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Information Processing Letters 86 (2003) 137–141

Information Processing Letters

www.elsevier.com/locate/ipl

How to detect a counterfeit coin: Adaptive versus non-adaptive solutions

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Received 20 August 2002; received in revised form 9 November 2002

Communicated by S. Albers

Abstract

In an old weighing puzzle, there are $n \geqslant 3$ coins that are identical in appearance. All the coins except one have the same weight, and that counterfeit one is a little bit lighter or heavier than the others, though it is not known in which direction. What is the smallest number of weighings needed to identify the counterfeit coin and to determine its type, using balance scales without measuring weights? This question was fully answered in 1946 by Dyson [The Mathematical Gazette 30 (1946) 231–234]. For values of n that are divisible by three, Dyson's scheme is non-adaptive and hence its later weighings do not depend on the outcomes of its earlier weighings. For values of n that are not divisible by three, however, Dyson's scheme is adaptive. In this note, we show that for all values $n \geqslant 3$ there exists an optimal weighing scheme that is non-adaptive.

Keywords: Non-adaptive algorithm; Algorithms; Weighing problem; Mathematical puzzle

1. Introduction

An old and well-known mathematical puzzle is the problem of the twelve coins: There are twelve coins that are exactly alike except for a counterfeit one, which weighs a bit more or a bit less than the others. The goal is to identify the counterfeit coin in three weighings with a balance scale (and no measuring weights), and to determine whether the counterfeit coin is underweight or overweight.

The standard solution in puzzle books (and in puzzle oriented news-groups on the web) first divides the coins into three groups $G_1 = \{a, b, c, d\}$, $G_2 = \{e, f, g, h\}$, and $G_3 = \{i, j, k, \ell\}$. In the first weighing, group G_2 is put on the left pan of the scales and G_3 is put on the right pan. Then there are three cases corresponding to the three possible outcomes of the first weighing: In case the pans balance, the eight coins in $G_2 \cup G_3$ are all genuine and the counterfeit coin is in $G_1 = \{a, b, c, d\}$. Then in the second weighing, one

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0020-0190/03/$ – see front matter
doi:10.1016/S0020-0190(02)00483-0

puts coins a, b, c on one pan and three (genuine!) coins from $G_2 \cup G_3$ on the other pan.

- If the pans balance, then coin d is counterfeit. The third weighing compares d against a genuine coin, and thus determines whether it is underweight or overweight.
- If the pan with a, b, c is heavier, then the counterfeit coin is one of a, b, c and it is overweight. The third weighing compares a to b. If the pans do not balance, then the heavier coin is counterfeit, and if the pans do balance, then c is counterfeit.
- If the pan with a, b, c is lighter, then the counterfeit coin is one of a, b, c and it is underweight. The third weighing compares a to b. If the pans do not balance, then the lighter coin is counterfeit, and if the pans do balance, then c is counterfeit.

In case the pans do not balance at the first weighing, then in the second weighing one puts coins e, f, i on one pan and coins g, h, j on the other pan. Then some additional case distinctions for the third weighing complete the solution; the tedious details are left to the reader. Clearly, the above approach is an *adaptive* approach: The second weighing depends on the outcome of the first weighing, and the third weighing depends on the outcomes of the first and second weighing.

Now let us discuss another approach where the later weighings do *not* depend on the outcomes of the earlier weighings; such an approach is called *non-adaptive*. The three weighings are as follows:

1st weighing: a, d, i, j versus b, e, g, k
2nd weighing: a, f, g, ℓ versus b, d, h, j
3rd weighing: c, d, g, k versus a, e, h, ℓ

If in the first two weighings the left pan is heavier and in the third weighing the right pan is heavier, then coin a is counterfeit and overweight. If in the first two weighings the left pan is heavier and in the third weighing the pans balance, then coin b is counterfeit and underweight. And so on, and so on, and so on. It can be verified that in all possible cases, the outcomes of the three weighings allow to uniquely identify the counterfeit coin and its type. The exact mechanism behind this procedure will become clear in Section 2 of this paper.

In a more general version of this weighing problem, there are $n \geqslant 3$ coins that are identical in appearance. All the coins except one have the same weight. What is the smallest number of weighings needed to identify the counterfeit coin and to determine whether it is overweight or underweight? Note that this question does not make sense for $n = 1$ and $n = 2$; in these two cases, a weighing does not provide us with any non-trivial information. Note furthermore that a solution for n coins in w weighings does not immediately imply a solution for $n - 1$ coins in w weighings (consider for instance the case with $n = 3$). A simple information theoretic argument shows that with w weighings, one can not solve the case with $\frac{1}{2}(3^w - 1)$ coins. In fact, the information theoretic bound is tight:

Theorem 1 [1]. *There exists a scheme that determines the counterfeit coin and its type out of n coins with w weighings on balance scales without measuring weights, if and only if* $3 \leqslant n \leqslant \frac{1}{2}(3^w - 3)$.

Dyson's scheme is non-adaptive, if n is a multiple of three. Dyson's scheme is adaptive, if n is not a multiple of three. In Section 2 we will prove the following slight strengthening of Dyson's result to non-adaptive schemes. Its proof is extremely simple and suitable for classroom use.

Theorem 2. *For any integer $w \geqslant 2$ and for any integer n with $3 \leqslant n \leqslant \frac{1}{2}(3^w - 3)$, there exists a* non-adaptive *scheme that determines the counterfeit coin and its type out of $n \geqslant 3$ coins with w weighings on balance scales without measuring weights.*

Our proof is fairly close to the original arguments of Dyson. The main contribution of this note is a clean presentation of the mathematical background in Section 2 that helps to remove the adaptive parts from Dyson's approach. In Section 3, we present a straightforward argument that a non-adaptive weighing scheme cannot determine the counterfeit coin out of $n = \frac{1}{2}(3^w - 1)$ coins with w weighings. This negative result is a special case of Dyson's more general negative result for *adaptive* weighing schemes, but our proof is short and easily comes out of the discussion in Section 2. For completeness, we also sketch Dyson's weighing scheme in Section 4. For more information on weighing problems and some of their variants, we refer the reader (for instance) to [2,3].

2. The proof

In this section we will prove Theorem 2. There is a natural correspondence between non-adaptive weighing schemes with w weighings and certain sets of w-dimensional vectors over the set $\{-1, 0, 1\}$ that we will call *Dyson sets*.

Definition 3. For $w \geqslant 2$, a set S of (pairwise distinct) vectors in $\{-1, 0, 1\}^w$ is called a Dyson set, if and only if

(D1) $\sum_{v \in S} v = 0$, and
(D2) whenever $v \in S$, then $-v \notin S$.

The following Lemmas 4 and 5 together constitute the proof of Theorem 2. Lemma 4 establishes the exact connection between Dyson sets and non-adaptive weighing schemes, and Lemma 5 proves that Dyson sets indeed exist for $3 \leqslant n \leqslant \frac{1}{2}(3^w - 3)$.

Lemma 4. *Let $w \geqslant 2$ be an integer. If there exists a Dyson set $S \subseteq \{-1, 0, 1\}^w$, then there exists a non-adaptive weighing scheme that determines the counterfeit coin and its type out of $n = |S|$ coins with w weighings on balance scales without measuring weights.*

Proof. Let $v_1, \ldots, v_n$ be an enumeration of the vectors in S. Consider the following non-adaptive weighing scheme with w weighings: In the ith weighing ($1 \leqslant i \leqslant w$), coin j is put on the left pan if the ith coordinate of vector v_j equals -1, it is put on the right pan if the ith coordinate of vector v_j equals 1, and it does not participate in the weighing if the ith coordinate of vector v_j equals 0.

If this scheme is applied to weigh n coins, the outcomes of the w weighings are collected in a w-dimensional vector $z \in \{-1, 0, 1\}^w$ in the following way: The ith coordinate z_i of z is set to -1 if in the ith weighing the left pan is heavier, z_i is set to 1 if the right pan is heavier, and z_i is set to 0 if the pans balance.

Now let us consider the case where coin j is counterfeit and overweight: By condition (D1), every weighing puts the same number of coins on the left and on the right balance. Therefore, whenever coin j is on the left pan, the left pan is heavier; whenever coin j is on the right pan, the right pan is heavier; whenever coin j is on neither pan, the two pans balance. This yields $z = v_j$. A symmetric argument shows that the case where coin k is counterfeit and underweight leads to $z = -v_k$. By condition (D2), one can distinguish the cases $z = v_j$ and $z = -v_k$ from each other and thus identify the counterfeit coin. □

Lemma 5. *For any $w \geqslant 2$ and n with $3 \leqslant n \leqslant \frac{1}{2}(3^w - 3)$, there exists a Dyson set $S \subseteq \{-1, 0, 1\}^w$ of cardinality n.*

Proof. Consider the bijection $f : \{-1, 0, 1\} \to \{-1, 0, 1\}$ with $f(0) = 1$, $f(1) = -1$, and $f(-1) = 0$. We extend f to the vectors $x = (x_1, \ldots, x_w)$ in $\{-1, 0, 1\}^w$ by defining $f(x) = (f(x_1), \ldots, f(x_w))$. Note that for any $x \in \{-1, 0, 1\}^w$ we have $f(f(f(x))) = x$ and

$$x + f(x) + f\big(f(x)\big) = 0. \tag{1}$$

Moreover, we observe that $-f(x) = f(f(-x))$ and $-f(f(x)) = f(-x)$. Next, we consider for an arbitrary $x \in \{-1, 0, 1\}^w$ the six vectors

$$x,\ f(x),\ f\big(f(x)\big),\ -x,\ -f(x),\ -f\big(f(x)\big). \tag{2}$$

The so-called *trivial* class consists of the all-zero vector, the all-one vector, and the all-minus-one vector. If x is from the trivial class, then the group of six vectors displayed in (2) boils down exactly to the trivial class. If x is not from the trivial class, then the six vectors in (2) are pairwise distinct and form a so-called *ordinary* class. This naturally yields a partition of the set $\{-1, 0, 1\}^w$ into $\frac{1}{6}(3^w - 3)$ ordinary classes and into the trivial class. Note that every class is closed under f.

Now let us prove the statement in the lemma. We distinguish three cases. In the first case $n = 3k$ holds with $1 \leqslant k \leqslant \frac{1}{6}(3^w - 3)$. We simply construct S by selecting k vector triples x, $f(x)$, $f(f(x))$ from k distinct ordinary classes. By (1) the sum of these $3k$ vectors equals 0, and hence S satisfies condition (D1). Moreover, from our discussion of (2) we see that for any selected vector triple the negative vectors $-x$,

$-f(x)$, $-f(f(x))$ are not selected. Hence, S also satisfies condition (D2).

In the second case $n = 3k + 1$ holds with $1 \leqslant k < \frac{1}{6}(3^w - 3)$. If $n = 4$, we let S contain the following four vectors (here a $+$ stands for $+1$ and a $-$ stands for -1):

$$\begin{array}{lccccccc}
v_1: & + & + & + & + & + & \cdots & + \\
v_2: & - & 0 & 0 & 0 & 0 & \cdots & 0 \\
v_3: & 0 & - & 0 & 0 & 0 & \cdots & 0 \\
v_4: & 0 & 0 & - & - & - & \cdots & -
\end{array}$$

The first vector v_1 comes from the trivial class, whereas v_2, v_3, v_4 come from three different ordinary classes. Clearly, S is a Dyson set. If $n \geqslant 7$, then we start by putting the following seven vectors into S:

$$\begin{array}{lccccccc}
v_1: & + & + & + & + & + & \cdots & + \\
v_2: & - & - & + & + & + & \cdots & + \\
v_3: & - & - & 0 & 0 & 0 & \cdots & 0 \\
v_4: & 0 & 0 & - & - & - & \cdots & - \\
v_5: & - & + & 0 & 0 & 0 & \cdots & 0 \\
v_6: & + & 0 & - & - & - & \cdots & - \\
v_7: & + & 0 & 0 & 0 & 0 & \cdots & 0
\end{array}$$

The first vector v_1 comes from the trivial class. Vectors v_2, v_3, v_4 all come from the same ordinary class, since $v_3 = -f(v_2)$ and $v_4 = f(f(v_2))$. Vectors v_5 and v_6 both come from the same ordinary class, since $v_6 = f(v_5)$, and vector v_7 comes from yet another ordinary class. Moreover, these seven vectors add up to 0. We complete the set S by selecting $k - 2$ vector triples x, $f(x)$, $f(f(x))$ from the $\frac{1}{6}(3^w - 3) - 3 \geqslant k - 2$ remaining ordinary classes. The resulting set S of $n = 3k + 1$ vectors satisfies conditions (D1) and (D2) in Definition 3, and thus is the desired Dyson set.

Finally, we consider the case $n = 3k + 2$ with $1 \leqslant k < \frac{1}{6}(3^w - 3)$. We start by putting the following five vectors into S:

$$\begin{array}{lccccccc}
v_1: & + & + & + & + & + & \cdots & + \\
v_2: & + & - & 0 & 0 & 0 & \cdots & 0 \\
v_3: & 0 & - & + & + & + & \cdots & + \\
v_4: & - & + & - & - & - & \cdots & - \\
v_5: & - & 0 & - & - & - & \cdots & -
\end{array}$$

The first vector v_1 comes from the trivial class. Vectors v_2 and v_3 both come from the same ordinary class, since $v_3 = -f(v_2)$, and vectors v_4 and v_5 both come from the same ordinary class, since $v_5 = -f(v_4)$. Moreover, these five vectors add up to 0. We complete the set S by selecting $k - 1$ vector triples x, $f(x)$, $f(f(x))$ from the $\frac{1}{6}(3^w - 3) - 2 \geqslant k - 1$ remaining ordinary classes. It is easily verified that the resulting set S of $n = 3k + 2$ vectors satisfies conditions (D1) and (D2). □

3. A lower bound argument

In this section we consider non-adaptive weighing schemes for $n = \frac{1}{2}(3^w - 1)$ coins, and we will show that for this case w weighings are not enough. (This statement immediately follows from the more general negative result of Dyson for adaptive weighing schemes. However, our proof is extremely short and nicely falls out of the concepts in Section 2.)

For the sake of contradiction let us suppose that there exists a non-adaptive weighing scheme with w weighings for $n = \frac{1}{2}(3^w - 1)$ coins. Let $v_1, \ldots, v_n \in \{-1, 0, 1\}^w$ be the corresponding Dyson set S for this weighing scheme. By condition (D2), the set S cannot contain the all-zero vector, and for any other vector $v \in \{-1, 0, 1\}^w$ exactly one of v and $-v$ must be in S. Therefore, S contains exactly $\frac{1}{2}(3^{w-1} - 1)$ vectors that have a 0 in the first coordinate and exactly 3^{w-1} vectors that have a 1 or -1 in the first coordinate. But then the first coordinate of $\sum_{v \in S} v$ cannot be 0 and hence condition (D1) is violated.

Observation 6. *Let $w \geqslant 2$ be an integer. Then there does not exist a non-adaptive weighing scheme that determines the counterfeit coin and its type out of $n = \frac{1}{2}(3^w - 1)$ coins with w weighings on balance scales without measuring weights.*

4. A sketch of Dyson's argument

Dyson [1] uses strings of length w over the alphabet $\{0, 1, 2\}$ instead of vectors in $\{-1, 0, 1\}^w$. Since these concepts clearly are isomorphic to each other, we will present Dyson's approach in the language of vectors; moreover, we will use the concepts introduced in the proof of Lemma 5. Dyson proceeds in two steps. In the first step he solves the case with $n = \frac{1}{2}(3^w - 3)$, and in the second step he solves the case with $3 \leqslant n < \frac{1}{2}(3^w - 3)$.

For the case with $n = \frac{1}{2}(3^w - 3)$, Dyson ignores the trivial class and scans each of the remaining vectors from left to right until he hits the first pair of unequal entries. He selects a vector (as a coin label) if and only if this pair equals $(1, 0)$ or $(0, -1)$ or $(-1, 1)$. In the language of our paper, he selects a vector triple x, $f(x)$, $f(f(x))$ from each of the $\frac{1}{6}(3^w - 3)$ ordinary classes and thus gets a Dyson set.

For the case with $3 \leqslant n < \frac{1}{2}(3^w - 3)$, Dyson distinguishes the ordinary class (hereinafter referred to as: the distinguished class) that contains the three vectors

$$\begin{array}{lcccccc} v_1: & + & + & \cdots & + & + & 0 \\ v_2: & 0 & 0 & \cdots & 0 & 0 & - \\ v_3: & - & - & \cdots & - & - & + \end{array}$$

If $n = 3k$, then Dyson uses vector triples x, $f(x)$, $f(f(x))$ from the non-distinguished ordinary classes as described above. If $n = 3k + 1$ then he furthermore uses the vector v_2 from the distinguished class, and if $n = 3k + 2$ then he furthermore uses the two vectors v_1 and v_3 from the distinguished class. The resulting set of vectors satisfies condition (D2) in Definition 3. However, it does not satisfy condition (D1), since the sum of these vectors is non-zero in the last coordinate.

Now the first $w - 1$ weighings in Dyson's scheme are done non-adaptively. If these $w - 1$ weighings do not all yield the same outcome (all $+1$, all -1, or all 0), then the (one or two) coins that belong to vectors from the distinguished group must be genuine. Hence, they can be disregarded for the last weighing. If the first $w - 1$ weighings all yield the same outcome, then the counterfeit coin must be among the (one or two) coins that belong to vectors from the distinguished group. The last weighing can be used to determine the type of this counterfeit coin.

The main difference between Dyson's approach and our approach in Section 2 is that Dyson does not use vectors from the trivial class, whereas we do use them. Without the trivial class, there is no chance for getting Dyson sets when n is not a multiple of three.

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