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ORIGINAL ARTICLE

On hamiltonicity of P_3-dominated graphs

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Abstract We introduce a new class of graphs which we call P_3-dominated graphs. This class properly contains all quasi-claw-free graphs, and hence all claw-free graphs. Let G be a 2-connected P_3-dominated graph. We prove that G is hamiltonian if $\alpha(G^2) \leq \kappa(G)$, with two exceptions: $K_{2,3}$ and $K_{1,1,3}$. We also prove that G is hamiltonian, if G is 3-connected and $|V(G)| \leq 5\delta(G) - 5$. These results extend known results on (quasi-)claw-free graphs.

Keywords Claw-free graph · Quasi-claw-free graph · Hamiltonian cycle · P_3-dominated graph

Mathematics Subject Classification (2000) 05C45 · 05C38

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1 Introduction

Throughout this paper, we consider only finite, undirected and simple graphs. Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. Throughout we use n for $|V|$ and we use $|G|$ as shorthand for $|V(G)|$. For a vertex u, we let $N(u) = \{v \in V(G) \mid uv \in E(G)\}$ and $N[u] = N(u) \cup \{u\}$. A set $A \subseteq V$ is *independent* if any vertices $x, y \in A$ are nonadjacent in G. The *independence number* $\alpha(G)$ of G is the cardinality of a maximum independent set in G.

The *square* G^2 of G is the graph with vertex set $V(G)$ and edge set $\{uv \mid u, v \in V(G)$ and $d(u, v) \le 2\}$, where $d(u, v)$ is the distance in G between u and v. A graph G is called *hamiltonian* if G contains a *Hamilton cycle*, i.e., a cycle containing all vertices of G. For terminology and concepts not defined here, we refer to Bondy and Murty (1976).

One of the most intensively investigated classes of graphs within hamiltonian graph theory is the class $\mathcal{CF}$ of *claw-free* graphs, i.e., graphs that do not contain an induced subgraph isomorphic to $K_{1,3}$. A large number of results have been obtained on claw-free graphs, while some interesting problems and some conjectures remain open (see Broersma 2002). During the last decade, several extensions of claw-free graphs have been introduced and many known results on claw-free graphs have been extended to these classes. We refer to Ainouche (1998), Ainouche et al. (1998), Broersma et al. (1996), Li et al. (1999) and Li (2000, 2001, 2003) for more details. We will repeat the definition of only one of these superclasses of claw-free graphs; the others are not relevant for our purposes.

Following Ainouche (1998), for each pair (a, b) of vertices at distance 2, we set $J(a, b) = \{u \in N(a) \cap N(b) \mid N[u] \subseteq N[a] \cup N[b]\}$. In 1998, Ainouche (1998) introduced the class $\mathcal{QCF}$ of *quasi-claw-free* graphs. A graph G is in $\mathcal{QCF}$, if $J(a, b) \ne \emptyset$ for each pair (a, b) of vertices at distance 2 in G.

The goal of this paper is to extend some known results on $\mathcal{QCF}$ (hence also on $\mathcal{CF}$) to a certain superclass of it, namely the class $\mathcal{P}3\mathcal{D}$ of P_3-*dominated* graphs, which are defined below.

Let (x, y) be a pair of vertices at distance 2 in G. We consider a common neighbor u of x and y with the following property.

$$\begin{aligned}&\text{If } v \in N(u)\backslash\{x, y\} \text{ is neither adjacent to}\\ &x \text{ nor to } y, \text{ then it is adjacent to all vertices of } N(x) \cup N(y) \cup N(u)\backslash\{x, y, v\}. \quad (1)\end{aligned}$$

For a pair (a, b) of vertices at distance 2 in G, analogous to $J(a, b)$, we set $J'(a, b) = \{u \in N(a) \cap N(b) \mid u$ satisfies (1)$\}$. We say that G is in the class $\mathcal{P}3\mathcal{D}$ of P_3-dominated graphs if $J(a, b) \cup J'(a, b) \ne \emptyset$ for every pair (a, b) of vertices at distance 2 in G. Clearly, by definition $\mathcal{QCF} \subseteq \mathcal{P}3\mathcal{D}$.

We denote by θ and θ^+, respectively, the complete bipartite graph $K_{2,3}$ and the complete tripartite graph $K_{1,1,3}$. Note that $K_{1,3}$, θ and θ^+ are P_3-dominated but not quasi-claw-free. It is easy to extend these graphs to infinite classes of graphs with the same property, by replacing some of the vertices by larger complete graphs and some edges by complete joins. One such class is, e.g., $\overline{K_3} \vee (K_p + K_q)$, where $+$ denotes the disjoint union, $\vee$ denotes the complete join, and $\overline{H}$ is the complement of H.

We note here that there are infinitely many graphs in $\mathcal{P}3D$ which are not in the class $\mathcal{A}CF$ of almost claw-free graphs that was introduced in Ryjáček (1994), and vice versa. Similarly, there are infinitely many graphs in $\mathcal{P}3D$ which are not in the class $\mathcal{D}CT$ of dominated claw toes graphs that was introduced in Ainouche et al. (1998), and vice versa. In fact, the graphs $\overline{K}_3 \vee (K_p + K_q)$ are not in $\mathcal{D}CT$ (and hence not in $\mathcal{A}CF$).

2 Properties of P_3-dominated graphs

Before we present some structural results on P_3-dominated graphs, we first introduce some more terminology and notation.

For a path P with end vertices x and y, we also write $P[x, y]$ and call P an (x, y)-path. The (sub)graph corresponding to P will occasionally be identified by P. Given a cycle C with a fixed cyclic orientation and $x, y \in V(C)$, we use $C[x, y]$, $C[x, y)$, $C(x, y]$ and $C(x, y)$ to denote the corresponding subpaths between x and y of C, respectively including both x and y (with possibly $x = y$), only x or only y (if x and y are distinct), and none of x and y (if there is at least a vertex between x and y on C). For a vertex $x \in V(C)$ we use x^+ and x^- to denote the successor and the predecessor of x on C, respectively. If $Z \subseteq V(C)$, then $Z^+ = \{u^+ \mid u \in Z\}$ and $Z^- = \{u^- \mid u \in Z\}$. As usual, we call a nontrivial connected graph separable if it has a cut vertex. For subgraphs H and K of G let $G - H$ denote the subgraph of G which is induced by $V(G)\backslash V(H)$, and let $N_H(K)$ denote the set of vertices in H that are adjacent to some vertex in K. Moreover, we let $N(K) := N_{G-K}(K)$. In particular, if K consists of one vertex v, we omit the brackets, and we use $d_H(v) = |N_H(v)|$ and $d(v) = |N(v)|$.

In all the proofs that follow, we assume that G is a k-connected nonhamiltonian graph ($k \geq 2$). Throughout we will use the following notation without repeating it. We denote by C a longest cycle in G with a fixed cyclic orientation and by H a component of $G - C$. For $s \geq k$, we label $N(H) = \{x_1, \ldots, x_s\}$ in cyclic order around C, where the subscripts are taken modulo s. Adopting a concept that was introduced by Ainouche (1992), we say that a vertex $u \in C(x_i, x_{i+1})$ is *insertible* if there exist vertices $v, v^+ \in C - C(x_i, x_{i+1})$ such that $uv, uv^+ \in E$. In Ainouche (1992) it is proved that $C(x_i, x_{i+1})$ contains a noninsertible vertex, for each $i = 1, \ldots, s$. We let y_i denote the first noninsertible vertex on $C(x_i, x_{i+1})$ and set $X = \{y_0, y_1, \ldots, y_s\}$, where $y_0 \in V(H)$. The following result is due to Ainouche (1998).

Lemma 1 *Let $u_i \in C(x_i, y_i]$, $i = 1, \ldots, s$ and $y \in V(H)$. Then*
(a) $N(u_i) \cap V(H) = \emptyset$;
(b) *there is no vertex $v \in C(y_i, y_j)$ such that $u_i v^+, u_j v \in E$;*
(c) *for $i \neq j$, $N(u_i) \cap N(u_j) \subseteq V(C) - \bigcup_{i=1}^{s} C(x_i, y_i)$;*
(d) *any set $W = \{y\} \cup \{w_i \in C(x_i, y_i] \mid 1 \leq i \leq s\}$ (in particular X) is independent.*

All the above observations can be proved by indicating a longer cycle than C if we assume the contrary to the observation. These arguments are nowadays well-known and pretty standard within the area of hamiltonian graph theory. In the later proofs, we will refer to these and similar arguments as standard long cycle arguments and abbreviate them as *SLCA*, omitting the tedious details of constructing the longer cycles.

In the next three lemmas, we present some properties of P_3-dominated graphs. The following lemma also holds for (quasi-)claw-free graphs [see Ainouche (1998)]. We use $L_G(a, b)$ to denote the length, i.e., the number of edges, of a longest (a, b)-path P in G [so $|V(P)| = L_G(a, b) + 1$].

Lemma 2 *Let $G \notin \{\theta, \theta^+\}$ be a 2-connected nonhamiltonian P_3-dominated graph. Then*
(a) $x^-x^+ \in E$ *for all* $x \in N(H)$*;*
(b) $N(H) \cap N(H') = \emptyset$ *for all pairs* H, H' *of distinct components of* $G - C$*;*
(c) $|C(x_i, x_{i+1}]| \geq 4 + L_H(v, v')$*, where* $v \in N_H(x_i)$ *and* $v' \in N_H(x_{i+1})$*,* $i = 1, \ldots, s$*.*

Proof (*a*). Suppose $xy \in E$, where $x \in V(C)$ and $y \in V(H)$. Clearly, $d(x^-, y) = d(x^+, y) = 2$. Assume $x^-x^+ \notin E$. Then $x^+ \notin N(x^-)$ and $x^+ \notin N(y)$. Hence $x \notin J(x^-, y)$. Similarly, $x \notin J(x^+, y)$.

Claim $x \in J'(x^-, y) \cap J'(x^+, y)$.

Proof of Claim By the definition of P_3-dominated graphs, $J(x^-, y) \cup J'(x^-, y) \neq \emptyset$. Suppose $w \neq x$ and $w \in J(x^-, y) \cup J'(x^-, y)$. Clearly, $w \in V(C)$. Using the definitions of J and J', we get that $w^- \in N(x^-) \cup N(y)$ or $w^-w^+ \in E$. In both cases, *SLCA* yield a contradiction to the choice of C. Hence in fact, $x \in J'(x^-, y)$. Analogously we have $x \in J'(x^+, y)$. □

Using the above claim, we deduce $H = \{y\}$; otherwise by definition, $x^+ \in N(x)$ is adjacent to some $y' \in V(H) \cap N(y)$, clearly contradicting the choice of C. Since G is 2-connected, there is a vertex $z \in N(H)\backslash\{x\}$. As $x^+ \notin N(x^-) \cup N(y)$, we infer by definition that $x^+z \in E$. A similar argument applied to the pair x^+, y yields $x^-z \in E$. Since $G \notin \{\theta, \theta^+\}$, $x^- \neq z^+$ or $x^+ \neq z^-$, say the former. By *SLCA*, it is clear that $z^-z^+ \notin E$. Thus $d(z^+, y) = 2$. By the above claim, we get $z \in J'(z^+, y)$. Since clearly $z^- \notin N(z^+) \cup N(y)$ and $z \in N(x^-)$, this in turn implies $z^-x^- \in E$. Now again *SLCA* yield a contradiction. The other case is similar. This settles (a).

Let H' be a component of $G - C$ other than H. To prove (b), we suppose that there exists some $x \in V(C)$ such that $x \in N(y_1) \cap N(y_2)$ for some $y_1 \in V(H)$ and $y_2 \in V(H')$. Note that $d(x^-, y_1) = 2$. Since $N(x^-) \cap N(y_1) \subseteq V(C)$ and $y_1y_2 \notin E$, *SLCA* yield $J(x^-, y_1) = \emptyset$. If $y \in J'(x^-, y_1)$ and $y \neq x$, then since $x^-x^+ \in E$ (by (a)), by definition we have $x^+y^+ \in E$, a contradiction (by *SLCA*). Hence indeed $J'(x^-, y_1) = \{x\}$. But then, since $x^-y_2, y_1y_2 \notin E$, we have $x^+y_2 \in E$, again a contradiction, which settles (b). Note that (c) is an immediate consequence of (a) and the fact that C is a longest cycle in G. □

Using Lemma 2, we can deduce a crucial property of X.

Lemma 3 *Let $G \notin \{\theta, \theta^+\}$ be a 2-connected nonhamiltonian P_3-dominated graph. Then for any distinct y_j, y_k in X, we have $N(y_j) \cap N(y_k) = \emptyset$.*

Proof Assuming the contrary, $d(y_j, y_k) = 2$ for some distinct $y_j, y_k \in X$. We distinguish two cases.

Case 1 $k = 0$.

Since $y_j \in C(x_j, x_{j+1})$ we have $J(y_0, y_j) \cup J'(y_0, y_j) \subseteq N(H)$. Without loss of generality, we may assume $x_1 \in J(y_0, y_j) \cup J'(y_0, y_j)$.

First assume $j \neq 1$. Since y_j is not insertible, we have $x_1^+ y_j \notin E$, and hence $x_1 \in J'(y_0, y_j)$. But then $x_1^+ y_j^- \in E(G)$, a contradiction to Lemma 1(d).

We are left with the case that $j = 1$. Since y_1 is not insertible, we have that $d(x_1^-, y_1) = 2$. It is again clear that $x_1 \notin J(x_1^-, y_1)$. But $x_1 \in J'(x_1^-, y_1)$ implies y_0 is adjacent to x_1^+, an obvious contradiction.

Case 2 $j, k \neq 0$.

Since all vertices in $C(x_i, y_i)$ $(i = j, k)$ are insertible, we may assume $J(y_j, y_k) \cup J'(y_j, y_k) \subseteq V(C)$. By Lemma 1(c) we have $(C(x_j, y_j) \cup C(x_k, y_k)) \cap (J(y_j, y_k) \cup J'(y_j, y_k)) = \emptyset$. So, without loss of generality, we may assume that there is a vertex u of $J(y_j, y_k) \cup J'(y_j, y_k)$ on $C[y_j^+, x_k^-]$. Since y_k is not insertible, we have $u^+ y_k \notin E$. Also $u^+ y_j \notin E$ by Lemma 1(b). Therefore, in fact, $u \in J'(y_j, y_k)$. But then $u^+ y_j^- \in E$, a contradiction to Lemma 1(b). This last contradiction settles Lemma 3. $\square$

In the next section, we will use the above lemmas to deduce some results on the hamiltonicity of P_3-dominated graphs.

3 Hamiltonicity of P_3-dominated graphs

We start this section by stating a sufficient condition for hamiltonicity in terms of the independence number and the connectivity, often referred to as a condition of Chvátal-Erdős type.

Lemma 3 immediately implies the following theorem, which was proved by Ainouche et al. (1990) for claw-free graphs and by Ainouche (1998) for quasi-claw-free graphs.

Theorem 1 *Let $G \notin \{\theta, \theta^+\}$ be a 2-connected P_3-dominated graph. Then G is hamiltonian if $\alpha(G^2) \leq \kappa(G)$.*

Note that the nonhamiltonian graphs θ and θ^+ clearly have to be excluded: their square graphs are complete and they are 2-connected.

We also note here that a similar result can be obtained for traceable graphs, i.e., graphs that contain a Hamilton path. This of course requires analogues of the previous lemmas. Starting with a longest path P instead of a longest cycle, and the assumption that G is not traceable, one can define a component H of $G - P$ and the vertices x_i and y_i in a similar way. One can obtain an additional pair from the first vertex and the first noninsertible vertex on P. For these pairs, the same observations as in Lemmas 1, 2(a), and 3 hold, if G is a connected P_3-dominated graph and $G \neq K_{1,3}$. It is routine to check the details by using analogous arguments as in the above proofs and using standard long path arguments. This way we can obtain the following result for the traceability of P_3-dominated graphs.

Theorem 2 *Let $G \neq K_{1,3}$ be a connected P_3-dominated graph. Then G is traceable if $\alpha(G^2) \leq \kappa(G) + 1$.*

It is clear that the nontraceable graph $K_{1,3}$ has to be excluded in the above theorem: its square graph is complete and its connectivity is one.

In the following theorem, we shall extend a result on the hamiltonicity of 3-connected claw-free graphs (see Li 1993) to P_3-dominated graphs. Theorem 3 was proved in Li (2003) for quasi-claw-free graphs.

Before we state and prove the theorem, we introduce some additional terminology and a useful lemma due to Jung (1986).

Let H be a connected graph and let $a, b \in V(H)$. Recall that $L_H(a, b)$ denotes the length of a longest (a, b)-path in H. If H is nonseparable and $|H| \geq 2$, we set $D(H) = \min\{L_H(a, b) \mid a, b \in V(H) \text{ and } a \neq b\}$. For $|H| = 1$, we set $D(H) = 0$. For our proof of Theorem 3 below, we need the following lemma of Jung (1986).

Lemma 4 *Let H be a 2-connected graph. There exist distinct vertices v_1, v_2 in H such that $D(H) \geq d_H(v_h)$ $(h = 1, 2)$.*

Our final result gives a sufficient Dirac-type condition in terms of the minimum degree δ for the hamiltonicity of 3-connected P_3-dominated graphs.

Theorem 3 *Let G be a 3-connected P_3-dominated graph on n vertices. If $n \leq 5\delta - 5$, then G is hamiltonian.*

Proof Clearly, since G is 3-connected, $G \notin \{\theta, \theta^+\}$. Suppose $n \leq 5\delta - 5$ and G is not hamiltonian. As before, let C be a longest cycle in G with a fixed cyclic orientation and let H be a component of $G - C$. By Lemma 3, there exists an independent set S in G with cardinality $|S| = |N(H)| + 1 \geq 4$ such that $N(x) \cap N(y) = \emptyset$ for any pair x, y of distinct vertices of S. This implies $n \geq (|N(H)| + 1)\delta + |N(H)| + 1$, and consequently we arrive at a contradiction unless $|N(H)| = 3$ (since G is 3-connected and $n \leq 5\delta - 5$).

We label $N(H) = \{x_1, x_2, x_3\}$ in cyclic order around C, where the subscripts are taken modulo 3. Since G is 3-connected we have either $|H| = 1$ or $|N_H(x_j) \cup N_H(x_k)| \geq 2$ for any distinct vertices x_j, x_k of $N(H)$. We may assume that H is nonseparable (for otherwise we can use similar arguments applied to an end block of H). Let w be a vertex with minimum degree in H. Then by Lemma 4, $D(H) \geq d_H(w) \geq d(w) - 3$, and hence by Lemma 2(c),

$$|C| \geq 3D(H) + 12 \geq 3d(w) + 3 \geq 3\delta + 3.$$

We will use this inequality repeatedly to obtain a contradiction with the assumption $n \leq 5\delta - 5$. For convenience, we abbreviate $d_{C_i}(v) := |N(v) \cap C(x_i, x_{i+1}]|$ for a vertex v of G and for $i = 1, 2, 3$.

We prove six claims before we complete the proof of the theorem. Our first claim states that H is hamiltonian-connected, i.e., there is a Hamilton path between any two distinct vertices of H. We will use the known fact that every graph G with $\delta(G) \geq (n + 1)/2$ is hamiltonian-connected [see, e.g., Chartrand and Lesniak (1996)].

Claim 1 H is hamiltonian-connected.

Proof of Claim 1 Assuming the contrary, we get $|H| \geq 2d_H(w) \geq 2d(w) - 6$. Then $n \geq |C| + |H| \geq 5d(w) - 3 \geq 5\delta - 3$, a contradiction. □

Using Claim 1, by Lemma 2(c) we obtain $|C| \geq 3(|H| - 1) + 12 = 3|H| + 9$. If there exists an $H' \neq H$ in $G - C$, then by similar arguments as before $|N(H')| = 3$ and hence $|H'| \geq \delta - 2$, yielding $n \geq |H| + |H'| + |C| \geq 2(\delta - 2) + 3\delta + 3 = 5\delta - 1$, a contradiction. Hence $G - C = H$. Since $\delta \leq |H| + 2$, $n \leq 5\delta - 5 \leq 5|H| + 5$. Combining this with $n = |C| + |H| \geq 4|H| + 9$, we obtain $|H| \geq 4$.

Suppose, for some $i \in \{1, 2, 3\}$, $|C(x_i, x_{i+1}]| \geq 2|H|$. Then, using Lemma 2(c), we obtain $|C| \geq 2|H| + 8 + 2|H| - 2 = 4|H| + 6$, implying $n \geq 5|H| + 6 \geq 5\delta - 4$, another contradiction. Hence we get the following:

$$|C(x_i, x_{i+1}]| < 2|H|, \quad i = 1, 2, 3. \tag{2}$$

Suppose $N(x_i) \cap C(x_{i+1}, x_{i+2}) \neq \emptyset$. Then let $u \in N(x_i) \cap C(x_{i+1}, x_{i+2})$. By considering the two cycles obtained from C by deleting $C(x_{i+1}, u^-)$ and, respectively $C(u^+, x_{i+2})$, and using the edge $x_i u$ and a Hamilton path through H, we obtain a longer cycle than C using (2). Hence we get $N(x_i) \cap C(x_{i+1}, x_{i+2}) = \emptyset$ and, by using similar long cycle arguments, $N(x_i^+) \cap N(x_{i+1}^-) \cap C(x_{i+1}, x_i) = \emptyset$, for $i = 1, 2, 3$. Moreover, using similar arguments, (2) yields the following claim.

Claim 2 Let x_j, x_k be distinct vertices of $N(H)$ such that $N(x_j^+) \cap C(x_k, x_{k+1}) \neq \emptyset$ and $N(x_{j+1}^-) \cap C(x_k, x_{k+1}) \neq \emptyset$. Let z and z' be the first and the last neighbors of x_j^+ and x_{j+1}^- on $C(x_k, x_{k+1})$, respectively. Then $z \in C(z', x_{k+1})$. In particular x_j^+ and x_{j+1}^- have no common neighbor on $C(x_k, x_{k+1})$.

If z is the first neighbor of x_i^+, and z' is the last neighbor of x_{i+1}^- on $C(x_{i+1}, x_{i+2})$, then consider the cycle obtained from C by deleting $C(z', z)$, $x_i x_i^+$ and $x_{i+1}^- x_{i+1}$, and using the edges $x_i^+ z$, $x_{i+1}^- z'$, and a Hamilton path P through H and edges from x_i and x_{i+1} incident with the two end vertices of P. Then, clearly $|C(z', z)| \geq |H|$. A similar cycle can be considered if we focus on the segment $C(x_{i+2}, x_i)$. Using these cycles and the choice of C, Claim 2 yields the following claim.

Claim 3 For $i = 1, 2, 3$,

$$|C(x_{i+1}, x_{i+2}]| \geq d_{C_{i+1}}(x_i^+) + d_{C_{i+1}}(x_{i+1}^-) + |H| + 1.$$

$$|C(x_{i+2}, x_i]| \geq d_{C_{i+2}}(x_i^+) + d_{C_{i+2}}(x_{i+1}^-) + |H|.$$

Now suppose $|C(x_i, x_{i+1}]| \geq d_{C_i}(x_i^+) + d_{C_i}(x_{i+1}^-) + 1$ for some $i \in \{1, 2, 3\}$. Then by Claim 3 we obtain $n \geq d(x_i^+) + d(x_{i+1}^-) + 3|H| + 2 \geq 5\delta - 4$. This contradiction shows that

$$|C(x_i, x_{i+1}]| \leq d_{C_i}(x_i^+) + d_{C_i}(x_{i+1}^-), \quad i = 1, 2, 3.$$

The last inequality clearly yields:

$$\text{There exists no vertex } y \text{ on } C(x_i, x_{i+1}) \text{ satisfying } N(x_i^+) \cap C(x_i, x_{i+1}) \subseteq C(x_i, y] \text{ and } N(x_{i+1}^-) \cap C(x_i, x_{i+1}) \subseteq C[y, x_{i+1}), \; i = 1, 2, 3. \tag{3}$$

Claim 4 $N(x_i^+) \cap (C(x_{i+1}, x_{i+2}^-) \cup C(x_{i+2}, x_i^-)) = \emptyset$, $i = 1, 2, 3$.

Proof of Claim 4 Assume the contrary, say $N(x_1^+) \cap C(x_2, x_3^-) \neq \emptyset$. Let z and z' be the first and the last neighbors of x_1^+ on $C(x_2, x_3^-)$. Using similar cycle constructions as before, the choice of C implies that $|C(x_2, z)| \geq |H|$. Using this in combination with (2), we get $N(x_2^+) \cap C(z, x_3] = \emptyset$; otherwise we can construct a longer cycle than C. By (3) there exists a vertex $u \in N(x_3^-) \cap C(x_2, z)$ and a vertex $u' \in N(x_2^+) \cap C(u, z]$ such that $C(u, u') \cap (N(x_1^+) \cup N(x_2^+)) = \emptyset$. Then, with a slight abuse of notation, $C[x_1^+, x_2] \cup C[x_2^+, u] \cup C[u', z'] \cup C[x_3^-, x_1] \cup x_1^+ z' \cup x_2^+ u' \cup x_3^- u$ gives rise to a cycle which contains all vertices of $H \cup (C - C(u, u') - C(z', x_3^-))$. Hence $|C(u, u') \cup C(z', x_3^-)| \geq |H|$. Combining this with the above observations, we get:

$$|C(x_2, x_3]| \geq d_{C_2}(x_1^+) + d_{C_2}(x_2^+) + |H| + 1.$$

Now, we consider the possible neighbors of x_1^+ and x_2^+ on the other segments in order to obtain a contradiction. First note that, if z is the first neighbor of x_1^+ or x_2^+ on $C(x_3, x_1)$, then $|C(x_3^+, z)| \geq |H|$, by similar arguments as before. If z and z' are two common neighbors of x_1^+ and x_2^+ on $C(x_3, x_1)$, then a cross-over argument similar to Lemma 1(b) yields that $|C(z, z')| \geq |H|$ (if we assume that $z' \in C(z, x_1)$). Then we obtain a contradiction with (2). Using these two observations, and the existence of the edges $x_1^- x_1^+$ and $x_3^- x_3^+$, we obtain the following inequality:

$$|C(x_3, x_1]| \geq d_{C_3}(x_1^+) + d_{C_3}(x_2^+) + |H| + 1.$$

Standard counting using cross-over arguments yields:

$$|C(x_1, x_2]| \geq d_{C_1}(x_1^+) + d_{C_1}(x_2^+).$$

Combining the three inequalities, we obtain $n = |C| + |H| \geq d(x_1^+) + d(x_2^+) + 3|H| + 2 \geq 5\delta - 4$, a contradiction. The other cases are similar. □

Claim 5 There is at most one edge of the form $x_i^+ x_{i+2}^-$, $i = 1, 2, 3$.

Proof of Claim 5 Assume the contrary, say $x_1^+ x_3^-, x_1^- x_2^+ \in E$. Then these two edges together with the segments $C[x_1^+, x_2]$, $C[x_2^+, x_3^-]$ and $C[x_3, x_1^-]$ give rise to a cycle C' which contains all vertices of H and $C - \{x_1\}$. Since C is a longest cycle we have $|H| = 1$, a contradiction. The other cases are similar. □

Claim 6 There exists no edge $e = z_j z_k$ with $z_j \in C(x_j^+, x_{j+1}^-)$ and $z_k \in C(x_k^+, x_{k+1}^-)$, where x_j, x_k are distinct vertices of $N(H)$.

Proof of Claim 6 Say $j = 1 = k - 1$. By (3), we infer $N(x_1^+) \cap C(z_1, x_2) \neq \emptyset$ or $N(x_2^-) \cap C(x_1, z_1) \neq \emptyset$, say the former. Let u_1 be the first vertex of $N(x_1^+)$ on $C(z_1, x_2)$. Again slightly abusing notation, we set $R = C[x_1^+, z_1] \cup x_1^+ u_1 \cup C[u_1, x_2^-]$. We define a (z_2, x_3^-)-path Q as follows. If $N(x_2^+) \cap C(z_2, x_3) \neq \emptyset$, then let u_2 denote the first neighbor of $N(x_2^+)$ on $C(z_2, x_3)$ and set $Q = C[x_2^+, z_2] \cup x_2^+ u_2 \cup C[u_2, x_3]$. If $N(x_2^+) \cap C(z_2, x_3) = \emptyset$, then by (3) there exist vertices $u_2' \in N(x_3^-) \cap C(x_2^+, z_2^+)$ and $u_2'' \in N(x_2^+) \cap C(u_2', z_2]$ such that $(N(x_1^+) \cup N(x_2^+)) \cap C(u_2', u_2'') = \emptyset$. In this event, we set $Q = C[u_2'', z_2] \cup u_2'' x_2^+ \cup C[x_2^+, u_2'] \cup u_2' x_3^-$. Now we set $L = (C(x_1, x_2) - R) \cup (C(x_2, x_1) - Q)$. Note that the segment $C[x_3^-, x_1]$ together with R and Q and the edge $z_1 z_2$ give rise to a cycle which contains all vertices of $H \cup C - L$. Hence $|L| \geq |H|$. Since $(N(x_1^+) \cup N(x_2^+)) \cap L = \emptyset$ and $x_3 \notin N(x_1^+) \cup N(x_2^+)$ we have

$$|C(x_1, x_3]| \geq |N(x_1^+) \cap C(x_1, x_3]| + |N(x_2^+) \cap C(x_1, x_3]| + |H| + 1 + \eta_{13},$$

where $\eta_{13} = 1$ if $x_1^+ x_3^- \notin E$, and $\eta_{13} = 0$ otherwise.

By Claim 4, $d_{C_3}(x_1^+) = 2$ and $d_{C_3}(x_2^+) = 0$ or 1 (if $x_1^- x_2^+ \in E$), hence we get

$$|C(x_3, x_1]| \geq |H| + 3 \geq d_{C_3}(x_1^+) + d_{C_3}(x_2^+) + |H| + \eta_{12},$$

where $\eta_{12} = 1$ if $x_1^- x_2^+ \notin E$ and $\eta_{12} = 0$ otherwise. Note that $\eta_{13} + \eta_{12} \geq 1$ by Claim 5, and therefore $n = |C| + |H| \geq |C \cup H| \geq d(x_1^+) + d(x_2^+) + 3|H| + 2 \geq 5\delta - 4$, a contradiction. The other cases are similar. $\square$

Now we are ready to complete the proof of Theorem 3. By (2) we have $N(x_3) \cap C(x_1, x_2) = \emptyset$. Since G is 3-connected, $\{x_1^+, x_2^-\}$ is not a cut set of G. Using Claim 6, clearly $(N(x_1) \cup N(x_2)) \cap C(x_1^+, x_2^-) \neq \emptyset$, say $N(x_1) \cap C(x_1^+, x_2^-) \neq \emptyset$. Let $z \in N(x_1) \cap C(x_1^+, x_2^-)$. By similar long cycle arguments as before, $|C(z, x_2)| \geq |H| + 1$ and $z^- \notin N(x_2^-) \cup N(x_2)$. If $N(z^-) \cap C(z, x_2^-) = \emptyset$, then $|C(x_1, x_2]| \geq d(z^-) + |H| + 1$ and $n \geq d(z^-) + 4|H| + 7 \geq 5\delta - 1$, a contradiction. Now suppose $N(z^-) \cap C(z, x_2^-) \neq \emptyset$, and let u be the last neighbor of z^- on $C(z, x_2]$. Since $u \notin \{x_2^-, x_2\}$ we have $|C(u, x_2]| \geq |H| + 2$. Thus again $|C(x_1, x_2]| \geq d(z^-) + |H| + 1$ and $n \geq 5\delta - 1$, a contradiction. The other cases are similar. $\square$

References

Ainouche A (1992) An improvement of Fraisse's sufficient condition for hamiltonian graphs. J Graph Theory 16:529–543

Ainouche A (1998) Quasi-claw-free graphs. Discrete Math 179:13–26

Ainouche A, Broersma HJ, Veldman HJ (1990) Remarks on Hamiltonian properties of claw-free graphs. Ars Combin 29C:110–121

Ainouche A, Favaron O, Li H (1998) Global insertion and hamiltonicity in DCT-graphs. Discrete Math 184:1–13

Bondy JA, Murty USR (1976) Graph theory with applications. Macmillan, London and Elsevier, New York

Broersma HJ (2002) On some intriguing problems in hamiltonian graph theory—a survey. Discrete Math 251:47–69

Broersma HJ, Ryjáček Z, Schiermeyer I (1996) Toughness and hamiltoncity in almost claw-free graphs. J Graph Theory 21:431–439

Chartrand G, Lesniak L (1996) Graphs and digraphs. Chapman & Hall

Jung HA (1986) Longest paths joining given vertices in a graph. Abh Math Sem Univ Hamburg 56:127–137
Li H, Chen C, Harkat-Benhamdine A (1999) Distance dominating cycles in quasi-claw-free graphs. Graphs Combin 15:279–285
Li MC (1993) Hamiltonian cycles in 3-connected claw-free graphs. J Graph Theory 17:303–313
Li MC (2000) Longest cycles in almost claw-free graphs. Graphs Combin 16:177–198
Li MC (2001) Hamiltonian cycles in almost claw-free graphs. Graphs Combin 17:687–706
Li R (2003) Hamiltonicity of 3-connected quasi-claw-free graphs. Discrete Math 265:393–396
Ryjáček Z (1994) Almost claw-free graphs. J Graph Theory 18:469–477