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Synthesis method for linkages with center of mass at invariant link point — Pantograph based mechanisms

Volkert van der Wijk *, Just L. Herder

Department of Mechanical Automation and Mechatronics, Fac. of Engineering Technology, University of Twente, P.O. Box 217, 7500 AE, Enschede, The Netherlands

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ABSTRACT

This paper deals with the synthesis of the motion of the center of mass (CoM) of linkages as being a stationary or invariant point at one of its links. This is of importance for the design of inherently shaking force balanced mechanisms, static balancing, and other branches of mechanical synthesis.
For this purpose Fischer's mechanism is investigated as being a composition of pantographs. It can be shown that linkages that are composed of pantographs and of which all links have an arbitrary CoM can be inherently balanced for which Fischer's method is a useful tool.
To calculate the principal dimensions for which linkages have their CoM at an invariant link point, an approach based on linear momentum is proposed. With this approach it is possible to investigate each degree-of-freedom individually. *Equivalent Linear Momentum Systems* are proposed to facilitate the calculations in order to use different convenient reference frames. The method is applied to planar linkages with revolute joints, however it also applies to linkages with other types of joints. As a practical example a shaking force and shaking moment balanced 2-DoF grasper mechanism is derived.

1. Introduction

Analysis of the motion of the center of mass (CoM) of a system of rigid bodies is important in many applications. For this analysis the computer is often of great help. However in the past, investigations had to be done by hand calculations. Geometric understanding in combination with graphic solutions then was highly important in order to facilitate the laborious calculations.

One illustrative example is presented in the studies by the German biomechanician Otto Fischer around 1900. He was involved in studying the motion of living creatures (human beings and animals) and approached them as systems of rigid bodies. To reduce the effort of calculating the internal forces and moments, he was particularly interested in the motion of the CoM of the system. He published various books considering these studies of which one published in 1906 [1] may be the most interesting.

For his studies, Fischer introduced a method to geometrically trace the CoM of multiple jointed rigid elements. This method was named the *method of principal vectors* by Lowen and Berkof [2]. Well known is the application of this method to the three degrees-of-freedom (DoF) chain shown in Fig. 1. The three principle links are jointed together with revolute pairs at $G_{1,2}$ and $G_{2,3}$. The CoMs of these links are S_1, S_2, and S_3. On each of these links a *principal point* H_i is determined. Subsequently with a geometric construction of lines that are parallel to lines $G_{1,2}H_1$, $G_{1,2}H_2$, $G_{2,3}H_2$, and $G_{2,3}H_3$, point S_0 is determined being the CoM of the three links. Physically, these parallel lines can be jointed (massless) links, leaving the DoFs of the linkage unaffected. For all motion, point S_0 is the location of the CoM of the linkage. Since this point is fixed in at least one of the links, in this case on both links $H_{1,2}S_0$ and $H_{2,3}S_0$, it is referred to in this paper as an invariant link point.

* Corresponding author.
E-mail addresses: v.vanderwijk@utwente.nl (V. van der Wijk), j.l.herder@utwente.nl (J.L. Herder).

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Fischer proved that his method works for a variety of simple and complex, open and closed chains of any number of elements. In [1], examples are shown of a chain of six links and of a spatial system of twenty elements with multiple open and closed chains. Fischer limited himself mainly to mass symmetric links, i.e. links of which the CoM is at the line connecting the joints.

Fischer's method is described by various authors. Wittenbauer in 1923 developed Fischer's method by applying it to some more complex closed chains and by illustrating the procedure of applying the method to parallel linkages [3]. Among others, Beyer [4] and Federhofer [5] summarized the method while Kreutziger [6] and Wunderlich [7] applied Fischer's method to show that the motion of the CoM of a 4R-four-bar mechanism describes a curve similar to a coupler-curve of the mechanism.

Nowadays Fischer's method remains interesting for the field of dynamic balancing and static balancing. In order to balance all shaking forces, i.e. to make the resultant dynamic reaction forces on the base of the mechanism zero, the CoM of the mechanism should remain stationary (or move with constant velocity). One approach to static balancing is also to have a stationary CoM. Fischer already related his method with shaking force balancing and studied the balancing of a crank-slider mechanism [1]. In 1957, Shchepetil'nikov [8] used Fischer's method and introduced the *method of double contour transformation* to find auxiliary mechanisms that describe the motion of the CoM of a linkage, not being restricted to four-bar mechanisms as in [6,7]. These auxiliary mechanisms were force balanced with additional counter-masses in order to make the CoM of the system stationary.

Hilpert in 1965 showed how in addition to Fischer's method a pantograph with counter-mass can be used to bring the CoM of a mechanism to a stationary position [9]. Shchepetil'nikov in 1975 extended the method of principle vectors by applying it to systems of unsymmetrical links, i.e. links with arbitrary CoM [10].

More recently, Agrawal et al. presented some articles, including notably [11], in which they mounted the mechanism of Fig. 1 on a pin at the CoM at S_0. The CoM then is stationary and the mechanism becomes a two degree-of-freedom (DoF) shaking force balanced manipulator. They also considered the mass of the parallel links and showed experimentally that this mass can be included such that point S_0 is the CoM of the complete linkage. However, theoretically the problem of including these masses was not solved. Their research was also restricted to mass symmetric links.

The advantage of applying Fischer's method for the purpose of force balancing is that no additional balancing elements such as counter-masses are needed. This results in balanced linkages that have a relatively low mass and inertia [12,13], which is advantageous for having, among others, low input torques, increased payload capabilities, and low bearing forces [14,15]. In fact, Fischer's mechanism is an example of what will here be referred to as an *inherently force balanced linkage*, i.e. a linkage that is kinematically arranged such that its CoM is at an invariant point in one of its links.

Although Fischer's method is well known and widely applied, the physical meaning of the principal points (and therefore the principal vectors) is not clear.

The purpose of this paper is twofold. It has the purpose to clarify the physical meaning of the principal points by showing that Fischer's method consists of a union of pantograph linkages, which proves that solutions exist even when all links have an arbitrary CoM. In addition, the paper has the purpose to propose a more fundamental approach to the calculations of the principal points. This approach is based on linear momentum equations, which for each DoF of the linkage can be investigated individually. The study concentrates on planar linkages with revolute joints, but this is not a fundamental restriction.

First the physical meaning of the principal points will be given scientific basis by studying the pantograph linkage. Then the derivation of Fischer's linkage of Fig. 1 from a union of pantograph linkages is presented. By using linear momentum equations and an equivalent linear momentum system, the principal points of the generalized Fischer's mechanism in which each link has an arbitrary CoM are calculated. Subsequently it is shown how this method can be applied to more complex linkages.

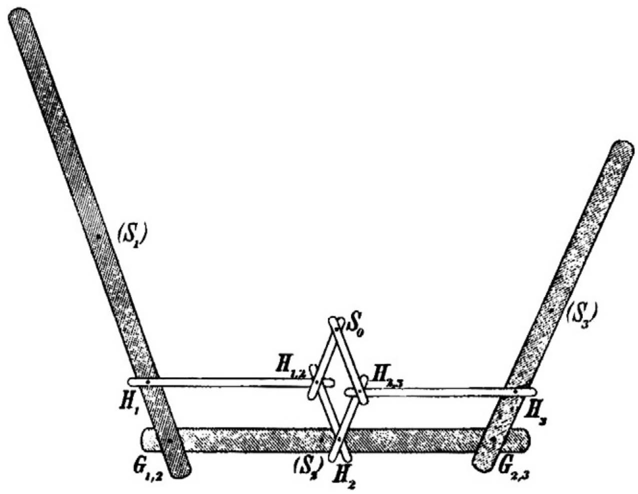

Fig. 1. Fischer's mechanism to trace the CoM of three links at S_0 by additional links [1].

2. CoM at invariant link point of a 2-DoF pantograph linkage

In this section the principal points of a pantograph are studied and determined. An approach by linear momentum equations is compared to a common approach to derive the balancing conditions of a pantograph.

Because of their properties of similarity, pantographs are well known for the purpose of shaking force balancing [9,16–18]. Fig. 2 shows the commonly applied pantograph which can be considered a specific version of Sylvester's pantograph or a modified version of Scheiner's pantograph [19]. Characteristic of a pantograph is that the links form a parallelogram.

In Fig. 2a a massless pantograph is shown of which point B traces the CoM of the two masses m_A and m_B for any motion of the pantograph if $m_A e_1 = m_B e_2$. This joint forms a revolute joint between links AB and BC and is therefore an invariant link point of each of these links. The pantograph has two DoFs with respect to any invariant link point of the mechanism. From the similarity $\Delta ABQ \sim \Delta CRB$ it follows that $e_1/e_2 = e_1'/e_2' = e_1''/e_2'' = k$ with k being a constant, which leads to the conditions $m_A e_1' = m_B e_2'$ and $m_A e_1'' = m_B e_2''$. These conditions are also obtained from Fischer's method, which states that the principle point on link DQ is found by calculating the CoM of m_A at Q and m_B considered to be at point D. The principle point on link DR then is found by calculating the CoM of m_A considered to be at D and m_B at R. The resulting principal points of the linkage are points A and C.

Fig. 2b shows a generalized version of the pantograph including arbitrary CoMs in each pantograph link. Points Q and R here can be interpreted as the effective mass locations of the pantograph. The common way to determine the conditions for which B is the CoM of the complete linkage is to develop the equations of the CoM position. Another possibility is to develop the linear momentum equations, which describe the motion of the CoM. These two approaches will be compared.

Assuming B to be the origin of the reference frame xy, the position of the CoM of the pantograph of Fig. 2b can be written as

$$
\begin{aligned}
\overline{CoM} = {} & m_2\begin{bmatrix} p_2 c\theta_2 - q_2 s\theta_2 \\ p_2 s\theta_2 + q_2 c\theta_2 \end{bmatrix} + m_3\begin{bmatrix} p_3 c\theta_1 + q_3 s\theta_1 \\ p_3 s\theta_1 - q_3 c\theta_1 \end{bmatrix} + m_1\begin{bmatrix} e_1'' c\theta_2 - p_1 c\theta_1 - q_1 s\theta_1 \\ e_1'' s\theta_2 - p_1 s\theta_1 + q_1 c\theta_1 \end{bmatrix} + m_A\begin{bmatrix} e_1'' c\theta_2 - p_A c\theta_1 - q_A s\theta_1 \\ e_1'' s\theta_2 - p_A s\theta_1 + q_A c\theta_1 \end{bmatrix} \\
& + m_4\begin{bmatrix} e_2' c\theta_1 - p_4 c\theta_2 + q_4 s\theta_2 \\ e_2' s\theta_1 - p_4 s\theta_2 - q_4 c\theta_2 \end{bmatrix} + m_B\begin{bmatrix} e_2' c\theta_1 - p_B c\theta_2 + q_B s\theta_2 \\ e_2' s\theta_1 - p_B s\theta_2 - q_B c\theta_2 \end{bmatrix}
\end{aligned} \tag{1}
$$

In order to find the conditions for which the CoM is stationary at B, these equations should be equal to zero for all motion, i.e. for any value of the time dependent parameters $\cos\theta_1$, $\sin\theta_1$, $\cos\theta_2$, and $\sin\theta_2$. Rearranging the equations by writing them in terms of these four time dependent variables then results in the four conditions that need to hold

$$m_2 q_2 = m_4 q_4 + m_B q_B \tag{2}$$

$$m_3 q_3 = m_1 q_1 + m_A q_A \tag{3}$$

$$(m_1 + m_A)e_1'' + m_2 p_2 = m_4 p_4 + m_B p_B \tag{4}$$

$$(m_4 + m_B)e_2' + m_3 p_3 = m_1 p_1 + m_A p_A \tag{5}$$

This procedure for this relatively simple linkage requires already a considerable effort, especially for the rearrangement of the equations.

By using the linear momentum equations per DoF independently, these conditions can be found more conveniently. The linear momentum for the first DoF θ_1 is written with respect to reference frame $x_1 y_1$ when $\dot{\theta}_2 = 0$ being

$$\begin{bmatrix} L_{x1} \\ L_{y1} \end{bmatrix} = \begin{bmatrix} (m_4 + m_B)e_2' + m_3 p_3 - m_1 p_1 - m_A p_A \\ -m_3 q_3 + m_1 q_1 + m_A q_A \end{bmatrix} \dot{\theta}_1 \tag{6}$$

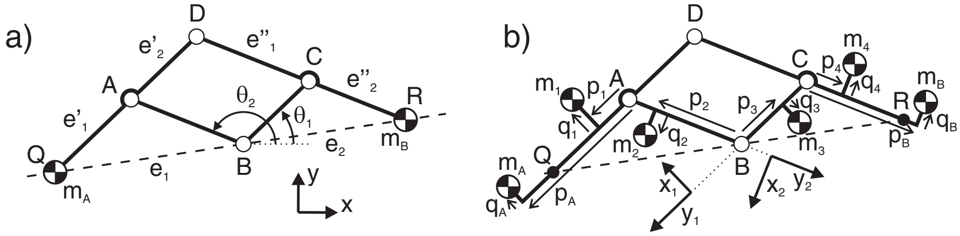

Fig. 2. (a) Pantograph mechanism with the CoM of masses m_A and m_B at B, which is an invariant point in both links AB and BC; (b) The CoM of the masses m_A and m_B and the pantograph link masses m_1, m_2, m_3, and m_4 is at B for any motion of the linkage.

The linear momentum for the second DoF θ_2 is written with respect to reference frame x_2y_2 when $\dot{\theta}_1 = 0$ and becomes

$$\begin{bmatrix} L_{x2} \\ L_{y2} \end{bmatrix} = \begin{bmatrix} (m_1+m_A)e_1'' + m_2p_2 - m_4p_4 - m_Bp_B \\ +m_2q_2 - m_4q_4 - m_Bq_B \end{bmatrix}\dot{\theta}_2 \tag{7}$$

This way, the conditions for which B is the mechanism CoM, for which the linear momentum equations need to be zero for all motion, are readily found.

Generally, the main advantages of using linear momentum equations are that solely the moving masses are considered for which constant terms (e.g. positions) do not appear, and that different reference frames for each DoF can be used. This results in having as many equations as conditions to find (two per DoF), while with the CoM position there are just two equations, independent of the linkage.

The complete set of principal dimensions of the pantograph of Fig. 2b are calculated with

$$e_1' = \frac{m_1p_1 + m_Ap_A - m_3p_3}{m_1 + m_A} \qquad e_2' = \frac{m_1p_1 + m_Ap_A - m_3p_3}{m_4 + m_B} \tag{8}$$

$$e_1'' = \frac{m_4p_4 + m_Bp_B - m_2p_2}{m_1 + m_A} \qquad e_2'' = \frac{m_4p_4 + m_Bp_B - m_2p_2}{m_4 + m_B} \tag{9}$$

Herewith the positions of Q and R are known.

3. CoM at invariant link point of a 3-DoF chain of 9 links

3.1. Derivation of Fischer's mechanism

The combined CoM of the three links of Fischer's mechanism in Fig. 1 can be traced by pantographs in various ways, as illustrated in Fig. 3. Fig. 3a shows a solution with two pantographs which both are fully auxiliary and can be applied in three different ways. Shown is the result when a first pantograph traces the CoM of m_1 and m_3 and a second pantograph traces the CoM of $m_1 + m_3$ and m_2, which is the CoM of all links. It is also possible (not shown) to have a first pantograph trace the CoM of m_1 and m_2 or the CoM of m_2 and m_3. For each of the three ways eight links are added to the mechanism.

Fig. 3b shows a solution with two pantographs of which one traces the CoM of m_1 and m_2 and has two links incorporated with principal links 1 and 2. The second pantograph is fully auxiliary and traces the CoM of all links. This solution can be applied in two different ways for which in each case a total of six links is added.

In Fig. 3c three pantographs are used to trace the CoM. Two of them have two links incorporated with the principal links 1, 2, and 3 and one is fully auxiliary. The two incorporated pantographs trace the CoM of m_1 and part of m_2, and the remaining part of m_2 and m_3, respectively, while the auxiliary pantograph traces the CoM of all links. In total eight links are added to the mechanism.

Fischer's mechanism can be derived from the configuration of Fig. 3c by unifying the three pantographs which will be explained graphically in three steps. The first step is to have the auxiliary pantograph be parallel to the other two for any motion of the linkage, as shown in Fig. 3d. In this intermediate result the kinematics of the auxiliary pantograph remain unchanged by an additional parallelogram at each side. This implies that point P_2 must be an invariant link point in link 2 and therefore the

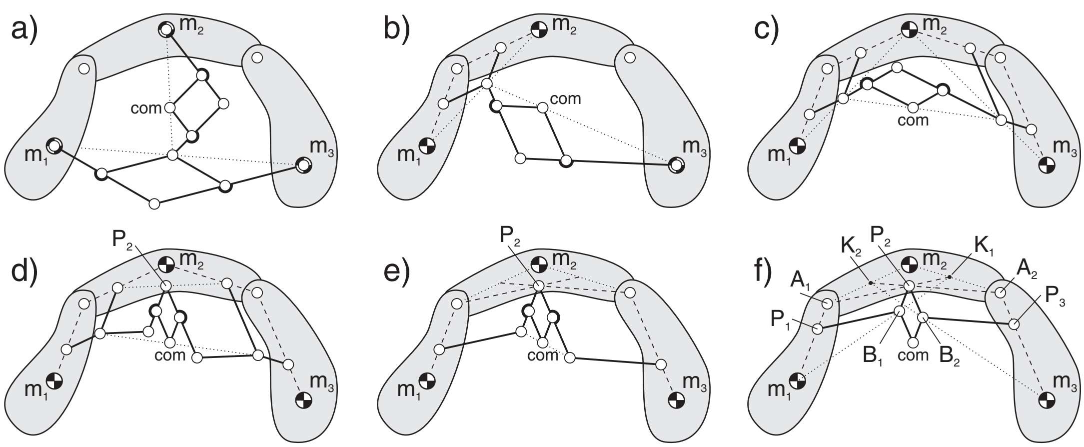

Fig. 3. (a) CoM of three links traced by two auxiliary pantographs; (b) CoM traced by one incorporated pantograph and one auxiliary pantograph; (c) CoM traced by two incorporated pantographs and one auxiliary pantograph; (d) Deduced from (c) when theauxiliary pantograph is parallel to the two incorporated pantographs; (e) Deduced from (d) when parallelograms are united; (f) Specific dimensions for which Fischer's configuration is obtained.

parallelograms at each side of the auxiliary pantograph can be united to a single parallelogram. This results in the configuration of Fig. 3e in which two new incorporated pantographs appear. Without effecting the kinematics, the dimensions of the links can be adapted to the configuration of Fig. 3f with which Fischer's mechanism is obtained.

The newly incorporated pantograph on the left is tracing the CoM of m_2+m_3 at K_1 and m_1. The right pantograph traces the CoM of m_1+m_2 at K_2 and m_3. K_1 represents the CoM of m_3 considered at A_2 and m_2, while K_2 represents the CoM of m_1 considered at A_1 and m_2.

The principal points of Fischer's mechanism are P_1, P_2, and P_3. Equivalent to the pantograph of Fig. 2, these points are the joints of the pantographs.

3.2. Fischer's mechanism with arbitrary CoMs in each link

Generalizing Fischer's mechanism by including arbitrary CoMs in each link results in the configuration of Fig. 4, which is a 3-DoF chain of 9 links. The masses of the principle links jointed at A_1 and A_2, m_1, m_2, and m_3, are located at positions M_1, M_2, and M_3, respectively. The masses m_{ij} of the other links are located at distances p_{ij} along and at distances q_{ij} normal to the line connecting the joints, respectively. The mechanism has three DoFs with respect to any invariant link point which are indicated with the angles of the principle links θ_1, θ_2, and θ_3.

When the locations of the principal points P_1, P_2, and P_3 are known, all principal dimensions of the mechanism are determined. In order to find the principle points, lengths a_1, a_{21}, a_{23}, and a_3 and angles α_1 and α_3 need to be calculated.

Length a_1 and angle α_1 can be calculated with the linear momentum equations for θ_1, having $\dot{\theta}_2=\dot{\theta}_3=0$. This can be interpreted as having parallelograms $A_1P_1B_1P_2$ and $P_2B_1SB_2$ move with respect to parallelogram $A_2P_2B_2P_3$. With S chosen as invariant link point to be the mechanism CoM for all motion, the linear momentum of the moving masses must be equal to the linear momentum of the total mass of the mechanism m_{tot} moving at S.

For an instantaneous position of the mechanism, a practical reference frame x_1y_1 is chosen to be perpendicular and parallel to the line a_1, respectively, as indicated in Fig. 4. With parallelogram $A_2P_2B_2P_3$ being stationary with respect to the reference frame, the linear momentum equations of the mechanism can be written as

$$L_{x1} = (m_1 s_1 cos\alpha_1 + (m_{11}+m_{33})a_1 + m_{12}p_{12} + m_{13}p_{13})\dot{\theta}_1 = m_{tot}a_1\dot{\theta}_1 \tag{10}$$

$$L_{y1} = (-m_1 s_1 sin\alpha_1 + m_{12}q_{12} + m_{13}q_{13})\dot{\theta}_1 = 0 \tag{11}$$

with $m_{tot}=m_1+m_2+m_3+m_{11}+m_{12}+m_{13}+m_{31}+m_{32}+m_{33}$. These linear momentum equations are constant for all motion for the conditions

$$m_1 s_1 cos\alpha_1 + (m_{11}+m_{33}-m_{tot})a_1 + m_{12}p_{12} + m_{13}p_{13} = 0 \tag{12}$$

$$-m_1 s_1 sin\alpha_1 + m_{12}q_{12} + m_{13}q_{13} = 0 \tag{13}$$

When the mass values and the CoMs of the links are known, α_1 and a_1 become

$$\alpha_1 = sin^{-1}\left(\frac{m_{12}q_{12}+m_{13}q_{13}}{m_1 s_1}\right) \tag{14}$$

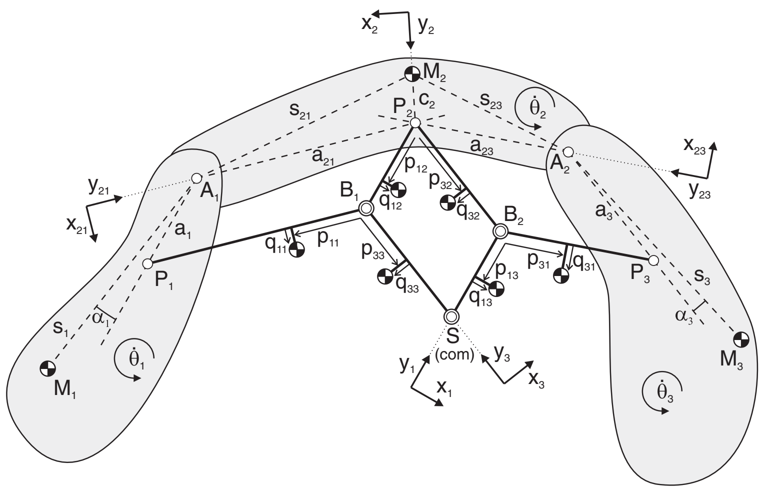

Fig. 4. A 3-DoF chain of 9 links with an arbitrary CoM in each link. Invariant link point *S* is the CoM of the complete mechanism.

$$a_1 = \frac{m_1 s_1 cos\alpha_1 + m_{12}p_{12} + m_{13}p_{13}}{m_{tot} - m_{11} - m_{33}} \tag{15}$$

Similarly α_3 and a_3 can be calculated with the linear momentum equations for θ_3, having $\dot{\theta}_1 = \dot{\theta}_2 = 0$. Then parallelograms $A_2P_2B_2P_3$ and $P_2B_1SB_2$ move with respect to parallelogram $A_1P_1B_1P_2$, which is stationary with respect to reference frame x_3y_3. The linear momentum equations of the mechanism then write

$$L_{x3} = (m_3 s_3 cos\alpha_3 + (m_{31} + m_{13})a_3 + m_{32}p_{32} + m_{33}p_{33})\dot{\theta}_3 = m_{tot}a_3\dot{\theta}_3 \tag{16}$$

$$L_{y3} = (m_3 s_3 sin\alpha_3 - m_{32}q_{32} - m_{33}q_{33})\dot{\theta}_3 = 0 \tag{17}$$

These equations are constant for all motion for the conditions

$$m_3 s_3 cos\alpha_3 + (m_{31} + m_{13} - m_{tot})a_3 + m_{32}p_{32} + m_{33}p_{33} = 0 \tag{18}$$

$$m_3 s_3 sin\alpha_3 - m_{32}q_{32} - m_{33}q_{33} = 0 \tag{19}$$

with which α_3 and a_3 are calculated with

$$\alpha_3 = sin^{-1}\left(\frac{m_{32}q_{32} + m_{33}q_{33}}{m_3 s_3}\right) \tag{20}$$

$$a_3 = \frac{m_3 s_3 cos\alpha_3 + m_{32}p_{32} + m_{33}p_{33}}{m_{tot} - m_{31} - m_{13}} \tag{21}$$

For the calculation of a_{21} and a_{23}, the linear momentum equations for θ_2 are written, having $\dot{\theta}_1 = \dot{\theta}_3 = 0$. Then parallelograms $A_1P_1B_1P_2$ and $A_2P_2B_2P_3$ move with respect to parallelogram $P_2B_1SB_2$. It turns out to be useful to first write the linear momentum of the mechanism with respect to the three different reference frames $x_{21}y_{21}$, $x_{23}y_{23}$, and x_2y_2 and unifying them afterwards. With $P_2B_1SB_2$ being stationary with respect to each reference frame, the linear momentum of the mechanism is written with the six equations

$$\begin{array}{ll} L_{x21} = (m_1 a_{21} + m_{11}p_{11})\dot{\theta}_2 & L_{y21} = (m_{11}q_{11})\dot{\theta}_2 \\ L_{x23} = (m_3 a_{23} + m_{31}p_{31})\dot{\theta}_2 & L_{y23} = (-m_{31}q_{31})\dot{\theta}_2 \\ L_{x2} = (m_2 c_2)\dot{\theta}_2 & L_{y2} = 0 \end{array} \tag{22}$$

These equations were obtained by writing the linear momentum of m_1 and m_{11} with respect to frame $x_{21}y_{21}$, m_3 and m_{31} with respect to frame $x_{23}y_{23}$, and m_2 with respect to frame x_2y_2. Since joint S is stationary, the term m_{tot} does not appear in the equations.

To unify the linear momentum equations to a single reference frame, the relations among the reference frames could be defined. However, this gives complications since the relations (e.g. angles among reference frames) depend on P_2, which is yet to be found. Another way is to project the linear momentum of the mechanism onto link A_1A_2. This means that the moving masses of the mechanism are projected on link A_1A_2 such that the linear momentum equations derived from this projection are equal to the original linear momentum equations. Since element A_1A_2 rotates about P_2 and since its linear momentum must be constant, P_2 then is the CoM of the projected masses.

Fig. 5 shows the mass projection for the linear momentum Eq. (22), which is referred to as the *Equivalent Linear Momentum System* (ELMS). Masses m_1, m_2, and m_3 are projected at A_1, M_2, and A_2, respectively, a mass m_{11} is projected at both I_1 and J_1, and a mass m_{31} is projected at both I_2 and J_2. It can be verified that the equations of the linear momentum of the ELMS with respect to the three reference frames are equal to Eq. (22).

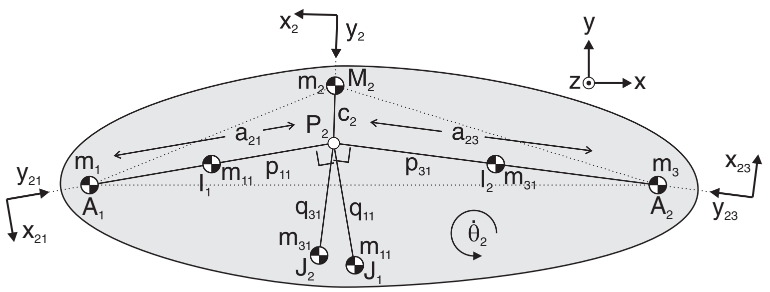

Fig. 5. Equivalent Linear Momentum System for θ_2 to find P_2; the moving masses are projected on link A_1A_2 by which P_2 is the CoM of the mass projection.

Since P_2 is the CoM of the projected masses, the linear momentum can be written from the ELMS in $[x\ y\ z]^T$ notation as

$$\vec{L} = \left(u_1 \cdot \overrightarrow{A_1P_2} \times \hat{z} + v_1 \cdot \overrightarrow{A_1P_2} + u_2 \cdot \overrightarrow{A_2P_2} \times \hat{z} - v_2 \cdot \overrightarrow{A_2P_2} + u_3 \cdot \overrightarrow{M_2P_2} \times \hat{z}\right)\dot{\theta}_2 = \vec{0} \tag{23}$$

with unit vector $\hat{z}=[0\ 0\ 1]^T$ and

$$\begin{aligned} u_1 &= m_1 + \frac{m_{11}p_{11}}{a_{21}} \quad & u_2 &= m_3 + \frac{m_{31}p_{31}}{a_{23}} \quad & u_3 &= m_2 \\ v_1 &= \frac{m_{11}q_{11}}{a_{21}} & v_2 &= \frac{m_{31}q_{31}}{a_{23}} \end{aligned} \tag{24}$$

in which $a_{21} = \left|\overrightarrow{A_1P_2}\right|$ and $a_{23} = \left|\overrightarrow{A_2P_2}\right|$. The only unknown in the equation is $\vec{P_2}$. The cross product with unit vector $\hat{z}$ is used to calculate perpendicular directions within the xy-plane.

When distances q_{11} and q_{31} are zero and the ratios $\frac{p_{11}}{a_{21}} = \lambda_1$ and $\frac{p_{31}}{a_{23}} = \lambda_2$ are known, for instance if m_{11} is halfway length a_{21} by which $\lambda_1 = \frac{1}{2}$, an algebraic solution for P_2 can be found which is

$$P_2 = \frac{u_1\vec{A_1} + u_2\vec{A_2} + u_3\vec{M_2}}{u_1 + u_2 + u_3} \tag{25}$$

with $u_1 = m_1 + m_{11}\lambda_1$, $u_2 = m_3 + m_{31}\lambda_2$, and $u_3 = m_2$. In fact this solution implies that P_2 is the CoM of masses with values u_1, u_2, and u_3 positioned at positions A_1, A_2, and M_2, respectively. When q_{11} and q_{31} are unequal to zero, finding an algebraic equation for P_2 becomes difficult because of the dependency of positions I_1, I_2, J_1, and J_2 and lengths a_{21}, a_{23}, and c_2 on P_2. P_2 then needs to be found numerically.

From Fischer's method, P_2 in Fig. 3f is the CoM of m_1 considered at A_1, m_2 at M_2, and m_3 considered at A_2, which is true when in the ELMS m_{11} and m_{31} are zero.

4. CoM at invariant link point of a 4-DoF chain of 16 links

In this section, the principal points are calculated of the 4-DoF chain of 16 links with arbitrary mass distribution of Fig. 6. It will be shown that the principal points of extended chains are determined in a similar way as the 3-DoF chain of 9 links.

As for Fischer's mechanism, the mechanism of Fig. 6 can also be derived from a union of pantographs. The pantograph representation is shown in Fig. 7a, consisting of in total six pantographs. Three of these pantographs have each two links incorporated with the principle links. Equivalently as for Fisher's mechanism in Fig. 3, the mechanism of Fig. 7b can be derived from Fig. 7a.

The masses of the four principle links are m_1, m_2, m_3, and m_4, which are located at positions M_1, M_2, M_3, and M_4, respectively. The masses m_{ij} of the other links are located at distances p_{ij} along and at distances q_{ij} normal to the line connecting the joints, respectively. The mechanism has four DoFs with respect to any invariant link point which are indicated with the angular velocities of the principle links $\dot{\theta}_1$, $\dot{\theta}_2$, $\dot{\theta}_3$, and $\dot{\theta}_4$.

Also for this mechanism holds that when the four principle points P_1, P_2, P_3, and P_4 are known, all principal dimensions of the mechanism are determined. Principle points P_1 and P_4 are calculated equivalently as P_1 and P_3 in Fig. 4, while P_2 and P_3 are obtained with an ELMS as P_2 in Fig. 5.

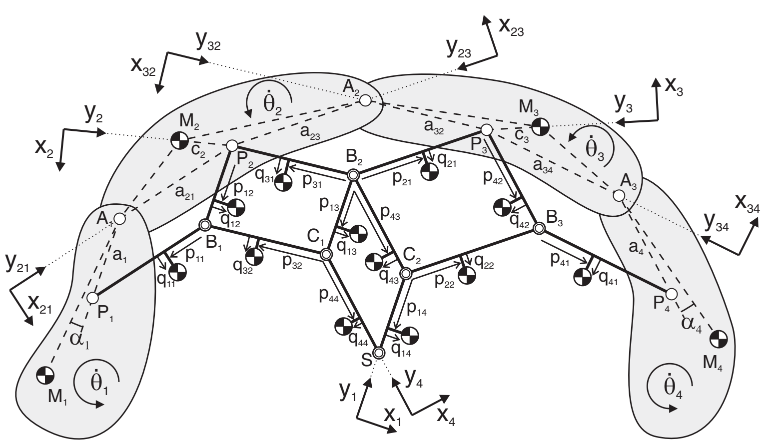

Fig. 6. A 4-DoF chain of 16 links with arbitrary CoMs of which invariant link point S is the CoM of the complete mechanism.

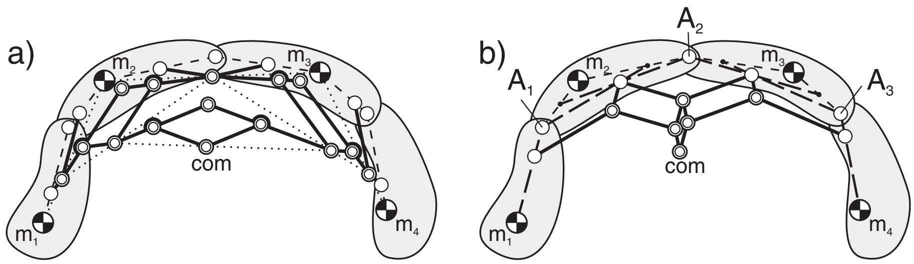

Fig. 7. (a) Six pantographs represent the configuration of Fig. 6; (b) Configuration when pantographs are unified.

P_1, which is defined by a_1 and α_1, can be calculated with the linear momentum equations for θ_1 with $\dot{\theta}_2 = \dot{\theta}_3 = \dot{\theta}_4 = 0$. This can be seen as having parallelograms $A_1P_1B_1P_2$, $P_2B_1C_1B_2$, and $B_2C_1SC_2$ move with respect to $A_2P_2B_2P_3$, $P_3B_2C_2B_3$, and $A_3P_3B_3P_4$ with the latter parallelograms being stationary with respect to reference frame x_1y_1. The linear momentum equations write

$$L_{x1} = (m_1s_1cos\alpha_1 + (m_{11} + m_{32} + m_{44})a_1 + m_{12}p_{12} + m_{13}p_{13} + m_{14}p_{14})\dot{\theta}_1 = m_{tot}a_1\dot{\theta}_1 \tag{26}$$

$$L_{y1} = (-m_1s_1sin\alpha_1 + m_{12}q_{12} + m_{13}q_{13} + m_{14}q_{14})\dot{\theta}_1 = 0 \tag{27}$$

with $m_{tot} = m_1 + m_2 + m_3 + m_4 + m_{11} + m_{12} + m_{13} + m_{14} + m_{21} + m_{22} + m_{31} + m_{32} + m_{41} + m_{42} + m_{43} + m_{44}$. These equations are constant for all motion for the conditions

$$m_1s_1cos\alpha_1 + (m_{11} + m_{32} + m_{44} - m_{tot})a_1 + m_{12}p_{12} + m_{13}p_{13} + m_{14}p_{14} = 0 \tag{28}$$

$$-m_1s_1sin\alpha_1 + m_{12}q_{12} + m_{13}q_{13} + m_{14}q_{14} = 0 \tag{29}$$

With the mass values and their locations at their links known, α_1 and a_1 become

$$\alpha_1 = sin^{-1}\left(\frac{m_{12}q_{12} + m_{13}q_{13} + m_{14}q_{14}}{m_1s_1}\right) \tag{30}$$

$$a_1 = \frac{m_1s_1cos\alpha_1 + m_{12}p_{12} + m_{13}p_{13} + m_{14}p_{14}}{m_{tot} - m_{11} - m_{32} - m_{44}} \tag{31}$$

Equivalently P_4 can be calculated for θ_4 with $\dot{\theta}_1 = \dot{\theta}_2 = \dot{\theta}_3 = 0$. Then parallelograms $A_3P_3B_3P_4$, $P_3B_2C_2B_3$, and $B_2C_1SC_2$ move with respect to $A_1P_1B_1P_2$, $P_2B_1C_1B_2$, and $A_2P_2B_2P_3$ with the latter parallelograms being stationary with respect to reference frame x_4y_4. The linear momentum writes

$$L_{x4} = (m_4s_4cos\alpha_4 + (m_{41} + m_{22} + m_{14})a_4 + m_{42}p_{42} + m_{43}p_{43} + m_{44}p_{44})\dot{\theta}_4 = m_{tot}a_4\dot{\theta}_4 \tag{32}$$

$$L_{y4} = (m_4s_4sin\alpha_4 - m_{42}q_{42} - m_{43}q_{43} - m_{44}q_{44})\dot{\theta}_4 = 0 \tag{33}$$

These equations are constant for all motion for the conditions

$$m_4s_4cos\alpha_4 + (m_{41} + m_{22} + m_{14} - m_{tot})a_4 + m_{42}p_{42} + m_{43}p_{43} + m_{44}p_{44} = 0 \tag{34}$$

$$m_4s_4sin\alpha_4 - m_{42}q_{42} - m_{43}q_{43} - m_{44}q_{44} = 0 \tag{35}$$

The equations for α_4 and a_4 then become

$$\alpha_4 = sin^{-1}\left(\frac{m_{42}q_{42} + m_{43}q_{43} + m_{44}q_{44}}{m_4s_4}\right) \tag{36}$$

$$a_4 = \frac{m_4s_4cos\alpha_1 + m_{42}p_{42} + m_{43}p_{43} + m_{44}p_{44}}{m_{tot} - m_{41} - m_{22} - m_{14}} \tag{37}$$

P_2 can be calculated with the ELMS shown in Fig. 8a. This is obtained by calculating the linear momentum of m_1 and m_{11} with respect to reference frame $x_{21}y_{21}$, the linear momentum of m_2 with respect to x_2y_2, and the linear momentum of m_3, m_4, m_{41}, m_{42}, m_{21}, and m_{22} with respect to $x_{23}y_{23}$.

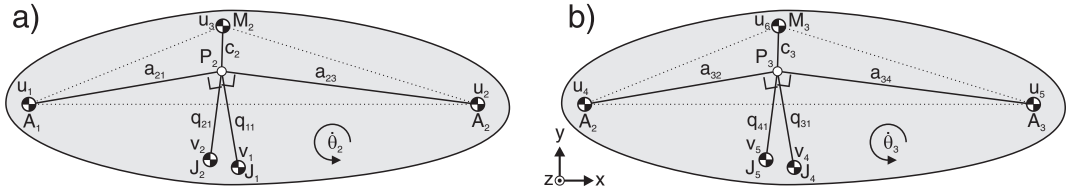

Fig. 8. Equivalent Linear Momentum Systems for (a) θ_2 to find P_2 (b) θ_3 to find P_3. P_2 and P_3 are the CoMs of their mass projections.

For P_2 with $\dot\theta_1 = \dot\theta_3 = \dot\theta_4 = 0$, for which parallelograms $P_2B_1C_1B_2$ and $B_2C_1SC_2$ are stationary with the reference frames, the linear momentum can be written as

$$\vec{L} = \left(u_1 \cdot \overrightarrow{A_1P_2} \times \hat{z} + v_1 \cdot \overrightarrow{A_1P_2} + u_2 \cdot \overrightarrow{A_2P_2} \times \hat{z} - v_2 \cdot \overrightarrow{A_2P_2} + u_3 \cdot \overrightarrow{M_2P_2} \times \hat{z}\right)\dot\theta_2 = \rightarrow 0 \tag{38}$$

with

$$\begin{aligned} u_1 &= m_1 + \frac{m_{11}p_{11}}{a_{21}} & v_1 &= m_{11} \\ u_2 &= m_3 + m_4 + m_{41} + m_{42} + \frac{m_{21}p_{21} + m_{22}p_{22}}{a_{23}} & v_2 &= m_{21} + \frac{m_{22}q_{22}}{q_{21}} \\ u_3 &= m_2 \end{aligned} \tag{39}$$

being the projected masses at A_1, A_2, M_2, J_1, and J_2, respectively.

P_3 can be calculated with the ELMS shown in Fig. 8b. This is obtained by calculating the linear momentum of m_1, m_2, m_{11}, m_{12}, m_{31}, and m_{32} with respect to reference frame $x_{32}y_{32}$, the linear momentum of m_3 with respect to x_3y_3, and the linear momentum of m_4, and m_{41} with respect to $x_{34}y_{34}$.

For P_3 with $\dot\theta_1 = \dot\theta_2 = \dot\theta_4 = 0$ with parallelograms $P_3B_2C_2B_3$ and $B_2C_1SC_2$ being stationary with the reference frames, the linear momentum can be written as

$$\vec{L} = \left(u_4 \cdot \overrightarrow{A_2P_3} \times \hat{z} + v_4 \cdot \overrightarrow{A_2P_3} + u_5 \cdot \overrightarrow{A_3P_3} \times \hat{z} - v_5 \cdot \overrightarrow{A_3P_3} + u_6 \cdot \overrightarrow{M_3P_3} \times \hat{z}\right)\dot\theta_3 = \rightarrow 0 \tag{40}$$

with

$$\begin{aligned} u_4 &= m_1 + m_2 + m_{11} + m_{12} + \frac{m_{31}p_{31} + m_{32}p_{32}}{a_{32}} & v_4 &= m_{31} + \frac{m_{32}q_{32}}{q_{31}} \\ u_5 &= m_4 + \frac{m_{41}p_{41}}{a_{34}} & v_5 &= m_{41} \\ u_6 &= m_3 \end{aligned} \tag{41}$$

being the projected masses at A_2, A_3, M_3, J_4, and J_5, respectively.

5. CoM at invariant link point of a 4-DoF chain of 19 links

A 4-DoF chain of 16 links can also be arranged with multiple chains as in Fig. 9a. This planar mechanism is derived from a union of six pantographs, resulting in six parallelograms. This can be applied in three different ways. Shown is the solution centered around principle link 11, while this is also possible with principle links 21 and 31. When the three solutions are combined, this results in the overconstraint mechanism of Fig. 9b.

Fig. 10 shows the mechanism of Fig. 9b including an arbitrary CoM of each link. The principle points are P_1, P_{11}, P_{21}, and P_{31}, which determine all principal dimensions of the mechanism and are calculated equivalently to the principal points of the linkages in Figs. 4 and 6.

The masses of the principle links are m_1, m_{11}, m_{21}, and m_{31} which are located at M_1, M_{11}, M_{21}, and M_{31}, respectively. The masses m_{ijk} of the other links are located at distances p_{ijk} along and at distances q_{ijk} normal to the line connecting the joints, respectively. The four DoFs are indicated with angular velocities $\dot\theta_1$, $\dot\theta_{11}$, $\dot\theta_{21}$, and $\dot\theta_{31}$.

P_{11}, which is defined by a_{11} and α_{11}, is calculated for θ_{11} with $\dot\theta_1 = \dot\theta_{21} = \dot\theta_{31} = 0$. With respect to reference frame $x_{111}y_{111}$ the linear momentum equations can be written with

$$\begin{aligned} L_{x111} &= (m_{11}s_{11}cos\alpha_{11} + (m_{111} + m_{214} + m_{215} + m_{313} + m_{314})a_{11} + m_{112}p_{112} + m_{113}p_{113} + m_{114}p_{114} + m_{115}p_{115})\dot\theta_{11} \\ &= m_{tot}a_{11}\dot\theta_{11} \end{aligned} \tag{42}$$

$$L_{y111} = (-m_{11}s_{11}sin\alpha_{11} + m_{112}q_{112} + m_{113}q_{113} + m_{114}q_{114} + m_{115}q_{115})\dot\theta_{11} = 0 \tag{43}$$

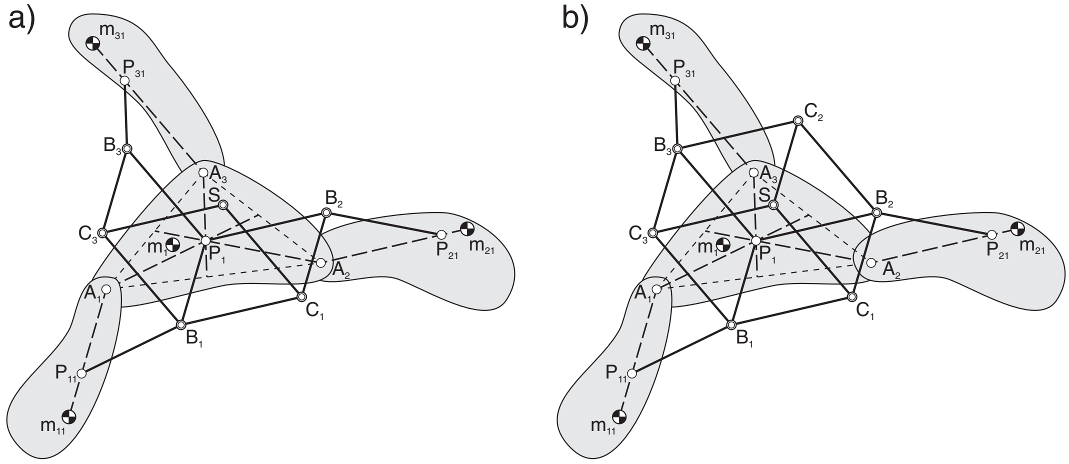

Fig. 9. (a) A 4-DoF chain of 16 links of which invariant link point S is the CoM of the mechanism. (b) Combination of the three possibilities gains a 4-DoF chain of 19 links, here joint S is overconstraint.

with $m_{tot} = m_1 + m_{11} + m_{21} + m_{31} + \Sigma_{i=1}^{5}(m_{11i} + m_{21i} + m_{31i})$. The conditions for which these equations are constant for all motion are

$$m_{11}s_{11}cos\alpha_{11} + m_{112}p_{112} + m_{113}p_{113} + m_{114}p_{114} + m_{115}p_{115} + (m_{111} + m_{214} + m_{215} + m_{313} + m_{314} - m_{tot})a_{11} = 0 \tag{44}$$

$$-m_{11}s_{11}sin\alpha_{11} + m_{112}q_{112} + m_{113}q_{113} + m_{114}q_{114} + m_{115}q_{115} = 0 \tag{45}$$

from which a_{11} and α_{11} are calculated with

$$\alpha_{11} = sin^{-1}\left(\frac{m_{112}q_{112} + m_{113}q_{113} + m_{114}q_{114} + m_{115}q_{115}}{m_{11}s_{11}}\right) \tag{46}$$

$$a_{11} = \frac{m_{11}s_{11}cos\alpha_{11} + m_{112}p_{112} + m_{113}p_{113} + m_{114}p_{114} + m_{115}p_{115}}{m_{tot} - m_{111} - m_{214} - m_{215} - m_{313} - m_{314}} \tag{47}$$

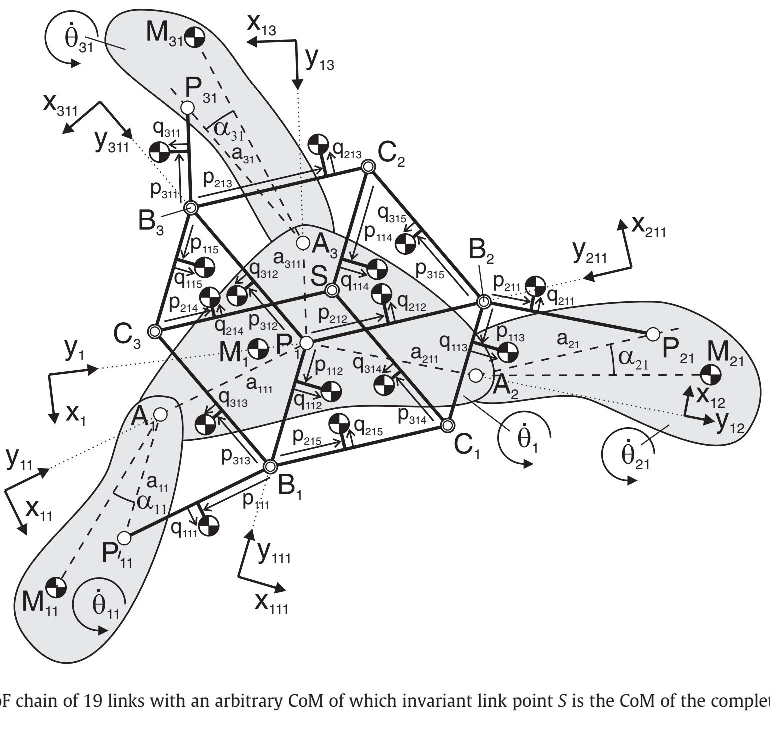

Fig. 10. A 4-DoF chain of 19 links with an arbitrary CoM of which invariant link point S is the CoM of the complete mechanism.

P_{21}, which is defined by a_{21} and α_{21}, is calculated for θ_{21} with $\dot{\theta}_1 = \dot{\theta}_{11} = \dot{\theta}_{31} = 0$. With respect to reference frame $x_{211}y_{211}$ the linear momentum equations can be written with

$$\begin{aligned} L_{x211} &= (m_{21}s_{21}cos\alpha_{21} + (m_{211} + m_{314} + m_{315} + m_{113} + m_{114})a_{21} + m_{212}p_{212} + m_{213}p_{213} + m_{214}p_{214} + m_{215}p_{215})\dot{\theta}_{21} \\ &= m_{tot}a_{21}\dot{\theta}_{21} \end{aligned} \tag{48}$$

$$L_{y211} = (-m_{21}s_{21}sin\alpha_{21} + m_{212}q_{212} + m_{213}q_{213} + m_{214}q_{214} + m_{215}q_{215})\dot{\theta}_{21} = 0 \tag{49}$$

The conditions for which these equations are constant for all motion are

$$m_{21}s_{21}cos\alpha_{21} + m_{212}p_{212} + m_{213}p_{213} + m_{214}p_{214} + m_{215}p_{215} + (m_{211} + m_{314} + m_{315} + m_{113} + m_{114} - m_{tot})a_{21} = 0 \tag{50}$$

$$-m_{21}s_{21}sin\alpha_{21} + m_{212}q_{212} + m_{213}q_{213} + m_{214}q_{214} + m_{215}q_{215} = 0 \tag{51}$$

from which a_{21} and α_{21} are calculated with

$$\alpha_{21} = sin^{-1}\left(\frac{m_{212}q_{212} + m_{213}q_{213} + m_{214}q_{214} + m_{215}q_{215}}{m_{21}s_{21}}\right) \tag{52}$$

$$a_{21} = \frac{m_{21}s_{21}cos\alpha_{21} + m_{212}p_{212} + m_{213}p_{213} + m_{214}p_{214} + m_{215}p_{215}}{m_{tot} - m_{211} - m_{314} - m_{315} - m_{113} - m_{114}} \tag{53}$$

P_{31}, which is defined by a_{31} and α_{31}, is calculated for θ_{31} with $\dot{\theta}_1 = \dot{\theta}_{11} = \dot{\theta}_{21} = 0$. With respect to reference frame $x_{311}y_{311}$ the linear momentum equations can be written with

$$\begin{aligned} L_{x311} &= (m_{31}s_{31}cos\alpha_{31} + (m_{311} + m_{114} + m_{115} + m_{213} + m_{214})a_{31} + m_{312}p_{312} + m_{313}p_{313} + m_{314}p_{314} + m_{315}p_{315})\dot{\theta}_{31} \\ &= m_{tot}a_{31}\dot{\theta}_{31} \end{aligned} \tag{54}$$

$$L_{y311} = (-m_{31}s_{31}sin\alpha_{31} + m_{312}q_{312} + m_{313}q_{313} + m_{314}q_{314} + m_{315}q_{315})\dot{\theta}_{31} = 0 \tag{55}$$

The conditions for which these equations are constant for all motion are

$$m_{31}s_{31}cos\alpha_{31} + m_{312}p_{312} + m_{313}p_{313} + m_{314}p_{314} + m_{315}p_{315} + (m_{311} + m_{114} + m_{115} + m_{213} + m_{214} - m_{tot})a_{31} = 0 \tag{56}$$

$$-m_{31}s_{31}sin\alpha_{31} + m_{312}q_{312} + m_{313}q_{313} + m_{314}q_{314} + m_{315}q_{315} = 0 \tag{57}$$

from which a_{31} and α_{31} are calculated with

$$\alpha_{31} = sin^{-1}\left(\frac{m_{312}q_{312} + m_{313}q_{313} + m_{314}q_{314} + m_{315}q_{315}}{m_{31}s_{31}}\right) \tag{58}$$

$$a_{31} = \frac{m_{31}s_{31}cos\alpha_{31} + m_{312}p_{312} + m_{313}p_{313} + m_{314}p_{314} + m_{315}p_{315}}{m_{tot} - m_{311} - m_{114} - m_{115} - m_{213} - m_{214}} \tag{59}$$

For the calculations of P_1, which is determined by a_{111}, a_{211}, and a_{311}, the ELMS of principle link 1 is used, which is drawn in Fig. 11. For $\dot{\theta}_{11} = \dot{\theta}_{21} = \dot{\theta}_{31} = 0$ the linear momentum of m_{11} and m_{111} are calculated with respect to reference frame $x_{11}y_{11}$, of m_{21} and m_{211} with respect to $x_{12}y_{12}$, of m_{31} and m_{311} with respect to $x_{13}y_{13}$, and of $m_1$with respect to x_1y_1. The parallelograms $P_1B_1C_1B_2$, $P_1B_2C_2B_3$, $P_1B_3C_3B_1$, $SC_3B_1C_1$, $SC_1B_2C_2$, $SC_2B_3C_3$ then are stationary with respect to the reference frames. The linear momentum of the mechanism is written as

$$\vec{L} = \left(u_1 \cdot \overrightarrow{A_1P_1} \times \hat{z} + v_1 \cdot \overrightarrow{A_1P_1} + u_2 \cdot \overrightarrow{A_2P_1} \times \hat{z} + v_2 \cdot \overrightarrow{A_2P_1} + u_3 \cdot \overrightarrow{A_3P_1} \times \hat{z} + v_3 \cdot \overrightarrow{A_3P_1} + u_4 \cdot \overrightarrow{M_1P_1} \times \hat{z}\right)\dot{\theta}_1 = \vec{0} \tag{60}$$

with

$$\begin{aligned} u_1 &= m_{11} + \frac{m_{111}p_{111}}{a_{111}} \quad & u_2 &= m_{21} + \frac{m_{211}p_{211}}{a_{211}} \quad & u_3 &= m_{31} + \frac{m_{311}p_{311}}{a_{311}} \quad & u_4 &= m_1 \\ v_1 &= m_{111} & v_2 &= m_{211} & v_3 &= m_{311} \end{aligned} \tag{61}$$

being the projected masses at A_1, A_2, A_3, M_1, J_1, J_2, and J_3, respectively. Also here P_1 must be the CoM of the ELMS since the linear momentum has to be zero and P_1 is the only stationary point.

6. Discussion

When the principal points need to be found for a set of links with known mass and CoM, each of them can be obtained independently with linear momentum equations. When the mass or CoM of the links would depend on the principle dimensions, i.e. they are not a

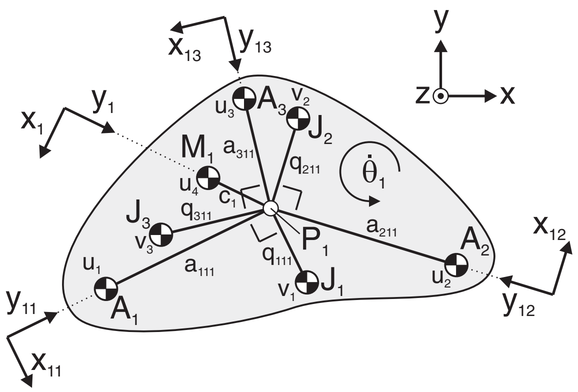

Fig. 11. Equivalent Linear Momentum System for θ_1 to find P_1. P_1 is the CoM of the mass projection.

priori known, the principle points need to be found in an iterative way. However, the iterative method will obtain solutions quickly since there is an equation for each parameter to be found.

Instead of choosing joint S as invariant link point being the mechanism CoM, other points on the linkage could be chosen to have this property. The presented approach for the calculations of the principle points is applicable for any chosen invariant link point.

The presented mechanisms are principal configurations of inherently balanced linkages. From these configurations, various mechanisms can be derived by e.g. changing link dimensions, exchanging links with gears, replacing joints with other types of joints,etc. As long as the pantograph relations are maintained, the property of being inherently balanced remains for the derived mechanisms.

One example of an engineering application is presented in Figs. 12 and 13. Fig. 12 shows a 2-DoF grasper mechanism which is derived from the configuration of Fig. 6. The CoM of the grasper at S is jointed to the base and by actuating the slider at A_2 the grasper opens and closes. By moving the joints C_1 and C_2, having joint B_2 slide along the symmetry line, the orientation of the fingers is changed to, for instance, the poses of Fig. 12b and c. Fig. 13 shows a prototype of a variation of the grasper mechanism in which B_2 is jointed to the base (the aluminium part) and also C_1 and C_2 are fixed but can be manually altered. Then the CoM at S is stationary with respect to the base without needing links C_1S and C_2S which therefore are left out. Without the slider at A_2 this joint can move freely with two DoF for which the prototype has 2-DoF motion allowing symmetric and asymmetric grasping.

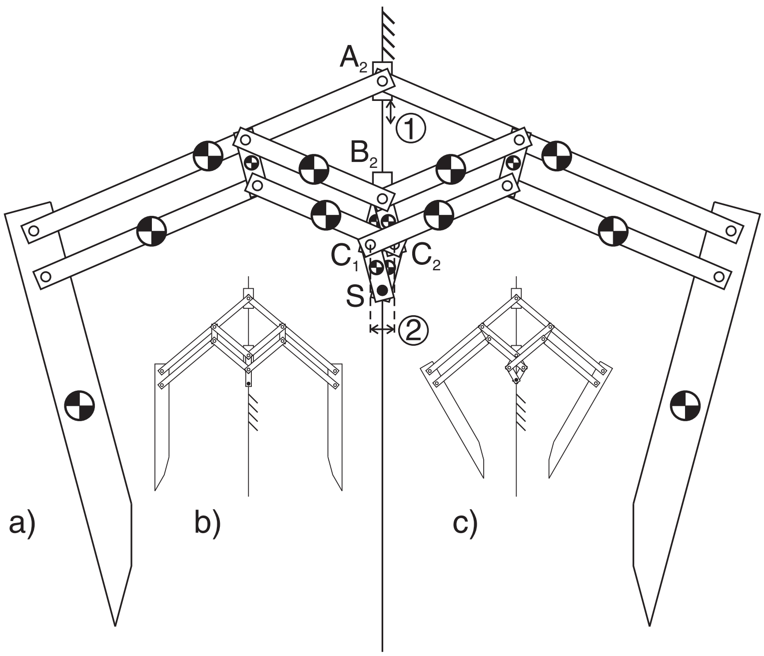

Fig. 12. 2-DoF grasper mechanism derived from the configuration of Fig. 6 which is both shaking force balanced and shaking moment balanced for all motion. The mechanism CoM is in S which is a fixed joint with the base. By moving slider A_2 the mechanism grasps with parallel motion of the fingers while by moving slider B_2 the mechanism grasps by rotating the fingers.

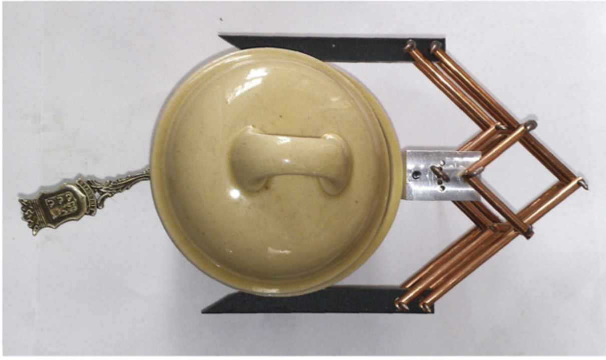

Fig. 13. Prototype of a 2-DoF grasper mechanism which is shaking force balanced for all motion and shaking moment balanced for synchronous motion of both fingers. Joints C_1 and C_2 are hidden inside the base block and can be manually altered to change the orientation of the fingers. Joint A_2 can move freely in two directions.

Both the mechanisms of Figs. 12 and 13 are shaking force-balanced for all motion. Although shaking moment balancing has not been considered in this paper, all motion of the grasper in Fig. 12 are perfectly shaking moment balanced too. This is due to the symmetric design and the synchronous opposite motion of the left and the right side which balance one another. For this synchronous motion, the prototype of Fig. 13 also exhibits perfect shaking moment balance, while for asymmetric grasping shaking moments exist.

The importance of dynamic balance for a grasper mechanism is that, when e.g. mounted on a manipulator, the manipulator motion does not affect the dynamics of the grasper, and vice versa. Unbalanced graspers tend to act as a pendulum, that tends to swing due to accelerations of the manipulator. Balanced graspers therefore help to reduce the pick and place cycle time, to increase the accuracy while actuators remain small since they don't need to counteract dynamic and also static forces [12,14,20].

Except for the fingers of the prototype, the links were designed mass symmetrical. An asymmetric design of the links is possible and may be used to adapt the range of motion, to improve the compactness or to include additional components in the balance such as electric cables or sensors attached to some of the links.

Although planar mechanisms were considered in this paper, the obtained results can be extended to spatial mechanisms. When the pantographs in this paper were 3-DoF spatial pantographs [21] with for instance spherical joints such that the parallelogram can also move out of plane (different from a 2-DoF planar pantograph being moved spatially), all presented linkages would be spatial linkages. Since spatial pantographs are balanced for equivalent balance conditions as planar pantographs [21], all linkages would become inherently balanced spatial linkages.

7. Conclusion

In this paper Fischer's mechanism was investigated as being a composition of pantographs. It was shown that linkages that are composed of pantographs, can be inherently balanced for all links having an arbitrary CoM. This is since any such pantograph can have a point at one of its links, an invariant link point, which is characterized as being the combined CoM of all links for all motion of the linkage. This aspect is useful for the synthesis of shaking force balanced mechanisms for which the invariant link point has to be a stationary point with respect to the base.

To calculate the principal dimensions for which pantograph based linkages have their CoM at an invariant link point, an approach based on linear momentum was proposed. With this approach it is possible to investigate each degree-of-freedom individually. *Equivalent Linear Momentum Systems* (ELMS) were proposed to facilitate the calculations in order to use different convenient reference frames. The method was applied to various planar linkages with revolute joints. The method also applies to related mechanisms with other types of joints and to spatial linkages that are based on spatial pantograph motion. As a practical example a shaking force and shaking moment balanced 2-DOF grasper mechanism was derived and presented.

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