamma_k(L_k)x(h_k), \tag{A.14}$$

$$\textit{where} \quad M_k = -\Xi_k(0). \tag{A.15}$$

Lemma A.2. *Consider LQR problem corresponding to the system*

$$\dot{x}(t) = Ax(t) + B_2\Phi_{j,N}(t), \quad x(h_N) = x_{j,N} \tag{A.16}$$

with the criterion function

$$\min_{\Phi_{j,N}} \int_{h_N}^{\infty} \|C_1x(t) + D_2\Phi_{j,N}(t)\|_2^2\,{\rm d}t, \tag{A.17}$$

where $x_{j,N}$ is known. Let M_N be the stabilizing solution of the Riccati equation:

$$\begin{aligned} 0 = Q_N &+ (A - B_2R_N^{-1}D_2^{\rm T}C_1)^{\rm T}M_N \\ &+ M_N(A-B_2R_N^{-1}D_2^{\rm T}C_1) \\ &- M_NB_2R_N^{-1}B_2^{\rm T}M_N, \end{aligned} \tag{A.18}$$

where $R_N = D_2^{\rm T}D_2$ and $Q_N = C_1^{\rm T}(I-D_2R_N^{-1}D_2^{\rm T})C_1$. Define

$$\begin{aligned} F_N &:= -R_N^{-1}(B_2^{\rm T}M_N + D_2^{\rm T}C_1), \\ S_N &:= A + B_2F_N, \end{aligned} \tag{A.19}$$

then the optimal control $\Phi_{j,N}(t)$ on $[h_N, \infty]$ is the impulse response of the transfer function

$$\Phi_{N,j,\mathrm{opt}}(s) = \mathrm{e}^{-sh_N}(F_N(sI - S_N)^{-1}x_{j,N}) \tag{A.20}$$

and the optimal cost is

$$x_{j,N}^{\mathrm{T}} M_N x_{j,N}. \tag{A.21}$$

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