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On the Integrability Conditions for Some Structures Related to Evolution Differential Equations

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Abstract. Using the result by D. Gessler, we show that any invariant variational bivector (resp., variational 2-form) on an evolution equation with nondegenerate right-hand side is Hamiltonian (resp., symplectic).

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Introduction

In [4], we described a method to construct Hamiltonian and symplectic structures on nonlinear evolution equations. The method was essentially based on the notions of *variational multivector* [3] and *variational differential form*. From technical viewpoint, for a given evolution equation $\mathcal{E}$, it consisted of two steps: (1) solving the linearized equation $\ell_{\mathcal{E}}\varphi = 0$ in the so-called $\ell^*_{\mathcal{E}}$*-covering* (resp., the equation $\ell^*_{\mathcal{E}}\psi = 0$ in the $\ell_{\mathcal{E}}$*-covering*) and (2) checking the Hamiltonianity condition $[\![\varphi, \varphi]\!] = 0$, where $[\![\cdot\,, \cdot]\!]$ denotes the *variational Schouten bracket* [3] (resp., the condition for ψ to be symplectic, i.e., closed with respect to a certain differential in the Vinogradov $\mathcal{C}$-spectral sequence [6]).

Surprisingly enough, it was found out in particular computations that the second condition always holds true 'by default' and we still do not know counterexamples (except for the case of first-order equations). On the other hand, a rather old result by Gessler [2] states that all terms $E_1^{p,n-1}(\mathcal{E})$ of the Vinogradov $\mathcal{C}$-spectral sequence vanish in the nondegenerate case for $p \geqslant 3$ (here n is the number of independent variables). This fact means exactly that all variational 2-forms on nondegenerate evolution equations are closed and thus symplectic. Since Gessler's proof almost literary works in the case of multivectors, we immediately obtain that all bivectors on such equations are Hamiltonian. These facts explain our experimental results.

We expose the details below. In Section 1, necessary introduction to the geometry of jet bundles and evolution equations is presented. Section 2 deals with the calculus of variational multivectors and forms on evolution equations. To make our exposition self-contained, we repeat Gessler's proof from [2]. Finally, in Section 3 we derive our main results on the integrability of Hamiltonian and symplectic structures on nondegenerate evolution equations of order > 1.

1. Generalities: Jet Bundles and Evolution Equations

Let us fix notation and recall briefly some definitions and results we will use. For explanations we refer to [1, 3, 5].

Let $\pi\colon E \to M$ be a vector bundle over an n-dimensional base manifold M and $\pi_\infty\colon J^\infty(\pi) \to M$ be the infinite jet bundle of local sections of the bundle π.

In coordinate language, if $x_1, \dots, x_n$, $u^1, \dots, u^m$ are coordinates on E such that x_i are base coordinates and u^j are fiber ones, then $\pi_\infty\colon J^\infty(\pi) \to M$ is an infinite-dimensional vector bundle with fiber coordinates u^j_σ, where $\sigma = i_1 \dots i_{|\sigma|}$ is a symmetric multi-index.

The basic geometric structure on $J^\infty(\pi)$ is the *Cartan distribution*. In coordinate language, the Cartan distribution is spanned by the *total derivatives*

$$D_i = \frac{\partial}{\partial x_i} + \sum_{j,\sigma} u^j_{\sigma i} \frac{\partial}{\partial u^j_\sigma}.$$

A differential operator on $J^\infty(\pi)$ is called $\mathcal{C}$*-differential operator* (or *horizontal operator*) if it can be written as a sum of compositions of $C^\infty(J^\infty(\pi))$-linear maps and vector fields that belong to the Cartan distribution. In coordinates, $\mathcal{C}$-differential operators are total derivatives operators.

Let P and Q be $C^\infty(J^\infty(\pi))$-modules of sections of some vector bundles over $J^\infty(\pi)$. All $\mathcal{C}$-differential operators from P to Q form a $C^\infty(J^\infty(\pi))$-module denoted by $\mathcal{C}\mathrm{Diff}(P, Q)$. More generally, a map $\Delta\colon P_1 \times \dots \times P_k \to Q$ is called a *multidifferential operator* (of degree k) if it is a $\mathcal{C}$-differential operator in each argument. Choose elements $p_i \in P_i$, $i = 1, \dots, k$, and consider the operators

$$\Delta_i = \Delta(p_1, \dots, p_{i-1}, \cdot, p_{i+1}, \dots, p_k)\colon\ P_i \to Q.$$

Let l_i be the order* of Δ_i. We define the *symbol*

$$\mathrm{smbl}(\Delta)\colon\ S^{l_1}(\Lambda^1(M)) \otimes P_1 \times \dots \times S^{l_k}(\Lambda^1(M)) \otimes P_k \to Q,$$

where S^l denotes the symmetric power, of Δ as follows. For any $f \in C^\infty(M)$, let us set

$$(\delta^{(i)}_f \Delta)(p_1, \dots, p_k) = f\Delta(p_1, \dots, p_k) - \Delta(p_1, \dots, p_{i-1}, fp_i, p_{i+1}, \dots, p_k)$$

* Of course, this number depends on the choice of p's; so we define the order as the maximum over all possible choices.

and $\delta^{(i)}_{f_1,\dots,f_l} = \delta^{(i)}_{f_1} \circ \dots \circ \delta^{(i)}_{f_l}$. If now $\omega^i = \mathrm{d}f^i_1 \dots \mathrm{d}f^i_{l_i}$, $i = 1, \dots, k$, are symmetric forms on M, we set

$$(\mathrm{smbl}\,\Delta)(\omega^1 \otimes p_1, \omega^2 \otimes p_2, \dots, \omega^k \otimes p_k)$$
$$= \delta^{(1)}_{f^1_1,\dots,f^1_{l_1}} \circ \dots \circ \delta^{(k)}_{f^k_1,\dots,f^k_{l_k}}(\Delta)(p_1, \dots, p_k).$$

Let $\rho \in J^\infty(\pi)$ and $x = \pi_\infty(\rho) \in M$. Then the value of the symbol at ρ is the map

$$\mathrm{smbl}(\Delta)|_\rho\colon\ T^*_x M \otimes P_{1,\rho} \times \dots \times T^*_x M \otimes P_{k,\rho} \to Q_\rho$$

polynomially dependent on points of $T^*_x M$ ($P_{i,\rho}$ and Q_ρ denote here the fibers of the corresponding vector bundles at the point ρ).

The lift of the de Rham complex on M to $J^\infty(\pi)$ is called *horizontal de Rham complex* and is denoted by

$$0 \to C^\infty(J^\infty(\pi)) \xrightarrow{\bar{\mathrm{d}}} \bar{\Lambda}^1(\pi) \xrightarrow{\bar{\mathrm{d}}} \dots \xrightarrow{\bar{\mathrm{d}}} \bar{\Lambda}^n(\pi) \to 0.$$

The cohomology of the horizontal de Rham complex are called *horizontal cohomology* and denoted by $\bar{H}^q(\pi)$.

The adjoint operator to a $\mathcal{C}$-differential operator $\Delta\colon P \to Q$ we denote by $\Delta^*\colon \hat{Q} \to \hat{P}$, where $\hat{P} = \mathrm{Hom}_{C^\infty(J^\infty(\pi))}(P, \bar{\Lambda}^n(\pi))$.

In coordinates,

$$\left\| \sum_\sigma a^\sigma_{ij} D_\sigma \right\|^* = \left\| \sum_\sigma (-1)^{|\sigma|} D_\sigma \circ a^\sigma_{ji} \right\|,$$

where $a^\sigma_{ij} \in C^\infty(J^\infty(\pi))$, and $D_\sigma = D_{i_1} \circ \dots \circ D_{i_{|\sigma|}}$ for $\sigma = i_1 \dots i_{|\sigma|}$.

Denote by $\mathcal{C}\mathrm{Diff}^{\mathrm{sk-ad}}_{(k)}(P, Q)$ the module of k-linear skew-symmetric and skew-adjoint in each argument $\mathcal{C}$-differential operators $P \times \dots \times P \to Q$.

A π_∞-vertical vector filed on $J^\infty(\pi)$ is called *evolutionary* if it preserves the Cartan distribution. The Lie algebra of evolutionary fields is denoted by $\varkappa(\pi)$. It is known that $\varkappa(\pi)$ is naturally isomorphic to the set of sections of the bundle $\pi^*_\infty(\pi)$; thus $\varkappa(\pi)$ is endowed with a structure of $C^\infty(J^\infty(\pi))$-module.

In local coordinates, the evolutionary field that corresponds to a section $\varphi = (\varphi^1, \dots, \varphi^m)$ has the form

$$\mathrm{Э}_\varphi = \sum_{j,\sigma} D_\sigma(\varphi^j) \frac{\partial}{\partial u^j_\sigma}.$$

We shall call elements of $\varkappa(\pi)$ *variational vectors*. Elements of the module $\mathcal{C}\mathrm{Diff}^{\mathrm{sk-ad}}_{(k-1)}(\hat{\varkappa}, \varkappa)$ will be called *variational k-vector*, while elements of $\mathcal{C}\mathrm{Diff}^{\mathrm{sk-ad}}_{(k-1)}(\varkappa, \hat{\varkappa})$ will be called *variational k-forms*.

One knows that standard constructions and formulas of the calculus of vector fields and forms on manifolds (the de Rham differential, inner product, the Lie derivative, the Schouten bracket) are also valid for their 'variational' counterparts, with elements of $\bar{H}^n(\pi)$ being regarded as 'functions'.

In particular, the Lie derivative on variational vectors $L_{\ni_\varphi}\colon \varkappa \to \varkappa$ takes the form $L_{\ni_\varphi} = \ni_\varphi - \ell_\varphi$, where the *linearization operator* ℓ_p is defined by the equality $\ell_p(\alpha) = \ni_\alpha(p)$, $\alpha \in \varkappa$.

The Lie derivative on variational forms $L_{\ni_\varphi}\colon \hat{\varkappa} \to \hat{\varkappa}$ is of the form $L_{\ni_\varphi} = \ni_\varphi + \ell_\varphi^*$.

The Lie derivative on variational k-vectors or k-forms satisfies the equality

$$L_{\ni_\varphi}(A)(\xi_1, \dots, \xi_{k-1}) \\ = L_{\ni_\varphi}(A(\xi_1, \dots, \xi_{k-1})) - \sum_i A(\xi_1, \dots, \xi_{i-1}, L_{\ni_\varphi}(\xi_i), \xi_{i+1}, \dots, \xi_{k-1}),$$

where A is a multivector or a form, while $\xi_1, \dots, \xi_{k-1}$ are elements of $\hat{\varkappa}$ in the former case and elements of $\varkappa$ in the latter one.

Consider a determined evolution equation

$$u_t^1 = f^1(t, x, u_\sigma^j), \\ \dots\dots\dots\dots\dots \\ u_t^m = f^m(t, x, u_\sigma^j),$$

where $x = (x_1, \dots, x_n)$. We shall interpret it in a geometric way as the space $\mathcal{E}^\infty = J^\infty(\pi) \times \mathbb{R}$ with the Cartan distribution generated by the Cartan fields on $J^\infty(\pi)$ and the vector field $D_t = \partial/\partial t + \ni_f$, where t is the coordinate along $\mathbb{R}$.

The linearization of $\mathcal{E}^\infty$ is of the form $\ell_{\mathcal{E}} = D_t - \ell_f$, while the adjoint linearization is $\ell_{\mathcal{E}}^* = -D_t - \ell_f^*$.

Note, that from the above we have

$$\ell_{\mathcal{E}} = L_{D_t}\colon \varkappa \to \varkappa, \\ \ell_{\mathcal{E}}^* = -L_{D_t}\colon \hat{\varkappa} \to \hat{\varkappa}. \tag{1}$$

2. Variational Multivectors and Forms on Evolution Equations

Let A be a (possibly dependent on t) variational multivector or form on $J^\infty(\pi)$. If $L_{D_t}(A) = 0$ then A is called a variational multivector or form *on the equation* $\mathcal{E}^\infty$.

Remark 1. Variational bivectors on evolution equations were considered in [4].

Remark 2. From (1) it follows that the set of variational 1-forms on $\mathcal{E}^\infty$ coincides with the term $E_1^{1,n-1}(\mathcal{E}) = \ker \ell_{\mathcal{E}}^*$ of the Vinogradov spectral sequence. Similarly, the terms $E_1^{p,n-1}(\mathcal{E})$ consist of variational p-forms.

The set of variational multivectors and forms on $\mathcal{E}^\infty$ is closed with respect to all operations that are defined on jet spaces whenever they are applicable: the differential on variational forms, inner product, the Schouten bracket, Lie derivative.

PROPOSITION 1. *Let $\mathcal{E}^\infty$ be an evolution equation $u_t = f$. For an operator A to be a variational k-vector or k-form on $\mathcal{E}^\infty$ it is necessary and sufficient to have*

$$\nabla(A(\xi_1,\dots,\xi_{k-1})) + \sum_i A(\xi_1,\dots,\xi_{i-1},\nabla^*(\xi_i),\xi_{i+1},\dots,\xi_{k-1}) = 0, \quad (2)$$

where $\nabla = \ell_{\mathcal{E}}$ if A is a multivector and $\nabla = \ell^*_{\mathcal{E}}$ if A is a form; here $\xi_1,\dots,\xi_{k-1}$ are elements of $\hat{\varkappa}$ in the case of multivectors and elements of $\varkappa$ in the case of forms.

Proof. We have

$$L_{D_t}(A)(\xi_1,\dots,\xi_{k-1}) = L_{D_t}(A(\xi_1,\dots,\xi_{k-1})) - \\ - \sum_i A(\xi_1,\dots,\xi_{i-1},L_{D_t}(\xi_i),\xi_{i+1}\dots,\xi_{k-1}) = 0.$$

Using (1), we get the result. □

PROPOSITION 2. *Let $\mathcal{E}^\infty$ be an evolution equation $u_t = f$. If operators A, A_1, $\dots$, A_{k-1} satisfy the equation*

$$\nabla(A(\xi_1,\dots,\xi_{k-1})) + \sum_i A_i(\xi_1,\dots,\xi_{i-1},\nabla^*(\xi_i),\xi_{i+1},\dots,\xi_{k-1}) = 0, \quad (3)$$

where $\nabla = \ell_{\mathcal{E}}$ or $\ell^*_{\mathcal{E}}$, then $A_1 = A_2 = \cdots = A_{k-1} = A$.

Proof. Denote the left-hand side of (3) by $\Omega(\xi_1,\dots,\xi_{k-1})$. Then we get

$$\Omega(\xi_1,\dots,\xi_{i-1},t\xi_i,\xi_{i+1}\dots,\xi_{k-1}) - t\Omega(\xi_1,\dots,\xi_{k-1}) \\ = \pm(A(\xi_1,\dots,\xi_{k-1}) - A_i(\xi_1,\dots,\xi_{k-1})) = 0.$$

□

Remark 3. The last proposition shows that computing variational multivectors and forms on an equation amounts to solving equation $\nabla(s) = 0$ on the ∇^*-covering (see [4] for the definition of Δ-coverings associated to a $\mathcal{C}$-differential operator Δ).

THEOREM 1. *Suppose that the symbol of the $\mathcal{C}$-differential operator ℓ_f is nonsingular on a dense open subset of $\mathcal{E}^\infty$ and the order of the operator ℓ_f is greater than* 1. *Then there are no nonzero operators A that satisfy Equation* (2) *for $k \geqslant 3$.*

Proof ([2, Th. 3]). Equation (2) can be written in the form

$$\pm D_t(A)(\xi_1,\dots,\xi_{k-1}) + \nabla'(A(\xi_1,\dots,\xi_{k-1})) + \\ + \sum_i A(\xi_1,\dots,\xi_{i-1},\nabla'^*(\xi_i),\xi_{i+1},\dots,\xi_{k-1}) = 0, \quad (4)$$

where $\nabla' = \ell_f$ if A is a multivector and $\nabla' = \ell_f^*$ if A is a form. Take a point $\rho \in \mathcal{E}^\infty$ such that the symbol $\lambda = \mathrm{smbl}(\nabla')|_\rho$ is nondegenerate at ρ. Let $\theta = \sum_{i=1}^n \theta_i \, \mathrm{d}x_i|_\rho$ be a covector, so that, in coordinates, λ is an $m \times m$ matrix $\lambda = \|\lambda_j^i\|$, where λ_j^i are homogeneous polynomials in θ's of degree $l = \mathrm{ord}(\nabla')$. Denote the components of the symbol $a = \mathrm{smbl}(A)|_\rho$ by $a^j_{i_1\ldots i_{k-1}}(\theta^1, \ldots, \theta^{k-1})$, $\theta^p = (\theta_1^p, \ldots, \theta_n^p)$. Then the symbol of Equation (4) takes the form

$$\sum_{j=1}^m \lambda_j^i(\theta^1 + \cdots + \theta^{k-1}) a^j_{i_1\ldots i_{k-1}} + (-1)^l \sum_{p=1}^{k-1} \sum_{j=1}^m a^i_{i_1\ldots i_{p-1} j i_{p+1} i_{k-1}} \lambda_j^{i_p}(\theta^p) = 0, \quad (5)$$

where $1 \leqslant i, i_1, \ldots, i_{k-1} \leqslant m$.

System (5) can be considered as a linear system of algebraic equations with polynomial coefficients over $\mathbb{C}$. Let us show that the determinant of this system does not vanish.

Since $\lambda = \lambda(\theta)$ is nonsingular, there exists $v \in \mathbb{C}^m$ such that $\det \lambda(v) \neq 0$. One can assume that $\lambda(v)$ has an upper triangular form, $\lambda_j^i(v) = 0$ if $i \geqslant j$ and $\lambda_i^i(v) \neq 0$. Then for any $\alpha \in \mathbb{C}$ the matrix $\lambda(\alpha v) = \alpha^l \lambda(v)$ has also an upper triangular form. Since $l = \mathrm{ord}\, \ell_f \geqslant 2$ and $k \geqslant 3$, there exist $\alpha_p \in \mathbb{C}$, $p = 1, \ldots, k-1$, such that for any $1 \leqslant i, i_1, \ldots, i_{k-1} \leqslant m$

$$A_{ii_1\ldots i_{k-1}} = \lambda_i^i(v)(\alpha_1 + \cdots + \alpha_{k-1})^l + (-1)^l \sum_{p=1}^{k-1} \lambda_{i_p}^{i_p}(\alpha_p)^l \neq 0. \quad (6)$$

Put $\theta^i = \alpha_i v$. Then system (4) is upper triangular with respect to the lexicographic order of indexes, with diagonal entries $A_{ii_1\ldots i_{k-1}} \neq 0$. Hence, the determinant of system (4) does not equal to zero, thus $a = 0$. Therefore, the symbol of A vanishes on a dense subset of $\mathcal{E}^\infty$, so that $A = 0$. □

3. Hamiltonian and Symplectic Structures

Recall that a bivector A on equation $\mathcal{E}^\infty$ is said to be *Hamiltonian* if $[\![A, A]\!] = 0$, where $[\![\cdot\,, \cdot]\!]$ is the *variational Schouten bracket* [3, 4]. Two structures A and B are *compatible* (or constitute a *pencil*) if $[\![A, B]\!] = 0$. Respectively, a *symplectic structure* on $\mathcal{E}^\infty$ is a closed variational form.

Now, our main result is obtained by reformulating the previous theorem.

THEOREM 2. *Assume that* $\mathcal{E} = \{u_t = f\}$ *is an evolution equation and* ℓ_f *satisfies the hypothesis of Theorem* 1. *Then any variational bivector on* $\mathcal{E}$ *is Hamiltonian, any two Hamiltonian structures are compatible and any variational* 2*-form is symplectic.*

Proof. Indeed, let A and B be bivectors. Then $[\![A, B]\!]$ is a 3-vector and thus vanishes. Similarly, the differential of any variational 2-form is a 3-form and therefore equals zero. □

Remark 4. The hypothesis of Theorem 1 comprises two conditions: (1) the order of $\mathcal{E}$ is to be > 1; (2) the symbol of ℓ_f is to be nondegenerate. The first one is really essential. As for the second condition, it seems that it may be weakened. At least in some computation (e.g., for the Boussinesq equation, see [4]) all bivectors are automatically Hamiltonian.

Remark 5. The proof of Theorem 1 does not use the fact that the operator A is skew-symmetric and holds also for symmetric $\mathcal{C}$-differential operators. This means that equations satisfying the hypothesis of the theorem may admit *linear* Hamiltonian and symplectic structures only.

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