Ultimate State of Thermal Convection

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The ultimate regime of thermal convection, the so-called Kraichnan regime [R. H. Kraichnan, Phys. Fluids **5**, 1374 (1962)], hitherto has been elusive. Here numerical evidence for that regime is presented by performing simulations of the bulk of turbulence only, eliminating the thermal and kinetic boundary layers and replacing them with periodic boundary conditions.

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Thermally driven turbulence is of tremendous importance in oceanography, geophysics, meteorology, astrophysics, or process technology. If the flow is confined to a box heated from below and cooled from above, the setup is called Rayleigh-Bénard (RB) convection. It is one of the classical problems in fluid dynamics [1,2]. The control parameters are the dimensionless temperature difference between bottom and top (called Rayleigh number Ra), the ratio between kinematic viscosity ν and heat diffusivity κ (called Prandtl number Pr), and the aspect ratio of the cell. The system answers with some heat transfer from the bottom to the top (called Nusselt number Nu in dimensionless form) and the degree of turbulence in the cell (expressed in terms of the Reynolds number Re).

All classical experiments on this system show (effective) power law dependences with Nu $\sim$ Ra$^{0.25}$ up to Nu $\sim$ Ra$^{0.33}$. However, for very large Ra, Kraichnan [3] has predicted an ultimate scaling law Nu $\sim$ Ra$^{1/2}$. In this ultimate state the heat flux and the turbulent velocity are expected to be independent of the viscosity and heat diffusivity. Indeed, for a system with infinite aspect ratio a mathematically strict upper bound Nu ≤ 0.167Ra$^{1/2} - 1$ could be found [4].

Hitherto, this ultimate state of thermal convection remained elusive [5], in spite of tremendous efforts to find it. Niemela *et al.* [6] measured up to Ra $\approx 10^{18}$ in helium gas close to the critical point (Pr ≈ 1), still finding an effective power law Nu $\sim$ Ra$^{0.31}$. Glazier *et al.* [7] measured up to Ra $\approx 10^{11}$ in mercury (Pr $= 0.025$), finding Nu $\sim$ Ra$^{0.28}$. The only evidence for a transition (at Ra $\approx 10^{11}$) towards the ultimate regime has been claimed by Chavanne *et al.* [8,9], but Niemela and Sreenivasan [10] argue that those data would be consistent with Nu $\sim$ Ra$^{1/3}$. In summary, it is therefore not clear (i) whether in today's experiments the Rayleigh number is only too small for the ultimate state, or (ii) whether it does not exist at all.

However, it is crucial to know which of these two alternatives is the correct one: (i) for practical reasons, in view of the above-mentioned applications of thermal

convection in geophysics, industry, etc., as upscaling from lab-scale models does not work if there is a transition to a new state of turbulence; (ii) for fundamental reasons, as is a "central dogma in turbulence" [5] that its effects become independent of viscosity and diffusivity for large enough Reynolds (or here Rayleigh) numbers [11]. Indeed, also within Grossmann and Lohse's unifying theory on thermal convection [12,13] the Kraichnan or ultimate regime follows, once the thermal and kinetic boundary layers either break down due to their expected instability at large Ra or hardly contribute to the global kinetic and thermal dissipation [14]. According to Ref. [13] the breakdown of the laminar kinetic boundary layer should happen at Ra $\approx 10^{14}$ for Pr $= 1$ and at Ra $\approx 10^{12}$ for Pr $= 0.025$ for an aspect ratio $\Gamma = 1$ cell; Niemela and Sreenivasan [10] find similar values. Then the total kinetic dissipation rate ϵ_u equals the bulk kinetic dissipation $\epsilon_{u,\text{bulk}}$ and the total thermal dissipation rate ϵ_θ equals the bulk thermal dissipation $\epsilon_{\theta,\text{bulk}}$ and the Kraichnan or ultimate regime with

$$\text{Nu} \sim \text{Ra}^{1/2}\text{Pr}^{1/2} \tag{1}$$

and

$$\text{Re} \sim \text{Ra}^{1/2}\text{Pr}^{-1/2} \tag{2}$$

results [12].

From the above it follows that an artificial destruction of the boundary layers should enhance the ultimate regime. Indeed, by performing Rayleigh-Bénard convection experiments in a cell with rough top and bottom walls, Roche *et al.* [15] find evidence for the onset of the Kraichnan regime.

Here we follow another approach: We perform numerical simulation *of the bulk of turbulence only,* eliminating the thermal and kinetic boundary layers and replacing them with periodic boundary conditions [16]. Then the Kraichnan regime should follow immediately. The thermal bulk turbulence is forced by a mean temperature profile $-z\Delta/L$, where L is the periodicity length and Δ the temperature difference on that length scale. The temperature fluctuations θ then obey the advection equation

$$\partial_t \theta + u_i \partial_i \theta = \kappa \partial_i \partial_i \theta + u_3 \frac{\Delta}{L}. \qquad (3)$$

The velocity field $u_i(\mathbf{x}, t)$ obeys the standard Boussinesq equation,

$$\partial_t u_i + u_j \partial_j u_i = -\partial_i p + \nu \partial_j^2 u_i + \beta g \delta_{i3} \theta. \qquad (4)$$

β is the thermal expansion coefficient, g gravity, and p the pressure. Ra and Pr are defined as usual as $\mathrm{Ra} = \beta g \Delta L^3/(\nu\kappa)$ and $\mathrm{Pr} = \nu/\kappa$, respectively. The Prandtl number is 1 in our simulations presented here. Also Nu is defined as usual as $\mathrm{Nu} = \langle u_3 \theta \rangle L/(\kappa\Delta) - \partial_3 \langle\theta\rangle_A L/\Delta$.

The Reynolds number is defined with the rms mean velocity fluctuation u', i.e., $\mathrm{Re} = u'L/\nu$. A similar simulation has earlier been performed by Borue and Orszag [17]. The focus of that work was on scaling properties and spectra, and thus hyperviscosity has been employed.

From Eqs. (3) and (4) one can derive by volume averaging and assuming statistically stationarity the exact [18] relations $\epsilon_\theta = \kappa L^{-2}\Delta^2 \mathrm{Nu}$ and $\epsilon_u = \nu^3 L^{-4}\mathrm{NuRaPr}^{-2}$. As in Refs. [12] we estimate $\epsilon_{u,\mathrm{bulk}} \sim u'^3/L$ and $\epsilon_{\theta,\mathrm{bulk}} \sim u'\Delta^2/L$, and the Kraichnan regime (1) and (2) results. Note that L is the relevant length scale for these estimates as for the Bolgiano length scale it holds $l_B \gtrsim L$.

This Kraichnan scaling is indeed seen in our numerical simulations of Eqs. (3) and (4). We run four different Ra numbers $\mathrm{Ra} = 8.64 \times 10^5$ (for 160 large eddy turnovers), $\mathrm{Ra} = 2.21 \times 10^6$ (for 60), $\mathrm{Ra} = 4.51 \times 10^6$ (for 180), and $\mathrm{Ra} = 1.38 \times 10^7$ (for 334) and average over space and time to obtain Nu and Re [19]. The Ra scaling is consistent with Eqs. (1) and (2); see Fig. 1. To our knowledge it is the first realization of the ultimate regime of thermal convection. It is remarkable that it is realized in spite of the relatively low Ra numbers of the simulations. The reason is that the boundary layers have been eliminated and the simulations focus on the bulk. Because of the lack of boundary layers and the more efficient driving, the Reynolds numbers achieved here are much larger than they would be in the standard Rayleigh-Bénard case. Vice versa, the required Rayleigh numbers to achieve the same degree of turbulence for the standard Rayleigh-Bénard case as here are much higher. For example, to achieve $\mathrm{Re} = 4996$ (here for $\mathrm{Ra} = 1.38 \times 10^7$) one needs a Rayleigh number of $\mathrm{Ra} = 7.2 \times 10^8$ in the standard case [20].

In conclusion, from the above alternatives the first one is favorable: Once Ra is so large that the laminar kinetic boundary layer breaks down, the ultimate regime of thermal convection, which has been so elusive in experiment, should finally show up. Obviously, the ultimate proof for the existence of this regime can come only from experiment.
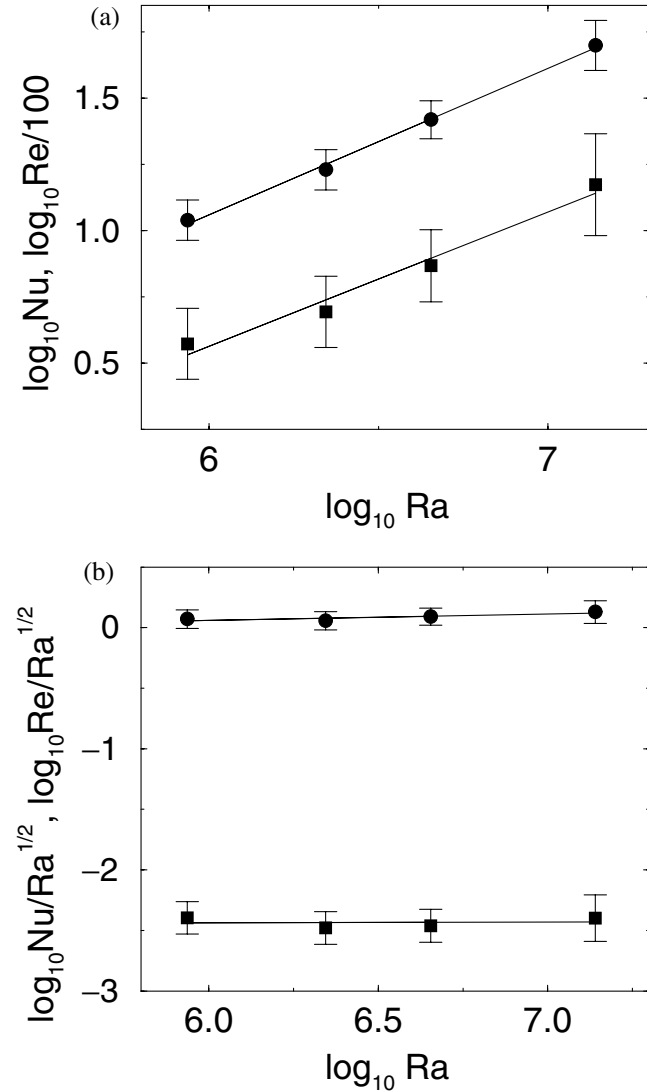
FIG. 1. (a) Nu (squares, lower curve) and Re (circles, upper curve) as a function of Ra from our simulation. The error bars are based on the variances of measured quantities which have been obtained by splitting the total time series into smaller pieces. Note that in this way we could also check that our simulations are statistically stationary. The straight lines are linear regressions giving $\mathrm{Nu} \sim \mathrm{Ra}^{0.51\pm0.06}$ and $\mathrm{Re} \sim \mathrm{Ra}^{0.55\pm0.02}$. (b) Same as in (a), but now compensated by $\mathrm{Ra}^{1/2}$.

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[18] Note that for the standard RB case the corresponding second relation reads $\epsilon_u = \nu^3 L^{-4}(\mathrm{Nu} - 1)\mathrm{RaPr}^{-2}$.

[19] These relatively long averaging times are required to achieve statistical stationarity. As already pointed out in Ref. [17], they have to be longer than in the standard Rayleigh-Bénard case, due to the huge fluctuations of large scale quantities. These resemble the ones found for homogeneous shear flow [see, e.g., A. Pumir and B. I. Shraiman, Phys. Rev. Lett. **75**, 3114 (1995)], to which our system indeed is comparable in some sense.

[20] Here we used the experimental finding for the standard case, $\mathrm{Re} = 0.186\mathrm{Ra}^{1/2}$, given, e.g., in J. J. Niemela, L. Skrbek, K. R. Sreenivasan, and R. J. Donnelly, J. Fluid Mech. **449**, 169 (2001).