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Accurate Test Limits Under Nonnormal Measurement Error

WILLEM ALBERS, WILBERT C.M. KALLENBERG and GERTJAN D. OTTEN

Department of Applied Mathematics, University of Twente, P.O. Box 217,7500 AE Enschede, The Netherlands

Abstract: When screening a production process for nonconforming items the objective is to improve the average outgoing quality level. Due to measurement errors specification limits cannot be checked directly and hence test limits are required, which meet some given requirement, here given by a prescribed bound on the consumer loss. Classical test limits are based on normality, both for the product characteristic and for the measurement error. In practice, often nonnormality occurs for the product characteristic as well as for the measurement error. Recently, nonnormality of the product characteristic has been investigated. In this paper attention is focussed on the measurement error.

Firstly, it is shown that nonnormality can lead to serious failure of the test limit. New test limits are therefore derived, which have the desired robustness property: a small loss under normality and a large gain in case of nonnormality when compared to the normal test limit.

Monte Carlo results illustrate that the asymptotic theory is in agreement with moderate sample behaviour.

Keywords and Phrases: specification limit, consumer loss, inspection, second order unbiasedness, density estimation, Monte Carlo experiments, Edgeworth expansion.

1 Introduction

Many statistical procedures are firstly developed assuming normality of the observations. Realizing that results may drastically change when normality fails and realizing that as a rule the claim of normality in practice is hardly reasonable, a lot of effort is invested to generate statistical procedures which can be applied in much more general situations. This tendency is also seen in setting test limits.

Test limits are used when during inspection items are compared with given specification limits. The presence of measurement errors forces producers to set test limits which are slightly more strict than the specification limits. It is important to set these accurately: test limits which are too strict cause unnecessary loss of yield, whereas those which are too liberal lead to consumer losses which exceed agreed bounds. (The consumer loss is the probability that a product is both nonconforming and accepted.) A test limit t_1 is called better or more accurate than another one given by t_2, if it comes closer to the prescribed bound on the consumer loss.

While nowadays statistical process control gets much attention, still the inspection approach remains of ongoing interest. This happens when rework costs are low or when relations between process characteristics and final product characteristics are not all that clear, as is often the case for example in semiconductor industry. Indeed, the problem has received attention in the literature over a long period, ranging (at least) from Grubbs and Coon (1954) to Mullenix (1990), Easterling et al. (1991), Albers et al. (1994a, b, 1997). When dealing with the topic of process screening one may use different models and different objectives as for example in the papers of Tang (1987), Bai et al. (1990), Bai and Lee (1993), Bai and Kwon (1995), Kim and Bai (1992), Owen et al. (1976, 1977, 1981).

Distributional assumptions in this area concern the distribution of the product characteristic and the distribution of the measurement error. In Albers et al. (1997) attention is devoted to nonnormality of the product characteristic, still assuming normality for the measurement error. However, in practice it turns out that also normality of the measurement error often fails. So, the logical next step is to investigate this situation and that is the aim of this paper.

The program is as follows. Firstly, we will investigate the effect of incorrectly assuming normality of the measurement error. It turns out that test limits derived under the condition of normality of the measurement error, may seriously fail in the sense that nonnormality can lead to severe violation of the prescribed bound on the consumer loss. Results are given in section 2.

Secondly, by nonparametric methods a new test limit is derived assuming that the distribution of the measured products is known, while the distribution of the measurement error is completely unknown (apart from some regularity). This corresponds to an often in practice occurring situation, where we have many production data and only a few observations of the measurement error. Note that it requires some effort to get observations of the measurement error. For instance, we can take the difference between the standard measurement of the product (in the factory) and a very precise measurement of the same product (for instance in a laboratory), where the latter one is considered as free of measurement error. The new test limits are presented and investigated in section 3. The proof of the main result is given in section 7. It is among others based on Edgeworth expansions.

Thirdly, in section 4 nonparametric density estimation is added, leading to new test limits in case both the distribution of the product characteristic and the distribution of the measurement error is unknown.

Fourthly, instead of the criterion of unbiasedness, which is used in sections 3 and 4, another criterion, related to the confidence interval concept, is studied in section 5. Simulations show that the asymptotic theory works very well in predicting finite sample behaviour of the test limits w.r.t. both criteria.

Finally, in section 6 an application of the results in semiconductor industry is presented.

2 The Normal Test Limit Under Nonnormal Distribution of the Measurement Error

Let X be the true value of the product characteristic and U the measurement error. Then we observe

$$\tilde{X} = X + U . \tag{2.1}$$

We assume, that X and U are independent. Moreover, U is small w.r.t. X in the sense that $\sigma^2 = \text{var}(U) \to 0$. Further let $\mu = E[U]$ and

$$V = -\left(\frac{U - \mu}{\sigma}\right) , \tag{2.2}$$

implying $E[V] = 0$ and $\text{var}(V) = 1$. It is assumed that V has a continuous density, denoted by g.

We consider a specification interval of the form $(-\infty, s]$. Denote by

$$\pi = P(X > s) \tag{2.3}$$

the probability that a product characteristic is nonconforming. For a given test limit t the *consumer loss* $CL(t)$ is defined by

$$CL(t) = P(X > s, \tilde{X} < t) . \tag{2.4}$$

Since we accept a product if $\tilde{X} < t$, $CL(t)$ is the probability that a product is both nonconforming and accepted. The producer and consumer have agreed that the consumer loss should be at most γ, which is typically quite small (10-100 ppm, parts per million). To achieve a maximal yield within this restriction the test limit should satisfy

$$CL(t) = \gamma .$$

The behaviour of the consumer loss under several distributions of the measurement error becomes more clear if $CL(t)$ is expanded in terms of σ. In order to make such an expansion we put some regularity conditions.

Let F_X and f_X be the distribution function and density respectively, of X. Assume

$$f_X(s) > 0 , \qquad f'_X \text{ is bounded} . \tag{A1}$$

Since test limits are as a rule more strict than specification limits, we have $t < s$ if $\mu = 0$ and $t < s + \mu$ for general μ. Hence

$$a = \frac{s + \mu - t}{\sigma} > 0 \tag{2.5}$$

and the test limit is written as

$$t = s + \mu - a\sigma . \tag{2.6}$$

Many of the results we are going to present remain true for, or can be adapted to negative a, but is does not seem worthwhile to bother about this. Further we tacitly assume that a is *bounded* as σ, $\gamma \to 0$. Denoting by 1_A the indicator function of the set A, we define, for $k = 0, 1, 2, \ldots$,

$$h_k(a) = E[(V - a)^k 1_{V>a}] = \int_a^\infty (v - a)^k g(v) dv . \tag{2.7}$$

Note that

$$h'_k(a) = -k h_{k-1}(a) . \tag{2.8}$$

Lemma 2.1: Assume (A1). Then

$$CL(t) = \sigma f_X(s) h_1(a)\{1 + O(\sigma)\} \tag{2.9}$$

as $\sigma \to 0$.

Proof: By definition

$$
\begin{aligned}
CL(t) &= P(X > s, X + U < t) = P(X > s, X + \mu - \sigma V < s + \mu - a\sigma) \\
&= P(X > s, X < s + \sigma(V - a)) \\
&= \int_a^\infty \{F_X(s + \sigma(v - a)) - F_X(s)\} g(v) dv \\
&= \int_a^\infty \{\sigma f_X(s)(v - a) + \tfrac{1}{2}\sigma^2 f'_X(\xi_v)(v - a)^2\} g(v) dv
\end{aligned}
$$

for some ξ_v between s and $s + \sigma(v - a)$. The result now easily follows. □

In the particular case of a normal measurement error distribution we get $g(v) = \phi(v)$ with ϕ the standard normal density. In that case we write g_1 instead of h_1. So,

$$
g_1(a) = \int_a^\infty (v - a)\phi(v) dv \, .
$$

Let

$$
t_0 = s + \mu - a_0\sigma
$$

be the test limit derived under the assumption of a normally distributed measurement error with mean μ and variance σ^2. Then, by Lemma 2.1,

$$
\gamma = \sigma f_X(s) g_1(a_0)\{1 + O(\sigma)\} \, .
$$

If U is not normally distributed, but still with expectation μ and variance σ^2, we get, again by Lemma 2.1,

$$
CL(t_0) = \sigma f_X(s) h_1(a_0)\{1 + O(\sigma)\}
$$

and hence we have the following theorem.

Theorem 2.1: Assume (A1). For the normal test limit t_0 we have

$$\frac{CL(t_0)}{\gamma} = \frac{h_1(a_0)}{g_1(a_0)}(1 + O(\sigma)) \tag{2.10}$$

as $\sigma \to 0$.

It is clear from (2.10) that the behaviour of $h_1(a_0)/g_1(a_0)$ determines the performance of the normal test limit t_0.

The following examples show that the ratio $h_1(a)/g_1(a)$ can differ much from one to both sides, even for symmetric and unimodal densities g of V and choices of a between 0 and 3.

Example 2.1: In this example it is shown that there exist continuous symmetric unimodal densities g for which $h_1(a)$ is arbitrarily small for any $a > 0$.

Consider densities of the form

$$g_\varepsilon(v) = \begin{cases} c + \left(\frac{d-c}{\varepsilon}\right)v & \text{if } 0 \le v < \varepsilon \\ \dfrac{d(\varepsilon^{-1} - v)}{\varepsilon^{-1} - \varepsilon} & \text{if } \varepsilon \le v \le \varepsilon^{-1} \\ 0 & \text{if } v \ge \varepsilon^{-1} \end{cases}$$

$$g_\varepsilon(-v) = g_\varepsilon(v)$$

with c and d such that $\int g_\varepsilon(v)dv = 1$, $\int g_\varepsilon(v)v^2 dv = 1$, implying $c = \varepsilon^{-1}(1 + o(1))$, $d = 6\varepsilon^3(1 + o(1))$ as $\varepsilon \to 0$. Hence $\int_\varepsilon^\infty (v - \varepsilon)g_\varepsilon(v)dv \to 0$ as $\varepsilon \to 0$ and thus $\int_a^\infty (v - a)g_\varepsilon(v)dv \to 0$ for every $a > 0$ as $\varepsilon \to 0$. □

Example 2.2: In this example we take the Laplace distribution with expectation 0 and variance 1. Its density is given by

$$g(v) = 2^{-1/2}\exp\{-\sqrt{2}|v|\} .$$

Taking $a_0 = 2$ we get $h_1(a_0)/g_1(a_0) = 2.46$ and for $a_0 = 3$ we obtain $h_1(a_0)/g_1(a_0) = 13.3$.

For the density

$$g(v) = \tfrac{1}{2}\sqrt{30}\exp\{-[\sqrt{120}|v|]^{1/2}\}$$

we get $h_1(2)/g_1(2) = 3.88$ and $h_1(3)/g_1(3) = 40.9$.

Note that the densities in this example are also continuous, symmetric and unimodal. □

From these examples we conclude that a test limit based on the assumption of normality of the measurement error indeed can lead to severe violation of the bound γ if normality fails, due to differences in the tails of the true distribution and the normal distribution.

Approximation by the normal distribution is often sufficiently well justified, if dealing for instance with the expectation of a measurable quality characteristic. If dealing with the nonconforming probability, i.e. with a quantity related to the tails of the distribution, then the normal approximation can be arbitrarily bad, thus leading to completely useless results. Therefore, often there are situations, where the normal approximation should not be used even for continuous. symmetric and unimodal distributions looking very similar to the normal distribution.

Remark 2.1: In case of a $N(0, \sigma^2)$-distributed measurement error the only information needed is the variance σ^2. It is seen from Theorem 2.1 and Examples 2.1 and 2.2 that information about the variance (and the expectation) of the measurement error distribution is not sufficient in the general case. To get a first order accurate test limit we also need $h_1(a)$. □

Remark 2.2: One may hope that some knowledge on the distribution of U is available. Although in particular cases this may happen, one should realize that some *global* information is not enough. In Remark 2.1 it is already stated that "classical" quantities as the expectation and variance are not sufficient to get a reasonable result. Moreover, it is rather dangerous to rely on a vague idea on the shape of the distribution of U. It is already seen Example 2.2 that for interesting values of a_0 like 2 or 3, the ratio $h_1(a_0)/g_1(a_0)$ (which determines the performance of the test limit, cf. (2.10)) may differ much from 1. It is well-known that by eye the logistic distribution is close to the normal distribution. Nevertheless, for the logistic distribution we get $h_1(2)/g_1(2) = 1.70$ and $h_1(3)/g_1(3) = 6.24$. So, assuming that the distribution of U is known can only be done if one is pretty sure about it, especially w.r.t. its influence on $h_1(a)$. Otherwise, large errors may occur. □

Remark 2.3: One may think that since U is small w.r.t. X, the form of the distribution of U is not that critical and to assume that this distribution is known is a rather weak assumption. However, this is not true. The fact that U is small is very important in determining the test limit adequately as is seen from the error term $O(\sigma^2)$ in Theorem 3.1: the smaller σ, the better the result. However, having used the fact that σ is small, it is seen from (2.7), (2.9) and (2.10) that V, the standardized U, is the crucial quantity and hence the *form* of the distribution is important too. □

3 Asymptotically Second Order Unbiased Test Limits, $f_{\bar{X}}$ Known

3.1 Stochastic Test Limits

It is clearly seen in Theorem 2.1 and Examples 2.1 and 2.2 that nonnormality of the measurement error distribution can not be ignored in determination of test limits.

A straightforward solution to the problem would be to estimate the distribution of U (and X if it is also unknown) sufficiently well, take the estimates for the exact values, and fix the test limit t according to the defining relation given by (2.4). However, firstly, without analyzing the resulting procedure, one has no check on its quality and properties. Secondly, it is well-known from similar situations, that inserting estimates in test limits causes serious bias, cf. Albers et al. (1994a, 1994b, 1997), and a correction is needed. The effect of inserting estimates can be ignored only if the estimates are based on very large samples. The reasoning that for moderately large sample sizes the distribution of U (and X if it is also unknown) can be estimated rather well and hence the estimated consumer loss is reasonably close to γ, turns out to be quite misleading. Even if normality holds, considerable sample sizes (much larger than in "classical" estimation problems) will be required before the correction can be neglected. This is explained in detail, e.g. in Albers et al. (1994a, Section 3; 1994b, Section 3). Further, it should be noted that often there is in advance not much knowledge on the distribution of the measurement error, cf. the discussion on p. 98 of Albers et al. (1994a). In the general case, not assuming normality, yet larger sample sizes are needed. Therefore, as a rule, in practice the bias should be corrected.

To analyse the procedure and to correct for the bias, insight on the influence of the estimates on the behaviour of the test limits is required. The above mentioned straightforward solution is much too untransparant to give such insight. Hence, it can hardly be analysed and it seems not feasible to make the necessary bias correction.

Moreover, even if it would be possible to give a bias correction, while the starting point of this solution looks rather straightforward, one ends up with a far more complicated procedure, when developing a bias correction. This is partly due to the definition of the test limit, which is of an implicit form. Therefore, to find a bias correction the obvious way is to simplify things by approximations. But then it seems far more promising to take the need of approximations into account from the very beginning. It is shown e.g. in Albers et al. (1994a), that second order approximations in this context perform strikingly well and are for all practical purposes almost equal to the exact solution, still having the advantage of being explicit and analytically tractable.

Lemma 2.1 indicates the first step in such an approach, giving the first order approximation (the final test limit will be based on a second order approximation, cf. (3.12) and (3.14)): we should take

$$t = s + \mu - a\sigma \tag{3.1}$$

with a given by

$$h_1(a) = \frac{\gamma}{\sigma f_X(s)} \ . \tag{3.2}$$

However, in general μ and σ are unknown and also the function h_1 is unknown. (Even also $f_X(s)$ may be unknown.) These unknown parameters and unknown function(s) should be estimated by available observations. This implies that the resulting test limit $\hat{t}$ depends on these observations, leading in turn to stochastic CL's.

A new problem arises, since we now deal with a stochastic variable, rather than with a single number. In the latter case the criterion to be used is evident: the closer this number is to γ, the better the test limit is. In the stochastic case more choices can be made. In this section and the next one we take the viewpoint of unbiased estimation. So we look for a test limit such that the resulting stochastic CL, denoted by $CL(\hat{t})$, makes

$$|E[CL(\hat{t})] - \gamma| \tag{3.3}$$

sufficiently small.

3.2 Model and Basic Assumptions

Let X be the true value of the product characteristic and U the measurement error. Then we observe

$$\tilde{X} = X + U \ .$$

In this section we consider the situation, often occurring in practice, that we have so many production data that the density $f_{\tilde{X}}$ (and its derivative) may be assumed to be known. With the notation as in section 2 we have

$$\tilde{X} = X + \mu - \sigma V \tag{3.4}$$

and hence

$$f_{\tilde{X}}(s+\mu) = \int_{-\infty}^{\infty} f_X(s+\sigma v)g(v)dv = f_X(s) + \tfrac{1}{2}\sigma^2 f_X''(s) + \cdots \ . \tag{3.5}$$

Therefore the following assumptions will typically hold ((A3) even with $O(\sigma^2)$):

$$f_{\tilde{X}}(s+\mu) = f_X(s) + O(\sigma^2) \qquad \text{as} \quad \sigma \to 0 \ ; \tag{A2}$$

$$f'_{\tilde{X}}(s+\mu) = f'_X(s) + O(\sigma) \qquad \text{as} \quad \sigma \to 0 \ . \tag{A3}$$

Further assume that

$$f''_{\tilde{X}} \text{ and } f''_X \text{ are bounded} \ ; \tag{A4}$$

$$E[|V|^r] < \infty \text{ for some } r > 6 \ . \tag{A5}$$

The parameters μ, σ and the density g of V are unknown. For estimation i.i.d. r.v.'s $U_1, \ldots, U_n$ are available, each U_i distributed according to U. Often in practice the random sample has limited size.

3.3 Definition of the New Test Limit

Define for $k = 0, 1, 2$

$$r_k(d) = E[(-U-d)^k 1_{\{-U-d>0\}}] \ . \tag{3.6}$$

The relation between r_k and h_k (cf. (2.2) and (2.7)) is

$$r_k(d) = \sigma^k h_k(\sigma^{-1}(\mu+d)) \ . \tag{3.7}$$

In view of (3.1) and (3.7) now (3.2) reads as

$$r_1(s-t) = \gamma / f_X(s) \ . \tag{3.8}$$

Estimating r_k by

$$\hat{r}_k(d) = n^{-1} \sum_{i=1}^{n} (-U_i - d)^k 1_{\{-U_i - d > 0\}} \tag{3.9}$$

and μ by

$$\hat{\mu} = n^{-1} \sum_{i=1}^{n} U_i , \tag{3.10}$$

(3.8) and (A2) suggest the first order accurate test limit

$$\hat{t}_1 = s - \hat{r}_1^{-1}(\gamma / f_{\bar{X}}(s + \hat{\mu})) = s - \hat{d}_1 , \qquad \text{say} . \tag{3.11}$$

Note that $\hat{r}_1$ is a piecewise linear decreasing function and hence $\hat{r}_1^{-1}$ is well defined. To get (3.3) up to second order we have to add an additional term for correcting the second order term in the Taylor expansion (one term further than in the proof of Lemma 2.1):

$$\hat{c} = \frac{1}{2} \frac{f'_{\bar{X}}(s + \hat{\mu})}{f_{\bar{X}}(s + \hat{\mu})} \frac{\hat{r}_2(\hat{d}_1)}{\hat{r}_0(\hat{d}_1)} . \tag{3.12}$$

Further we have to correct for the bias involved by taking estimators at several places. It will turn out that the appropriate correction term for this effect equals

$$\hat{c}_u = \frac{\hat{r}_1(\hat{d}_1)}{n} \frac{\{1 - \hat{r}_0(\hat{d}_1)\}}{\hat{r}_0^2(\hat{d}_1)} \tag{3.13}$$

and the resulting new test limit is

$$\hat{t} = \hat{t}_1 - \hat{c} - \hat{c}_u = s - (\hat{d}_1 + \hat{c} + \hat{c}_u) . \tag{3.14}$$

3.4 Second Order Unbiasedness if μ is Unknown

Although the measurement error distribution may by nonnormal in many practical situations, still its mean may be zero or otherwise known. Therefore, we here first discuss the situation that μ is known. In that case at all places where $\hat{\mu}$ occurs, implicit or explicit, in (3.11)–(3.14) one should replace it by μ.

Let

$$a_1 = h_1^{-1}\left(\frac{\gamma}{\sigma f_{\tilde{X}}(s+\mu)}\right) \tag{3.15}$$

and

$$d_1 = r_1^{-1}\left(\frac{\gamma}{f_{\tilde{X}}(s+\mu)}\right) , \tag{3.16}$$

then

$$a_1 = \frac{d_1 + \mu}{\sigma} . \tag{3.17}$$

We put the following mild assumptions

$$\begin{aligned} &a_1 \text{ is bounded as } \sigma, \gamma \to 0 \\ &g(a_1) > 0 \\ &g'(a) \text{ is bounded on } |a - a_1| \leq \varepsilon \text{ for some } \varepsilon > 0 . \end{aligned} \tag{A6}$$

The following theorem expresses that $\hat{t}$ is a second order unbiased test limit, thus satisfying the aim as mentioned in (3.3).

Theorem 3.1: Assume (A1)–(A6). Then

$$E[CL(\hat{t})] = \gamma(1 + o(n^{-1}) + O(\sigma^2)) + O(n^{-r/4}) \tag{3.18}$$

as $n \to \infty$ and σ, $\gamma \to 0$.

The proof of Theorem 3.1 is given in section 7.

3.5 *Simulation Results*

To demonstrate Theorem 3.1 for finite sample sizes, simulations have been carried out. For each replication a sample of size n of measurement errors $U_1, \ldots, U_n$ is simulated, the test limit $\hat{t}$ is computed according to (3.14), and the corresponding consumer loss $CL(\hat{t})$ is computed numerically. This total procedure is executed 10000 times. Based on the 10000 replications, the average and standard deviation of the consumer loss is computed if observations on the measurement error are from a normal or a Γ-distribution. In order to see in what way the bias correction $\hat{c}_u$, cf. (3.13), contributes, we also computed the 'uncorrected' test limit,

$$\hat{t}_2 = \hat{t}_1 - \hat{c} \tag{3.19}$$

(again with $\hat{\mu}$ replaced by the known μ).

Further, for comparison, the normal test limit $\hat{t}_N$ is presented.

Firstly, it is seen in Table 1 and Table 2 that Theorem 3.1, which is based on asymptotics, gives good approximation results for finite sample sizes. This is seen by comparing the simulated $E[CL(\hat{t})]$'s with γ. Moreover, the approximation for $E[CL(\hat{t}_2)]$ works very well also (compare theoretical $E[CL(\hat{t}_2)]$ with simulated $E[CL(\hat{t}_2)]$.

Secondly, it is seen that already $\hat{t}_2$ gives quite accurate results with still a significant improvement by the bias correction. As a consequence $E[CL(\hat{t})]$ is almost equal to γ, except when $r_0(d_1)$ and n are too small. (Note that by definition $\hat{r}_0(\hat{d}_1) \geq n^{-1}$ and hence $r_0(d_1)$ and consequently the correction term c_u cannot be accurately estimated if r_0 and n are too small.)

Thirdly, in the normal case we are loosing not much compared to the normal test limit (which is especially constructed for that case), while with the gamma distribution we get a large gain. This is exactly what is wanted.

Finally, both for $\hat{t}$, $\hat{t}_2$ and the normal test limit the standard deviation is rather high. This is no problem if all parts are shipped to the same consumer. If this is not the case, we can hardly expect that a consumer who complains about receiving parts with 215 ppm rather than 100 ppm, will be soothed much if he is told that his competitor received only 30 ppm, thus making the average more correct! This point is further discussed in section 5.

3.6 *Estimation of μ*

When μ is unknown, it is estimated by (3.10) leading to the test limit $\hat{t}$ as given in subsection 3.3. It can be shown that, to the order considered, no additional

Table 1. Test limit $\hat{t}$ with a $N(0,\sigma^2)$-distributed measurement error

	theory	simulated		
$(\gamma,\sigma,\pi) = (100, 0.01, 0.01)$, $r_0(d_1) = 0.481$				
n	$\mathrm{E}[CL(\hat{t}_2)]$	$\mathrm{E}[CL(\hat{t}_2)]$	$\mathrm{E}[CL(\hat{t})]$	$\mathrm{E}[CL(\hat{t}_N)]$
40	102.7	102.7 (24.2)	100.0 (24.1)	100.0 (11.6)
80	101.4	101.4 (16.9)	100.1 (16.8)	100.0 (8.2)
500	100.2	100.2 (6.8)	100.0 (6.8)	100.0 (3.3)
2000	100.1	100.1 (3.4)	100.0 (3.4)	100.0 (1.6)
$(\gamma,\sigma,\pi) = (100, 0.10, 0.01)$, $r_0(d_1) = 0.082$				
n	$\mathrm{E}[CL(\hat{t}_2)]$	$\mathrm{E}[CL(\hat{t}_2)]$	$\mathrm{E}[CL(\hat{t})]$	$\mathrm{E}[CL(\hat{t}_N)]$
40	127.9	125.3 (77.4)	101.9 (73.3)	100.3 (43.6)
80	113.9	112.3 (51.5)	99.5 (50.0)	99.4 (30.0)
500	102.2	102.1 (19.5)	99.9 (19.4)	100.0 (12.0)
2000	100.6	100.7 (9.7)	100.1 (9.7)	100.1 (6.0)
$(\gamma,\sigma,\pi) = (20, 0.01, 0.15)$, $r_0(d_1) = 0.023$				
n	$\mathrm{E}[CL(\hat{t}_2)]$	$\mathrm{E}[CL(\hat{t}_2)]$	$\mathrm{E}[CL(\hat{t})]$	$\mathrm{E}[CL(\hat{t}_N)]$
40	41.3	43.2 (40.8)	31.9 (36.3)	20.3 (15.6)
80	30.6	31.0 (24.8)	23.5 (22.9)	20.1 (10.4)
500	21.7	21.7 (8.2)	20.0 (8.1)	19.9 (4.0)
2000	20.4	20.4 (4.0)	20.0 (4.0)	20.0 (2.0)

The table shows simulated mean and standard deviation (between brackets) of $CL(\hat{t}_2)$, $CL(\hat{t})$ and of $CL(\hat{t}_N)$, based on 10000 replications. Moreover the approximation to the mean of $CL(\hat{t}_2)$ is shown. The values of γ are in ppm, π denotes $P(X > s)$, the probability that a product is nonconforming, and n is the number of observations on the measurement error.
The characteristic is standard normally distributed.

correction term is needed for estimating μ. Therefore, Theorem 3.1 continuous to hold in this more general situation. We do not present here the technical details, but refer to section 6.3.4 in Otten (1995).

Simulations as the ones presented in Table 1 and 2 but with estimation of μ by $\hat{\mu}$, indeed yield results which are practically the same as the results obtained when μ is known. Numerical examples are therefore omitted.

4 Asymptotically Second Order Unbiased Test Limits, $f_{\tilde{X}}$ Unknown

We start with investigating the situation where μ is *known*. When the density of $\tilde{X}$ is estimated, an additional correction term is needed to the test limit

Table 2. Test limit $\hat{t}$ with a $\Gamma(8)$-distributed measurement error

	theory	simulated		
$(\gamma, \sigma, \pi) = (100, 0.01, 0.01)$, $r_0(d_1) = 0.533$				
n	$E[CL(\hat{t}_2)]$	$E[CL(\hat{t}_2)]$	$E[CL(\hat{t})]$	$E[CL(\hat{t}_N)]$
40	102.2	102.1 (20.7)	99.9 (20.5)	98.9 (14.7)
80	101.1	101.2 (14.6)	100.1 (14.6)	98.9 (10.4)
500	100.2	100.1 (5.7)	99.9 (5.7)	98.7 (4.2)
2000	100.0	100.0 (2.9)	100.0 (2.9)	98.7 (2.1)
$(\gamma, \sigma, \pi) = (100, 0.10, 0.01)$, $r_0(d_1) = 0.120$				
n	$E[CL(\hat{t}_2)]$	$E[CL(\hat{t}_2)]$	$E[CL(\hat{t})]$	$E[CL(\hat{t}_N)]$
40	118.2	116.6 (60.2)	99.6 (57.0)	46.9 (39.0)
80	109.1	107.6 (40.0)	98.9 (38.6)	43.7 (26.6)
500	101.5	101.1 (15.3)	99.7 (15.2)	40.6 (10.5)
2000	100.4	100.2 (7.6)	99.9 (7.6)	40.2 (5.2)
$(\gamma, \sigma, \pi) = (20, 0.01, 0.15)$, $r_0(d_1) = 0.040$				
n	$E[CL(\hat{t}_2)]$	$E[CL(\hat{t}_2)]$	$E[CL(\hat{t})]$	$E[CL(\hat{t}_N)]$
40	32.1	33.4 (29.3)	24.4 (26.6)	2.7 (5.9)
80	26.0	26.1 (17.2)	20.8 (16.0)	1.7 (2.9)
500	21.0	21.0 (5.9)	20.1 (5.8)	1.0 (0.7)
2000	20.2	20.3 (2.8)	20.0 (2.8)	0.8 (0.3)

This table summarizes the same simulation as in Table 1, however, with a measurement error which has a Γ-distribution with shape parameter 8 and with the location and scale parameter such that the mean is 0 and the variance σ^2.

which has been derived in section 3. It will be seen by a simple heuristic argument that estimation of the measurement error distribution and estimation of $f_{\tilde{X}}$ can be dealt with separately. (The same phenomenon of no mix-up of errors is present in Albers et al. (1994b, cf. formula (3.11)).) To avoid too many technicalities we do not give a rigorous proof.

It is intuitively clear that correcting for estimating $f_{\tilde{X}}$ may be restricted to the first order term of $E[CL(\hat{t})]$ as $\hat{c}_u$ in (3.13) also only is concerned with the first order term of $E[CL(\hat{t})]$, cf. (7.28) and (7.34).

To derive the additional correction term, let $\hat{f}_{\tilde{X}}(s+\mu)$ be the estimator of $f_{\tilde{X}}(s+\mu)$ and define

$$q = \frac{\gamma}{\sigma f_{\tilde{X}}(s+\mu)} , \qquad \hat{q} = \frac{\gamma}{\sigma \hat{f}_{\tilde{X}}(s+\mu)} , \tag{4.1}$$

and

$$a_1 = h_1^{-1}(q) , \qquad \hat{a}_1 = \hat{h}_1^{-1}(\hat{q}) . \tag{4.2}$$

With test limit $\hat{t}_1 = s + \mu - \hat{a}_1\sigma$ the consumer loss $CL(\hat{t}_1)$ is approximated by, cf. Lemma (2.1) and condition (A2),

$$\sigma f_{\bar{X}}(s+\mu)h_1(\hat{a}_1) \ ,$$

and its relative error by

$$\frac{CL(\hat{t}_1)-\gamma}{\gamma} \approx \frac{h_1(\hat{a}_1)-h_1(a_1)}{h_1(a_1)} = \frac{\hat{h}_1(\hat{a}_1)-h_1(a_1)}{h_1(a_1)} + \frac{h_1(\hat{a}_1)-\hat{h}_1(\hat{a}_1)}{h_1(a_1)} \ . \tag{4.3}$$

The first term in (4.3) is equal to

$$\frac{\hat{q}-q}{h_1(a_1)} = \frac{f_{\bar{X}}(s+\mu)}{\hat{f}_{\bar{X}}(s+\mu)} - 1 = -\left(\frac{\hat{f}_{\bar{X}}(s+\mu)}{f_{\bar{X}}(s+\mu)} - 1\right) + \left(\frac{\hat{f}_{\bar{X}}(s+\mu)}{f_{\bar{X}}(s+\mu)} - 1\right)^2 + \cdots \ . \tag{4.4}$$

For the second term in (4.3) we write

$$\begin{aligned} h_1(\hat{a}_1) - \hat{h}_1(\hat{a}_1) &= h_1(\hat{h}_1^{-1}(\hat{q})) - \hat{q} \\ &= h_1(\hat{h}_1^{-1}(q)) - q + (\hat{q}-q)\left(\frac{h_1'(\hat{h}_1^{-1}(\xi))}{\hat{h}_1'(\hat{h}_1^{-1}(\xi))} - 1\right) , \end{aligned} \tag{4.5}$$

with ξ between q and $\hat{q}$. Note that $\hat{h}_1'$ is the derivative of $\hat{h}_1$ and not some newly defined estimator of h_1'.

The first term in (4.5) is already studied in the proof of Theorem 3.1, cf. (7.28), (7.26) and (7.3). The second factor in the last term of (4.5) will be small and hence its expectation will be of smaller order than $E[\hat{q}-q]$, appearing in the first term of (4.3). Together with (4.3) and (4.4) we therefore obtain

$$\frac{E[CL(\hat{t}_1)]-\gamma}{\gamma} \approx E\left[\frac{\hat{q}-q}{h_1(a_1)} + \frac{h_1(\hat{h}_1^{-1}(q))-q}{h_1(a_1)}\right] \ . \tag{4.6}$$

In view of (4.6) we see that indeed estimation of $f_{\bar{X}}$ and estimation of the measurement error distribution can be done separately. The first term on the right-hand side of (4.6) concerns estimation of $f_{\bar{X}}$ (cf. (4.4)), the second term concerns the estimation of the measurement error distribution with known $f_{\bar{X}}$. For the latter the correction term was already derived in section 3. What remains is an additional correction term in $\hat{c}_u$ to cancel

$$\frac{E[\hat{q}-q]}{h_1(a_1)} .$$

Hence, cf. (7.28), the additional correction term should be equal to

$$\frac{h_1(a_1)}{h_0(a_1)} \frac{E[\hat{q}-q]}{h_1(a_1)} .$$

To estimate the density (and its derivative) we assume that we have independent observations $\tilde{X}_1, \ldots, \tilde{X}_m$ from $\tilde{X}$, which are also independent of the observations $U_1, \ldots, U_n$ from the measurement error. We take Rosenblatt's estimator (by $(\hat{f'}_{\tilde{X}})$ we denote the estimator of $f'_{\tilde{X}}$ to avoid possible confusion with the derivative of $\hat{f}_{\tilde{X}}$, which is denoted by $\hat{f}'_{\tilde{X}}$)

$$\hat{f}_{\tilde{X}}(s+\mu) = \frac{1}{2mh}\sum_{i=1}^{m} Z_i \qquad \text{and} \qquad (\hat{f'}_{\tilde{X}})(s+\mu) = \frac{1}{m\bar{h}^2}\sum_{i=1}^{m} \bar{Z}_i , \tag{4.7}$$

where

$$Z_i = \begin{cases} 1 & \text{if} \quad \tilde{X}_i \in [s+\mu-h, s+\mu+h] \\ 0 & \text{otherwise} , \end{cases}$$

$$\bar{Z}_i = \begin{cases} -1 & \text{if} \quad \tilde{X}_i \in [s+\mu-\bar{h}, s+\mu] \\ 1 & \text{if} \quad \tilde{X}_i \in (s+\mu, s+\mu+\bar{h}] \\ 0 & \text{otherwise} , \end{cases} \tag{4.8}$$

$$h = \hat{\tau}\{m\phi((s+\mu-\hat{\nu})/\hat{\tau})\}^{-1/2}$$

and

$$\bar{h} = \hat{\tau}\{m\phi((s+\mu-\hat{\nu})/\hat{\tau})\}^{-1/4} .$$

Here in the bandwidths $\hat{\nu}$ and $\hat{\tau}$ are the sample mean and sample standard deviation of $\tilde{X}$. (For a discussion on the choice of the estimator and the bandwidths we refer to Albers et al. (1997).)

As a consequence we have

$$\frac{E[\hat{q} - q]}{h_1(a_1)} \approx \frac{1}{2mhf_{\bar{X}}(s+\mu)} - \frac{1}{m} \; . \tag{4.9}$$

The test limit then becomes, cf. (3.14),

$$\hat{t} = s - (\hat{d}_1 + \hat{c} + \hat{c}_u) \; , \tag{4.10}$$

with

$$\begin{aligned} \hat{d}_1 &= \hat{r}_1^{-1}\left(\frac{\gamma}{\hat{f}_{\bar{X}}(s+\mu)}\right) \\ \hat{c} &= \frac{1}{2}\,\frac{(\hat{f}'_{\bar{X}})(s+\mu)}{\hat{f}_{\bar{X}}(s+\mu)}\,\frac{\hat{r}_2(\hat{d}_1)}{\hat{r}_0(\hat{d}_1)} \\ \hat{c}_u &= \frac{\hat{r}_1(\hat{d}_1)}{n}\,\frac{\{1-\hat{r}_0(\hat{d}_1)\}}{\hat{r}_0^2(\hat{d}_1)} + \frac{\hat{r}_1(\hat{d}_1)}{\hat{r}_0(\hat{d}_1)}\left(\frac{1}{2mh\hat{f}_{\bar{X}}(s+\mu)} - \frac{1}{m}\right) . \end{aligned} \tag{4.11}$$

To see how this test limit behaves for finite sample sizes, simulations are carried out for the same situations as in Table 1 and Table 2, see Table 3 and Table 4.

The results indicate that $m = 100$ observations are not enough to estimate the density in the tail of the distribution. If a sufficiently large number of observations is available the consumer losses are quite close to γ. Estimation of the measurement error works very well, except when $r_0(d_1)$ and n are too small. This is seen by noting that there is only a minor difference between $n = 80$ and $n = 2000$. If μ is *unknown*, it is estimated by (3.10). Replacing in (4.10) and (4.11) μ by $\hat{\mu}$ (and of course also in (4.7) and (4.8)) the desired test limit is obtained. As in section 3 no further correction term is needed.

Remark 4.1: The distribution of the product characteristic X may be not constant in time in certain practical situations. This issue is not specific for the new test limit, derived in this paper. The question whether one may assume that the distribution of X (and also of the measurement error U) does not depend on time is more general and can be put for all test limits.

Test limits are not used for one single new item, but a whole series of new items is judged. After some time the test limit will be updated. The longer one is going to use a certain limit, the better it should be and hence the more observations are needed for the estimation part. This intuitive feeling is made

Table 3. Test limit $\hat{t}$ with a $N(0, \sigma^2)$-distributed measurement error

$(\gamma, \sigma, \pi) = (100, 0.01, 0.01)$, $r_0(d_1) = 0.481$

	($m = 100$)		($m = 400$)		($m = 1600$)	
n	$E[CL(\hat{t})]$	nr	$E[CL(\hat{t})]$	nr	$E[CL(\hat{t})]$	nr
40	67.0 (28.9)	43	90.5 (39.7)	37	98.0 (36.7)	1
80	67.0 (25.4)	33	90.1 (36.2)	27	98.2 (33.6)	1
500	66.8 (21.9)	34	90.4 (33.5)	5	97.8 (29.8)	1
2000	66.4 (21.4)	21	89.7 (32.3)	4	98.1 (29.9)	–

$(\gamma, \sigma, \pi) = (100, 0.10, 0.01)$, $r_0(d_1) = 0.082$

	($m = 100$)		($m = 400$)		($m = 1600$)	
n	$E[CL(\hat{t})]$	nr	$E[CL(\hat{t})]$	nr	$E[CL(\hat{t})]$	nr
40	74.6 (64.9)	23	96.9 (81.7)	4	99.6 (77.5)	–
80	69.7 (47.8)	32	92.4 (63.6)	3	97.9 (58.9)	–
500	67.9 (30.4)	25	93.1 (50.7)	2	97.5 (39.1)	–
2000	68.2 (27.0)	25	92.7 (48.5)	3	97.6 (34.5)	–

$(\gamma, \sigma, \pi) = (20, 0.01, 0.15)$, $r_0(d_1) = 0.023$

	($m = 100$)		($m = 400$)		($m = 1600$)	
n	$E[CL(\hat{t})]$	nr	$E[CL(\hat{t})]$	nr	$E[CL(\hat{t})]$	nr
40	32.0 (37.4)	–	32.5 (37.6)	–	32.7 (37.1)	–
80	23.9 (23.3)	–	24.1 (23.5)	–	24.0 (23.0)	–
500	19.9 (10.2)	–	20.1 (9 3)	–	20.2 (8.8)	–
2000	19.8 (7.5)	–	20.0 (6.1)	–	20.0 (5.2)	–

The table shows simulated mean and standard deviation (between brackets) of $CL(\hat{t})$, with $\hat{t}$ from (4.10) based on 10000 replications. To estimate the density and its derivative Rosenblatt's estimators are applied. The number of observations to estimate the density is denoted by m. In the table, nr denotes the number of replications for which the test limit could not be determined (caused by a very small estimate of the density). The values of γ are in ppm, π denotes $P(X > s)$, the probability that a product is nonconforming, and n is the number of observations on the measurement error. The characteristic is standard normally distributed.

more precise in Albers et al. (1994b). It has been shown there that, at first order, there is no loss in working with a number of smaller samples, each leading to its own estimates, as compared to using one single, very large sample. Moreover, from the perspective of robustness, it is even quite attractive to work with a number of separate steps, as this will provide better protection against deviations from the assumption that the production process is stationary. For more details we refer to Albers et al. (1994b), Section 5.

Of course, in cases of strong time-dependence, the dependence of time should be inserted more directly. However, one should have an idea of the form of this dependence, since it should be modelled explicitly to derive test limits. If the dependence is strong enough and can be modelled adequately, it may be useful to do it. Investigations in this area may lead to new interesting problems. We do not treat such models in this paper.

Table 4. Test limit $\hat{t}$ with a $\Gamma(8)$-distributed measurement error

$(\gamma, \sigma, \pi) = (100, 0.01, 0.01)$, $r_0(d_1) = 0.533$

	($m = 100$)		($m = 400$)		($m = 1600$)	
n	$\mathrm{E}[CL(\hat{t})]$	nr	$\mathrm{E}[CL(\hat{t})]$	nr	$\mathrm{E}[CL(\hat{t})]$	nr
40	66.0 (26.5)	35	90.0 (37.5)	28	97.8 (34.9)	1
80	66.1 (23.7)	19	90.4 (35.0)	16	97.6 (32.5)	1
500	66.0 (20.9)	24	89.8 (32.8)	4	98.0 (29.9)	–
2000	65.7 (20.4)	27	90.3 (32.1)	4	97.8 (28.7)	1

$(\gamma, \sigma, \pi) = (100, 0.10, 0.01)$, $r_0(d_1) = 0.120$

	($m = 100$)		($m = 400$)		($m = 1600$)	
n	$\mathrm{E}[CL(\hat{t})]$	nr	$\mathrm{E}[CL(\hat{t})]$	nr	$\mathrm{E}[CL(\hat{t})]$	nr
40	70.6 (53.4)	26	94.6 (70.2)	3	99.5 (65.3)	–
80	68.2 (39.4)	34	93.7 (58.8)	9	99.4 (51.1)	–
500	68.1 (27.3)	24	93.5 (45.9)	7	99.4 (35.8)	–
2000	68.4 (25.4)	40	92.7 (42.3)	3	99.2 (33.6)	–

$(\gamma, \sigma, \pi) = (20, 0.01, 0.15)$, $r_0(d_1) = 0.040$

	($m = 100$)		($m = 400$)		($m = 1600$)	
n	$\mathrm{E}[CL(\hat{t})]$	nr	$\mathrm{E}[CL(\hat{t})]$	nr	$\mathrm{E}[CL(\hat{t})]$	nr
40	24.1 (26.8)	–	24.4 (26.9)	–	24.5 (27.1)	–
80	21.0 (17.5)	–	21.2 (17.2)	–	21.0 (16.6)	–
500	19.8 (8.4)	–	19.9 (7.3)	–	20.1 (6.6)	–
2000	19.7 (6.9)	–	20.0 (5.5)	–	20.0 (4.3)	–

This table summarizes the same simulation as in Table 3, however, with a measurement error which has a Γ-distribution with shape parameter 8 and with the location and scale parameter such that the mean is 0 and the variance σ^2.

Even if one assumes no change in time, choosing n and m constitutes an important problem. The effect on the accuracy w.r.t. n when $f_{\tilde{X}}$ is known is given in Theorem 3.1. Estimating $f_{\tilde{X}}$ results in case of normal measurement errors in an additional error term of order $O(\gamma m^{-1})$, cf. Theorem 3.3 in Albers et al. (1997). The simulation results in Table 3 and 4 show that also here the improvement from $m = 100$ to 400 and from $m = 400$ to 1600 is twice about a factor 4 in most cases. □

5 Test Limits for Which γ is Violated With Small Probability

It is seen from Tables 1–4 that the new test limits accurately correct for the bias. However, the standard deviation is not small and hence $CL(\hat{t})$ will vary widely around γ. For certain applications, for example one and the same consumer receiving all batches, this will be no serious problem, since the long run average of the consumer loss will indeed tend to γ, notwithstanding the considerable variation between batches.

In other applications a stronger condition will be more satisfactory. A test limit $\hat{t}_v$ (with v from 'violated') is required such that, for some small positive α,

$$P(CL(\hat{t}_v) > \gamma) = \alpha \tag{5.1}$$

with sufficient precision. This approach, related to the well-known confidence-interval concept, guarantees that in the long run in a fraction of 1-α of the cases the consumer loss will be at most γ.

From (7.27), (7.28) and the definition of $\hat{a}_1$ in (7.3) it follows by Lemma 7.2 that $CL(\hat{t})$ is asymptotically normal. For the unbiased test limit this would imply (5.1) with $\alpha = 0.5$. To get smaller α we modify the test limit by introducing a negative bias.

Let

$$\hat{t}_v = s + \mu - \hat{a}_1\sigma - \hat{c} - \hat{c}_v \tag{5.2}$$

with

$$\hat{a}_1 = \hat{h}_1^{-1}\left(\frac{\gamma}{\sigma\hat{f}_{\tilde{X}}(s+\mu)}\right), \tag{5.3}$$

the correction term $\hat{c}$ as in (4.11) and $\hat{c}_v$ such that

$$P(CL(\hat{t}_v) \geq \gamma) = \alpha \tag{5.4}$$

is obtained to sufficient precision, for a given α.

From (7.27) and (7.28), neglecting higher order terms, in combination with (4.3), (4.4) and (4.5) it follows that

$$\frac{CL(\hat{t}_v) - \gamma}{\gamma} \approx (\hat{a}_1 - a_1)\frac{h_1'(a_1)}{h_1(a_1)} - \left(\frac{\hat{f}_{\tilde{X}}(s+\mu)}{f_{\tilde{X}}(s+\mu)} - 1\right)$$
$$+ \left(\frac{\hat{f}_{\tilde{X}}(s+\mu)}{f_{\tilde{X}}(s+\mu)} - 1\right)^2 + c_v\frac{h_1'(a_1)}{h_1(a_1)} .$$

Rosenblatt's estimator is applied to estimate the density, cf. (4.7) and (4.8). It follows from Lemma 7.2 and section 4 that the right-hand side is asymptotically normal $AN(\mu_{CL}, \sigma^2_{CL})$ with (neglecting terms of order n^{-1}, h^2 and $(mh)^{-1}$ in μ_{CL})

$$\mu_{CL} = c_v \frac{h_1'(a_1)}{h_1(a_1)}$$

and, cf. (7.17) and (7.24),

$$\sigma^2_{CL} = \frac{h_2(a_1) - h_1^2(a_1)}{nh_1^2(a_1)} + \frac{1}{2\,mhf_{\bar{X}}(s+\mu)} \; .$$

Hence c_v should be taken

$$c_v = -u_\alpha \frac{h_1(a_1)}{h_1'(a_1)} \sqrt{\frac{h_2(a_1) - h_1^2(a_1)}{nh_1^2(a_1)} + \frac{1}{2\,mhf_{\bar{X}}(s+\mu)}} \; ,$$

where $u_\alpha = \Phi^{-1}(1-\alpha)$.

The test limit thus becomes

$$\hat{t}_v = s - \hat{d}_1 - \hat{c} - \hat{c}_v \; ,$$

with

$$\hat{d}_1 = \hat{r}_1^{-1}\left(\frac{\gamma}{\hat{f}_{\bar{X}}(s+\mu)}\right)$$

$$\hat{c} = \frac{1}{2} \frac{(\hat{f'}_{\bar{X}})(s+\mu)}{\hat{f}_{\bar{X}}(s+\mu)} \frac{\hat{r}_2(\hat{d}_1)}{\hat{r}_0(\hat{d}_1)}$$

$$\hat{c}_v = u_\alpha \frac{\hat{r}_1(\hat{d}_1)}{\hat{r}_0(\hat{d}_1)} \sqrt{\frac{\hat{r}_2(\hat{d}_1) - \hat{r}_1^2(\hat{d}_1)}{n\hat{r}_1^2(\hat{d}_1)} + \frac{1}{2\,mh\hat{f}_{\bar{X}}(s+\mu)}} \; .$$

When μ is unknown, it should be estimated by $\hat{\mu}$, given in (3.10). Our experience with the approach in this section is that it works very well for finite samples, cf. e.g. Albers et al. (1994a, Section 3).

6 An Application in Semiconductor Industry

As an example of the theory we consider a television color decoder TDA9162/N1 manufactured at Philips' consumer IC plant at Nijmegen. From several characteristics specified we choose one that should be below $s = 670.0$. For $n = 44$ products both a standard measurement and a precise laboratory measurement is carried out. From production $m = 2732$ observations are available. Two histograms summarize the data.

The histograms indicate that the assumption of normality for neither the characteristic nor the measurement error is justified. A plot (not presented

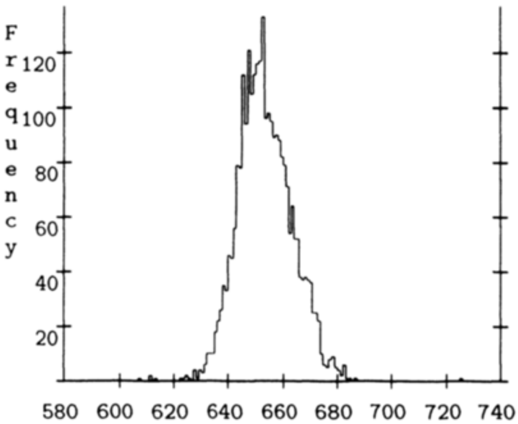

Fig. 1. Histogram for production data.

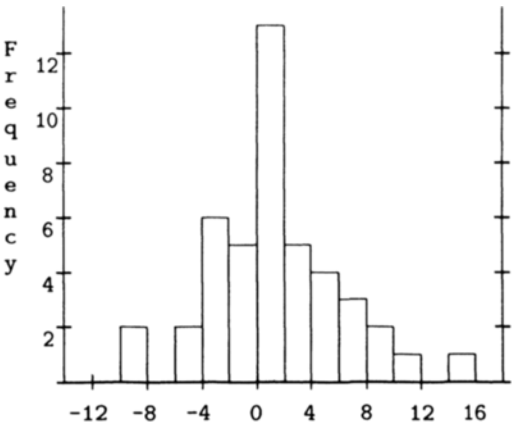

Fig. 2. Histogram for the measurement error.

here) of the values of X_i against the values of $U_i (i = 1, \dots, n)$ shows that it is reasonable to assume that the measurement error and the inspected characteristic are independent.

The estimated mean of the measurement error is equal to 1.38, therefore we estimate the density (and its derivative) at $s + \hat{\mu} = 671.38$. To apply Rosenblatt's estimators, first we determine the bandwidths, cf. (4.8). The sample mean and sample standard deviation of the $\tilde{X}$-observations are 653.6 and 10.02, respectively, leading to $h = 0.666$ and $\bar{h} = 2.584$. We find $\hat{f}_{\tilde{X}}(s + \hat{\mu}) = 0.0096$ and $(\hat{f}'_{\tilde{X}})(s + \hat{\mu}) = -0.002$.

Suppose the bound on the consumer loss is $\gamma = 100$ ppm. Then in case of unbiased estimation we find $\hat{t} = 661.92$, cf. (4.10). Since m is very large, the part of the correction $\hat{c}_u$, cf. (4.11) which corrects for the estimation of $f_{\tilde{X}}$ may be omitted. (Its value is 0.006.) If the consumer loss should exceed γ with probability $\alpha = 0.10$ only, we find $\hat{t}_v = 661.70$, cf. (5.2). (Again we omit the part of the correction term with m in the denominator.)

About the reliability of the test limit we remark the following. The simulation results show that the accuracy of the test limit depends on the value of $r_0(d_1) = P(U < -d_1)$. In the present situation there are two observations to the left of $-\hat{d}_1 = -7.98$. The simulation results indicate that two observations in expectation to the left of the true value of $-d_1$ is sufficient.

7 Proof of Theorem 3.1

It is assumed in this section that the conditions of Theorem 3.1 hold. The proof starts with showing that we may restrict ourselves to a neigbourhood of a_1, where we can make expansions.

Let, for $k = 0, 1, 2$,

$$\hat{h}_k(a) = n^{-1} \sum_{i=1}^{n} (V_i - a)^k 1_{\{V_i > a\}} \tag{7.1}$$

with

$$V_i = -\sigma^{-1}(U_i - \mu) , \qquad i = 1, \dots, n . \tag{7.2}$$

Let

$$\hat{a}_1 = \hat{h}_1^{-1}\left(\frac{\gamma}{\sigma f_{\tilde{X}}(s + \mu)}\right) = \frac{\hat{d}_1 + \mu}{\sigma} , \tag{7.3}$$

and for every $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ let

$$A = A(\varepsilon_1, \varepsilon_2, \varepsilon_3)$$
$$= \left\{ |\hat{a}_1 - a_1| \le \varepsilon_1, \left| \frac{\hat{h}_2(\hat{a}_1)}{\hat{h}_1'(\hat{a}_1)} - \frac{h_2(a_1)}{h_1'(a_1)} \right| \le \varepsilon_2 , \right.$$
$$\left. \left| \frac{1 + \hat{h}_1'(\hat{a}_1)}{\{\hat{h}_1'(\hat{a}_1)\}^2} - \frac{1 + h_1'(a_1)}{\{h_1'(a_1)\}^2} \right| \le \varepsilon_3 \right\} . \tag{7.4}$$

Note that $\hat{h}_1'$ is the derivative of $\hat{h}_1$ and not some newly defined estimator of h_1'. Since obviously

$$E[CL(\hat{t})] = E[CL(\hat{t})1_A] + O(P(A^c)) , \tag{7.5}$$

the error caused by restriction to A is determined by $P(A^c)$.

Lemma 7.1: For every $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$

$$P(A^c) = O(n^{-r/4}) \qquad as \quad n \to \infty . \tag{7.6}$$

Proof: Since $\hat{h}_1$ is nonincreasing, $\hat{h}_1(\hat{a}_1) = h_1(a_1)$ by (3.15) and (7.3) and $h_1' = -h_0$ by (2.8), $\hat{a}_1 - a_1 \ge \varepsilon_1$ implies

$$\hat{h}_1(a_1 + \varepsilon_1) - h_1(a_1 + \varepsilon_1) \ge h_1(a_1) - h_1(a_1 + \varepsilon_1) \ge \varepsilon_1 h_0(a_1 + \varepsilon_1) > 0 .$$

(Note that the smaller ε_1, ε_2, ε_3 the larger A^c. Hence w.l.o.g. we may assume that ε_1 is sufficiently small to ensure that $h_0(a_1 + \varepsilon_1) > 0$ by (A6) and the continuity of g.) In view of (A5)

$$E|\hat{h}_1(a_1 + \varepsilon_1) - h_1(a_1 + \varepsilon_1)|^r = O(n^{-r/2})$$

as $n \to \infty$ and hence

$$P(\hat{a}_1 - a_1 \ge \varepsilon_1) = O(n^{-r/2}) \qquad \text{as} \quad n \to \infty . \tag{7.7}$$

Similarly we get

$$P(\hat{a}_1 - a_1 \leq -\varepsilon_1) = O(n^{-r/2}) \quad \text{as} \quad n \to \infty . \tag{7.8}$$

Again by (A5)

$$E[|\hat{h}_2(a_1 + \varepsilon_1) - h_2(a_1 + \varepsilon_1)|^r] = O(n^{-r/4}) \quad \text{as} \quad n \to \infty .$$

On the set $|\hat{a}_1 - a_1| \leq \varepsilon_1$ monotonicity of $\hat{h}_2$ implies

$$\hat{h}_2(a_1 + \varepsilon_1) \leq \hat{h}_2(\hat{a}_1) \leq \hat{h}_2(a_1 - \varepsilon_1) .$$

For any $\eta > 0$ we may take ε_1 sufficiently small such that

$$(1 - \eta)h_2(a_1) \leq (1 - \eta/2)h_2(a_1 + \varepsilon_1) .$$

Hence, for any $\eta > 0$,

$$\begin{aligned} &P(|\hat{a}_1 - a_1| \leq \varepsilon_1, \hat{h}_2(\hat{a}_1) \leq (1 - \eta)h_2(a_1)) \\ &\qquad \leq P(\hat{h}_2(a_1 + \varepsilon_1) \leq (1 - \eta/2)h_2(a_1 + \varepsilon_1)) = O(n^{-r/4}) \quad \text{as } n \to \infty . \end{aligned} \tag{7.9}$$

Similarly, for any $\eta > 0$,

$$P(|\hat{a}_1 - a_1| \leq \varepsilon_1, \hat{h}_2(\hat{a}_1) \geq (1 + \eta)h_2(a_1)) = O(n^{-r/4}) \quad \text{as} \quad n \to \infty . \tag{7.10}$$

Since $\hat{h}_1'(a)$ is a sum of bounded i.i.d. r.v.'s, probabilities like

$$P(\hat{h}_1'(\hat{a}_1) \geq (1 + \eta)h_1'(a_1)) \tag{7.11}$$

are even exponentially small for any $\eta > 0$. In combination with (7.7)–(7.10) the result is now easily obtained. □

Remark 7.1: If (A5) is replaced by

$$E[e^{tV^2}] < \infty \qquad \text{for some} \quad t > 0$$

we obtain $P(A^c) = O(e^{-\delta n})$ for some $\delta > 0$ by standard large deviation results. □

The next step in the proof is to make expansions. We give an Edgeworth expansion of the distribution of $\hat{a}_1$ in Lemma 7.2, leading to expansions of moments in Lemma 7.6. In the rest of this section ε_1 is small enough to ensure that $g(a) > 0$ and $g'(a)$ is bounded on $|a - a_1| \leq \varepsilon$, cf. (A6).

We define the following

$$
\begin{aligned}
H_n(y) &= P((\hat{a}_1 - a_1)\sqrt{n} \leq y) = P(\hat{h}_1(a_1 + y/\sqrt{n}) \leq h_1(a_1)) \\
\mu_n(y) &= h_1(a_1 + y/\sqrt{n}) \\
\sigma_n^2(y) &= h_2(a_1 + y/\sqrt{n}) - h_1^2(a_1 + y/\sqrt{n}) \\
\rho_n(y) &= E[\{(V - (a_1 + y/\sqrt{n})) \cdot 1_{\{V > a_1 + y/\sqrt{n}\}} - \mu_n(y)\}^3]/\sigma_n^3(y) \\
z_n(y) &= \frac{h_1(a_1) - \mu_n(y)}{\sigma_n(y)} \sqrt{n} \; .
\end{aligned}
\tag{7.12}
$$

Lemma 7.2: With H_n and z_n as in (7.12), uniformly for $|y| \leq \varepsilon_1\sqrt{n}$ we have

$$\{1 + |z_n(y)|^3\}|H_n(y) - H_n^*(y)| = o(n^{-1/2}) \; , \tag{7.13}$$

with

$$H_n^*(y) = \Phi(z_n(y)) - \frac{\rho_n(y)}{6} \phi(z_n(y))\{z_n^2(y) - 1\}n^{-1/2} \; . \tag{7.14}$$

Proof: Direct application of Theorem 20.6 in Bhattacharya and Rao (1976), taking, in their notation, $s = 3$ and

$$f(y) = \{1 + |x|^3\} \cdot 1_{(-\infty,x]}(y) \qquad \text{if} \quad x < 0$$

and

$$f(y) = \{1 + |x|^3\} \cdot 1_{[x,\infty)}(y) \qquad \text{if} \quad x \geq 0 \; ,$$

respectively. □

Lemma 7.3: With z_n, ρ_n from (7.12) we have, as $n \to \infty$, uniformly in $|y| \le \varepsilon_1\sqrt{n}$,

$$z_n(y) = A_1 y + A_2 y^2/\sqrt{n} + O(|y|^3/n) \tag{7.15}$$

$$\rho_n(y) = A_3 + O(|y|/\sqrt{n}) \ , \tag{7.16}$$

with

$$\begin{aligned} A_1 &= -h_1'(a_1)b \\ A_2 &= -\tfrac{1}{2}h_1''(a_1)b - h_1'(a_1)\{1 + h_1'(a_1)\}h_1(a_1)b^3 \\ A_3 &= \{h_3(a_1) - 3h_2(a_1)h_1(a_1) + 2h_1^3(a_1)\}b^3 \\ b &= \{h_2(a_1) - h_1^2(a_1)\}^{-1/2} \ . \end{aligned} \tag{7.17}$$

Proof: By Taylor expansion we have, uniformly in $|y| \le \varepsilon_1\sqrt{n}$,

$$\mu_n(y) = h_1(a_1) + h_1'(a_1)y/\sqrt{n} + \tfrac{1}{2}\ h_1''(a_1)y^2/n + O(|y|^3/n^{3/2})$$

$$\sigma_n^2(y) = h_2(a_1) - h_1^2(a_1) + \{h_2'(a_1) - 2h_1'(a_1)h_1(a_1)\}y/\sqrt{n} + O(y^2/n) \ ,$$

as $n \to \infty$. The results follow by noting that $h_2' = -2h_1$. □

Lemma 7.4: With H_n^* from (7.14) and A and ε_1 from (7.4) we have

$$E[\sqrt{n}(\hat{a}_1 - a_1)1_A] = \int_0^{\varepsilon_1\sqrt{n}} 1 - H_n^*(y) - H_n^*(-y)dy + o(n^{-1/2}) \tag{7.18}$$

$$E[n(\hat{a}_1 - a_1)^2 1_A] = \int_0^{\varepsilon_1\sqrt{n}} 2y\{1 - H_n^*(y) + H_n^*(-y)\}dy + o(n^{-1/2}) \ , \tag{7.19}$$

as $n \to \infty$.

Proof: Let $B = \{|\hat{a}_1 - a_1| \le \varepsilon_1\}$. It is seen from the proof of Lemma 7.1, cf. (7.9)–(7.11), that $P(A \cap B^c) = O(n^{-r/4})$ as $n \to \infty$.

Therefore

$$E[n(\hat{a}_1 - a_1)^2 1_A] = E[n(\hat{a} - a_1)^2 1_B] + O(n^{-r/4+1}) \tag{7.20}$$

and

$$E[\sqrt{n}(\hat{a}_1 - a_1)1_A] = E[\sqrt{n}(\hat{a} - a_1)1_B] + O(n^{-r/4+1/2}) \tag{7.21}$$

as $n \to \infty$. Note that for $r > 6$, $n^{-r/4+1} = o(n^{-1/2})$ as $n \to \infty$.

Partial integration yields

$$\begin{aligned} E[n(\hat{a}_1 - a_1)^2 1_B] &= \int_{-\varepsilon_1\sqrt{n}}^{\varepsilon_1\sqrt{n}} y^2 dH_n(y) \\ &= -n\varepsilon_1^2\{1 - H_n(\varepsilon_1\sqrt{n}) + H_n(-\varepsilon_1\sqrt{n})\} \\ &\quad + 2\int_0^{\varepsilon_1\sqrt{n}} y\{1 - H_n(y) + H_n(-y)\}dy \,. \end{aligned} \tag{7.22}$$

The first term on the right-hand side is sufficiently small. To see this we write

$$\begin{aligned} \varepsilon_1^2 n\{1 - H_n(\varepsilon_1\sqrt{n}) + H_n(-\varepsilon_1\sqrt{n})\} &= \varepsilon_1^2 n\{1 - H_n^*(\varepsilon_1\sqrt{n}) + H_n^*(-\varepsilon_1\sqrt{n})\} \\ - \varepsilon_1^2 n\{H_n(\varepsilon_1\sqrt{n}) - H_n^*(\varepsilon_1\sqrt{n})\} &+ \varepsilon_1^2 n\{H_n(-\varepsilon_1\sqrt{n}) - H_n^*(-\varepsilon_1\sqrt{n})\} \,. \end{aligned}$$

There exists $\varepsilon_4 > 0$ (cf. also (7.15)) such that

$$|z_n(y)| \ge \varepsilon_4|y| \,, \tag{7.23}$$

for $|y| \le \varepsilon_1\sqrt{n}$. Hence, for some $\delta > 0$ we have that

$$\varepsilon_1^2 n\{1 - H_n^*(\varepsilon_1\sqrt{n})\} = O(e^{-\delta n})$$

$$\varepsilon_1^2 n H_n^*(-\varepsilon_1\sqrt{n}) = O(e^{-\delta n}) \,.$$

By Lemma 7.2 we get

$$\varepsilon_1^2 n\{H_n(\varepsilon_1\sqrt{n}) - H_n^*(\varepsilon_1\sqrt{n})\} = o(n^{-1})$$

$$\varepsilon_1^2 n\{H_n(-\varepsilon_1\sqrt{n}) - H_n^*(-\varepsilon_1\sqrt{n})\} = o(n^{-1})$$

and therefore the first term on the right-hand side of (7.22) is $o(n^{-1})$ as $n \to \infty$. By (7.13) and (7.23) we obtain

$$\int_{-\varepsilon_1\sqrt{n}}^{\varepsilon_1\sqrt{n}} |y||H_n(y) - H_n^*(y)|dy \leq o(n^{-1/2}) \int_{-\varepsilon_1\sqrt{n}}^{\varepsilon_1\sqrt{n}} \frac{|y|}{1+\varepsilon_4^3|y|^3}\, dy$$
$$= o(n^{-1/2})$$

as $n \to \infty$ and (7.19) is established.

In the same way (7.18) is proved. □

Lemma 7.5: If $y = o(n^{1/2})$, then for some $\tilde{A} \neq 0$

$$H_n^*(y) = \Phi(A_1 y) + A_2\phi(A_1 y)y^2/\sqrt{n} - \frac{1}{6}\frac{A_3}{\sqrt{n}}\phi(A_1 y)\{(A_1 y)^2 - 1\}$$
$$+ O(\phi(\tilde{A}y)/n) ,$$

as $n \to \infty$.

Proof: In view of Lemma 7.3 Taylor expansion of functions like $\Phi(A_1 y(1+x))$ around $x = 0$ yields the result. □

Lemma 7.6: We have

$$E[\sqrt{n}(\hat{a}_1 - a_1)1_A] = -\frac{A_2}{A_1^3}\frac{1}{\sqrt{n}} + o(n^{-1/2})$$

$$E[n(\hat{a}_1 - a_1)^2 1_A] = \frac{1}{A_1^2} + o(n^{-1/2}) , \tag{7.24}$$

as $n \to \infty$.

Proof: Combination of Lemma 7.4 and Lemma 7.5 and direct calculation yields the result. □

Next we expand $CL(\hat{t})$ on the set A. By a similar Taylor-argument as in the proof of Lemma 2.1, but going one step further, we get

$$CL(\hat{t})1_A = \{\sigma f_X(s)h_1(\hat{a}) + \tfrac{1}{2}\,\sigma^2 f_X'(s)h_2(\hat{a})\}\{1 + O(\sigma^2)\}1_A \ , \tag{7.25}$$

where

$$\begin{aligned}
\hat{a} &= \frac{s+\mu-\hat{t}}{\sigma} = \hat{a}_1 + \hat{c}^{(a)} + \hat{c}_u^{(a)} \ , \\
\hat{c}^{(a)} &= \frac{\hat{c}}{\sigma} = -\frac{\sigma f_{\tilde{X}}'(s+\mu)}{2 f_{\tilde{X}}(s+\mu)} \frac{\hat{h}_2(\hat{a}_1)}{\hat{h}_1'(\hat{a}_1)} \ , \\
\hat{c}_u^{(a)} &= \frac{\hat{c}_u}{\sigma} = \frac{\hat{h}_1(\hat{a}_1)}{n} \frac{\{1+\hat{h}_1'(\hat{a}_1)\}}{\{\hat{h}_1'(\hat{a}_1)\}^2} \ .
\end{aligned} \tag{7.26}$$

Application of (A2) and (A3) yields in combination with (7.25)

$$CL(\hat{t})1_A = \{\sigma f_{\tilde{X}}(s+\mu)h_1(\hat{a}) + \tfrac{1}{2}\sigma^2 f_{\tilde{X}}'(s+\mu)h_2(\hat{a})\}\{1 + O(\sigma^2)\}1_A \ . \tag{7.27}$$

Taylor expansion of $h_1(\hat{a})$ around a_1 yields

$$\begin{aligned}
h_1(\hat{a}) &= h_1(a_1) + \{(\hat{a}_1 - a_1) + (\hat{c}^{(a)} + \hat{c}_u^{(a)})\}h_1'(a_1) \\
&\quad + \tfrac{1}{2}(\hat{a}_1 - a_1)^2 h_1''(a_1) + \tfrac{1}{2}(\hat{a}_1 - a_1)^2\{h_1''(\xi) - h_1''(a_1)\} \\
&\quad + \{(\hat{a}_1 - a_1)(\hat{c}^{(a)} + \hat{c}_u^{(a)}) + \tfrac{1}{2}(\hat{c}^{(a)} + \hat{c}_u^{(a)})^2\}h_1''(\xi)
\end{aligned} \tag{7.28}$$

for some ξ between a_1 and $\hat{a}$.

Writing $c^{(a)}$ and $c_u^{(a)}$ for the correction terms with a_1 and h_k instead of $\hat{a}_1$ and $\hat{h}_k$, we have that $c^{(a)} = O(\sigma)$ and $c_u^{(a)} = O(n^{-1})$ as $n \to \infty, \sigma \to 0$.

In view of (7.4) and (7.26) it is seen that $\hat{c}^{(a)}1_A = O(\sigma)$ and $\hat{c}_u^{(a)}1_A = O(n^{-1})$ as $n \to \infty, \sigma \to 0$ and hence

$$E[1_A(\hat{c}^{(a)})^2] = O(\sigma^2) \ , \qquad E[1_A(\hat{c}_u^{(a)})^2] = O(n^{-2}) \ . \tag{7.29}$$

Moreover, using e.g. results on the oscillation modulus of the empirical process (cf. Mason et al. (1983); we omit the technical details)

$$E[1_A\hat{c}^{(a)}] = c^{(a)} + O(\sigma n^{-3/4}\log n)\ , \qquad E[1_A\hat{c}_u^{(a)}] = c_u^{(a)} + O(n^{-7/4}\log n)\ . \tag{7.30}$$

Together with Lemma 7.6 this gives

$$E[1_A(\hat{a}_1 - a_1)(\hat{c}^{(a)} + \hat{c}_u^{(a)})] = O(\sigma n^{-1} + n^{-2})\ . \tag{7.31}$$

Again by Lemma 7.6 and the definition of ξ we obtain

$$E[1_A(\hat{a} - a_1)^2\{h_1''(\xi) - h_1''(a_1)\}] = o(n^{-1})\ . \tag{7.32}$$

Further, combination of (7.17), (7.24) and (7.26) yields

$$E[1_A\{(\hat{a}_1 - a_1)h_1'(a_1) + c_u^{(a)}h_1'(a_1) + \tfrac{1}{2}(\hat{a}_1 - a_1)^2 h_1''(a_1)\}] = o(n^{-1})\ . \tag{7.33}$$

By (7.27)–(7.33) we get, using that $h_1(a_1)\sigma f_{\bar{X}}(s+\mu)E[1_A] = \gamma\{1 + O(n^{-r/4})\}$

$$E[CL(\hat{t})1_A] = \gamma\{1 + o(n^{-1}) + O(\sigma^2)\} + c^{(a)}h_1'(a_1)\sigma f_{\bar{X}}(s+\mu)$$
$$+ \tfrac{1}{2}\sigma^2 f_{\bar{X}}'(s+\mu)E[h_2(\hat{a})1_A]\ .$$

Noting that

$$E[h_2(\hat{a})1_A] = h_2(a_1) + O(\sigma + n^{-1}) \tag{7.34}$$

it is seen that indeed $c^{(a)}$ compensates the second order term of $CL(\hat{t})$. This completes the proof of Theorem 3.1.

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References

Albers W, Kallenberg WCM, Otten GD (1997) Robust test limits. Mathematical Methods of Statistics, to appear

Albers W, Kallenberg WCM, Otten GD (1994a) Accurate test limits with estimated parameters. Technometrics 36:92–101

Albers W, Kallenberg WCM, Otten GD (1994b) Setting test limits under prescribed consumer loss. Metrika 41:163–181

Bai DS, Kim SB, Riew MC (1990) Economic screening procedures based on correlated variables. Metrika 37:263–280

Bai DS, Kwon HM (1995) Economic design of a two-stage screening procedure with a prescribed outgoing quality. Metrika 42:1–18

Bai DS, Lee MK (1993) Economic designs of single and double screening procedures. Metrika 40:95–113

Bhattacharya RN, Rao RR (1976) Normal approximation and asymptotic expansions. Wiley, New York

Easterling RG, Johnson ME, Bement TR, Nachtsheim CJ (1991) Statistical tolerancing based on consumer's risk considerations. Journal of Quality Technology 23:1–11

Grubbs FE, Coon HJ (1954) On setting test limits relative to specification limits. Industrial Quality Control 10:15–20

Kim SB, Bai DS (1992) Economic design of one-sided screening procedures based on a correlated variable with all parameters unknown. Metrika 39:85–93

Mason DM, Shorack GR, Wellner JA (1983) Strong limit theorems for oscillation moduli of the uniform empirical process. Z. Wahrsch. Verw. Gebiete 65:83–97

Mullenix P (1990) The capability of capability indices with an application to guardbanding in a test environment. 1990 International Test Conference IEEE: 907–915

Otten GD (1995) Statistical test limits in quality control. Ph.D. dissertation, University of Twente, The Netherlands

Owen DB, Boddie JW (1976) A screening method for increasing acceptable product with some parameters unknown. Technometrics 18:195–199

Owen DB, Su YH (1977) Screening based on normal variables. Technometrics 19:65–68

Owen DB, Li L, Chou YM (1981) Prediction intervals for screening using a measured correlated variate. Technometrics 23:165–170

Tang K (1987) Economic design of a one-sided screening procedure using a correlated variable. Technometrics 28:477–485

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