PERGAMON

Automatica 38 (2002) 2085–2092

automatica

www.elsevier.com/locate/automatica

Brief Paper

# Some applications of randomized algorithms for control system design[☆]

Vijay V. Patel[a], Girish Deodhare[a,*], T. Viswanath[b]

[a] Centre for Artificial Intelligence and Robotics, Raj Bhavan Circle, High Grounds, Bangalore 560 001, India
[b] Faculty of Mathematical Sciences, University of Twente, P.O. Box 217, 7500 AE, Enschede, Netherlands

Received 16 March 2000; received in revised form 10 April 2002; accepted 10 June 2002

**Abstract**

In this paper a few "difficult" problems related to simultaneous stabilization of three plants (equivalent to a certain problem related to unit interpolation in $H_\infty$) have been addressed through the framework of randomized algorithms. These problems which were proposed by Blondel (Simultaneous Stabilization of Linear Systems, Springer, Berlin, 1994) and Blondel and Gevers (Math. Control Signals Systems 6 (1994) 135) concern the existence of a controller.


*Keywords:* Simultaneous stabilization of three plants; Unit interpolation in $H_\infty$; Randomized algorithms

## 1. Introduction

It is known that several problems in control system design are either NP-complete or NP-hard, see Blondel and Tsitsiklis (2000). For example, the simultaneous stabilization of three plants is a difficult problem. This problem stems from the practical consideration of stabilizing a plant with a controller not only under normal operating conditions but under plant sensor/actuator failures as well. Thus, one needs to stabilize more than one plant with the same controller. From a practical point of view it is also desirable that the controller is stable. It can be shown that simultaneous stabilization of $N$ plants with an arbitrary controller is equivalent to simultaneous stabilization of $N-1$ plants with a *stable* controller (Vidyasagar, 1985). As an illustration of the difficulty of solving this problem a numerical design problem was posed in Blondel, Gevers, Mortini, and Rupp (1994) and a bottle of good French champagne was offered for its solution (Blondel & Gevers, 1994). Similarly, finding a unit controller (i.e., stable and inverse stable) for a given plant is also a difficult open problem. In Blondel (1994), a numerical problem was proposed and 1 kg of Belgian chocolates was offered for determining the existence of a unit controller for this problem or for a proof that no such controller exists.

In the face of various negative results, one is forced to make some compromises to the notion of "solving" a problem. An approach that is recently gaining popularity is the use of randomized algorithms, which are not required to work "all" the time, only "most" of the time as proposed in Vidyasagar (2001). Hence, one can make a "reasonably confident" statement regarding the non-existence of solutions to the tough problems mentioned above. Moreover, in case one can actually find a solution using randomized algorithms, this can be used to develop analytical methods to solve these problems. Randomized algorithms have been used recently in the literature to search for controllers guaranteeing probabilistic robustness with real and complex structured uncertainty in Calafiore, Dabbene, and Tempo (2000) and for robustness analysis and design of uncertain systems (Tempo & Dabbene, 2001; Stengel & Ray, 1991).

The paper is organized as follows. In Section 2, the problem of determining a simultaneously stabilizing controller for three given plants (champagne problem) and the problem of designing a unit compensator for a given plant (Belgian chocolates problem) is tackled. Both these problems are equivalent to satisfying a simple interpolation condition with rational functions. This problem is cast in a suitable frame-

[☆] This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor Shinji Hara under the direction of Editor Roberto Tempo.

* Corresponding author.
*E-mail addresses:* vvp@cair.res.in (V.V. Patel), deodhare@mantraonline.com (G. Deodhare), t.viswanath@math.utwente.nl (T. Viswanath).


PII: S0005-1098(02)00112-7

work in order to solve it using randomized algorithms. Detailed results of the numerical problems addressed in Section 2 are presented in Appendices A–C. Section 3 contains the conclusions.

## 2. The champagne problem and the Belgian chocolates problem

The following numerical problems were proposed as open problems in Blondel et al. (1994) as an illustration of the difficulty of simultaneous stabilization of three plants in general.

*Problem statement A*1: Does there exist a controller that simultaneously stabilizes the following three plants:

$$P_0 = \frac{2}{17}\frac{s-1}{s+1}, \quad P_1 = \frac{(s-1)(s-1)}{(9s-8)(s+1)} \quad \text{and} \quad P_2 = 0. \tag{1}$$

Patel (1999) solved this problem where it was shown that there *does not* exist a controller.

A generalization of the above problem (which remained unsolved) is considered in this paper. The problem statement is as follows:

*Problem statement A*2: Let the three plants to be simultaneously stabilized be given by

$$P_0 = 2\delta\frac{s-1}{s+1}, \quad P_1 = \frac{2\delta(s-1)(s-1)}{((1+\delta)s-(1-\delta))(s+1)} \quad \text{and} \quad P_2 = 0. \tag{2}$$

What is the minimum $\delta$ for which there exists a controller?

**Remarks.** (1) We can rewrite

$$P_1 = P_0\frac{1}{(1+\delta)}\frac{(s-1)}{\left(s-\frac{1-\delta}{1+\delta}\right)}.$$

As $\delta \to 0$, a right half-plane pole-zero cancellation occurs for plant $P_1$. Therefore, it is difficult to stabilize these three plants for very small values of $\delta$.

(2) The problem A1 is a special case of A2 for $\delta = \frac{1}{17}$.

(3) As discussed in Patel (1999), it can be proved analytically that for any $\delta < \frac{1}{16}$ a controller for problem A2 does not exist. However, the minimum value of $\delta$ for which a controller exists is not known.

(4) In Leizarowitz, Kogan, and Zeheb (1999) the conjectured minimum value of $\delta$ for problem A2 is $0.5/e(=1/5.4366)$. They have determined a controller for this value of $\delta$. However, the problem of determining whether a controller exists for any value of $\delta$ in the interval $[\frac{1}{16}, 1/5.4366]$ remains open.

*Problem statement B*1: Can the continuous-time second-order system

$$P(s) = \frac{s^2-1}{s^2-1.8s+1} \tag{3}$$

be stabilized by a stable and inverse stable controller?

This problem appears in Blondel (1994, p. 149) (Note that that there is a misprint in the problem stated in Blondel, 1994). One kilogram of Belgian chocolates have been offered in this paper to determine a bistable controller for this system (or for a proof that no such controller exists).

*Problem Statement B*2: Another kilogram of chocolate is offered in Blondel (1994) for the more difficult problem of finding the *range* of $\delta$ for which the system

$$P(s) = \frac{s^2-1}{s^2-2\delta s+1} \tag{4}$$

is stabilizable by a stable and inverse stable controller?

**Remarks.** (1) For $\delta = 0.9$, Problem B2 reduces to Problem B1. Hence, we shall solve Problem B2 first.

(2) *Results in the literature*: Problem B2 was tackled first in Blondel et al. (1994) where it was shown that there exists a positive $\delta^*$ such that the system in B2 is stabilizable by a unit controller when $\delta < \delta^*$, and is not stabilizable by such a controller when $\delta > \delta^*$. However, the value of $\delta^*$ was not determined in this paper. A bound for $\delta^*$ was first given by Rupp (1994) and improved further in Blondel, Rupp, and Shapiro (1995). These bounds were given for an equivalent problem in the $z$-domain. The corresponding result in the $s$-domain is given below. It is known that $\delta^*$ lies in the following range:

$$\underset{\text{Stabilizable by a unit}}{0.7615941559557649} < \delta^* < \underset{\text{Not stabilizable a unit}}{0.9999800001999982}. \tag{5}$$

Problem B2 is finding the exact value of $\delta^*$ whereas Problem B1 asks whether or not $\delta^* > 0.9$. Further details about the (equivalent $z$-domain) bounds are also available Blondel, Sontag, Vidyasagar, and Willems (1999).

**Notation**. We limit ourselves here to real rational functions and therefore we define the Hardy space $RH_\infty$ as the space of transfer functions in the Laplace variable $s$ that are proper and analytic for $\mathrm{Re}\, s \geqslant 0$, i.e., bounded-input bounded-output stable transfer functions. A unit in $RH_\infty$ is a function whose inverse is also in $RH_\infty$. Henceforth, whenever we refer to a unit, it is a unit in $RH_\infty$.

*Overview of the solution methodology*: The above problems, i.e., Problems A2 and B2 are solved using randomized algorithms. For this purpose, the problems are first cast as the following equivalent unit interpolation problems. We shall prove that Problems A2 and B2 are equivalent to Problems EA and EB (stated below), respectively.

### 2.1. Conversion to unit interpolation problems

*Problem EA*: Find a unit $U(s)$ satisfying the following:

$$U(1) = 1 \tag{6a}$$

and

$$\left[U(s) - \frac{(s-1)}{\delta(s+1)}\right] \tag{6b}$$

is a unit.

*Problem EB*: Find a unit $U(s)$ satisfying the following:

$$U(1) = 1, \tag{7a}$$

$$U - \left(\frac{2}{1-\delta}\right)\left(\frac{s^2 - 2\delta s + 1}{(s+1)^2}\right) = \left(\frac{2}{1-\delta}\right)\left(\frac{s-1}{s+1}\right)R(s), \tag{7b}$$

where $R(s)$ is a unit.

The following theorem will be used to convert the problems in (2) and (4) to those in (6) and (7), respectively.

**Theorem 1** (Vidyasagar, 1985). *Given a stable plant $P_0$ and an unstable plant $P_1$. Then a controller $C$ which will stabilize both the plants simultaneously (if it exists) is given by $C = R/(1 - RP_0)$ where $R = (U - D)/N$ and $N, D$ are the stable coprime factors of the plant $P = P_1 - P_0 = N/D$.*

*$U$ is a unit that interpolates the values of $D$ at the unstable zeros of $N$.*

**Proof for equivalence of A2 and EA.** Using Theorem 1, we first establish conditions for the existence of a controller to simultaneously stabilize the plants $P_0$ and $P_1$ in (2). For these plants the plant $P$ (Theorem 1) is given by

$$P = P_1 - P_0 = -2\delta^2 \frac{(s-1)}{(1+\delta)s - (1-\delta)}.$$

Let $P = N/D$, where

$$N = -2\delta\frac{(s-1)}{(s+1)} \quad \text{and} \quad D = \frac{(1+\delta)s - (1-\delta)}{\delta(s+1)}.$$

The stabilizing controller (if it exists) for the plant $P$ can be obtained from Theorem 1 as follows:

$$R = \frac{U-D}{N} = \frac{U - \dfrac{(1+\delta)s - (1-\delta)}{\delta(s+1)}}{-2\delta\dfrac{(s-1)}{(s+1)}},$$

where $U$ is a unit which interpolates the value of $D$ at the non-minimum phase zeros of $N$. Thus one has to find a unit $U$ such that $U(1) = 1$.

The controller $C$ that stabilizes $P_0$ and $P_1$ is given by

$$C = \frac{R}{1 - RP_0} = \frac{U - \dfrac{(1+\delta)s - (1-\delta)}{\delta(s+1)}}{-2\delta\dfrac{(s-1)}{(s+1)}\left[1 + \left[U - \dfrac{(1+\delta)s - (1-\delta)}{\delta(s+1)}\right]\right]}.$$

Further, since $P_2 = 0$ also has to be stabilized, $C$ has to be stable. The factor $(s-1)$ in the denominator of $C$ is cancelled with the numerator since $U$ interpolates the values of $D$ at the non-minimum phase zeros of $N$ (i.e., $U(1) = D(1) = 1$). Therefore for $C$ to be stable, the term

$$G(s) = \left[1 + \left[U - \frac{(1+\delta)s - (1-\delta)}{\delta(s+1)}\right]\right]$$

must be a unit.

Then after rewriting $G(s)$ the problem in (2) is equivalent to finding the smallest $\delta$ for which (6a) and (6b) are satisfied. □

**Proof for equivalence of B2 and EB.** Let $P = N/D$, be a stable coprime factorization of the plant in Problem (4), where

$$N = \left(\frac{2}{1-\delta}\right)\left(\frac{s-1}{s+1}\right) \quad \text{and}$$

$$D = \left(\frac{2}{1-\delta}\right)\left(\frac{s^2 - 2\delta s + 1}{(s+1)^2}\right).$$

Then using Theorem 1, a stable controller (if it exists) for the plant $P$ (denoted by $R$) is given by

$$R = \frac{U-D}{N} = \frac{U - \left(\dfrac{2}{1-\delta}\right)\left(\dfrac{s^2 - 2\delta s + 1}{(s+1)^2}\right)}{\left(\dfrac{2}{1-\delta}\right)\left(\dfrac{s-1}{s+1}\right)},$$

where $U$ is a unit which interpolates the values of $D$ at the non-minimum phase zeros of $N$. Thus one has to find a unit $U$ such that $U(1) = 1$. This will give a stable controller, but it may not be inverse stable. To obtain a stable and inverse stable controller, $R$ must be a unit. Therefore, we must have

$$G(s) = U - \left(\frac{2}{1-\delta}\right)\left(\frac{s^2 - 2\delta s + 1}{(s+1)^2}\right) = \left(\frac{2}{1-\delta}\right)\left(\frac{s-1}{s+1}\right)R(s),$$

where $R(s)$ is a unit.

*Convert to determination of Hurwitz polynomials*: Let the unit $U(s)$ have order $n$ (to be determined later). Consider a polynomial factorization of $U(s)$,

$$U(s,q) = x(s,q)/y(s,q),$$

where $x(s,q)$ and $y(s,q)$ are Hurwitz polynomials. Let the vector $q$ consist of the parameters $\{a_i, b_i, c_i, d_i, e, f\}$, where

$i = 1, 2, \ldots, (n/2)$ if $n$ is even and $i = 1, 2, \ldots, \text{floor}(n/2)$ if $n$ is odd as defined in (8) and (9) below. Let these be represented as

$$x(s,q) = \frac{\prod_1^{n/2}(s^2 + b_i s + a_i)}{\prod_1^{n/2}(1 + b_i + a_i)} \quad \text{if } n \text{ is even} \tag{8a}$$

or

$$x(s,q) = \frac{\prod_1^{\text{floor}}(n/2)(s^2 + b_i s + a_i) \times (cs + 1)}{\prod_1^{\text{floor}}(n/2)(1 + b_i + a_i) \times (c + 1)} \quad \text{if } n \text{ is odd} \tag{8b}$$

and

$$y(s,q) = \frac{\prod_1^{n/2}(s^2 + e_i s + d_i)}{\prod_1^{n/2}(1 + e_i + d_i)} \quad \text{if } n \text{ is even} \tag{9a}$$

or

$$y(s,q) = \frac{\prod_1^{\text{floor}}(n/2)(s^2 + e_i s + d_i) \times (fs + 1)}{\prod_1^{\text{floor}}(n/2)(1 + e_i + d_i) \times (f + 1)} \quad \text{if } n \text{ is odd.} \tag{9b}$$

The above representation ensures that $U(s)$ is a unit and the condition in (6a) or (7a), i.e., $U(1) = 1$ is satisfied. □

*EA*: Now satisfying (6b) is equivalent to ensuring that

$$l(s,q,\delta) = \delta x(s,q)(s+1) - y(s,q)(s-1)$$

is a Hurwitz polynomial. Thus, solving problem (2) can be reduced to finding a solution to the following:

$$\min_{x(s),y(s)} \left\{ \begin{array}{l} \delta : l(s,q,\delta) = \delta x(s,q)(s+1) - y(s,q)(s-1), \\ \text{is a Hurwitz polynomial} \end{array} \right\}. \tag{10}$$

Note that the minimization is performed over all $x(s,q)$ and $y(s,q)$ satisfying (8) and (9) above. Further, if all the elements of $q$ are constrained to be greater than zero then $U$ is a unit.

*EB*: Now satisfying (7b) is equivalent to ensuring that

$$\frac{x(s,q)}{y(s,q)} - \left(\frac{2}{1-\delta}\right)\left(\frac{s^2 - 2\delta s + 1}{(s+1)^2}\right) = \frac{(1-\delta)(s+1)^2 x(s,q) - 2(s^2 - 2\delta s + 1)y(s,q)}{(1-\delta)(s+1)^2 y(s,q)}.$$

Define a polynomial $l(s,q,\delta)$ as follows:

$$l(s,q,\delta) = (1-\delta)(s+1)^2 x(s,q) - 2(s^2 - 2\delta s + 1)y(s,q) = (s-1)m(s,q\delta).$$

Note that $l(1) = 0$, and therefore $s = 1$ is a zero of the polynomial $l(s,q,\delta)$. Therefore, we can write $l(s,q,\delta) = (s-1)m(s,q,\delta)$. Thus, $R(s)$ is unit if and only if $m(s,q,\delta)$ is a Hurwitz polynomial. (Then $R(s) = m(s,q,\delta)/2(s+1)y(s,q)$ is a unit controller for the plant $P(s)$).

Thus, this problem is equivalent to solving the following:

$$\max_{\substack{x(s,q),y(s,q)\\ \text{Hurwitz}}} \left\{ \begin{array}{l} \delta : l(s,q,\delta) \\ \quad = (1-\delta)(s+1)^2 x(s,q) \\ \qquad -2(s^2 - 2\delta s + 1)y(s,q) \\ \quad = (s-1)m(s,q,\delta) \\ \text{where } m(s,q,\delta) \text{ is a Hurwitz} \\ \text{polynomial} \end{array} \right\}.$$

One can convert this to a minimization problem by replacing $\delta = 1/\gamma$ as follows:

$$\min_{\substack{x(s,q),y(s,q)\\ \text{Hurwitz}}} \left\{ \begin{array}{l} \gamma : l_1(s,q,\gamma) \\ \quad = (\gamma - 1)(s+1)^2 x(s,q) \\ \qquad -2(\gamma s^2 - 2s + \gamma)y(s,q) \\ \quad = (s-1)m_1(s,q,\gamma) \\ \text{where } m_1(s,q,\gamma) \text{ is a Hurwitz} \\ \text{polynomial} \end{array} \right\} \tag{11}$$

In the next section, a randomized algorithm is used to find $U(s)$ for the smallest value of $\delta$ for which (6) is satisfied and similarly to find $U(s)$ for the largest value of $\delta$ for which (7) is satisfied.

*Definition of cost functions for randomized optimization*:

(A) Define the cost function to be minimized as

$$\psi(q) = \min_{\delta} \psi_1(q,\delta) \tag{12}$$

where

$$\begin{array}{ll} \psi_1(q,\delta) = 1e8 & \text{if } \; l(s,q,\delta) \text{ is not Hurwitz} \\ \delta & \text{if } \; l(s,q,\delta) \text{ is Hurwitz.} \end{array}$$

(B) The cost function to be minimized is

$$\psi(q) = \min_{\gamma} \psi_1(q,\gamma) \tag{13}$$

where

$$\begin{array}{ll} \psi_1(q,\gamma) = 1e8 & \text{if } m_1(s,q,\gamma) \text{ is not Hurwitz} \\ \gamma & \text{if } m_1(s,q,\gamma) \text{ is Hurwitz.} \end{array}$$

These optimization problems are solved using randomized algorithms. Then one can state with a predefined confidence and level parameter that there does not exist a stabilizing controller for the value of $\delta$ or $\gamma$ less than the above minimum. This can be achieved by considering various orders $n$ of the unit randomly and letting each component of $q$ lie with equal probability either in the compact set $[\varepsilon, 1]$ or $[1, 1/\varepsilon]$ for some (small) $\varepsilon > 0$.

Design using randomized algorithms: The MATLAB 5.1 random number generator using function "rand.m" can generate floating-point numbers in the closed interval $[2^{-53}, 1-2^{-53}]$. Theoretically, it can generate over $2^{1492}$ values before repeating itself.

Now, for solving (10) the following steps are involved:

1. Choosing the degree of the unit at random according to the probability distribution $A$ (defined subsequently).
2. Choosing the elements of $q$ randomly either in the range $[\varepsilon, 1]$ or $[1, 1/\varepsilon]$ according to the probability distribution $B$ (as explained in Step 2 below).
3. Solving the optimization problem iteratively by fixing some of the poles, i.e., roots of $y(s)$ and zeros, i.e., roots of $x(s)$ (obtained in previous iterations) at each stage.

We will now describe each of the probability distributions mentioned above.

*Step* 1: The distribution $A$ for choosing the degree of the unit at random. The probability density for the degree $n$ of the unit chosen is given by

$$p = f(n) \begin{cases} f(n) = 0.09 & \text{for } n = 1, 2, \ldots, 10, \\ f(n) = 0.09 r^{n-10} & \text{for } n = 11, 12, \ldots \\ & \text{where } r = \dfrac{10}{19}. \end{cases}$$

This distribution assigns a uniform probability for each integer order up to 10 and a reduced probability for orders above 10. See Fig. 1 for a plot of the probability density.

The order $n$ was chosen according to the above distribution using the random number generator as follows. Let $\beta$ be a random number chosen uniformly in the interval $(0.0, 1.0)$. Then

$$n = \begin{cases} \text{ceil}\left(\dfrac{\beta}{0.09}\right) & \text{for } \beta \leqslant 0.9, \\ 10 + \text{ceil}\left(\log_r\left(1 - \dfrac{1-r}{0.09r}(\beta - 0.9)\right)\right) & \text{for } \beta > 0.9. \end{cases}$$

This probability distribution is chosen because it is impossible to search units of all order. From a practical point of view lower order controllers are preferred. Moreover, for a first-order plant (see equation for $l(s, q, \delta)$) a 10th order controller is of a reasonably high order. Based on this it was decided to assign a uniform probability for orders up to 10 and a reduced probability for orders above 10.

*Step* 2: The distribution $B$ for choosing the elements of $q$ at random.

Suppose that in Step 1 we get an integer value $n_1$. If $n_1$ is odd then one root is real and other roots are the roots of the second-order polynomials in (8) and (9) in which case both the roots could be real or complex conjugate. Choose directly the coefficients of the second-order polynomials. For this consider one second-order polynomial $x(s) = (s^2 + bs + a)$. In this case, we could choose $a, b \in (0, \infty)$. However, a

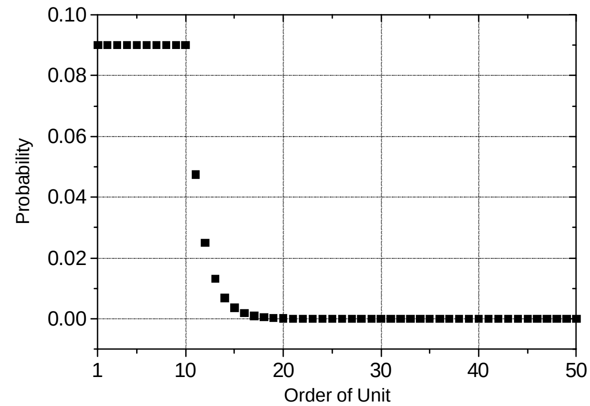


Fig. 1. Probability density for the order of the unit.

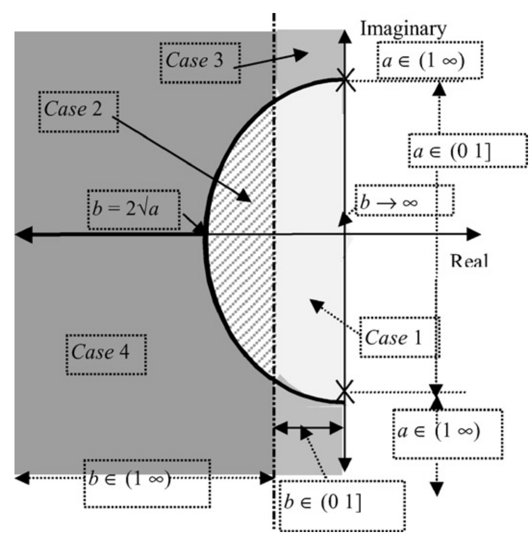


Fig. 2. The roots of the second-order polynomial for different four cases.

random number can be generated on compact interval using "rand.m" (which generates numbers over (0 1] interval). Hence to cover the entire left of the $s$-plane using this random number generator, we can choose the coefficients $a, b$ in the following four ways:

(i) $a, b \in (0, 1]$, (ii) $a \in (0, 1]$, $b \in (1, \infty)$,
(iii) $a \in (1, \infty)$, $b \in (0, 1]$, (iv) $a, b \in (1, \infty)$.

One can easily map regions in the complex plane for the above four cases (which have been shown in Fig. 2). It can be seen that entire left half-plane is covered.

*Results*: Using Algorithm 2 in Vidyasagar (2001) it can be seen that one can obtain a result with a confidence of at least $1 - \lambda$ and a level parameter $\alpha$ if one tests for $\eta$ randomly generated controllers where

$$\eta = \text{ceil}\left(\frac{lg(1/\lambda)}{lg[1/(1-\alpha)]}\right). \quad (14)$$

For $\alpha = 1e-5$ and $\lambda = 1e-5$, we have $\eta = 1151287$.

(A) As a first step, the algorithm was run a few times for 100000 randomly chosen units to obtain information about any structure of the problem. During these experiments, it was noticed that the optimal unit had a zero close to the origin and a pole at $-1$. In the next step, the algorithm was run for 1151287 units with one pole and zero constrained as above. The minimum value of $\delta$ obtained was 1/(5.7531). *This is an improvement over the conjecture in* Leizarowitz et al. (1999).

Finally, a local search was performed around the optimal unit obtained in the above step. Let $q_{\min}$ denote the parameters of the optimal unit obtained above. The coefficients were now chosen randomly only in the range $((1-\beta)q_{\min},\ (1+\beta)q_{\min})$, where $\beta$ was chosen as 0.3. After this iteration, the minimum value of $\delta$ could be further improved to $\delta_{\min} = 1/(6.719367588932806)$.

The details of the unit and the corresponding controller for this value of $\delta_{\min}$ are given in Appendix A.

(B) For Problem B2 no specific pole-zero structure was noticed in the preliminary iterations. Hence, we did not assign any specific pole-zero structure. As above to obtain a result with a confidence of at least $1–10^{-5}$ with a level factor of $10^{-5}$ one needs to test for 1151287 randomly generated controllers.

After the algorithm was run for 1151287 units, the maximum value of $\delta$ obtained was 0.8771929824561(=1/1.14). *This is an improvement over the bound* (5), *given by* Rupp (1994) and Blondel et al. (1995). Thus using randomized algorithms a better bound is obtained.

Finally, as before, a local search was performed around the optimal unit obtained above. Let $q_{\min}$ denote the parameters of the optimal unit obtained above. The coefficients were now chosen randomly only in the range $((1-\beta)q_{\min},\ (1+\beta)q_{\min})$, where $\beta$ was chosen as 0.3. With this iteration, the maximum value of $\delta$ could be further improved to $\delta_{\max} = 0.9372071227741330$ (=1/1.067). *This in fact solves the Problem B1*. That is, there exists a unit controller for $\delta = 0.9$. However, the controller obtained for $\delta_{\max} = 0.9372071227741330$ does not stabilize the plant for $\delta = 0.9$. Therefore a local search was performed only for $\delta = 0.9$and a controller was found. The controller for Problem B1 ($\delta = 0.9$), is given in Appendix B and the controller for $\delta_{\max} = 0.9372071227741330$ is given in Appendix C. The gap between the $\delta_{\max}$ achieved using randomized algorithms and the upper bound in (7) on $\delta^*$ i.e. 0.99998 (above which there does not exist a controller) is very small. It is conjectured that there exists an analytic controller in the limit, which may be approximated by a very high-order controller. Since the probability distribution that we put on the order of the unit is concentrated up to 10th order, we found the best possible value with this distribution. (In fact, the order of the optimal unit is 10 in Appendices B and C, which implies that with a higher order unit this bound could be further improved).

## 3. Conclusions

In this paper, we have demonstrated how randomized algorithms can be used to solve some difficult problems. It was shown that the conjecture about the Champagne problem mentioned in Leizarowitz et al. (1999) is invalid and in fact a much better bound was obtained in this paper. Similarly, the bounds on the Belgian chocolates problem were improved. Moreover, it could be shown that there exists a controller for $\delta = 0.9$ which solved one of the Belgian chocolate problems. The gap between the $\delta_{\max}$ (i.e., 0.9372071) achieved using randomized algorithms and the upper bound (i.e., 0.99998) beyond which there does not exist a unit controller is very small. We conjecture that there exists an analytic controller in the limit, which may be approximated, by a very high-order controller.

## Appendix A. Champagne problem

The value of $\delta$ obtained for (10) using randomized algorithms is $\delta = 1/6.719367588932806$. The polynomials $x(s)$ and $y(s)$ are given in the following table. Conjectured value of $\delta$ in Leizarowitz et al. (1999) is $\delta = 1/(2e) = 1/5.436563656918090$. The following controller stabilizes all the three plants for $\delta = 1/6.719367588932806$.

| $x(s)$ | $y(s)$ | Coeff. of |
|---|---|---|
| 0.017269 | 0.00257 | $s^9$ |
| 0.096007 | 0.01942 | $s^8$ |
| 0.22176 | 0.066494 | $s^7$ |
| 0.28208 | 0.13967 | $s^6$ |
| 0.2243 | 0.20547 | $s^5$ |
| 0.11614 | 0.22558 | $s^4$ |
| 0.036696 | 0.18576 | $s^3$ |
| 0.0047318 | 0.10862 | $s^2$ |
| 0.0010134 | 0.039704 | $s^1$ |
| $4.2464e-006$ | 0.0067069 | $s^0$ |

Controller (stable) polynomials

| Numerator | Denominator | Coeff. of |
|---|---|---|
| 0.00863435 | $5208e-013$ | $s^9$ |
| 0.0823285 | $5208e-005$ | $s^8$ |
| 0.36609 | 0.0013991 | $s^7$ |
| 1.0181 | 0.010714 | $s^6$ |
| 1.9617 | 0.053547 | $s^5$ |
| 2.72 | 0.15181 | $s^4$ |
| 2.6896 | 0.26857 | $s^3$ |
| 1.7981 | 0.29117 | $s^2$ |
| 0.72129 | 0.17767 | $s^1$ |
| 0.12889 | 0.04507 | $s^0$ |

In the following table the value of $\delta$ achieved is tabulated along with the iteration no. In the Main search 1151286 iterations were carried out. However, from the following

(Main Search) table it can be seen that after iteration number 3729 there was very little improvement.

The Local Search was carried out around the answer obtained for iteration number 3729. The local search was carried out for 1,00,000 iterations. The value of $\delta$ obtained after 9368 iterations did not improve further.

| Local search | | Local search | |
|---|---|---|---|
| Iteration no. | $\delta$ | Iteration no. | $\delta$ |
| 9 | 0.15882 | 9354 | 0.15382 |
| 933 | 0.15882 | 9368 | 0.14882 |
| 1295 | 0.15382 | 97421 | 0.14882 |

## Appendix B. Belgian chocolates problem—Problem B1

For $\delta = 0.9$ ($\gamma = 1.1111$) polynomials $x(s)$ and $y(s)$ were obtained. The controller polynomials are also given in the following table. It can be verified that the controller is a unit controller.

| $x(s)$ | $y(s)$ | Coeff. of |
|---|---|---|
| $6.3871e-007$ | 0.0017213 | $s^{10}$ |
| $2.3653e-005$ | 0.014867 | $s^9$ |
| 0.00037603 | 0.056272 | $s^8$ |
| 0.0032792 | 0.12653 | $s^7$ |
| 0.018224 | 0.19681 | $s^6$ |
| 0.069511 | 0.22401 | $s^5$ |
| 0.17293 | 0.19016 | $s^4$ |
| 0.27216 | 0.11926 | $s^3$ |
| 0.26723 | 0.053283 | $s^2$ |
| 0.15476 | 0.015087 | $s^1$ |
| 0.041499 | 0.001991 | $s^0$ |
| Controller polynomials | | |
| Num. | Den. | Coeff. of |
| −0.003825 | 0.003825 | $s^{11}$ |
| −0.029974 | 0.036862 | $s^{10}$ |
| −0.099333 | 0.15809 | $s^9$ |
| −0.18802 | 0.40623 | $s^8$ |
| −0.24149 | 0.71854 | $s^7$ |
| −0.2211 | 0.93516 | $s^6$ |
| −0.14831 | 0.92039 | $s^5$ |
| −0.074098 | 0.68761 | $s^4$ |
| −0.028653 | 0.38343 | $s^3$ |
| −0.0072546 | 0.15193 | $s^2$ |
| −0.0010411 | 0.037952 | $s^1$ |
| −0.00018647 | 0.0044245 | $s^0$ |

## Appendix C. Belgian chocolates problem—Problem B2

For $\delta = 0.9372071227741330$ ($\gamma = 1/1.067$) polynomials $x(s)$ and $y(s)$ given below were obtained. The controller polynomials are given in the following table:

| $x(s)$ | $y(s)$ | Coeff. of |
|---|---|---|
| $7.795868100869428e-8$ | $2.994917878556120e-3$ | $s^{10}$ |
| $3.109819223185664e-6$ | $1.824035209582114e-2$ | $s^9$ |
| $6.049131715241308e-5$ | $5.733888770171772e-2$ | $s^8$ |
| $6.917352527378138e-4$ | $1.211455773171276e-1$ | $s^7$ |
| $5.199789268090457e-3$ | $1.862356348992394e-1$ | $s^6$ |
| $2.665053210021064e-2$ | $2.160360268234199e-1$ | $s^5$ |
| $9.323004034218772e-2$ | $1.908174186813875e-1$ | $s^4$ |
| $2.169548855688423e-1$ | $1.261631337576180e-1$ | $s^3$ |
| $3.161584615383594e-1$ | $5.995889103258438e-2$ | $s^2$ |
| $2.556074792332202e-1$ | $1.838879040598354e-2$ | $s^1$ |
| $8.544339760129475e-2$ | $2.680369406544572e-3$ | $s^0$ |
| Controller numerator | Controller denominator | Coeff. of |
| $-6.391149529607131e-3$ | $6.391154752838759e-3$ | $s^{11}$ |
| $-3.333617058351376e-2$ | $4.531606612532108e-2$ | $s^{10}$ |
| $-8.912262845127680e-2$ | $1.612860977279479e-1$ | $s^9$ |
| $-1.571619905553188e-1$ | $3.808858483502160e-1$ | $s^8$ |
| $-1.919225811941725e-1$ | $6.559515068697271e-1$ | $s^7$ |
| $-1.639968811585718e-1$ | $8.584477261161548e-1$ | $s^6$ |
| $-9.431802032063310e-2$ | $8.682252527072589e-1$ | $s^5$ |
| $-3.248696588533808e-2$ | $6.764364989048376e-1$ | $s^4$ |
| $-6.490091492260812e-3$ | $3.971844009022918e-1$ | $s^3$ |
| $-1.101421239170584e-3$ | $1.671939521899039e-1$ | $s^2$ |
| $-5.971461272927941e-5$ | $4.496158703993500e-2$ | $s^1$ |
| $-4.799325720644676e-6$ | $5.719908313566116e-3$ | $s^0$ |

| Main search | | Main search | |
|---|---|---|---|
| Iteration no. | $\delta$ | Iteration no. | $\delta$ |
| 1 | 0.55382 | 851 | 0.26882 |
| 6 | 0.31382 | 2662 | 0.20382 |
| 469 | 0.29882 | 3729 | 0.17882 |
| 717 | 0.28382 | $1.8408e+005$ | 0.17882 |
| 741 | 0.27882 | $4.3112e+005$ | 0.17382 |

## References


Blondel, V. (1994). *Simultaneous stabilization of linear systems*. Berlin: Springer.

Blondel, V., & Gevers, M. (1994). Simultaneous stabilization of three linear systems is rationally undecidable. *Mathematics of Control, Signals Systems*, *6*, 135–145.

Blondel, V., Gevers, M., Mortini, R., & Rupp, R. (1994). Simultaneous stabilization of three or more systems: Conditions on the real axis do not suffice *SIAM Journal of Control and Optimization*, *32*(2), 572–590.

Blondel, V., Rupp, R., & Shapiro, H. (1995). On zero and one points of analytic functions. *Complex Variables: Theory and Applications*, *28*, 189–192.

Blondel, V., Sontag, E. D., Vidyasagar, M., & Willems, J. C. (1999). *Open problems in mathematical systems and control theory*. London: Springer.

Blondel, V., & Tsitsiklis, J.N. (2000). A survey of computational complexity results in systems and control. *Automatica*, *36*(9), 1249–1274.

Calafiore, G., Dabbene, F., & Tempo, R. (2000). Randomized algorithms for probabilistic robustness with real and complex structured uncertainty. *IEEE Transactions in Automatic Control*, *45*(12), 2218–2235.

Leizarowitz, A., Kogan, J., & Zeheb, E. (1999). On simultaneous stabilization of linear plants. *Latin American Applied Research*, *29*(03–04), 167–174.

Patel, V. V. (1999). Solution to the "Champagne Problem" on the simultaneous stabilization of three plants. *Systems and Control Letters*, *37*, 173–175.

Rupp, R. (1994). A covering theorem for a composite class of analytic functions. *Complex Variables: Theory and Applications*, *25*, 35–41.

Stengel, R. F., & Ray, L. R. (1991). Stochastic robustness of linear time-invariant control systems. *IEEE Transactions on Automatic Control*, *36*, 82–87.

Tempo, R., & Dabbene, F. (2001). Randomized algorithms for analysis and control of uncertain systems: An overview. In: *Perspectives in robust control*, London: Springer.

Vidyasagar, M. (1985). *Control system synthesis: A factorization approach*. Cambridge, MA: MIT Press.

Vidyasagar, M. (2001). Randomized algorithms for robust controller synthesis using statistical learning theory. *Automatica*, *37*(10), 1515–1528.


**Vijay V. Patel** was born on 1st April 1969 in Nandurbar, Maharashtra, India. He completed his schooling in Dhadgaon and Shahada, (dist. Nandurbar). Subsequently he obtained his B.E. in Instrumentation in 1990 from Govt. College of Engg. Pune. He completed his M. Tech. and Ph. D. in Control Systems in 1992 and 1996 respectively from IIT Kharagpur. His field of specialization is in Unit interpolation in $H_\infty$, simultaneous stabilization of plants, robust and optimal control synthesis techniques. Since 1995, Dr. Patel has been working as a Scientist at the Centre for Artificial Intelligence and Robotics, Bangalore. He is selected as Associate of Indian Academy of Sciences (2000–2004). In 2001, he received the prestigious INSA (Indian National Science Academy) Young Scientist award in Engineering Sciences. His current research interest is to evolve control law synthesis techniques for a high performance aircraft.

**Girish Deodhare** was born in Pune, India on 9th October 1963. He completed his B. Tech. (Electrical Engg.) and M. Tech. (Control and Instrumentation) in 1984 and 1986 respectively from the Indian Institute of Technology, Bombay. Subsequently, he obtained his Ph. D. in Electrical Engineering from the University of Waterloo, Ontario, Canada in 1990. Since 1990, he has been a Scientist at the Centre for Artificial Intelligence and Robotics, Bangalore, India where he currently heads the Control Systems Group. Dr. Deodhare's research interests are in robust and optimal control with a special emphasis on designing "practical" controllers. He is also currently working on design of control laws for a modern high performance aircraft.

**Talasila Viswanath** received his Bachelor of Engineering degree from Bangalore University (Bangalore, India) in 1997. He was working at the Institute of Robotics and Intelligent Systems, (Bangalore, India) during the years 1997 to 2000. Since 2000 he has been pursuing his Ph.D. at the Faculty of Applied Mathematics at the University of Twente, Netherlands. His research interests are in port-Hamiltonian modeling of distributed parameter systems and in Hamiltonian discretizations of such system models.