Duality transformations for generalized WDVV in Seiberg–Witten theory

L.K. Hoevenaars

Afdeling Toegepaste Wiskunde, Faculteit EWI, Universiteit Twente, 7500 AE Enschede, The Netherlands

Abstract

In Seiberg–Witten theory the solutions to these equations come in certain classes according to the gauge group. We show that the duality transformations transform solutions within a class to another solution within the same class, by using a subset of the Picard–Fuchs equations on the Seiberg–Witten family of Riemann surfaces. The electric–magnetic duality transformations can be thought of as changes of a canonical homology basis on the surfaces which in our derivation is clearly responsible for the covariance of the generalized WDVV system.
© 2004 Elsevier B.V. All rights reserved.

1. Introduction

In 1994, Seiberg and Witten [1] solved the low energy behaviour of pure $N = 2$ super-Yang–Mills theory by giving the solution of the prepotential $\mathcal{F}$. The essential ingredients in their construction are a family of Riemann surfaces Σ, a meromorphic differential λ_{SW} on it and the definition of the prepotential in terms of period integrals of λ_{SW}

$$a_I = \int_{A_I} \lambda_{SW}, \qquad \frac{\partial \mathcal{F}}{\partial a_I} = \int_{B_I} \lambda_{SW}. \tag{1.1}$$

E-mail address: l.k.hoevenaars@ewi.utwente.nl (L.K. Hoevenaars).

The cycles A_I and B_I belong to a subset of a canonical homology basis on the surface Σ and the a_I are the moduli parameters of the family of surfaces. These formulae define the prepotential $\mathcal{F}(a_1, \ldots, a_r)$ implicitly, where r denotes the rank of the gauge group under consideration.

A link between the prepotential and the Witten–Dijkgraaf–Verlinde–Verlinde equations [2,3] was proven to exist in [4] where it was found that the prepotential $\mathcal{F}(a_1, \ldots, a_r)$ for pure $N = 2$ SYM theory for classical Lie algebras (those of type A, B, C, D) satisfies the generalized WDVV equations

$$\mathcal{F}_I \left[\sum_M \gamma_M \mathcal{F}_M \right]^{-1} \mathcal{F}_J = \mathcal{F}_J \left[\sum_M \gamma_M \mathcal{F}_M \right]^{-1} \mathcal{F}_I \quad \forall I, J, K = 1, \ldots, r, \tag{1.2}$$

where the $\mathcal{F}_I$ are matrices given by $(\mathcal{F}_I)_{JK} = \frac{\partial^3 \mathcal{F}}{\partial a_I \partial a_J \partial a_K}$ and the γ_M may depend on the a_I. If $\mathcal{F}$ satisfies (1.2) for some set of γ_M it will automatically satisfy them for any other set $\tilde{\gamma}_M$ as long as $\sum_M \tilde{\gamma}_M \mathcal{F}_M$ is invertible. These equations are indeed a generalization of the original WDVV equations, since we no longer demand the matrix $\sum_M \gamma_M \mathcal{F}_M$ to be flat and constant.

In an alternative approach to [4], Ito and Yang [5] give a proof which is valid[1] for Lie algebras of type A, D, E_6 and it was shown in [6] that this method can be adapted to give a proof also in the case of Lie algebras of type B, C. The approach used by Ito and Yang consists of two main ingredients: an associative algebra with structure constants C_{IJ}^K, together with a relation between the structure constants and the third order derivatives of the function $\mathcal{F}$

$$\mathcal{F}_I = C_I \cdot \left(\sum_{M=1}^r \gamma_M \mathcal{F}_M \right) \tag{1.3}$$

for some γ_M. To derive this relation they used a subset of the Picard–Fuchs equations on the Riemann surfaces. If the matrix $\sum_{M=1}^r \gamma_M \mathcal{F}_M$ is invertible then we can substitute (1.3) into the WDVV equations (1.2) which then express associativity of the algebra through a relation on the structure constants: $[C_I, C_J] = 0$.

In the physical context of Seiberg–Witten theory, electric–magnetic duality transformations are very important and therefore it is a natural question if these are symmetries of the generalized WDVV equations. The duality transformations form a subgroup of $Sp(2r, \mathbb{Z})$ acting as linear transformations on the vector $(\vec{a}, \vec{\nabla} F)$. These transformations are indeed symmetries, as was shown in [7] for a pure S-duality transformation and recently in [8] for general symplectic transformations.

We will show in this Letter that the classes of solutions of the WDVV system coming from Seiberg–Witten theory are invariant under duality transformations, by giving the duality transformations their natural interpretation in the Seiberg–Witten context. The reason for the invariance is that the Picard–Fuchs equations leading to the relation (1.3) have solutions in terms of period integrals of λ_{SW}. We require of our solutions (1.1) only that the cycles be canonical, so they are invariant under symplectic transformations of the cycles. These are precisely the duality transformations. In the derivation of (1.3) we will rely heavily on the connection between the Seiberg–Witten and Landau–Ginzburg theories which is quite straightforward in the simply laced (ADE) cases. For B and C Lie algebras this connection is less clear [6] but we will show that one can use the same method as in the other cases. For F_4 it is not rigorously proven that the prepotential satisfies the WDVV equations, so we cannot say anything definite about duality in this case.[2]

2. Duality transformations

Duality transformations play an important role in $N = 2$ super-Yang–Mills theory, where in the classical theory they exchange the Bianchi identity and the equations of motion of the Yang–Mills field strength. In the quantum

---

[1] Algebras E_7 and E_8 are not considered in [5] because of computational difficulty, but they are expected to follow the same pattern.

[2] Since G_2 has rank 2, the prepotential depends only on 2 variables and it trivially satisfies the WDVV equations.

field theory, these transformations were studied extensively in the context of supergravity (for a review, see [9]) and they turn out to form a subgroup of $Sp(2r, \mathbb{Z})$ where r is the rank of the gauge group. The action on $(a_I, \mathcal{F}_I)$ of (1.1) is given by

$$\begin{pmatrix} \vec{a} \\ \vec{\mathcal{F}} \end{pmatrix} \rightarrow \begin{bmatrix} U & Z \\ W & V \end{bmatrix} \begin{pmatrix} \vec{a} \\ \vec{\mathcal{F}} \end{pmatrix}, \qquad \begin{bmatrix} U & Z \\ W & V \end{bmatrix} \in Sp(2r, \mathbb{Z}). \tag{2.1}$$

There is some terminology used by physicists for some special transformations

- An example of S-duality is a transformation for which

$$\begin{pmatrix} \vec{a} \\ \vec{\mathcal{F}} \end{pmatrix} \rightarrow \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{pmatrix} \vec{a} \\ \vec{\mathcal{F}} \end{pmatrix}$$

  and in general S-duality transformations exchange the strong and weak coupling regimes of the physical theory.
- An example of T-duality is a transformation for which

$$\begin{pmatrix} \vec{a} \\ \vec{\mathcal{F}} \end{pmatrix} \rightarrow \begin{bmatrix} I & 0 \\ W & I \end{bmatrix} \begin{pmatrix} \vec{a} \\ \vec{\mathcal{F}} \end{pmatrix}$$

  and in general T-duality transformations are perturbative, so they go from weak coupling to weak coupling.

Seiberg and Witten used the fact that duality transformations can be thought of as changes of a canonical basis of a family of auxiliary Riemann surfaces in the following way. Let a canonical basis $\{A_I, B_I\}$ of homology on a Riemann surface be given. The transformations that take this canonical basis into another one are known to be (see, e.g., [10]) symplectic transformations $Sp(2g, \mathbb{Z})$ acting like

$$\begin{pmatrix} \vec{A} \\ \vec{B} \end{pmatrix} \rightarrow \begin{bmatrix} U & Z \\ W & V \end{bmatrix} \begin{pmatrix} \vec{A} \\ \vec{B} \end{pmatrix}. \tag{2.2}$$

For Lie algebra A_r the genus of the curves taken in Seiberg–Witten theory equals the rank of the gauge group, so (2.2) generate transformations on $a_I, \mathcal{F}_I$ since for example

$$\int_{A_1+A_2} \lambda_{SW} = \int_{A_1} \lambda_{SW} + \int_{A_2} \lambda_{SW} = a_1 + a_2$$

so

$$\begin{pmatrix} \vec{a} \\ \vec{\mathcal{F}} \end{pmatrix} = \begin{pmatrix} \int_{\vec{A}} \lambda_{SW} \\ \int_{\vec{B}} \lambda_{SW} \end{pmatrix} \rightarrow \begin{pmatrix} \int_{U\vec{A}+Z\vec{B}} \lambda_{SW} \\ \int_{W\vec{A}+V\vec{B}} \lambda_{SW} \end{pmatrix} = \begin{bmatrix} U & Z \\ W & V \end{bmatrix} \begin{pmatrix} \vec{a} \\ \vec{\mathcal{F}} \end{pmatrix}.$$

In other words, electric–magnetic duality transformations in the Seiberg–Witten context can be thought of as changes of canonical bases of homology on families of Riemann surfaces.

For other gauge groups, the rank is always less than the genus of the curve and we have to take a subset of the homology basis in such a way that this subset has canonical intersection numbers $A_I \circ A_J = B_I \circ B_J = 0$, $A_I \circ B_J = \delta_{IJ}$. If we call the linear subspace of $H^1(\Sigma, \mathbb{Z})$ spanned by these cycles X, then duality transformations are generated by changes of the canonical homology basis which leave X invariant.

3. Picard–Fuchs equations and duality

In our derivation of the generalized WDVV equations, we consider a system of differential equations for the periods $\oint_\Gamma \lambda_{SW}$ (Picard–Fuchs like equations) and then substitute $a_I = \oint_{A_I} \lambda_{SW}$ and $\mathcal{F}_I = \oint_{B_I} \lambda_{SW}$ in them. This way, the derivation of the WDVV equations is manifestly invariant under changes of the canonical homology basis and therefore under electric–magnetic duality transformations.

The Riemann surfaces for the ADE cases read [11]

$$z + \frac{1}{z} = W(x, u_i), \tag{3.1}$$

where W can be though of as the one-dimensional version of the Landau–Ginzburg superpotential. In deriving the Picard–Fuchs equations, use is made of the flat coordinates of the corresponding Landau–Ginzburg theories. For the B, C Lie algebras, the methods of [5] are no longer directly applicable: we have to 'twist' the affine Lie algebra to construct the surfaces.[3] This leads to surfaces

$$z + \frac{1}{z} = \tilde{W}(x, u_i) \tag{3.2}$$

where $\tilde{W}$ need not have a direct relation to Landau–Ginzburg theory. For the B, C cases, where the relation with the corresponding superpotential is still quite straightforward, it is shown in [6] that the method of using Picard–Fuchs equations can be adapted in order to give the WDVV equations.

In the next sections we will review the derivation of the WDVV equations in order to see that duality transformations are symmetries of them.

3.1. The ADE cases

The family of Riemann surfaces associated with Seiberg–Witten theory with Lie algebras of A, D, E type is given by [11]

$$z + \frac{1}{z} = W(x, u_i). \tag{3.3}$$

Here W can be thought of as a one variable Landau–Ginzburg superpotential for the corresponding Lie algebra. For instance, $W_{A_r}(x, u_i) = x^{r+1} - \sum_{i=1}^{r-1} u_i x^{r-i}$ is the superpotential for Lie algebra A_r. However W_{E_6} contains a square root of a polynomial, but the structure constants of the chiral ring are the same as in the three-variable situation [12]. From the Landau–Ginzburg theory it is known (see, e.g., [13] and references therein) that we can pass from the moduli u_i to flat coordinates t_i in terms of which the Gauss–Manin connection is set to zero. The Landau–Ginzburg product structure reads

$$\frac{\partial W}{\partial t_i} \frac{\partial W}{\partial t_j} = \sum_{k=1}^{r} C_{ij}^k(t) \frac{\partial W}{\partial t_k} \frac{\partial W}{\partial t_r} + Q_{ij} \frac{\partial W}{\partial x} \tag{3.4}$$

which leads to the algebra

$$\frac{\partial W}{\partial t_i} \frac{\partial W}{\partial t_j} = \sum_{k=1}^{r} C_{ij}^k(t) \frac{\partial W}{\partial t_k} \tag{3.5}$$

and the flat coordinates[4] satisfy

$$\frac{\partial Q_{ij}}{\partial x} = \frac{\partial^2 W}{\partial t_i \partial t_j}. \tag{3.6}$$

---

[3] The ADE algebras are invariant under this twisting.

[4] The t_i are flat coordinates for the *multivariable* Landau–Ginzburg superpotentials. In the cases of A, D Lie algebras these superpotentials are respectively not and not much different from the one variable case presented here. For the Lie algebras of type E_6 it was shown explicitly in [12] that the known flat coordinates are indeed a solution to (3.6) for the one-variable superpotential, and that the algebra is the same as in the multivariable case.

Now that the Riemann surfaces (3.3) are introduced, we need a meromorphic differential on them

$$\lambda_{SW} = x\frac{dz}{z} \tag{3.7}$$

whose derivatives with respect to the moduli are

$$\frac{\partial \lambda_{SW}}{\partial t_i} = d[\cdots] - \frac{\partial W}{\partial t_i}\frac{dx}{z - \frac{1}{z}}. \tag{3.8}$$

It is stated often in the literature that the forms

$$-\frac{\partial W}{\partial t_i}\frac{dx}{z - \frac{1}{z}}$$

are holomorphic and linearly independent. For all classical Lie algebras, the Riemann surfaces are hyperelliptic and this statement can be easily checked. For all exceptional Lie algebras however, the Riemann surfaces are not hyperelliptic and it is still not proven that the forms are holomorphic.

As a final ingredient for the Seiberg–Witten theory, we introduce a third set of coordinates a_I on the moduli space of the family of surfaces and objects $\mathcal{F}_I$

$$a_I = \oint_{A_I} \lambda_{SW}, \qquad \mathcal{F}_I = \oint_{B_I} \lambda_{SW}, \tag{3.9}$$

where we take a subset of a canonical basis for the homology. The holomorphic parts of the differentials $\frac{\partial \lambda_{SW}}{\partial a_I}$ are canonical with respect to the cycles and therefore $\frac{\partial \mathcal{F}_I}{\partial a_J}$ is a submatrix of the period matrix, which is symmetric. So $\mathcal{F}_I$ is a gradient and there exists a function $\mathcal{F}(a_I)$ with derivatives $\mathcal{F}_I$ and this so-called prepotential solves the low energy behaviour of $N = 2$ super-Yang–Mills theory.

The product structure (3.4) can also be expressed in terms of the a_I as follows

$$\frac{\partial W}{\partial a_I}\frac{\partial W}{\partial a_J} = \sum_{K=1}^{r} C_{IJ}^{K}(a)\frac{\partial W}{\partial a_K}\left[\sum_{M=1}^{r}\frac{\partial a_M}{\partial t_r}\frac{\partial W}{\partial a_M}\right] + \tilde{Q}_{IJ}\frac{\partial W}{\partial x}, \tag{3.10}$$

where the structure constants are related through $C_{IJ}^{K}(a) = \sum_{i,j,k}\frac{\partial t_i}{\partial a_I}\frac{\partial t_j}{\partial a_J}C_{ij}^{k}(t)\frac{\partial a_K}{\partial t_k}$. Here we assume that the transformation from t_i to a_I is invertible, which can be justified: in the case of type A Lie algebras, the number of moduli equals the genus and therefore the Jacobian

$$\frac{\partial a_I}{\partial t_j} = \oint_{A_I} \frac{\partial \lambda_{SW}}{\partial t_j}, \quad I, j = 1, \ldots, g \tag{3.11}$$

is indeed invertible, since $\frac{\partial \lambda_{SW}}{\partial t_j}$ form a basis of holomorphic forms (modulo exact forms). For the other Lie algebras, we have to take a subset of a canonical basis. Suppose we supplement the forms $\frac{\partial \lambda_{SW}}{\partial t_j}$ with more forms ω_i to form a basis of holomorphic forms, then we know that the $g \times g$ matrix

$$\begin{bmatrix} \oint_{A_1}\frac{\partial \lambda_{SW}}{\partial t_1} & \cdots & \oint_{A_1}\frac{\partial \lambda_{SW}}{\partial t_r} & \oint_{A_1}\omega_1 & \cdots & \oint_{A_1}\omega_{g-r} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \oint_{A_g}\frac{\partial \lambda_{SW}}{\partial t_1} & \cdots & \oint_{A_g}\frac{\partial \lambda_{SW}}{\partial t_r} & \oint_{A_g}\omega_1 & \cdots & \oint_{A_g}\omega_{g-r} \end{bmatrix}$$

has rank g. So the submatrix

$$\begin{bmatrix} \oint_{A_1} \frac{\partial \lambda_{SW}}{\partial t_1} & \cdots & \oint_{A_1} \frac{\partial \lambda_{SW}}{\partial t_r} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \oint_{A_g} \frac{\partial \lambda_{SW}}{\partial t_1} & \cdots & \oint_{A_g} \frac{\partial \lambda_{SW}}{\partial t_r} \end{bmatrix}$$

has rank r, and we can always find a square submatrix of rank r by choosing the proper cycles. Therefore we can always choose cycles in such a way that the transformation from t_i to a_I is invertible. In Seiberg–Witten theory, there is a specific prescription [11] of what cycles one has to take and inevitability of the corresponding submatrix should be checked.

Following [5] we write down a subset of the Picard–Fuchs[5] equations

$$\left(\frac{\partial^2}{\partial t_i \partial t_j} - \sum_{k=1}^{r} C_{ij}^k(t) \frac{\partial^2}{\partial t_k \partial t_r} \right) \oint_\Gamma \lambda_{SW} = 0 \tag{3.12}$$

which hold for integrals along any closed cycle Γ.

We will now prove the following theorem

Theorem 1. *The following formula holds*: $\mathcal{F}_{IJK} = \sum_{L=1}^{r} C_{IJ}^L(a) \left[\sum_{M=1}^{r} \frac{\partial a_M}{\partial t_r} \mathcal{F}_{MKL} \right]$.

Proof. Making a change of coordinates in (3.12) from t_i to a_I we get

$$\left(\frac{\partial a_I}{\partial t_i} \frac{\partial a_J}{\partial t_j} \frac{\partial^2}{\partial a_I \partial a_J} - \sum_{k=1}^{r} C_{ij}^k(t) \frac{\partial a_K}{\partial t_k} \frac{\partial a_M}{\partial t_r} \frac{\partial^2}{\partial a_K \partial a_M} \right) \oint_\Gamma \lambda_{SW} = 0,$$

where we made use of the fact that the a_I themselves are solutions to (3.12). Now we can substitute $\Gamma = B_K$ and find

$$\frac{\partial^3 \mathcal{F}}{\partial a_I \partial a_J \partial a_K} = \sum_{L=1}^{r} C_{IJ}^L(a) \left[\sum_{M=1}^{r} \frac{\partial a_M}{\partial t_r} \mathcal{F}_{MKL} \right] \tag{3.13}$$

which is the relation (1.3) between the third order derivatives of $\mathcal{F}(a_I)$ and the structure constants $C_{IJ}^L(a)$. $\quad\square$

To obtain the generalized WDVV equations for the function $\mathcal{F}(a_I)$ we need the matrix $\sum_{M=1}^{r} \frac{\partial a_M}{\partial t_r} \mathcal{F}_M$ to be invertible, which is generically true. Otherwise small perturbations can make it so. Since the derivation of the theorem does not depend on the specific choice of canonical homology basis we started with, and since duality transformations are changes of canonical homology, we have the following

Corollary 2. *Take a set of solutions of the WDVV system coming from Seiberg–Witten theory with gauge group of ADE type. Duality transformations leave the set of solutions invariant.*

---

[5] If a basis of cohomology on a family of Riemann surfaces is given by $\{\omega_i\}$ where $i = 1, \ldots, 2g$ then the Picard–Fuchs equations express

$$\frac{\partial}{\partial t_i} \oint_\Gamma \omega_j = \sum_k A_{ij}^k \oint_\Gamma \omega_k,$$

where Γ is some closed cycle and the t_i are the moduli of the family. By a subset of Picard–Fuchs equations, we mean that we can express derivatives of some (not all) period integrals into a linear combination of some others.

In other words, after a duality transformation there exists a function $\tilde{\mathcal{F}}(\tilde{a}_I)$ which is in general different[6] from $\mathcal{F}(a_I)$ but still satisfies the generalized WDVV system.

3.2. The B and C cases

Let us stress that for B, C Lie algebras, we no longer have

$$z + \frac{1}{z} = W$$

with W a one variable version of the Landau–Ginzburg superpotential. This will be reflected in the Picard–Fuchs equations. In fact, if we still work with the flat coordinates of the Landau–Ginzburg superpotential W_{BC} one can derive the Picard–Fuchs equations [5]

$$\left(\frac{\partial^2}{\partial t_i \partial t_j} - \sum_{k=1}^{r} C_{ij}^k \frac{\partial^2}{\partial t_k \partial t_r} - \sum_{k=1}^{r} \sum_{n=1}^{r} \frac{d_n t_n}{h^\vee} D_{ij}^k \frac{\partial^2}{\partial t_k \partial t_n} + \sum_{k=1}^{r} D_{ij}^k \frac{1}{h^\vee} (1 - d_k) \frac{\partial}{\partial t_k} \right) \oint_\Gamma \lambda_{\text{SW}} = 0, \tag{3.14}$$

where the C_{ij}^k are structure constants of the W_{BC} Landau–Ginzburg theory, the d_n are so-called degrees of the Lie algebra (the exponents $+1$), D_{ij}^k depend on the t_i and $h^\vee$ is the dual Coxeter number. Making a change of coordinates to the a_I just like we did for ADE algebras and using the fact that the a_I satisfy (3.14), we get

$$\left[\frac{\partial a_I}{\partial t_i} \frac{\partial a_J}{\partial t_j} - \sum_k C_{ij}^k \frac{\partial a_I}{\partial t_k} \frac{\partial a_J}{\partial t_r} - \sum_{k,n} D_{ij}^k \frac{d_n t_n}{h^\vee} \frac{\partial a_I}{\partial t_n} \frac{\partial a_J}{\partial t_k} \right] \frac{\partial^2}{\partial a_I \partial a_J} \oint_\Gamma \lambda_{\text{SW}} = 0. \tag{3.15}$$

These equations are not of such a form that we can derive a relation between $\mathcal{F}_{IJK}$ and the structure constants C_{ij}^k in the same way as before. However, in [6] it was shown that some other constants $\tilde{C}_{ij}^k$ form structure constants of an associative algebra and are related to C_{ij}^k through

$$C_{ij}^k = \tilde{C}_{ij}^k - \sum_{l,n=1}^{r} D_{ij}^l \frac{2n t_n}{h^\vee} \tilde{C}_{nl}^k, \qquad C_i = \tilde{C}_i - D_i \cdot \left(\sum_{n=1}^{r} \frac{2n t_n}{h^\vee} \tilde{C}_n \right), \tag{3.16}$$

where the second line is in matrix form. Substituting this into (3.15) we get

$$\left[\frac{\partial a_I}{\partial t_i} \frac{\partial a_J}{\partial t_j} - \sum_k \tilde{C}_{ij}^k \frac{\partial a_I}{\partial t_k} \frac{\partial a_J}{\partial t_r} \right] \frac{\partial^2}{\partial a_I \partial a_J} \oint_\Gamma \lambda_{\text{SW}}$$
$$+ \sum_n D_{ij}^l \frac{2n t^n}{h^\vee} \left[\frac{\partial a_I}{\partial t_n} \frac{\partial a_J}{\partial t_l} - \sum_l \tilde{C}_{nl}^k \frac{\partial a_I}{\partial t_k} \frac{\partial a_J}{\partial t_r} \right] \frac{\partial^2}{\partial a_I \partial a_J} \oint_\Gamma \lambda_{\text{SW}} = 0 \tag{3.17}$$

and in [6] further information about the D_{ij}^l was used to conclude

$$\left[\frac{\partial a_I}{\partial t_i} \frac{\partial a_J}{\partial t_j} - \sum_k \tilde{C}_{ij}^k \frac{\partial a_I}{\partial t_k} \frac{\partial a_J}{\partial t_r} \right] \frac{\partial^2}{\partial a_I \partial a_J} \oint_\Gamma \lambda_{\text{SW}} = 0. \tag{3.18}$$

From this point we can proceed with the same reasoning as in the ADE case and conclude that the following theorem holds

---

[6] The transformations are often too difficult to perform explicitly.

Theorem 3. *The following formula holds*: $\mathcal{F}_{IJK} = \sum_{L=1}^{r} \tilde{C}_{IJ}^{L}(a)\left[\sum_{M=1}^{r} \frac{\partial a_M}{\partial t_r} \mathcal{F}_{MKL}\right]$.

Corollary 4. *Take a set of solutions of the WDVV system coming from Seiberg–Witten theory with gauge group of B, C type. Duality transformations leave the set of solutions invariant.*

4. Conclusion and outlook

It was shown in [8] that elements of the *continuous* symplectic group $Sp(2n, \mathbb{C})$ are symmetries of the generalized WDVV system. Because of the relation between Seiberg–Witten theory and Riemann surfaces, this leads naturally to the study of the *discrete* subgroup $Sp(2n, \mathbb{Z})$ in the Seiberg–Witten context, and we have shown how the interpretation of this discrete subgroup as changes of a canonical homology basis leads automatically to the invariance of classes of solutions to the WDVV system.

Other solutions of the generalized WDVV system found so far either come from the original WDVV equations or from the context of tau functions of conformal mappings [14]. Interpretations of the symplectic transformations can be looked for in both contexts. This might show whether the discrete subgroup is a natural object there as well.

Other interesting symmetries of the WDVV system have been found [15,16], but no systematic investigation of symmetries has been undertaken. We believe such an investigation would be interesting.

References

[1] N. Seiberg, E. Witten, Nucl. Phys. B 426 (1994) 19, hep-th/9407087.
[2] E. Witten, Two-dimensional gravity and intersection theory on moduli space, in: Surveys in Differential Geometry, Cambridge, MA, 1990, Lehigh Univ. Press, Bethlehem, PA, 1991, p. 243.
[3] R. Dijkgraaf, H. Verlinde, E. Verlinde, Nucl. Phys. B 352 (1991) 59.
[4] A. Marshakov, A. Mironov, A. Morozov, Phys. Lett. B 389 (1996) 43, hep-th/9607109.
[5] K. Ito, S.-K. Yang, Phys. Lett. B 433 (1998) 56, hep-th/9803126.
[6] L.K. Hoevenaars, R. Martini, Lett. Math. Phys. 57 (2001) 175.
[7] Y. Ohta, J. Math. Phys. 41 (2000) 6042.
[8] A. Marshakov, B. de Wit, hep-th/0105289.
[9] B. de Wit, Nucl. Phys. B (Proc. Suppl.) 101 (2001) 154, hep-th/0103086.
[10] H.M. Farkas, I. Kra, Riemann Surfaces, second ed., Springer-Verlag, New York, 1992.
[11] E. Martinec, N. Warner, Nucl. Phys. B 459 (1996) 97, hep-th/9509161.
[12] T. Eguchi, S.-K. Yang, Phys. Lett. B 394 (1997) 315, hep-th/9612086.
[13] N.P. Warner, hep-th/9301088.
[14] A. Boyarsky, A. Marshakov, O. Ruchayskiy, P. Wiegmann, A. Zabrodin, Phys. Lett. B 515 (2001) 483, hep-th/0105260.
[15] B. Dubrovin, Geometry of 2D topological field theories, in: Integrable Systems and Quantum Groups, Montecatini Terme, 1993, Springer, Berlin, 1996, p. 120, hep-th/9407018.
[16] A. Mironov, A. Morozov, Phys. Lett. B 424 (1998) 48.