This document is the online-only appendix to:

Indeterministic Handling of Uncertain Decisions in Deduplication

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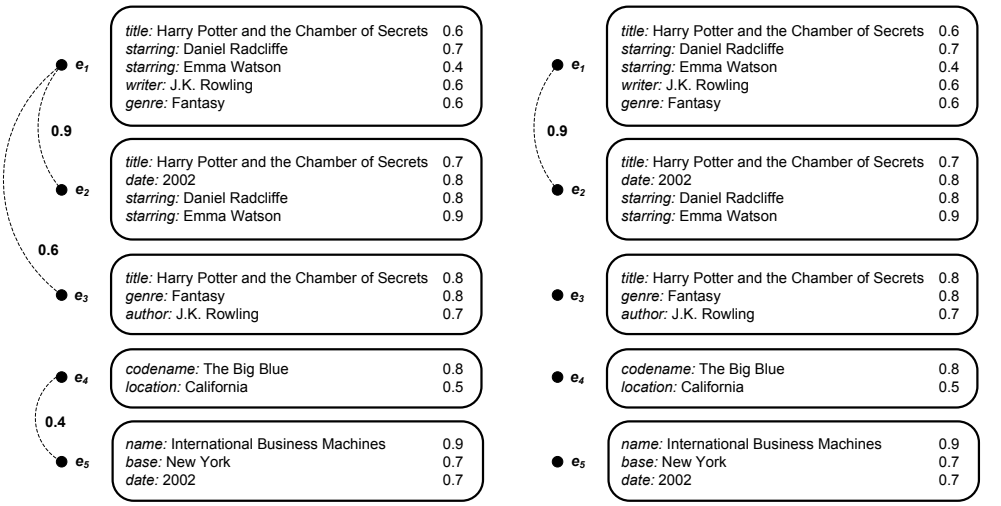
Journal of the ACM, Vol. V, No. N, Month 20YY, Pages 1–0??.

A. DEMONSTRATING EXAMPLE FOR SOME DISCUSSION ON RELATED WORK

As mentioned in Section 2, Ioannou et al. cannot ensure that from their approach always the most probable worlds result. Moreover, this approach disregards negative information (information that two entities are not duplicates for sure) which makes the result more inaccurate and which makes an introduction of negative domain knowledge during the initially offline performed linkage creation impossible.

As an example consider the probabilistic linkage database $PLDB = <\mathcal{E}, \mathcal{L}, p^a, p^l>$ with the five entities $\mathcal{E} = \{e_1, e_2, e_3, e_4, e_5\}$ which has been already used in [Ioannou et al. 2010], but let us assume that the linkage between e_4 and e_5 has a probability of $p^l(e_4, e_5) = 0.4$ instead of 0.8 and let us assume that e_2 and e_3 are not linked because a domain expert know with absolute certainty that both entities are no duplicates. Nevertheless, the probabilities of all remaining linkages which are not presented in this example can be greater than 0. The modified database is graphically illustrated in Figure 15 (i).

We understand the approach of Ioannou et al. in a way that linkages between some entities, e.g. the linkage between e_3 and e_4, are not present in Figure 1 in [Ioannou et al. 2010], because their probabilities (e.g. $p^l(e_3, e_4)$) are too small for being of interest. As a consequence, we assume that the presented linkages result from using a threshold T_l which demarcates all the linkages being of interest (all linkages with a probability greater than or equal to T_l) from all remaining linkages. Our modified example (Figure 15 (i)) would for example result by using a threshold $T_l = 0.4$, because the lowest probability of all presented linkages is 0.4. The probabilities of all missing linkages must be smaller than 0.4, because otherwise (for reasons of consistency) these linkages must be present in this figure as well. By increasing T_l from 0.4 to 0.8 the linkage between e_1 and e_3 as well as the linkage between e_4 and e_5 are further removed (see Figure 15 (ii)).



(i) with threshold $T_l = 0.4$ (ii) with threshold $T_l = 0.8$

Fig. 15. The marginally modified sample probabilistic linkage database from Figure 1 in [Ioannou et al. 2010], with the threshold (i) $T_l = 0.4$ and (ii) $T_l = 0.8$

Let Q be a query which condition is satisfied by all five entities. Thus, from applying Q with the threshold $T_l = 0.8$ (Figure 15 (ii)), the two possible worlds $I_1 = \{e_1, e_2, e_3, e_4, e_5\}$ or $I_2 = \{\mu(\{e_1, e_2\}), e_3, e_4, e_5\}$ result. However, since the case of nonexistence for the linkage $p^l(e_1, e_2)$ is more unlikely than the case of existence of the linkage $p^l(e_4, e_5)$, the ignored possible world $I_3 = \{\mu(\{e_1, e_2\}), e_3, \mu(\{e_4, e_5\})\}$ is more probable than the considered world I_1. For that reason in our (α, β)-restrictions, we always restrict the decision considered to be uncertain at both ends of the probability range.

Furthermore, let the threshold be $T_l = 0.4$. As mentioned above, this configuration leads to all the linkages explicitly presented in Figure 15 (i). Because for each linkage specification $\mathcal{L}^{sp} \supset \mathcal{L}$ the transitive closure is performed, among others the possible world $I_4 = \{\mu(\{e_1, e_2, e_3\}), e_4, e_5\}$ results (compare with the possible world I_2 in Example 2 in [Ioannou et al. 2010]), Nevertheless, since e_2 and e_3 are not linked, because both entities are known to be definitely no duplicates, this world is definitely not possible. As a consequence, information on non-duplicates is ignored during query evaluation and hence cannot be incorporated by the user (or domain expert) during linkage creation. In order to avoid such situations, we firstly consider all matching results (whether possible or not), then create all possible combination of uncertain edges (worlds) and finally remove impossible ones (see Section 5.1.2.3).

B. INDETERMINISTIC DEDUPLICATION WITH MAYBMS:

MayBMS is a probabilistic database management system developed at the Cornell University [Koch 2008; 2009]. For a succinct representation of probabilistic data with attribute value dependencies and tuple correlations MayBMS uses the concept of U-relations (U-relational databases) which is based on probabilistic conditional

tables [Suciu et al. 2011]. This means that tuples can be assigned with boolean conditions defined on a finite set of variables[7]. Each variable can be interpreted as an independent probabilistic event. U-relational databases are restricted to a finite set of possible worlds. Thus, each variable is defined on a finite domain (its set of possible outcomes). Correlations are modeled by using same variables for attribute value combinations (attribute value dependencies) or for different tuples (tuple correlations). To increase succinctness further on, MayBMS uses vertical partitioning to decompose tuples in several probabilistic independent set of attributes (also denoted as normalization [Suciu et al. 2011]). A U-relation consists of a set of attributes to be stored (*value columns*) and, in addition, a set of pairs $(\mathbf{V}_i, \mathbf{D}_i)$ (*condition columns*) for modeling variable assignments which serves as atomic conditions for tuple existence. The possible outcomes of variables along with their probabilities are separately stored in a ternary relation **W**, called *world-table*. As ULDB, U-relational databases are a complete and closed representation system for finite probabilistic databases. From a U-relational database, an arbitrary possible world can be derived by (1) mapping each variable in **W** to one of its possible outcomes (the world's corresponding variable assignment θ), (2) joining each U-relation with **W** on the condition columns, (3) selecting all tuples whose conditions satisfy θ, and (4) projecting away the condition columns.

A succinct U-relational database representation whose set of possible worlds is equivalent to this of the ULDB database shown in Figure 3 is presented in Figure 16.

$U_{\mathcal{R}_1[\mathbf{name}]}$	**V**	**D**	**TID**	**name**
	v	1	t_1	Tim
	v	2	t_1	Jim
	w	1	t_2	Tim

$U_{\mathcal{R}_1[\mathbf{firm}]}$	**V**	**D**	**TID**	**firm**
	v	1	t_1	Oracle
	v	2	t_1	Nokia
	w	1	t_2	IBM

$U_{\mathcal{R}_2}$	**V**	**D**	**TID**	**firm**	**industry**
	x	1	t_3	IBM	software
	y	1	t_4	Oracle	software
	z	1	t_5	Nokia	cell-phone

W	**V**	**D**	**P**
	v	1	0.3
	v	2	0.7
	w	1	0.6
	x	1	1.0
	y	1	1.0
	z	1	1.0

Fig. 16. Succinct representation of the sample probabilistic database of Figure 3 with U-relational databases

B.1 Modeling Tuple Dependencies in MayBMS

Since U-relational databases are principally based on the concept of conditional tables, tuple dependencies can be simply modeled by variables. To model the complementation $\mathfrak{cpl}(A_1, \ldots, A_k)$, a new U-relation $U_{\mathcal{R}[\text{corr.}]}$ and one variable x with k possible outcomes is required. The newly created U-relation $U_{\mathcal{R}[\text{corr.}]}$ needs only one pair of condition columns, and hence has the following three attributes: (1) **V** for storing the variable x, (2) **D** for storing the possible outcome of x being

[7]Note, in U-relational databases variables are not allowed as attribute values [Koch 2009].

the condition that the considered tuple exists, and (3) **TID** for storing the identifier of the considered tuple. Each tuple $t \in \bigcup_{i \in [1,k]} A_i$ is stored in $U_{\mathcal{R}[\text{corr.}]}$ with the same variable $\mathbf{V} = x$. To each tuple in A_i the possible outcome i is assigned. Finally, the newly created variable x along with the probabilities of its k possible outcomes are stored in the world-table $\mathbf{W}$.

For demonstration, we consider the same example as already used in Section 3.2.2, i.e. the complementation $\mathfrak{cpl}(\{t_1, t_2\}, \{t_{12}\})$. This dependency is modeled by creating the U-relation $U_{\mathcal{R}[\text{corr.}]}$ and extending the world table $\mathbf{W}$ as illustrated in Figure 17.

$U_{\mathcal{R}[\text{corr.}]}$	**V**	**D**	**TID**
	x	1	t_1
	x	1	t_2
	x	2	t_{12}

W	**V**	**D**	**P**
	x	1	0.4
	x	2	0.6

Fig. 17. Tuple correlations modeled with U-relational databases

B.2 Generation of Probabilistic Result Data with U-Relations

For representing the set of possible worlds $W = \{I_1, \ldots, I_k\}$ resulting from an indeterministic deduplication as a U-relational database, a new variable x of the world-table $\mathbf{W}$ having $|W|$ possible outcomes (one for each world) is required. For each tuple t_i of the possible world I_j, a new tuple (x, j, t_i) is inserted into the U-relation $U_{\mathcal{R}[\text{corr.}]}$, where j is the index number of the considered possible world.

For the purpose of demonstration, we consider our example already used for x-relation generation. We create a variable x with one possible value for each of the five resultant possible worlds $W = \{I_1, I_2, I_3, I_4, I_8\}$ and generate the new tuples in the way described above (prior condition variables are assumed to be non-existent). The resultant U-relation $U_{\mathcal{R}[\text{corr.}]}$ and the extended world-table $\mathbf{W}$ are shown in Figure 18.

$U_{\mathcal{R}[\text{corr.}]}$	**V**	**D**	**TID**
	x	1	t_1
	x	1	t_2
	x	1	t_3
	x	2	t_{12}
	x	2	t_3
	x	3	t_{13}
	x	3	t_2
	x	4	t_{23}
	x	4	t_{12}
	x	5	t_{123}

W	**V**	**D**	**P**
	x	1	0.138
	x	2	0.553
	x	3	0.092
	x	4	0.059
	x	5	0.158

Fig. 18. U-relation $U_{\mathcal{R}[\text{corr.}]}$ (left) and world-table $\mathbf{W}$ (right)

C. PROOFS FOR THEOREMS 1 AND 2

In this section, we prove Theorems 1 and 2 which are introduced in Section 5.1.2.3.

Theorem 1: *A* W-graph $G = (N, E, P)$ *is consistent, if and only if* G *is equivalent to its transitive closure:* $G = G^*$.

PROOF. ($\Rightarrow$) Assumption: $G \neq G^*$, but G is consistent.
$\Rightarrow (\exists t_1, t_2, t_3 \in N) : \{t_1, t_2\}, \{t_1, t_3\} \in E \wedge \{t_2, t_3\} \notin E$
$\Rightarrow$ the world $I = \{t_1, t_2, t_3, \ldots\}$ is impossible
$\Rightarrow G$ is inconsistent $\square$

PROOF. ($\Leftarrow$) Assumption: G is inconsistent, but $G = G^*$.
$\Rightarrow$ the world $I = N$ is impossible
$\Rightarrow (\exists t_1, t_2, t_3 \in N) : \{t_1, t_2\}, \{t_1, t_3\} \in E \wedge \{t_2, t_3\} \notin E$
$\Rightarrow G \neq G^*$ $\square$

Theorem 2:*An* M-graph $M = (N, E, \gamma)$ *is consistent, if and only if:*
$(\forall t_1, t_2, t_3 \in N) : \gamma(\{t_1, t_2\}) = \gamma(\{t_1, t_3\}) = 1 \Rightarrow \gamma(\{t_2, t_3\}) > 0$.

PROOF. ($\Rightarrow$) Assumption: $(\exists t_1, t_2, t_3 \in N) :$
$\gamma(\{t_1, t_2\}) = \gamma(\{t_1, t_3\}) = 1 \wedge \gamma(\{t_2, t_3\}) = 0$, but M is consistent.
$\Rightarrow (\forall G = (N, E, P) \in \nu(M)) : \{t_1, t_2\}, \{t_1, t_3\} \in E \wedge \{t_2, t_3\} \notin E$
$\Rightarrow (\forall G = (N, E, P) \in \nu(M)) : G$ is inconsistent
$\Rightarrow M$ is inconsistent $\square$

PROOF. ($\Leftarrow$) Assumption: M is inconsistent, but
$(\forall t_1, t_2, t_3 \in N) : \gamma(\{t_1, t_2\}) = \gamma(\{t_1, t_3\}) = 1 \Rightarrow \gamma(\{t_2, t_3\}) > 0$.
$\Rightarrow (\forall G = (N, E, P) \in \nu(M)) : G$ is inconsistent
$\Rightarrow (\exists t_1, t_2, t_3 \in N) : (\forall G = (N, E, P) \in \nu(M)) :$
$\{t_1, t_2\}, \{t_1, t_3\} \in E \wedge \{t_2, t_3\} \notin E$
$\Rightarrow (\exists t_1, t_2, t_3 \in N) : \gamma(\{t_1, t_2\}) = \gamma(\{t_1, t_3\}) = 1 \Rightarrow \gamma(\{t_2, t_3\}) = 0$ $\square$

D. ALGORITHM

In this section, first we present an algorithm for possible world creation and then two algorithms for generating probabilistic result data (one for the ULDB model and one for U-relational databases).

D.1 Algorithm for Possible World Creation

An algorithm for possible world creation is shown in Figure 19. The input of the algorithm is a set of consistent *W-graphs* (*WSet*). Based on these *W-graphs*, a set of possible worlds (denoted as W) is generated (one world for each consistent *W-graph*). For each *W-graph* an initially empty world (I) is defined (Step 2.1). Then, for each of the *W-graph*'s components a tuple is added to the possible world by merging the tuples belonging to the component's nodes (Step 2.2). Finally, the resultant world is added to the set of possible worlds (Step 2.3) and its probability is defined as the probability of the corresponding *W-graph* (Step 2.4).

D.2 Algorithm for Generating X-Relations

A complete algorithm for x-relation generation is shown in Figure 20. The input of the algorithm is W, a set of possible worlds and P a probability distribution over

Input: Set of consistent *W-graphs* $WSet$
1. $W = \emptyset$
2. For each graph $G = (N, E, P) \in WSet$
   - 2.1 $I = \emptyset$
   - 2.3 For each component $G_i = (N_i, E_i)$
     $I = I \cup \{\mu(N_i)\}$
   - 2.4 $W = W \cup \{I\}$
   - 2.5 $P(I) = P$

Output: **Set of possible worlds** $W = \{I_1, I_2, \ldots, I_k\}$, **Probability distribution** $P : W \mapsto [0, 1]$

Fig. 19. Algorithm for possible world creation

these worlds. First, a new indicator tuple is created (Step 1). Second, for each possible world an alternative of the indicator tuple is generated (Step 2.1). Then we iterate over all tuples of the considered world (Step 2.2). If a tuple already belongs to the output x-relation $\mathcal{R}_X$, the lineage and probability of this tuple is adapted. Otherwise, the tuple along with its new lineage and probability is inserted into $\mathcal{R}_X$. Finally (Step 3), prior lineage is taken into account. For merged tuples, we consider the prior lineage generation as a part of the tuple merging step.

Input: Set of possible worlds $W = \{I_1, I_2, \ldots, I_k\}$, **Probability distribution** $P : W \mapsto [0, 1]$
1. Create an indicator tuple $i \in \mathcal{I}_{td}$
2. For each world $I_j \in W$
   - 2.1 Create the alternative i^j with probability $p(i^j) = P(I_j)$
   - 2.2 For each tuple $t \in I_j$
     - If $t \in \mathcal{R}_X$
       - $\lambda(t) = \lambda(t) \vee (i, j)$
       - $p(t) = p(t) + P(I_j)$ *//if probability is not only computed at query time*
     - Else
       - Insert t into $\mathcal{R}_X$
       - $\lambda(t) = (i, j)$
       - $p(t) = P(I_j)$ *//if probability is not only computed at query time*
3. For each tuple $t \in \mathcal{R}_X$
   - 3.1 $\lambda(t) = \lambda'(t) \wedge \lambda(t)$

Output: **Probabilistic X-relations** $\mathcal{R}_X$ **and** $\mathcal{I}_{td}$

Fig. 20. Algorithm for x-relation generation

D.3 Algorithm for Generating a U-relational Database

A complete algorithm for U-relation generation is shown in Figure 21. The input of the algorithm is a set of possible worlds $W = \{I_1, I_2, \ldots, I_k\}$. First, the variable x is created (Step 1). Second, for each possible world I_j the tuple $(x, j, P(I_j))$ is inserted in the world-table $\mathbf{W}$ (Step 2.1). Finally, for each tuple t_i of the considered world I_j, the tuple $(x, j, t_i.TID)$ is inserted into the resultant U-relation $U_{\mathcal{R}[\text{corr.}]}$ (Step 2.2).

Input: Set of possible worlds $W = \{I_1, I_2, \ldots, I_k\}$
1. Create a variable x
2 For each world $I_j \in W$
  2.1 Insert the tuple $(x, j, P(I_j))$ into **W**
  2.2 For each tuple $t \in I_j$
    Insert the tuple $(x, j, t.TID)$ into $U_{\mathcal{R}[\text{corr.}]}$
Output: U-relation $U_{\mathcal{R}[\text{corr.}]}$**, world table W**

Fig. 21. Algorithm for U-relation generation

E. FURTHER SEMI-INDETERMINISTIC APPROACH

In this Section, we introduce a fifth semi-indeterministic approach, called knowledge-based-restriction.

E.1 Knowledge-Based-Restrictions

During an integration process additional information on the given sources can be available. Sometimes, this information can be used to restrict the set of *uncertain decision*. For instance, one or more source relations can be known (no heuristic, because certain knowledge is used) or assumed (heuristic) to be duplicate-free. Thus, w.r.t. these relations instead of intra-source duplicates only inter-source duplicates need to be detected. In this case, tuples originating from same sources do not need to be compared and corresponding matching probabilities can be automatically set to 0. This enormously decreases the number of worlds which have to be considered.

As an example, we consider the two relations $\mathcal{R}_1 = \{t_1, t_2, t_3\}$ and $\mathcal{R}_2 = \{t_4, t_5\}$. Both relations are known to be duplicate-free. For that reason, in the corresponding *M-graph* M_1 (see Figure 22) all edges connecting two nodes representing tuples of the same source are weighted with 0 (for simplification, these edges are removed in Figure 22). Thus, by using the available knowledge the initial *M-graph* can be restricted to a bipartite graph. As a consequence, each *W-graph* is a bipartite graph, too, which is only consistent, if each node is only connected with at most one other node. This in turn reduces the number of considered *W-graphs* enormously.

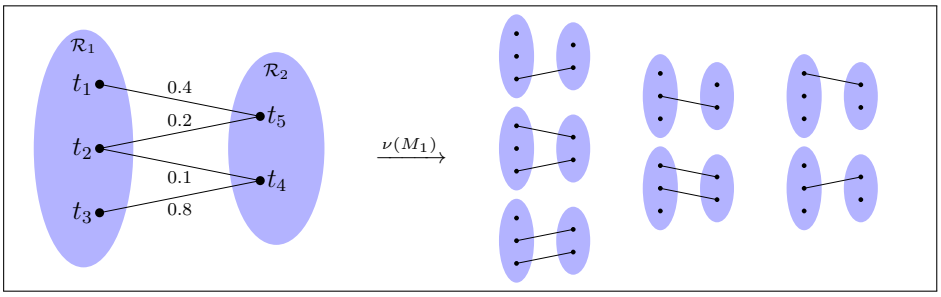


Fig. 22. Bipartite *M-graph* M_1 resulting from a knowledge-based restriction on the two duplicate-free relations $\mathcal{R}_1$ and $\mathcal{R}_2$ (left) and the resultant set of its bipartite *W-graphs* (right)

F. COMPLEXITY

In this section, we formally discuss the complexity of our full-indeterministic approach.

As known from other techniques which are based on the possible world semantics, a full-indeterministic approach has, in theory, a very high complexity. The number of *W-graphs* which can be generated from an *M-graph* with k uncertain edges is:

$$N_{W\text{-}graph}(k) = 2^k$$

A fully connected *M-graph* with n nodes has $|E| = n(n-1)/2$ edges. As a consequence, given a source relation with n tuples, the number of resultant *W-graphs* ($k = |E|$) is at most:

$$N^{\max}_{W\text{-}graph} = N_{W\text{-}graph}(|E|) = 2^{n(n-1)/2}$$

The number of consistent possible worlds resulting from a certain source relation with n tuples, where each duplicate decision is uncertain, is equal to the number of possible partitions of the relation's tuples. Thus, the maximal possible number of resultant possible worlds can be reduced to the complexity of set partitioning and results in:

$$N^{\max}_{PW} = B_n = \frac{1}{e}\sum_{i=1}^{\infty}\frac{i^n}{i!}$$

where B_n is the nth bell number [Rota 1964].

If each edge is uncertain, the resultant x-relation can be mapped to the power set of the source relation's tuples without the empty set. Thus, in the worst case, the number of resultant x-tuples is:

$$N^{\max}_{|\mathcal{R}_X|} = |2^{\mathcal{R}}| - 1 = 2^{|\mathcal{R}|} - 1$$

In order to get an idea of the dramatic complexity scale, we assume a source relation $\mathcal{R}$ with 10 tuples. The number of *W-graphs* which can be generated from the initial *M-graph* is maximal:

$$N^{\max}_{W\text{-}graph} = 2^{45} \simeq 3.5184 \cdot 10^{13}$$

The number of resultant possible worlds and hence the number of indicator alternatives is at most:

$$N^{\max}_{PW} = B_{10} = 115,975$$

Finally, the maximal number of resultant x-tuples is:

$$N^{\max}_{|\mathcal{R}_X|} = 2^{10} - 1 = 1,023 \simeq 100 \cdot |\mathcal{R}|$$

In summary, the complexity of the indeterministic deduplication algorithm, as well as the size of the resultant data increases dramatically with the number of uncertain edges. As a consequence, a full-indeterministic approach is in general only of theoretical value and not usable in practice.

G. EQUIVALENT AND WORKING QUERIES

The queries $Q1$-$Q3$ presented in Section 6 are only of a theoretical nature and do not work on the current Trio prototype (query $Q4$ works as it is). In this Section, we

present sequences of TriQL commands which are working on the current prototype and which lead to same results as the presented ones.

G.1 Working Implementation of Query $Q1$

Query $Q1$ does not work, because it uses the IN-predicate and a vertical subquery. The IN-predicate was realized by a join and the subquery by an auxiliary table *MaxIndAlternative*. Thus, in the first step the table *MaxIndAlternative* was created by Statement $S1$:

$S1$: **CREATE TABLE** MaxIndAlternative
AS SELECT * **FROM** Ind i **WHERE** Conf(i)=[max(Conf(*))];

The final result is produced by Statement $S2$:

$S2$: **SELECT** * **FROM** R_X t, Ind i, MaxIndAlternative mia
WHERE Lineage(t,i) **AND** i.id=mia.id**AND** i.val=mia.val **COMPUTES** Confidences;

G.2 Working Implementation of Query $Q2$

Query $Q2$ does not work for the same reasons as $Q1$ (EXISTS-predicate and vertical subquery). The EXISTS-predicate was realized by a join and the subquery by an auxiliary table *PersonWithNonUniqueName*. Thus, in the first step the table *PersonWithNonUniqueName* was created by Statement $S3$:

$S3$: **CREATE TABLE** PersonWithNonUniqueName
AS SELECT DISTINCT t1.tid **FROM** Person t1, Person t2
WHERE t1.tid!=t2.tid **AND** t1.name=t2.name;

The final result is produced by Statement $S4$:

$S4$: **SELECT** * **FROM** PersonWithNonUniqueName **WHERE** Conf(*)=1.0;

G.3 Working Implementation of Query $Q3$

Query $Q3$ does not work for the same reasons as $Q1$ and $Q2$ (IN-predicate and vertical subquery). The IN-predicate was realized by a join and the subquery by an auxiliary table *LessCertainDecision*. Thus, in the first step the table *LessCertainDecision* was created by Statement $S5$:

$S5$: **CREATE TABLE** LessCertainDecision
AS SELECT id **FROM** Ind i **WHERE** [max(Conf(*))]<0.6;

The final result is produced by Statement $S6$:

$S6$: **SELECT** * **FROM** R_X t, Ind i, LessCertainDecision lcd
WHERE Lineage(t,i) **AND** i.id=lcd.id **ORDER BY** Confidences **ASC**;

Note, NOT IN- and NOT EXISTS-predicates cannot be realized by using theta-joins only, but also require a set minus operator (EXCEPT or MINUS) or an outer join (LEFT OUTER JOIN or RIGHT OUTER JOIN). Outer joins are not implemented in the current Trio Prototype even if they are listed in the TriQL manual. In contrast, a set minus operator is part of the currently implemented TriQL, but did not work correctly. Therefore, using queries based on one of these subquery predicates is more complicated at the moment, but will not pose a problem, when all the query features which are currently listed in the TriQL manual will be implemented in the Trio prototype.

H. EXPERIMENTAL RESULTS

In this section, we present the results of our experiments on efficiency in more detail. All these experiments were performed on machine with an Intel(R) 2.8GHz dual core processor, 4GB main memory, and a 64-bit operating system. We conducted a set of $(\alpha, 1-\alpha)$-restrictions with different threshold settings and the original *W-graph*-generation mapping ν. Moreover, we consider the improvement of efficiency by additionally using a HC-restriction (*W-graph*-generation mapping ν_2) in Section H.4. In all experiments, we use *M-graph* decomposition for making our indeterministic deduplication approach feasible in practice. The setting $\alpha = 0.0165$ was the lowest threshold for which we were able to perform an $(\alpha, 1-\alpha)$-restriction with the *W-graph*-generation mapping ν. In contrast, by using the mapping ν_2 (HC-restriction), we were able to perform an $(\alpha, 1-\alpha)$-restriction up to the threshold setting $\alpha = 0.002$.

H.1 Algorithm Complexity

In Table IV, we present our results on algorithm complexity for $(\alpha, 1-\alpha)$-restrictions with the *W-graph*-generation mapping ν (without HC-restriction). We measure complexity by the number of definite positive edges (matches, i.e. edges weighted with $\gamma = 1$), the number of uncertain edges (possible matches, i.e. edges weighted with $0 < \gamma < 1$), the number of *partial M-graphs* in which the initial *M-graph* could be decomposed, the number of *partial W-graphs* which could be derived from these *partial M-graphs*, the number of consistent *partial W-graphs*, and runtime. As mentioned in Section 7, runtime started from processing the initial *M-graph* and hence did not include tuple matching.

α	**#definite positive edges**	**#uncertain edges**	**#partial *M-graphs***	**#partial *W-graphs***	**#consistent partial *W-graphs***	**runtime [sec.]**
0	9	577004	$\rightarrow \infty$[8]	$\rightarrow \infty$[8]	$\rightarrow \infty$[8]	$\rightarrow \infty$[8]
.0165	9	98	1905	133296	3085	18.519
.0175	9	95	1907	4288	2119	10.549
.02	9	84	1915	2354	2041	8.644
.03	9	60	1935	2122	2015	4.926
.04	9	43	1951	2018	1997	2.581
.05	9	42	1952	2011	1996	2.519
.06	9	41	1953	2011	1996	2.406
.07	10	37	1955	2003	1992	2.232
.08	10	33	1958	1997	1990	2.013
.09	10	33	1958	1997	1990	1.984
.1	10	30	1961	1996	1990	1.845
.2	11	13	1977	1991	1989	1.131
.3	15	4	1982	1986	1986	1.009
.4	16	2	1983	1985	1985	0.969
.5	17	0	1984	1984	1984	0.966

Table IV. Algorithm Complexity of several $(\alpha, 1-\alpha)$-restrictions with decomposition and the *W-graph*-mapping ν

[8]Due to our limited resources, processing a full-indeterministic approach was not feasible.

The results presented in Table IV show that the number of partial *W-graphs* only linearly increased with a growing number of uncertain edges, if this number remained low, but increased exponentially, if this number exceeds 50.

Moreover, in Table V, we present the absolute frequencies of the different sizes of *partial M-graphs* (frequency distribution of partial *M-graph* size) of several $(\alpha, 1-\alpha)$-restrictions. Note, a *M-graph* of size 1 represents a unicum. The largest *M-graph* size we measured in our experiments for restrictions with non-hierarchical clusterings with the settings $\alpha \in [0.0165, 0.5]$ was 13. In contrast, the largest *M-graph* size we measured in our experiments for restrictions with hierarchical clusterings with the settings $\alpha \in [0.002, 0.5]$ was 213. Per definition, these frequencies are independent from the use of a HC-restriction. Nevertheless, additionally using a HC-restriction allowed us to perform experiments also for extreme small values of α ($\alpha < 0.01$) and hence supply us more results on frequency distribution. However, distributions having frequencies up to the size of 213 are hard to present in a single table. For that reason, Table V only shows the results for the indeterministic approach performed with the non-hierarchical *W-graph*-mapping ν.

α	**frequency of partial *M-graph* size**										
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**...**	**13**
.0165	1848	42	8	2	2	1	1	0	0	...	1
.0175	1850	41	8	3	2	1	1	0	1	...	0
.02	1858	42	8	4	2	0	0	1	0	...	0
.03	1886	40	6	1	1	0	1	0	0	...	0
.04	1912	31	7	0	1	0	0	0	0	...	0
.05	1913	31	7	1	0	0	0	0	0	...	0
.06	1915	30	7	1	0	0	0	0	0	...	0
.07	1918	29	8	0	0	0	0	0	0	...	0
.08	1922	30	6	0	0	0	0	0	0	...	0
.09	1922	30	6	0	0	0	0	0	0	...	0
.1	1927	29	5	0	0	0	0	0	0	...	0
.2	1955	21	1	0	0	0	0	0	0	...	0
.3	1965	16	1	0	0	0	0	0	0	...	0
.4	1967	15	1	0	0	0	0	0	0	...	0
.5	1969	14	1	0	0	0	0	0	0	...	0

Table V. Absolute frequencies of different partial *M-graph* sizes of several $(\alpha, 1-\alpha)$-restrictions with decomposition and the *W-graph*-mapping ν

In large restrictions ($\alpha > 0.6$) the initial *M-graph* could be completely decomposed in partial *M-graphs* with one node (unicum) or two nodes respectively. In contrast, the smaller the restrictions, the less the number of *partial M-graphs* the initial *M-graph* could be decomposed into and the greater the resultant partial *M-graphs* became.

H.2 Data Complexity

In Table VI, we present our results on the complexity of the resultant data measured by the number of resultant tuples, the number of indicator alternatives required to model all resultant complementations, and the number of unicums, i.e. tuples which are no duplicates with absolute certainty.

α	#result tuples	#indicator alternatives	#unicums
0	$\to \infty^8$	$\to \infty^8$	12
.0165	5739	1194	1848
.0175	2349	227	1850
.02	2156	140	1858
.03	2091	88	1886
.04	2039	53	1912
.05	2036	51	1913
.06	2035	50	1915
.07	2027	44	1918
.08	2022	37	1922
.09	2022	37	1922
.1	2019	33	1927
.2	2001	12	1955
.3	1990	4	1965
.4	1987	2	1967
.5	1984	0	1969

Table VI. Data (Storage) complexity resulting from several $(\alpha, 1-\alpha)$-restrictions with decomposition and the *W-graph*-mapping ν

From $\alpha = 0.5$ to $\alpha = 0.04$ the increase of resultant tuples and the amount of required indicator alternatives remained low, with or without a HC-restriction. Moreover, the most tuples were unicums. For smaller values of α, storage complexity increased dramatically. For example, for $\alpha = 0.0165$ from 2000 input tuples 5739 output tuples and 1194 indicator alternatives resulted.

H.3 Data (Un)certainty

In Table VII, we present the (un)certainty of the resultant data measured by the number of resultant possible worlds, *Uncertainty Density*, and *Answer Decisiveness*.

α	#possible worlds	Uncertainty Density	Answer Decisiveness
0	$\to \infty^8$	$\to 1^8$	$\to 0^8$
.0165	$3.599 \cdot 10^{23}$	0.0151	0.9915
.0175	$1.045 \cdot 10^{23}$	0.0153	0.9911
.02	$1.205 \cdot 10^{21}$	0.0149	0.9915
.03	$4.696 \cdot 10^{15}$	0.0117	0.9934
.04	$2.378 \cdot 10^{11}$	0.0088	0.9939
.05	$1.698 \cdot 10^{11}$	0.0087	0.9939
.06	$8.493 \cdot 10^{10}$	0.0084	0.9940
.07	$7.644 \cdot 10^{9}$	0.0078	0.9942
.08	$1.019 \cdot 10^{9}$	0.0073	0.9947
.09	$1.019 \cdot 10^{9}$	0.0073	0.9947
.1	$1.698 \cdot 10^{8}$	0.0067	0.9951
.2	4,096	0.0030	0.9973
.3	16	0.0010	0.9989
.4	4	0.0005	0.9994
.5	1	0	1

Table VII. Data (un)certainty resulting from several $(\alpha, 1-\alpha)$-restrictions with decomposition and the *W-graph*-mapping ν

The number of modeled possible world increased exponentially with a growing area of indeterministically handled decisions. However, *Uncertainty Density* as well as *Answer Decisiveness* remained close to the optimal value of complete certainty, even if α was very low.

H.4 Efficiency Improvement by HC-Restrictions

For demonstrating the ability of HC-restrictions to improve efficiency, we conduct the same set of experiments as before, but now with the hierarchical *W-graph*-generation mapping ν_2. The results on algorithm complexity, data complexity and data (un)certainty are presented in Table VIII and Table IX. Note, per definition the number of definite positive edges, the number of uncertain edges and the number of *partial M-graphs* (as well as its frequency distribution) are not affected by the hierarchical clustering and hence remained unchanged.

α	#definite positive edges	#uncertain edges	#partial *M-graphs*	#partial *W-graphs*	#consistent partial *W-graphs*	runtime [sec.]
0	9	577004	$\rightarrow \infty^8$	$\rightarrow \infty^8$	$\rightarrow \infty^8$	$\rightarrow \infty^8$
.002	9	618	1725	1891	1766	497.867
.003	9	359	1779	1923	1828	232.122
.004	9	207	1830	1970	1883	69.568
.005	9	169	1851	1984	1912	46.942
.006	9	161	1857	1984	1918	41.839
.007	9	155	1862	1983	1921	35.488
.008	9	149	1867	1983	1926	31.518
.009	9	148	1868	1983	1927	32.467
.01	9	137	1872	1980	1928	25.104
.0165	9	98	1905	1991	1963	10.886
.0175	9	95	1907	1993	1965	11.143
.02	9	84	1915	1992	1970	8.575
.03	9	60	1935	1991	1980	4.986
.04	9	43	1951	1991	1983	2.620
.05	9	42	1952	1991	1984	2.484
.06	9	41	1953	1991	1984	2.374
.07	10	37	1955	1989	1983	2.267
.08	10	33	1958	1990	1985	1.999
.09	10	33	1958	1990	1985	1.982
.1	10	30	1961	1990	1986	1.874
.2	11	13	1977	1989	1989	1.118
.3	15	4	1982	1986	1986	1.028
.4	16	2	1983	1985	1985	0.937
.5	17	0	1984	1984	1984	0.902

Table VIII. Algorithm Complexity of several $(\alpha, 1-\alpha)$-restrictions with decomposition and the *W-graph*-mapping ν_2

The presented results show that the additional HC-restriction improved efficiency enormously for small $(\alpha, 1-\alpha)$-restrictions, even not in terms of runtime. That is because the decomposition of the initial *M-graph* is in general one of the most time consuming steps, especially if the size of the resultant *partial W-graphs* is relatively small. Since the complexity of decomposition depends on the number of uncertain

edges which is not affected by the HC-restriction, the runtime of an $(\alpha, 1-\alpha)$-restriction combined with a HC-restriction did not decrease noticeably compared to the runtime of an $(\alpha, 1-\alpha)$-restriction without a HC-restriction. The reason that runtime significantly grew further on for small values of α was the enormous sizes of the created *W-graphs* (up to 213).

In contrast, the number of *partial W-graphs* and the number of consistent *partial W-graphs* did not grow exponentially anymore and was always lower than 1994 graphs. An interesting observation is that the number of consistent *partial W-graphs* initially grew from 1984 to 1989, if the area of indeterministically handled decisions was increased from $\alpha = 0.5$ to $\alpha = 0.2$, but then shrank up to 1766 for $\alpha < 0.2$. Instead the number of *partial W-graphs* initially increased from 1984 ($\alpha = 0.5$) to 1993 ($\alpha = 0.0175$), then varied between the values 1980 and 1991 for the settings $\alpha \in (0.005, 0.02)$, and finally shrank noticeably up to 1891 for $\alpha = 0.002$. Data complexity and data uncertainty (Table IX) stayed low even for small values of α. Moreover, instead of dramatically increasing, the number of required indicator alternatives shrank surprisingly from 69 to 45, if α was lower than 0.005.

α	#result tuples	#indicator alternatives	#unicums	#possible worlds	Uncertainty Density	Answer Decisiveness
0	$\to \infty^8$	$\to \infty^8$	≥ 12	$\to \infty^8$	$\to 1^8$	$\to 0^8$
.002	2064	45	1901	$1.359 \cdot 10^{11}$	0.0096	0.9953
.003	2070	53	1883	$2.505 \cdot 10^{14}$	0.0123	0.9948
.004	2062	59	1875	$2.630 \cdot 10^{16}$	0.0135	0.9936
.005	2069	69	1861	$1.818 \cdot 10^{19}$	0.0156	0.9927
.006	2067	68	1863	$9.089 \cdot 10^{18}$	0.0155	0.9931
.007	2063	64	1867	$5.112 \cdot 10^{18}$	0.0154	0.9931
.008	2063	64	1867	$2.272 \cdot 10^{18}$	0.0152	0.9934
.009	2063	64	1867	$2.272 \cdot 10^{18}$	0.0151	0.9934
.01	2060	59	1872	$2.840 \cdot 10^{17}$	0.0143	0.9935
.0165	2057	65	1869	$7.213 \cdot 10^{17}$	0.0143	0.9927
.0175	2055	66	1870	$1.154 \cdot 10^{18}$	0.0144	0.9924
.02	2051	62	1875	$2.886 \cdot 10^{17}$	0.0140	0.9929
.03	2040	50	1895	$1.113 \cdot 10^{14}$	0.0114	0.9940
.04	2023	35	1921	$2.446 \cdot 10^{10}$	0.0085	0.9945
.05	2023	35	1921	$2.446 \cdot 10^{10}$	0.0085	0.9945
.06	2022	34	1923	$1.223 \cdot 10^{10}$	0.0082	0.9946
.07	2018	30	1926	$3.057 \cdot 10^{9}$	0.0077	0.9947
.08	2017	27	1927	$1.019 \cdot 10^{9}$	0.0073	0.9949
.09	2017	27	1927	$1.019 \cdot 10^{9}$	0.0073	0.9949
.1	2015	25	1931	$1.699 \cdot 10^{8}$	0.0067	0.9951
.2	2001	12	1955	4,096	0.0030	0.9973
.3	1990	4	1965	16	0.0010	0.9989
.4	1987	2	1967	4	0.0005	0.9994
.5	1984	0	1969	1	0	1

Table IX. Data (Storage) complexity and data (un)certainty resulting from several $(\alpha, 1-\alpha)$-restrictions with decomposition and the *W-graph*-mapping ν_2

Another difference results in the number of unicums, e.g. 1848 unicums for mapping ν and 1869 unicums for mapping ν_2 for the setting $\alpha = 0.0165$. Whereas,

by using the *W-graph*-generation mapping ν the number of unicums is equivalent to the number of *partial M-graphs* with exactly one node, by using the mapping ν_2 unicums can also result from *partial M-graphs* with more than one node. An example of such a situation is illustrated in Figure 23. The initial *M-graph* M_1 has one definite negative edge $\{t_1, t_2\}$ and two uncertain edges ($\{t_1, t_3\}$ and $\{t_2, t_3\}$). From applying the mapping ν_2 the three *W-graphs* G_1, G_2 and G_3 result. Only two of these *W-graphs*, namely G_1 and G_2, are consistent. Since t_1 is not connected with any other tuple in one of these two consistent *W-graphs*, t_1 is a unicum, despite of the fact that it is involved in an *M-graph* with more than one node. In conclusion, the number of unicums resulting from an $(\alpha, 1-\alpha)$-restriction is always equal or greater by using an additional HC-restriction than by using without.

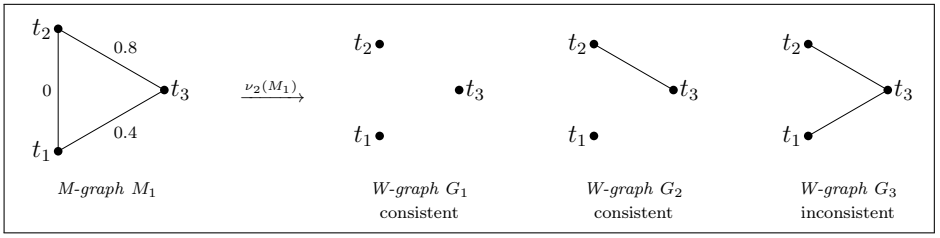


Fig. 23. The initial *M-graph* M_1, its the three *W-graphs* G_1 to G_3 and the unicum t_1.

A further interesting aspect is that the number of resultant possible worlds was dramatically increasing (up to $3.6 \cdot 10^{23}$) for a shrinking value of α in our non-hierarchical approach (*W-graph*-generation mapping ν), but was always lower than $2 \cdot 10^{19}$ by using a HC-restriction. Moreover, by additionally using a HC-restriction, the number of possible worlds decreased for settings $\alpha < 0.005$.

To illustrate the improvement of efficiency further on, we compare the number of generated *partial W-graphs*, the number of generated consistent *partial W-graphs*, the number of resultant tuples, the number of resultant indicator alternatives, the number of resultant unicums, and the number of resultant possible worlds with and without using an additional HC-restriction in Figure 24-29.

All six figures show that the complexity (algorithm complexity in terms of number of *partial W-graphs* and number of consistent *partial W-graphs*, and data (storage) complexity in terms of number of resultant tuples and number of resultant indicator alternatives) as well as data uncertainty (in term of number of possible worlds) dramatically increased for $\alpha \leq 0.03$, if the non-hierarchical *W-graph*-generation mapping ν was used. In contrast, by using the hierarchical *W-graph*-generation mapping ν_2 the results remained stable, even for small settings of α. Moreover, as already discussed above, the number of unicums resulting from an $(\alpha, 1-\alpha)$-restriction combined with a HC-restriction was always higher than 1980. In contrast, for small settings of α the number of unicums became less than 1700 (and theoretically only 12 for a full-indeterministic approach), if no HC-restriction was used.

In summary, the benefits of the HC-restriction were that $(\alpha, 1-\alpha)$-restrictions became computable even for small values of α, because of the reduced number of *partial M-graphs*, and that the data (storage) complexity was manageable even for large areas of indeterministically handled decisions ($\alpha < 0.01$).

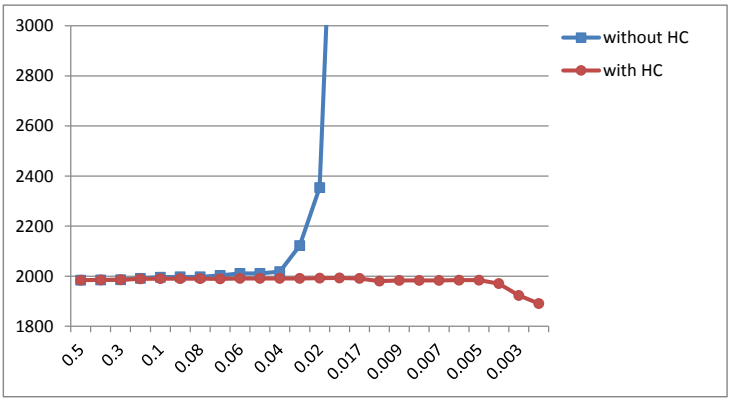


Fig. 24. Number of *partial W-graphs* for $\alpha \in (0.5 - 0.002)$ with and without using a HC-restriction

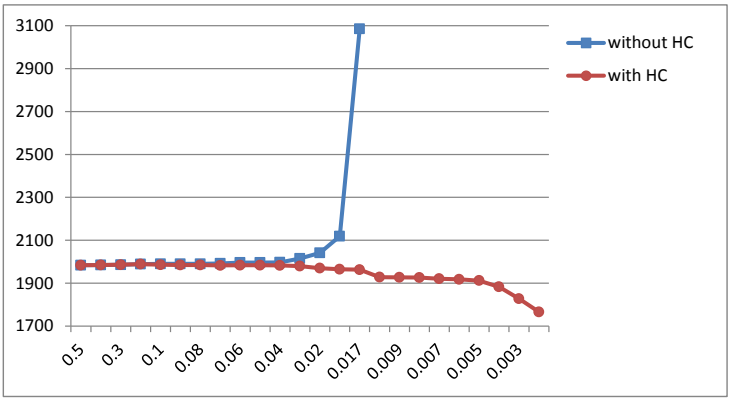


Fig. 25. Number of consistent *partial W-graphs* for $\alpha \in (0.5 - 0.002)$ with and without using a HC-restriction

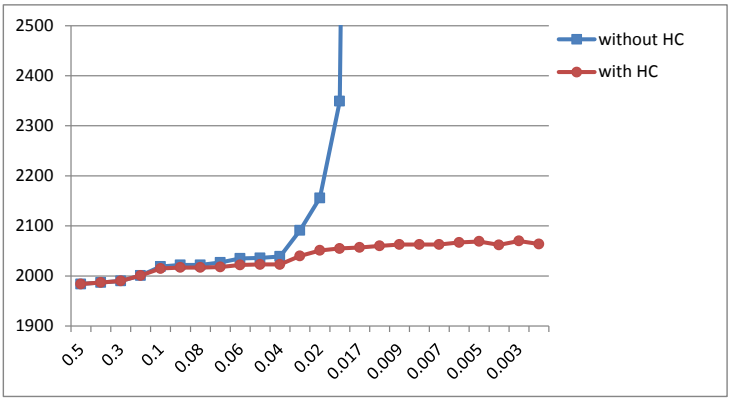


Fig. 26. Number of resultant tuples for $\alpha \in (0.5 - 0.002)$ with and without using a HC-restriction

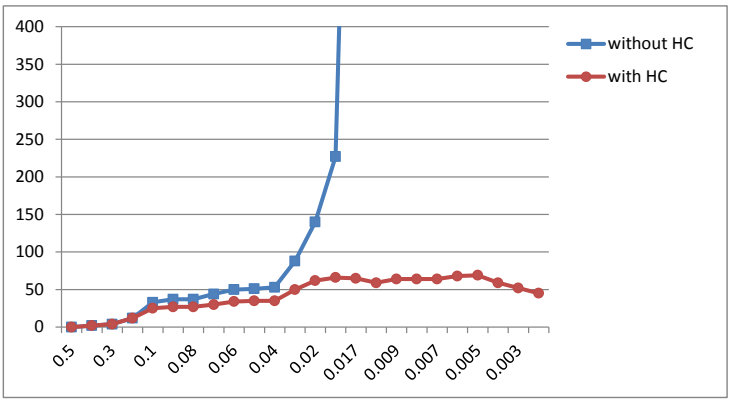


Fig. 27. Number of indicator alternatives for $\alpha \in (0.5 - 0.002)$ with and without using a HC-restriction

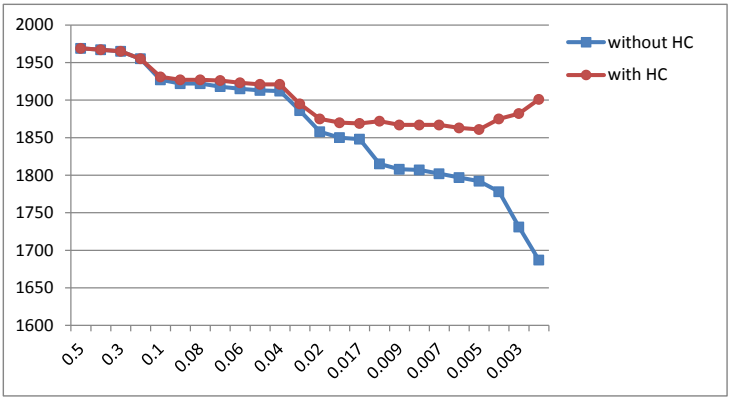


Fig. 28. Number of resultant unicums for $\alpha \in (0.5-0.002)$ with and without using a HC-restriction

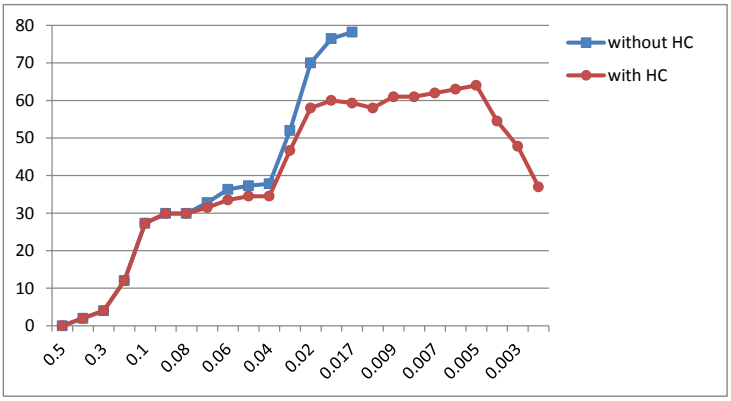


Fig. 29. Number of possible worlds (measured in lb(#possible worlds)) for $\alpha \in (0.5 - 0.002)$ with and without using a HC-restriction