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Constrained stabilization problems for discrete-time linear plants

Ali Saberi[1,‡], Guoyong Shi[1,§], Anton A. Stoorvogel[2,3,*,†] and Jian Han[1,¶]

[1] *School of Electrical Engineering and Computer Science, Washington State University, Pullman, WA 99164-2752, U.S.A.*
[2] *Department of Mathematics and Computing Science, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands*
[3] *Department of Electrical Engineering, Mathematics and Computer Science, Delft University of Technology, P.O. Box 5031, 2600 GA Delft, The Netherlands*

SUMMARY

In this paper we study discrete-time linear systems with full or partial constraints on both input and state. It is shown that the solvability conditions of stabilization problems are closely related to important concepts, such as the right-invertibility of the constraints, the location of constraint invariant zeros and the order of constraint infinite zeros. The main results show that for right-invertible constraints the order of constrained infinite zeros cannot be greater than one in order to achieve global or semi-global stabilization. This is in contrast to the continuous-time case. Controllers for both state feedback and measurement feedback are constructed in detail. Issues regarding non-right invertible constraints are discussed as well. Copyright © 2004 John Wiley & Sons, Ltd.

KEY WORDS: discrete-time systems; stabilization; constraints; solvability conditions

1. INTRODUCTION

Two of the most commonly encountered *constraints* in control engineering are actuator constraints and state constraints. References [1] and [2] capture some recent research activities regarding constraints on actuators, i.e. on inputs. Besides actuator constraints, state constraints are a major concern in control engineering. However, the state constraints, unlike the actuator constraints, have not received much attention from a structural point of view. There have been some efforts on dealing with state and input constraints utilizing the concept of positive invariant sets [3] and techniques of model predictive control [4–6]. However, the available tools

*Correspondence to: Professor A. A. Stoorvogel, Department of Electrical Engineering, Mathematics and Computer Science, Delft University of Technology, P.O. Box 5031, 2600 GA Delft, The Netherlands.
† E-mail: a.a.stoorvogel@tue.nl, a.a.stoorvogel@ewi.tudelft.nl
‡ E-mail: saberi@eecs.wsu.edu
§ E-mail: gshi@eecs.wsu.edu
¶ E-mail: jhan@eecs.wsu.edu

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along these lines are computationally very demanding and the resulting controllers are highly complicated.

During the last decade several aspects of control design problems for linear systems with magnitude and rate constraints on control variables have been studied among others by the first and third author and their students and collaborators. A number of powerful analysis and design methods such as low gain, low-high gain, scheduled low gain, scheduled low-high gain and many variations of them have been developed for several core control design problems including global and semi-global internal stabilization, external stabilization, output regulation, and disturbance rejection. We have studied stabilization (continuous-time in Reference [7] and discrete-time in Reference [8]) and output regulation problems (continuous-time in Reference [9] and discrete-time in Reference [10]) associated with magnitude and rate constraints on control variables. Many of these issues have also been addressed in Refeence [11]. The research thrust of the first and third author and their students has broadened to include additionally magnitude constraints on state variables. In connection with stabilization, whenever amplitude and rate constraints on both state as well as input variables exist, a taxonomy of all possible constraints is introduced, and several fundamental results on global, semi-global and regional stabilization are developed in a recent paper [12]. The work of Saberi *et al.* [12] focuses on continuous-time systems and generalizes, extends and covers all existing results including those developed in Reference [13]. The focus of this paper is to address the same issues for discrete-time systems. Output regulation for systems with both input and state constraints has in the mean time also been studied in Reference [14] (continuous-time) and [15] (discrete-time).

It is becoming evident that the taxonomy of constraints developed in Reference [12] plays dominant roles in every type of control design problem, not only for continuous-time systems but also for discrete-time systems. The taxonomy of constraints is developed by appropriately modelling the constraints in terms of what is called a constraint output (of the given system) with its magnitude and rate subject to some prescribed constraint sets. It turns out that the structural properties of the mapping from the input to the constrained output vector play dominant roles in dictating what is feasible and what is not feasible. Such structural properties have been categorized in three directions. The first direction of categorization is based on the right invertibility of the mapping from the input to the constraint output vector. This direction of categorization delineates the constraints into two mutually exclusive categories: (1) right invertible constraints representing the case when the mapping from the input to the constraint output vector is right invertible and (2) non-right invertible constraints representing the case when the mapping from the input to the constraint output vector is not right invertible. The second direction of categorization is based on the so called constraint invariant zeros of the plant, i.e. the invariant zeros of the mapping from the input to the constrained output vector. Like in the first categorization, this second categorization also delineates the constraints into two mutually exclusive main categories: (1) at most weakly non-minimum phase constraints representing the case when the constraint invariant zeros are in the closed left-half complex plane for continuous-time systems or in the closed unit disc for discrete-time systems and (2) strongly non-minimum phase constraints representing the case when one or more of the constraint invariant zeros are in the open right half complex plane for continuous-time systems or outside the unit disc for discrete-time systems. The third direction of categorization is based on the order of constraint infinite zeros, i.e. the infinite zeros of the mapping from the input to the constrained output.

Based on such a taxonomy of constraints, two main features emerge:

- Neither the constrained semi-global nor the constrained global stabilization problem is solvable whenever the constraints are strongly non-minimum phase.
- There exists a perceptible demarkation line between the right and non-right invertible constraints. In particular, the solvability conditions for the constrained semi-global and global stabilization problems via state feedback do not depend on the shape of the constraint sets for right invertible constraints, whereas for non-right invertible constraints they indeed do so.

This paper, which deals with discrete-time systems, focuses on the above aspects. Although the development for discrete-time systems parallels somewhat that in continuous-time systems, there are several fundamental differences between continuous- and discrete-time systems: (1) the solvability conditions for semi-global stabilization, unlike in continuous-time systems, requires that the order of the constraint infinite zeros be less than or equal to one, (2) the methods of constructing appropriate controllers need to be revised as needed and (3) some new issues arise, which do not exist in continuous-time systems.

Following the problem formulation in the next section, a taxonomy of constraints is presented in Section 3. This taxonomy facilitates the statements of the main results of global and semi-global stabilization for right invertible constraints in Section 4. In this section control laws are designed in detail for both state feedback and measurement feedbacks. Some new issues regarding non-minimum phase constraints are discussed as well. Such issues were not observed in the continuous-time case. Systems with non-right invertible constraints are studied in Section 5, where some necessary conditions for the global and semi-global stabilization are developed. In the same section the shape dependence on the constraint sets is also carefully examined. This paper concludes in Section 6.

2. PROBLEM FORMULATION

Consider a discrete-time linear system:

$$\Sigma : \begin{cases} x(k+1) & = Ax(k) + Bu(k) \\ y(k) & = C_y x(k) + D_y u(k) \\ z(k) & = C_z x(k) + D_z u(k) \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^r, z \in \mathbb{R}^p$ are respectively state, input, measurement output and constrained output. The constrained output z is subject to both amplitude and rate constraints in that for two *a priori* given sets $\mathcal{S} \subset \mathbb{R}^p$ and $\mathcal{T} \subset \mathbb{R}^p$ we require

$$\begin{aligned} & z(k) \in \mathcal{S}, \quad \forall k \geqslant 0 \\ & z(k+1) - z(k) \in \mathcal{T}, \quad \forall k \geqslant 0 \end{aligned} \tag{2}$$

Figure 1 shows the basic structure of the system model.

Given the system described above, our goal is to obtain necessary and sufficient conditions for the possibility of stabilization. Whenever it is possible, we will present design methodologies for constrained stabilization.

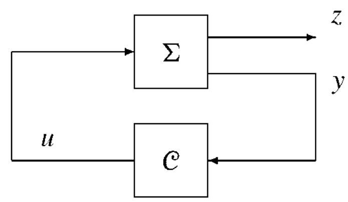

Figure 1. Closed-loop system with constrained output.

Remark
Note that in (2) we only require the rate constraint to be satisfied for all $k \geq 0$ by ignoring the rate constraint on $z(0) - z(-1)$. This avoids the technicality of requiring a non-abrupt transition at the beginning. However, if we incorporate a deadbeat operator in the controller, this issue can be easily resolved [16].

We make the following fundamental assumption on the structure of the constrained output and the nature of the constraint sets $\mathcal{S}$ and $\mathcal{T}$.

Assumption 2.1

(i) The sets $\mathcal{S}$ and $\mathcal{T}$ are closed, convex, and contain 0 as an interior point.
(ii) $\mathcal{S} \cap \mathcal{T}$ is bounded.
(iii) The matrices C_z and D_z satisfy $C_z^{\mathrm{T}} D_z = 0$ and moreover $\mathcal{S}$ and $\mathcal{T}$ satisfy the following decomposition:

$$\mathcal{S} = (\mathcal{S} \cap \operatorname{im} C_z) + (\mathcal{S} \cap \operatorname{im} D_z)$$
$$\mathcal{T} = (\mathcal{T} \cap \operatorname{im} C_z) + (\mathcal{T} \cap \operatorname{im} D_z)$$

Remark
We observe that $\operatorname{im} C_z$ reflects the state constraints while $\operatorname{im} D_z$ reflects the input constraints. Therefore the decomposition of $\mathcal{S}$ and $\mathcal{T}$ as required in (iii) reflects the fact that the constraints can be on states and/or inputs.

Clearly, the set of initial conditions has to be restricted, otherwise one can never avoid constraint violation if the system starts from certain initial conditions. We introduce the concept of admissible set of initial conditions in the next definition.

Definition 2.2
Let the system (1) and constraint sets $\mathcal{S}$ and $\mathcal{T}$ be given. We define

$$\mathcal{A}(\mathcal{S}, \mathcal{T}) := \{x(0) \in \mathbb{R}^n \mid \exists\, u(0) \text{ such that } C_z x(0) + D_z u(0) \in \mathcal{S} \text{ and } C_z[Ax(0) + Bu(0)] - C_z x(0) \in \mathcal{T}\}$$

as the *admissible set of initial conditions*.

Remark
In defining the admissible set, one might expect the rate constraint condition to be

$$z(1) - z(0) = C_z\{[Ax(0) + Bu(0)] - x(0)\} + D_z[u(1) - u(0)] \in \mathscr{T}$$

However, we have omitted the second term because by the decomposition in item (iii) of Assumption 2.1 these two constraints are equivalent if we can ensure $D_z[u(1) - u(0)] \in \mathscr{T}$. But this can be done by design.

Remark
A concept related to the admissible set of initial conditions is the so called *maximal output admissible set*, which was discussed extensively in Reference [17]. The definition of maximal output admissible set was based on a given linear control law. However, our work in this paper is mainly focused on characterizing solvability conditions under which appropriate control laws can be synthesized so that the constraint violation is avoided during the whole control process.

The following problems are formulated either in global or in semi-global setting.

Problem 2.3 (Global constrained stabilization via state feedback)
Consider the system (1) with constraint sets $\mathscr{S} \subset \mathbb{R}^p$ and $\mathscr{T} \subset \mathbb{R}^p$. Find, if possible, a state feedback (possibly nonlinear) $u(k) = f(x(k), k)$ such that the following conditions hold:

(i) The equilibrium point $x = 0$ of the closed-loop system is asymptotically stable with $\mathscr{A}(\mathscr{S}, \mathscr{T})$ contained in its domain of attraction.
(ii) For any $x(0) \in \mathscr{A}(\mathscr{S}, \mathscr{T})$, we have $z(k) \in \mathscr{S}$ and $z(k+1) - z(k) \in \mathscr{T}$ for all $k \geqslant 0$.

Problem 2.4 (Semi-global constrained stabilization via state feedback)
Consider the system (1) with constraint sets $\mathscr{S} \subset \mathbb{R}^p$ and $\mathscr{T} \subset \mathbb{R}^p$. For any *a priori* given compact set $\mathscr{A}_0$ contained in the interior of $\mathscr{A}(\mathscr{S}, \mathscr{T})$ find, if possible, a state feedback (possibly nonlinear) $u(k) = f(x(k), k)$ such that the following conditions hold:

(i) The equilibrium point $x = 0$ of the closed-loop system is asymptotically stable with $\mathscr{A}_0$ contained in its domain of attraction.
(ii) For any $x(0) \in \mathscr{A}_0$, we have $z(k) \in \mathscr{S}$ and $z(k+1) - z(k) \in \mathscr{T}$ for all $k \geqslant 0$.

Problem 2.5 (Global constrained stabilization via measurement)
Consider the system (1) with constraint sets $\mathscr{S} \subset \mathbb{R}^p$ and $\mathscr{T} \subset \mathbb{R}^p$. Find, if possible, a measurement feedback (possibly nonlinear and time-varying) of the form,

$$v(k+1) = g(v(k), y(k), k), \quad v \in \mathbb{R}^q$$
$$u(k) = h(v(k), y(k), k)$$

such that the following conditions hold:

(i) The equilibrium point $(x, v) = (0, 0)$ of the closed-loop system is asymptotically stable with $\mathscr{A}(\mathscr{S}, \mathscr{T}) \times \mathbb{R}^q$ contained in its domain of attraction.
(ii) For any $(x(0), v(0)) \in \mathscr{A}(\mathscr{S}, \mathscr{T}) \times \mathbb{R}^q$, we have $z(k) \in \mathscr{S}$ and $z(k+1) - z(k) \in \mathscr{T}$ for all $k \geqslant 0$.

Problem 2.6 (Semi-global constrained stabilization via measurement)
Consider the system (1) with constraint sets $\mathcal{S} \subset \mathbb{R}^p$ and $\mathcal{T} \subset \mathbb{R}^p$. Find (if possible) a family of measurement feedbacks (possibly nonlinear and time-varying) of the form

$$v(k+1) = g(v(k), y(k), k), \quad v \in \mathbb{R}^q$$
$$u(k) = h(v(k), y(k), k)$$

such that for any compact set $\mathcal{A}_0$ contained in the interior of the set $\mathcal{A}(\mathcal{S}, \mathcal{T})$ and any compact set $\mathcal{V}_0 \subset \mathbb{R}^q$ there exists a measurement feedback in this family such that the following conditions hold:

(i) The equilibrium point $(x, v) = (0, 0)$ of the closed-loop system is asymptotically stable with $\mathcal{A}_0 \times \mathcal{V}_0$ contained in its domain of attraction.
(ii) For any $(x(0), v(0)) \in \mathcal{A}_0 \times \mathcal{V}_0$, we have $z(k) \in \mathcal{S}$ and $z(k+1) - z(k) \in \mathcal{T}$ for all $k \geqslant 0$.

3. TAXONOMY OF CONSTRAINTS

We let $\mathbb{C}$, $\mathbb{C}^{\oplus}$, $\mathbb{C}^{\ominus}$ and $\mathbb{C}^{\bigcirc}$ denote respectively the set of complex numbers in the entire complex plane, outside the unit circle, inside the unit circle and on the unit circle.

The following notions are fundamental to the taxonomy of constraints given below.

Definition 3.1
The system:

$$x(k+1) = Ax(k) + Bu(k)$$
$$y(k) = Cx(k) + Du(k)$$

is said to be *right invertible* if for any sequence $y_{\text{ref}}(k)$ defined for $k \geqslant 0$ there exists an input u and a choice of $x(0)$ such that $y(k) = y_{\text{ref}}(k)$ for all $k \geqslant 0$.

Definition 3.2
The *invariant zeros* of a linear system with a realization (A, B, C, D) are those points $\lambda \in \mathbb{C}$ for which

$$\text{rank}\begin{pmatrix} \lambda I - A & -B \\ C & D \end{pmatrix} < \text{normrank}\begin{pmatrix} sI - A & -B \\ C & D \end{pmatrix}$$

where 'normrank' denotes normal rank.

The first categorization is based on whether the subsystem from u to z is right invertible or not. We have the following definition:

Definition 3.3
The constraints are said to be

- *right invertible constraints* if the subsystem $\Sigma_{zu} : (A, B, C_z, D_z)$ is right invertible.
- *non-right invertible constraints* if the subsystem $\Sigma_{zu} : (A, B, C_z, D_z)$ is not right invertible.

It turns out that the location of the invariant zeros of the subsystem Σ_{zu} is also important in characterizing the solvability of stabilization problems. We refer to these invariant zeros as constraint invariant zeros:

Definition 3.4
The invariant zeros of the subsystem characterized by the quadruple (A, B, C_z, D_z) are called *constraint invariant zeros* of the given system Σ.

The second categorization of constraints is based on the location of the constraint invariant zeros. We have the following definition:

Definition 3.5
The constraints are said to be

- *minimum phase constraints* if all the constraint invariant zeros are in $\mathbb{C}^{\ominus}$.
- *weakly minimum phase constraints* if all the constraint invariant zeros are in $\mathbb{C}^{\ominus} \cup \mathbb{C}^{\bigcirc}$ with the restriction that any invariant zero in $\mathbb{C}^{\bigcirc}$ is simple,
- *weakly non-minimum phase constraints* if all the constraint invariant zeros are in $\mathbb{C}^{\ominus} \cup \mathbb{C}^{\bigcirc}$ with at least one non-simple invariant zero in $\mathbb{C}^{\bigcirc}$.
- *at most weakly non-minimum phase constraints* if all the constraint invariant zeros are in $\mathbb{C}^{\ominus} \cup \mathbb{C}^{\bigcirc}$.
- *strongly non-minimum phase constraints* if at least one constraint invariant zeros is in $\mathbb{C}^{\oplus}$.

The third categorization is based on the order of the infinite zeros of the subsystem Σ_{zu}. See Reference [18] for a definition of infinite zeros of a system. Because of their importance, we specifically label the infinite zeros of the subsystem Σ_{zu} as the constraint infinite zeros of the plant.

Definition 3.6
The infinite zeros of the subsystem Σ_{zu} are called the *constraint infinite zeros* of the plant associated with the constrained output z.

We have the following definition regarding the third categorization of constraints.

Definition 3.7
The constraints are said to be *type one constraints* if the order of all constraint infinite zeros is less than or equal to one.

4. RIGHT INVERTIBLE CONSTRAINTS

In this section we provide necessary and sufficient conditions for the solvability of Problems 2.3–2.6, under the assumption that the subsystem (A, B, C_z, D_z) is right invertible, i.e. system (1) has right invertible constraints. We leave the case that (A, B, C_z, D_z) is non-right invertible to Section 5, where we clarify some extra difficulties involved in the general case.

It is worth pointing out that for the discrete-time systems the solvability conditions for the global and semi-global constrained stabilization are the same. This is in contrast to the continuous-time case [12]. For an easy presentation, we separate this section in three subsections. In the first subsection we present the main results for constrained global and semi-global stabilization via state and measurement feedback. In the second subsection we introduce the special coordinate basis [18,19], which is a tool for the later development and present results for systems with non-minimum phase constraints. In the third subsection we prove the results appeared in the first two subsections.

4.1. Main results for global and semi-global stabilization

The first result is about the solvability conditions for the constrained global or semi-global stabilization via state feedback.

Theorem 4.1

Consider the plant Σ as given by (1) with the constraint sets $\mathscr{S}$ and $\mathscr{T}$ satisfying Assumption 2.1. Assume that the constraints are right-invertible and the set $\mathscr{S}$ is bounded. Then the global or semi-global constrained stabilization problem via state feedback as defined in Problem 2.3 or Problem 2.4 is solvable if and only if:

(i) (A, B) is stabilizable.
(ii) The constraints are at most weakly non-minimum phase.
(iii) The constraints are of type one.

Remark

A fundamental consequence of Theorem 4.1 is that the conditions are independent of any specific shapes of the given constraint set. That is, for the case of a right invertible system Σ, if the semi-global or global constrained stabilization problems are solvable for some given constraint set satisfying Assumption 2.1, then these problems are also solvable for any other constraint sets satisfying Assumption 2.1.

Note that the controller needs in general to be nonlinear. However, in the semi-global case, the controller can be chosen either as a time-invariant nonlinear controller or as a time-varying linear controller.

For the case of measurement feedback, we have the following theorem.

Theorem 4.2

Consider the plant Σ as given by (1) with the constraint sets $\mathscr{S}$ and $\mathscr{T}$ satisfying 2.1. Assume that the constraints are right-invertible. Then, the global and semi-global constrained stabilization problem via measurement feedback as defined in Problems 2.5 and 2.6 are solvable if the following conditions hold:

(i) (A, B) is stabilizable.
(ii) The constraints are at most weakly non-minimum phase.
(iii) The constraints are of type one.
(iv) The pair (C_y, A) is observable.

(v) $\ker C_z \subset \ker C_z A$.
(vi) $\ker (C_y \;\; D_y) \subset \ker (C_z \;\; D_z)$.

Moreover, conditions (i) to (iii) are necessary.

Remark
Note that condition (vi) states that the constrained output is part of the measurements. The following example shows that conditions (v) and (vi) in Theorem 4.2 are needed for discrete-time systems. Consider the system

$$\begin{aligned} x_1(k+1) &= x_1(k) + x_2(k) \\ x_2(k+1) &= u(k) + x_1(k) \\ y(k) &= x_1(k) + 2x_2(k) \\ z(k) &= x_2(k) \end{aligned}$$

It is easy to see that all the conditions except (v) and (vi) in the theorem are satisfied. Suppose we have a constraint $z(k) \in [-1, 1]$ for all $k \geqslant 0$. There exists a deadbeat observer which gives an exact state estimate for $x_1(k)$ and $x_2(k)$ for $k \geqslant 1$.

The set of admissible initial conditions is characterized by the set of all initial conditions satisfying $|x_2(0)| \leqslant 1$. Using this information together with the first measurement $y(0)$ we can only conclude that

$$x_1(0) \in [y_1(0) - 2, y_1(0) + 2] \tag{3}$$

But then there is no choice for $u(0)$ to ensure that

$$x_2(1) = x_1(0) + u(0) \in [-1, 1]$$

for all possible values of $x_1(0)$ satisfying (3), i.e. there is no guarantee that at time $k = 1$ the constraint is not violated. Hence, the semi-global constrained stabilization via measurement feedback is in general not possible without conditions (v) and (vi).

Remark
The solvability conditions as given by Theorems 4.2 are independent of any specific features of the given constraint sets. But the solvability of the semi-global or global constrained stabilization problems in the measurement feedback case is in general dependent on the shape of the constraint sets even for the case of right-invertible constraints. But, this is not in contradiction with the above theorem since we only obtained sufficient conditions for solvability. For example, consider the system

$$\begin{aligned} x_1(k+1) &= u_1(k) \\ x_2(k+1) &= u_2(k) + x_1(k) \\ y(k) &= x_2(k) \\ z_1(k) &= x_1(k) \\ z_2(k) &= x_2(k) \end{aligned}$$

Semi-global and global constrained stabilization problems are trivially solved by the state feedback $u_1 = 0$ and $u_2 = -x_1$. But in the measurement case we can only implement this feedback for $k \geqslant 1$. At time $k = 0$ we have no information available about $x_1(0)$ except for the fact that the state must be in the admissible set of initial conditions.

- For the constraint set $|z_1| \leqslant 1$ and $|z_2| \leqslant 2$ the controller $u_1 = 0$ and $u_2 = 0$ trivially solves the global stabilization problem.
- For the constraint set $|z_1| \leqslant 1$ and $|z_2| \leqslant 1/2$, no measurement based feedback can guarantee that the constraint is not violated at time $k = 1$, because the controller lacks information about $x_1(0)$.

Note that the above is in contrast with continuous time where the solvability is always independent of the constraint set for right-invertible systems.

4.2. Main results for non-minimum phase constraints

From Theorems 4.1 and 4.2 we see that the global and semi-global constrained stabilization problems are solvable only for a system Σ which has at most weakly non-minimum phase constraints. If the given system Σ has *strictly non-minimum phase* constraints, the domain of attraction cannot be enlarged arbitrarily, that is, the domain of attraction is bounded at least in some directions of the state space. Our next goal is to characterize a maximally achievable domain of attraction, in the sense that the given system cannot be stabilized for those initial conditions outside of such a set.

To present such a characterization we need to introduce special coordinate basis for the subsystem represented by the quadruple (A, B, C_z, D_z) which is related to the mapping from the input u to the constrained output z. For simplicity, we denote this subsystem as Σ_{zu} (A, B, C_z, D_z). The special coordinate basis as developed in References [18,19] clearly reveal most of the system properties involving invariant zeros and infinite zeros.

Under the assumption that the subsystem Σ_{zu} (A, B, C_z, D_z) is right invertible, we can choose coordinate basis in the state space, input space and output space such that the subsystem Σ_{zu} can be rewritten in term of scb in the following form:

$$\begin{aligned}
x_a(k+1) &= A_{aa}x_a(k) + K_a z(k)\\
x_c(k+1) &= A_{cc}x_c(k) + B_c[u_c(k) + J_a x_a(k)] + K_c z(k)\\
x_d(k+1) &= A_{dd}x_d(k) + B_d[u_d(k) + E_a x_a(k) + E_c x_c(k) + E_d x_d(k)] + K_d z(k)\\
y(k) &= C_{ya}x_a(k) + C_{yc}x_c(k) + C_{yd}x_d(k) + \tilde{D}_y\tilde{u}(k)\\
z_0(k) &= u_0(k)\\
z_d(k) &= C_d x_d(k)
\end{aligned} \tag{4}$$

where

$$\begin{pmatrix} x_a \\ x_c \\ x_d \end{pmatrix} = \tilde{x} = \Gamma_x^{-1}x, \quad \begin{pmatrix} u_0 \\ u_c \\ u_d \end{pmatrix} = \tilde{u} = \Gamma_u^{-1}u, \quad \begin{pmatrix} z_0 \\ z_d \end{pmatrix} = \Gamma_z^{-1}z$$

and x_a, x_c, x_d, u_0, u_c, u_d are of appropriate dimensions, while Γ_x, Γ_u and Γ_z are transformation matrices. This decomposition renders the subsystem characterized by the state variables x_c and x_d strongly controllable without finite zeros (see Reference [18]). Moreover, the pair (A_{aa}, K_a) is stabilizable.

The zero dynamics of the subsystem Σ_{zu} are the dynamics of x_a in (9) and given by

$$\Sigma_1 : \begin{cases} x_a(k+1) &= A_{aa}x_a(k) + K_a z(k) \\ &= A_{aa}x_a(k) + K_{a0}z_0(k) + K_{ad}z_d(k) \end{cases} \tag{5}$$

where $K_a = (K_{a0} \;\; K_{ad})$. By viewing z as the input to this subsystem, we have a system with input constraints in the sense that $z(k) \in \mathscr{S}$ and $z(k+1) - z(k) \in \mathscr{T}$ for all $k \geq 0$. Since $\mathscr{S}$ and $\mathscr{T}$ satisfy Assumption 2.1 there exist appropriate sets $\mathscr{S}_0$, $\mathscr{S}_d$, $\mathscr{T}_0$, and $\mathscr{T}_d$ such that

$$z \in \mathscr{S} \quad \text{if and only if } z_0 \in \mathscr{S}_0 \text{ and } z_d \in \mathscr{S}_d \tag{6a}$$

$$\bar{z} \in \mathscr{T} \quad \text{if and only if } \bar{z}_0 \in \mathscr{T}_0 \text{ and } \bar{z}_d \in \mathscr{T}_d \tag{6b}$$

Next we introduce the second subsystem:

$$\Sigma_2 : \begin{cases} x_c(k+1) &= A_{cc}x_c(k) + B_c[u_c(k) + J_a x_a(k)] + K_c z(k) \\ x_d(k+1) &= A_{dd}x_d(k) + B_d[u_d(k) + E_a x_a(k) + E_c x_c(k) + E_d x_d(k)] + K_d z(k) \\ z_0(k) &= u_0(k) \\ z_d(k) &= C_d x_d(k) \end{cases} \tag{7}$$

The decomposition of the original system into two subsystems makes it possible to design a controller in two layers.

Now we are at a position to characterize the maximal domain of attraction of the closed-loop system with constraints in terms of the two subsystems obtained above. We define the admissible set for the subsystem Σ_2 as

$$\mathscr{A}(\Sigma_2, \mathscr{S}, \mathscr{T}) = \left\{ x_2 = \begin{pmatrix} x_c \\ x_d \end{pmatrix} \in \mathbb{R}^{n_2} \,\middle|\, C_d x_d \in \mathscr{S}_d \text{ and } \exists\, u_0 \in \mathscr{S}_0 \text{ and } u_d \text{ such that} \right.$$
$$\left. C_d \left[A_{dd}x_d + B_d u_d + K_d \begin{pmatrix} u_0 \\ C_d x_d \end{pmatrix} - x_d \right] \in \mathscr{T}_d \right\}$$

where $x_2 := (x_c^{\mathrm{T}}, x_d^{\mathrm{T}})^{\mathrm{T}}$ and n_2 is the dimension of x_2. Note that x_a has no effect on $\mathscr{A}(\Sigma_2, \mathscr{S}, \mathscr{T})$ as can be seen from the scb structure.

Next we introduce the notion of null-controllability region of a linear system with constrained input. Given any two sets $\bar{\mathscr{S}} \subset \mathbb{R}^m$ and $\bar{\mathscr{T}} \subset \mathbb{R}^m$, we define the set

$$\mathscr{U}(\bar{\mathscr{S}}, \bar{\mathscr{T}}) := \{\bar{u}\colon\ \bar{u}(k) \in \bar{\mathscr{S}} \text{ and } \bar{u}(k+1) - \bar{u}(k) \in \bar{\mathscr{T}},\ \forall k \geq 0\}.$$

Definition 4.3
Consider the system:

$$\bar{\Sigma} : \left\{ \bar{x}(k+1) = \bar{A}\bar{x}(k) + \bar{B}\bar{u}(k) \right. \tag{8}$$

subject to constraint $\bar{u} \in \mathscr{U}(\bar{\mathscr{S}}, \bar{\mathscr{T}})$, where $\bar{\mathscr{S}}$ and $\bar{\mathscr{T}}$ are two closed convex sets containing zero as an interior point and satisfying that $\bar{\mathscr{S}} \cap \bar{\mathscr{T}}$ is bounded. The set

$$\mathscr{R}(\bar{\Sigma}, \bar{\mathscr{S}}, \bar{\mathscr{T}}) = \left\{ \bar{x}(0) \in \mathbb{R}^n \colon \exists\ \bar{u} \in \mathscr{U}(\bar{\mathscr{S}}, \bar{\mathscr{T}}) \text{ such that } \lim_{k\to\infty} \bar{x}(k) = 0 \right\}$$

is said to be the *region of asymptotic null-controllability with input constraint sets* $\bar{\mathscr{S}}$ *and* $\bar{\mathscr{T}}$.

We will connect the domain of attraction of the first subsystem to the domain of attraction of the full system but only for the case without rate constraints. The general case is not much more difficult but involves some additional techniqualities. Note that if we want to control the state x_a of the first subsystem then we can do so through u_0 and z_d. However, we cannot choose z_d arbitrarily. We can control z_d arbitrarily after a delay of at most k steps where k is the maximal order of the constraint infinite zeros. Therefore we need to able to make sure that in the first k steps the system dynamics behave appropriately. We define:

$$\mathscr{V}_0 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \middle| \ x_1 \in \mathscr{R}(\Sigma_1, \mathscr{S}, \mathbb{R}^p), x_2 \in \mathscr{A}(\Sigma_2, \mathscr{S}, \mathbb{R}^p) \right\}$$

and recursively:

$$\mathscr{V}_{i+1} = \{ \tilde{x} \in \mathbb{R}^n \mid \exists\, \tilde{u} \text{ such that} \tilde{A}\tilde{x} + \tilde{B}\tilde{u} \in \mathscr{V}_i \text{ and } \tilde{C}\tilde{x} + \tilde{D}\tilde{u} \in \mathscr{S} \}$$

where $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ are the system matrices of the system (A, B, C_z, D_z) after transforming it to the scb, i.e. $\tilde{A} = \Gamma_x^{-1} A \Gamma_x$, $\tilde{B} = \Gamma_x^{-1} B \Gamma_u$, $\tilde{C} = \Gamma_z^{-1} C_z \Gamma_x$ and $\tilde{D} = \Gamma_z^{-1} D_z \Gamma_u$.

If the order of the infinite zeros is less than or equal to k then we have for $i \geqslant k$:

$$\mathscr{V}_{i+1} = \mathscr{V}_i$$

We define:

$$\bar{\mathscr{R}}(\Sigma, \mathscr{S}, \mathbb{R}^p) = \mathscr{V}_i$$

where $i > k$.

The following theorem characterizes the maximum domain of attraction of a system Σ when the constraints are non-minimum phase. Let $u = f(x)$ be any stabilizing control law for system Σ subject to constraints (2) and let $\mathscr{R}_f(\Sigma, \mathscr{S}, \mathscr{T})$ denote the domain of attraction of the zero equilibrium of the closed-loop system with no violation of the constraints.

Theorem 4.4
Consider the plant Σ as given by (1) with constraint sets $\mathscr{S}$ and $\mathscr{T}$ satisfying Assumption 2.1. Let the constraints be right invertible and non-minimum phase. Then, for any given stabilizing controller $u = f(x)$, the domain of attraction $\mathscr{R}_f(\Sigma, \mathscr{S}, \mathscr{T})$ of the zero equilibrium under this control law satisfies

$$\mathscr{R}_f(\Sigma, \mathscr{S}, \mathscr{T}) \subseteq \mathscr{R}(\Sigma_1, \mathscr{S}, \mathscr{T}) \times \mathscr{A}(\Sigma_2, \mathscr{S}, \mathscr{T})$$

For the case without rate constraints we can strengthen the above inclusion and obtain:

$$\mathscr{R}_f(\Sigma, \mathscr{S}, \mathbb{R}^p) \subseteq \bar{\mathscr{R}}(\Sigma, \mathscr{S}, \mathbb{R}^p)$$

For the case without rate constraints, we call the set

$$\bar{\mathscr{M}} := \bar{\mathscr{R}}(\Sigma_1, \mathscr{S}, \mathbb{R}^p)$$

the *maximum achievable domain of attraction*. In the original coordinate system $\bar{\mathscr{M}}$ becomes $\mathscr{M} = \Gamma_x \bar{\mathscr{M}}$. Following the similar philosophy of the semi-global stabilization inside the admissible set $\mathscr{A}(\mathscr{S}, \mathscr{T})$ as defined in Problem 2.4, we can define a semi-global stabilization problems inside the maximum achievable domain of attraction $\mathscr{M}$.

Problem 4.5 (Semi-global stabilization for non-minimum phase constraints via state feedback) Consider the system (1) with constraint sets $\mathscr{S} \subset \mathbb{R}^p$ and $\mathscr{T} = \mathbb{R}^p$. For any *a priori* given compact set $\mathscr{W}$ contained in the interior of the maximum achievable domain of attraction $\mathscr{M}$ find, if possible, a state feedback (possibly nonlinear) $u(k) = f(x(k), k)$ such that the following conditions hold:

(i) The equilibrium point $x = 0$ of the closed-loop system is asymptotically stable with $\mathscr{W}$ contained in its domain of attraction.
(ii) For any $x(0) \in \mathscr{W}$, we have $z(k) \in \mathscr{S}$ for all $k \geqslant 0$.

Remark
Note that the above semi-global stabilization problem reduces to Problem 2.4 whenever the constraints are at most weakly non-minimum phase. In this case the maximum achievable domain of attraction $\mathscr{M}$ is equal to the admissible set $\mathscr{A}(\mathscr{S}, \mathbb{R}^p)$.

Note that Problem 4.5 can also be defined for the case of rate constraints but this requires the appropriate definition of the set $\bar{M}$. The next theorem provides solvability conditions for the semi-global stabilization with non-minimum phase constrains.

Theorem 4.6
Consider the plant Σ as given by (1) with constraint sets $\mathscr{S}$ and $\mathscr{T} = \mathbb{R}^n$ satisfying Assumption 2.1. Let the constraints be right invertible. The semi-global stabilization problem as defined in Problem 4.5 is solvable. More specifically, for any compact set $\mathscr{K}$ contained in the interior of the maximal achievable domain of attraction $\mathscr{M}$, there exists a stabilizing controller $u = g(x)$ for the whole system Σ such that

$$\mathscr{K} \subseteq \mathscr{R}_g(\Sigma, \mathscr{S}, \mathbb{R}^p)$$

where $\mathscr{R}_g(\Sigma, \mathscr{S}, \mathbb{R}^p)$ denotes the domain of attraction of the zero equilibrium of the closed-loop system with the constraints enforced.

4.3. *Proofs and construction of controllers*

The solvability conditions in Theorems 4.1 and 4.2 both require that the constraints be at most weakly nonminimum phase and of type one. Once the constraints are of type one, the scb representation of system Σ in (4) can be simplified. More specifically, the equations for x_d and z_d have a simpler structure because of the first order relative degree. To facilitate the proofs of

Theorems 4.1 and 4.2, we rewrite (4) after simplification as

$$\begin{aligned} x_a(k+1) &= A_{aa}x_a(k) + K_a z(k) \\ x_c(k+1) &= A_{cc}x_c(k) + B_c[u_c(k) + J_a x_a(k)] + K_c z(k) \\ x_d(k+1) &= u_d(k) + G_a x_a(k) + G_c x_c(k) + G_d x_d(k) \\ y(k) &= C_{ya}x_a(k) + C_{yc}x_c(k) + C_{yd}x_d(k) + \tilde{D}_y \tilde{u}(k) \\ z_0(k) &= u_0(k) \\ z_d(k) &= x_d(k) \end{aligned} \tag{9}$$

where G_a, G_c and G_d are matrices with appropriate dimensions.

Proof of Theorem 4.1

Necessity: The necessity of conditions (i) and (ii) is obvious. By the decomposition obtained above, the constrained variable z becomes the input to the zero dynamics (5), hence the system has to be at most weakly non-minimum phase, i.e. the poles of the zero dynamics must be in the closed unit disc. Next, we show the necessity of condition (iii).

We consider the global case first. Since the system is right invertible, having no infinite zeros of order greater than one is equivalent to $(C_zB \;\; D_z)$ being surjective. Therefore, if the system has infinite zeros of order greater than one, then there exists a vector $c \neq 0$ such that

$$c^{\mathrm{T}} D_z = 0 \quad \text{and} \quad c^{\mathrm{T}} C_z B = 0 \tag{10}$$

Moreover, since $\mathscr{T}$ contains zero in its interior, we can guarantee that $c \in \mathscr{T}$. Let $\zeta_0 \in \mathscr{S}$ be such that

$$\langle z, c \rangle \leqslant \langle \zeta_0, c \rangle$$

for all $z \in \mathscr{S}$. Since $\mathscr{S}$ is a compact and convex set, such a ζ_0 always exists at the boundary of $\mathscr{S}$. Next, because (A, B, C_z, D_z) is right invertible there exist an initial condition $x(0) = \xi_0$ and an input $u(0) = \mu_0$ such that the output z satisfies $z(0) = \zeta_0$ and $z(1) - z(0) = c$. Clearly $\xi_0 \in \mathscr{A}(\mathscr{S}, \mathscr{T})$. But if the system starts at time 0 from ξ_0 then we have

$$\langle c, z(0) \rangle = \langle c, C_z \xi_0 \rangle = \langle c, \zeta_0 \rangle \quad \text{and} \quad \langle c, z(1) - z(0) \rangle = \langle c, c \rangle > 0$$

for any input signal u because of property (10). Hence, $\langle c, z(1) \rangle > \langle c, \zeta_0 \rangle$ for any input u. By definition of ζ_0 this implies $z(1) \notin \mathscr{S}$ for any input u. Therefore, there exist initial conditions in $\mathscr{A}(\mathscr{S}, \mathscr{T})$ which cannot be stabilized without violating the constraints. This yields a contradiction.

The necessity of condition (iii) for the semi-global case follows by a mild modification of the above argument. Choose a λ close to 1 from below such that $\langle c, c \rangle > (1-\lambda)\langle c, \zeta_0 \rangle$, where ζ_0 is chosen as before. Let $z(0) = \lambda\zeta_0$. By the right invertibility as above, there exist an initial condition ξ_0 and an input $u(0) = \mu_0$ such that the output z satisfies $z(0) = \lambda\zeta_0$ and $z(1) - z \times (0) = c$. Then we can choose a compact set $\mathscr{A}_0$ in the interior of $\mathscr{A}(\mathscr{S}, \mathscr{T})$ so that $\xi_0 \in \mathscr{A}_0$. Since $\langle c, z(1) - z(0) \rangle = \langle c, c \rangle > 0$, we get $\langle c, z(1) \rangle = \langle c, z(0) \rangle + \langle c, c \rangle > \langle c, \zeta_0 \rangle$. By the same argument as in the global case, this implies that $z(1) \notin \mathscr{S}$ for any input u, which is a contradiction.

Sufficiency: The proof of sufficiency is constructive. It follows from two steps.

Step 1: We first design for the first subsystem Σ_1 in (5) while viewing z as an input variable. Let $v = z - \phi$, where the functions v and ϕ will become clear shortly. Then, (5) becomes

$$x_a(k+1) = A_{aa}x_a(k) + K_a\phi(k) + K_a v(k) \tag{11}$$

Note that the conditions of the theorem require that all eigenvalues of A_{aa} be in the closed unit disc. Viewing ϕ as an input to this subsystem, we can construct a state feedback law $\phi(k) = f(x_a(k))$ for the system (11), which has the following properties:

(a) It satisfies the constraints:

$$f(x_a(k)) \in \mathcal{S}, \quad f(x_a(k+1)) - f(x_a(k)) \in \mathcal{T}, \quad k \geqslant 0$$

(b) It renders the zero equilibrium point of the closed-loop system of (11) globally or semi-globally attractive in the presence of any signal satisfying

$$\|v(k)\| \leqslant M\lambda^k, \quad \lambda \in (0,1) \tag{12}$$

for some $M > 0$, i.e. $x_a(k) \to 0$ as $k \to \infty$.

(c) It renders the zero equilibrium point of the closed-loop system with $v = 0$ locally exponentially stable.

Note that the two parameters M and λ in (12) only depend on the size of the constraint sets $\mathcal{S}$ and $\mathcal{T}$. Whenever $\mathcal{S}$ and $\mathcal{T}$ are known, M and λ can be chosen *a priori* following the way specified in the design of *Step* 2. Knowing these facts, we are assured that the ℓ_2 norm of v signal is uniformly upper bounded. For completeness the details of designing such a state feedback for the first subsystem in the global or semi-global sense are presented in Appendix A. Note that in the global case the function $f(x_a)$ must be nonlinear; however, in the semi-global case $f(x_a)$ can be linear.

Step 2: In this step we design a control law for the second subsystem Σ_2 given in (7), so that the closed-loop system of the interconnection of the two subsystems with the control law is asymptotically stable and without constraint violation.

Choose $\lambda \in (0,1)$ such that

$$(1-\lambda)\bar{\mathcal{S}}_d \subset \mathcal{T}_d, \tag{13}$$

where $\bar{\mathcal{S}}_d := \{\xi - \eta : \xi \in \mathcal{S}_d,\ \eta \in \mathcal{S}_d\}$. The control law is designed as follows. Partition f and v compatibly with the decomposition of z as

$$f(x_a) = \begin{pmatrix} f_0(x_a) \\ f_d(x_a) \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} v_0 \\ v_d \end{pmatrix}$$

Then choose

$$u_c(k) = F_c x_c(k) - J_a x_a(k) \tag{14}$$

where F_c is such that $A_{cc} + B_c F_c$ is Schur-stable. Choose

$$u_0(k) = f_0(x_a(k)) \tag{15}$$

$$u_d(k) = \lambda[x_d(k) - f_d(x_a(k))] + f_d(x_a(k+1)) - \lambda^{k+1}[f_d(x_a(k+1)) - f_d(x_a(k))] \\ - G_a x_a(k) - G_c x_c(k) - G_d x_d(k) \tag{16}$$

where $x_a(k+1) = A_{aa}x_a(k) + K_a z(k)$. Note that the control law for u_d is time-varying, and is nonlinear in the global case and linear in the semi-global case. It remains to show that for the control law given above we have $z(k) \in \mathscr{S}$ and $z(k+1) - z(k) \in \mathscr{T}$ for all $k \geqslant 0$, moreover, $v(k) = z(k) - f(x_a(k))$ satisfies (12) for a suitably chosen $M > 0$.

Given the feedback for u_d, we obtain

$$x_d(k+1) - f_d(x_a(k+1)) = \lambda[x_d(k) - f_d(x_a(k))] - \lambda^{k+1}[f_d(x_a(k+1)) - f_d(x_a(k))] \tag{17}$$

Solving this difference equation yields that

$$x_d(k) = \lambda^k x_d(0) + (1-\lambda^k) f_d(x_a(k)) \tag{18}$$

Since both $x_d(0)$ and $f_d(x_a(k))$ are in the convex set $\mathscr{S}_d$, we have $z_d(k) = x_d(k) \in \mathscr{S}_d$. On the other hand,

$$\begin{aligned} x_d(k+1) - x_d(k) = {} & \lambda^k\{(1-\lambda)[f_d(x_a(k+1)) - x_d(0)]\} \\ & + (1-\lambda^k)[f_d(x_a(k+1)) - f_d(x_a(k))] \end{aligned}$$

Hence, by (13) we get

$$z_d(k+1) - z_d(k) = x_d(k+1) - x_d(k) \in \mathscr{T}_d$$

From (18) we see that

$$v_d(k) = x_d(k) - f_d(x_a(k)) = \lambda^k[x_d(0) - f_d(x_a(k))] \tag{19}$$

Noting that both $x_d(0)$ and $f_d(x_a(k))$ are in the bounded set $\mathscr{S}_d$ and that $z_0 = u_0 = f_0(x_a)$, we find that there exists $M > 0$ such that (12) holds.

So far we have shown that the equilibrium point $x = 0$ of the overall closed-loop system is globally attractive. Since we have used a time-varying control law and the control law is nonlinear in the global case, the asymptotical stability of the equilibrium point $x = 0$ needs a careful verification. First note that, according to the design of $f(x_a)$ presented in Appendix A, the feedback $f(x_a)$ is globally Lipschitz and locally linear in terms of x_a. Then, it can be shown that for sufficiently small initial conditions $x_{a0} = x_a(0)$, $x_{c0} = x_c(0)$, and $x_{d0} = x_d(0)$ we have $\|x_a(k)\| \leqslant \kappa_1(\|x_{a0}\| + \|x_{d0}\|)$ for some constant $\kappa_1 > 0$ and all $k \geqslant 0$. This part of proof is presented in Appendix B. From (18) we see that $\|v(k)\| \leqslant \kappa_2(\|x_{a0}\| + \|x_{d0}\|)$ for some constant $\kappa_2 > 0$ and all $k \geqslant 0$. From (17) it is straightforward that $\|x_d(k)\| \leqslant \kappa_3(\|x_{a0}\| + \|x_{d0}\|)$ for some constant $\kappa_3 > 0$ and all $k \geqslant 0$. Finally, viewing the dynamics of x_c as a Schur stable system with disturbance $K_c z(k) = K_c[f(x_a(k)) + v(k)]$ we obtain that $\|x_c(k)\| \leqslant \kappa_4(\|x_{a0}\| + \|x_{c0}\| + \|x_{d0}\|)$ for some constant $\kappa_4 > 0$ and all $k \geqslant 0$. In conclusion, we have shown the local stability of the equilibrium point $x = 0$. This completes the proof. □

Proof of Theorem 4.2

Note that condition (v) $\ker C_z \subset \ker C_z A$ implies that $G_a = G_c = 0$ in (7). Moreover, condition (vi) $\ker(C_y \;\; D_y) \subset \ker(C_z \;\; D_z)$ ensures that we can decompose y in a suitable basis such that

$$y = \begin{pmatrix} \tilde{y} \\ x_d \end{pmatrix} = \begin{pmatrix} \tilde{C}_{ya} & \tilde{C}_{yc} & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} x_a \\ x_c \\ x_d \end{pmatrix} + \begin{pmatrix} \tilde{D}_{yu} \\ 0 \end{pmatrix} \tilde{u}$$

which clearly indicates that the state x_d is directly determined by y. We get the following system

$$
\begin{aligned}
x_a(k+1) &= A_{aa}x_a(k) + K_a z(k) \\
x_c(k+1) &= A_{cc}x_c(k) + K_c z(k) + B_c[u_c(k) + J_a x_a(k)] \\
x_d(k+1) &= u_d(k) + G_d x_d(k) \\
\tilde{y} &= \tilde{C}_{ya}x_a + \tilde{C}_{yc}x_c + \tilde{D}_{yu}\tilde{u} \\
z_0(k) &= u_0(k) \\
z_d(k) &= x_d(k)
\end{aligned}
\tag{20}
$$

Since (C_y, A) is an observable pair, the scb decomposition guarantees that the pair

$$
\left(\begin{pmatrix} A_{aa} & 0 \\ B_c J_a & A_{cc} \end{pmatrix}, \quad (\tilde{C}_{ya} \quad \tilde{C}_{yc}) \right)
$$

is also observable. That is, there exist matrices L_a and L_c such that

$$
\tilde{A} = \begin{pmatrix} A_{aa} - L_a\tilde{C}_{ya} & -L_a\tilde{C}_{yc} \\ B_c J_a - L_c\tilde{C}_{ya} & A_{cc} - L_c\tilde{C}_{yc} \end{pmatrix}
$$

is Schur-stable. For the above system, we use a reduced-order observer for the state variables (x_a, x_c):

$$
\begin{aligned}
\hat{x}_a(k+1) &= A_{aa}\hat{x}_a(k) + K_a z(k) + L_a[\tilde{y}(k) - \tilde{C}_{ya}\hat{x}_a(k) - \tilde{C}_{yc}\hat{x}_c(k) - \tilde{D}_{yu}\tilde{u}(k)] \\
\hat{x}_c(k+1) &= A_{cc}\hat{x}_c(k) + K_c z(k) + B_c[u_c(k) + J_a\hat{x}_a(k)] \\
&\quad + L_c[\tilde{y}(k) - \tilde{C}_{ya}\hat{x}_a(k) - \tilde{C}_{yc}\hat{x}_c(k) - \bar{D}_{yu}\tilde{u}(k)]
\end{aligned}
$$

Note that the measurement error is exponentially decaying.

The remaining design procedure follows the state feedback controller design presented in the proof of Theorem 4.1 with (x_a, x_c) replaced by $(\hat{x}_a, \hat{x}_c)$ in the controller, except that we have an additional exponentially decaying error perturbation as a result of the replacement. Note that this additional error disturbance can be accommodated in the error signal v in the state feedback design, which is taken care of by a properly designed feedback $z_0 = f(\hat{x}_a)$ for the first subsystem. From the construction of state feedback, it can be verified that with the states (x_a, x_c) replaced by their measurements $(\hat{x}_a, \hat{x}_c)$ the constraints remain not violated. This completes the proof. □

Proof of Theorem 4.4

Let $x(0) = (x_a^{\mathrm{T}}(0), x_2^{\mathrm{T}}(0))^{\mathrm{T}}$ be any an initial condition in the domain of attraction $\mathscr{R}_f(\Sigma, \mathscr{S}, \mathscr{T})$ where $x_2^{\mathrm{T}} = (x_c^{\mathrm{T}}, x_d^{\mathrm{T}})$. It is clear that $x_2(0) \in \mathscr{A}(\Sigma_2, \mathscr{S}, \mathscr{T})$ because $x_2(0)$ must be in the admissible set of initial conditions for subsystem Σ_2. On the other hand the first subsystem Σ_1 can be viewed as being controlled by z with initial condition $x_a(0)$. Since $x(0)$ is in the domain of attraction, we know that $x(k) \to 0$ as $k \to \infty$, which implies that the state $x_a(k)$ converges to zero as $k \to \infty$ while z satisfies the constraints. This means that $x_a(0) \in \mathscr{R}(\Sigma_1, \mathscr{S}, \mathscr{T})$. □

Proof of Theorem 4.6
Since $\mathscr{K} \subset \mathscr{M}$, by the definition of $\mathscr{M}$, we can find an input u which guarantees that at time k, $x_a(k) = \bar{x}_a \in \mathscr{R}(\Sigma_1, \mathscr{S}, \mathbb{R}^p)$. Clearly, there exists a feedback $z = f(x_a)$ from time k onward which guarantees stability of the first subsystem when starting inside the set $\mathscr{R}(\Sigma_1, \mathscr{S}, \mathbb{R}^p)$. On the other hand, we can find an input from time 0 onward that guarantees that we still have $x_a(k) = \bar{x}_a(k)$ but additionally $z(i) = f(x_a(i))$ for $i \geqslant k$. Choosing u such that $z(i)$ has the desired value is initially a noncausal feedback since the subsystem from u to z contains delays. But since this system does not contain external disturbances we can implement this feedback in a causal way since the trajectory from time i onward is completely determined by $x(i)$. $\square$

5. NON-RIGHT INVERTIBLE CONSTRAINTS

For a system with non-right invertible constraints, according to scb, there exists a transformation matrix Γ_z such that $\tilde{z} = \Gamma_z^{-1} z$ yields the following decomposition

$$\tilde{z} = \begin{pmatrix} z_0 \\ z_b \\ z_d \end{pmatrix} = \begin{pmatrix} 0 \\ C_{zb} \\ 0 \end{pmatrix} x_b + \begin{pmatrix} 0 \\ 0 \\ I_{n_d} \end{pmatrix} x_d + \begin{pmatrix} I_{n_0} \\ 0 \\ 0 \end{pmatrix} u_0 \tag{21}$$

The decomposition of state has to be modified as $\tilde{x} = \Gamma_x^{-1} x = (x_a^{\mathrm{T}}, x_b^{\mathrm{T}}, x_c^{\mathrm{T}}, x_d^{\mathrm{T}})^{\mathrm{T}}$. Then system Σ in scb becomes (see References [18,19] for details):

$$\begin{aligned}
x_a(k+1) &= A_{aa} x_a(k) + K_a \tilde{z}(k) \\
x_b(k+1) &= A_{bb} x_b(k) + K_b \tilde{z}(k) \\
x_c(k+1) &= A_{cc} x_c(k) + K_c \tilde{z}(k) + B_c[u_c(k) + J_a x_a(k)] \\
x_d(k+1) &= u_d(k) + G_a x_a(k) + G_b x_b(k) + G_c x_c(k) + G_d x_d(k) \\
y(k) &= C_{ya} x_a(k) + C_{yb} x_b(k) + C_{yc} x_c(k) + C_{yd} x_d(k) + \tilde{D}_y \tilde{u}(k) \\
z_0(k) &= u_0(k) \\
z_b(k) &= C_{zb} x_b(k) \\
z_d(k) &= x_d(k)
\end{aligned} \tag{22}$$

Note that choosing a basis in the output space affects our sets $\mathscr{S}$ and $\mathscr{T}$. Therefore, we obtain new constraint sets $\tilde{\mathscr{S}} = \Gamma_z^{-1} \mathscr{S}$ and $\tilde{\mathscr{T}} = \Gamma_z^{-1} \mathscr{T}$. Since $C_z^{\mathrm{T}} D_z = 0$, it is guaranteed that these new constraint sets still satisfy Assumption 2.1.

Consider our original system in the special coordinate basis as given in (9) together with the extra output transformation in (21). Defining

$$\tilde{A}_1 = \begin{pmatrix} A_{aa} & 0 \\ 0 & A_{bb} \end{pmatrix}, \quad \tilde{B}_1 = \begin{pmatrix} K_a \\ K_b \end{pmatrix}, \quad \tilde{x}_1 = \begin{pmatrix} x_a \\ x_b \end{pmatrix}$$

$\tilde{C}_1 = C_{zb}$, $\tilde{v}_1 = \tilde{z}$, and $\tilde{z}_1 = z_b$, we obtain for $i = 1$ the following system:

$$\tilde{\Sigma}_i : \begin{cases} \tilde{x}_i(k+1) = \tilde{A}_i \tilde{x}_i(k) + \tilde{B}_i \tilde{v}_i(k) \\ \tilde{z}_i(k) = \tilde{C}_i \tilde{x}_i(k) \end{cases} \tag{23}$$

where both $\tilde{v}_1$ and $\tilde{z}_1$ are constrained. Temporarily dropping the constraint on $\tilde{v}_1$, we can repeat the same procedure to obtain $\tilde{\Sigma}_2$ from $\tilde{\Sigma}_1$ and so on. At each step of the construction we should make sure that the matrix $\tilde{B}_i$ has full column rank and the matrix $\tilde{C}_i$ has full row rank. This can be done without loss of generality. This chain ends if we obtain a subsystem $\tilde{\Sigma}_i$ which is right invertible in the sense that $\tilde{\Sigma}_{i+1}$ satisfies $\tilde{C}_{i+1} = 0$. Another possibility of termination is that at some step we get $\tilde{B}_i = 0$, which obviously implies that we can end the chain. It can be shown easily that if the pair (A, B) of the given system Σ is stabilizable, then all the systems $\tilde{\Sigma}_i$ as defined in (23) are stabilizable.

The following theorem contains some necessary conditions for constrained global or semi-global stabilization when the system is not right invertible.

Theorem 5.1

Consider the system Σ as given by (1) . Let the sets $\mathscr{S}$ and $\mathscr{T}$ satisfy Assumption 2.1. Moreover, let the chain of systems $\tilde{\Sigma}_i$ $(i = 1, \ldots, s)$ be as described above. Then the global and semi-global constrained stabilization problems formulated in Problems 2.3 and 2.4 are solvable *only if* the following conditions are satisfied:

(i) (A, B) is stabilizable.
(ii) The constraints of system Σ are at most weakly non-minimum phase.
(iii) The constraints of system Σ are of type one.
(iv) All the subsystems $\tilde{\Sigma}_i$ $(i = 1, \ldots s)$ have at most weakly non-minimum phase constraints.
(v) The subsystems $\tilde{\Sigma}_i$ $(i = 1, \ldots, s)$ with realization (23) satisfy:

$$\ker \tilde{C}_i \subset \ker \tilde{C}_i \tilde{A}_i \tag{24}$$

Proof

The necessity of these conditions except (v) is self-evident by considering each subsystem as an independent system with input and output constraints and recalling the necessary conditions in Theorem 4.1 for systems with output constraints. To see that the condition (v) is also necessary, we go back to the scb decomposition used earlier in the proof of Theorem 4.1. As an illustration, let us look at the x_d equation in (7) at time 0. We must have

$$x_d(1) = u_d(0) + G_a x_a(0) + G_c x_c(0) + G_d x_d(0) \in \mathscr{S}_d \tag{25}$$

for all possible initial conditions, but keep in mind that now u_d is constrained following the way we obtain the decomposition of $\tilde{\Sigma}_i$. Since x_a and x_c are completely unconstrained whereas $u_d(0)$ and $x_d(0)$ are constrained, condition (25) can be guaranteed only if G_a and G_c both equal 0. This is a condition equivalent to condition (v). □

The following example indicates that the conditions given in Theorem 5.1 are just necessary but not sufficient conditions for solving the constrained stabilization problems. Also, this example shows that the solvability conditions for global and semi-global stabilization in the case of non-right invertible constraints in general depend on the *particular* choice of constraint sets $\mathcal{S}$ and $\mathcal{T}$, unlike the case of right invertible constraints.

Example 5.2
Consider the system:

$$\begin{aligned} x_1(k+1) &= x_2(k) \\ x_2(k+1) &= u(k) \\ z_1(k) &= x_1(k) \\ z_2(k) &= x_2(k) \end{aligned} \tag{26}$$

Note that the transfer matrix from u to z is non-right invertible and all the conditions in Theorem 5.1 are satisfied. If the constraint set is defined as

$$\mathcal{S} = \{z : |z_1| \leqslant 1, |z_2| \leqslant 2\} \quad \text{and} \quad \mathcal{T} = \mathbb{R}^2$$

Then for any initial condition with $x_1(0) = 0$ and $x_2(0) > 1$, we find that $x_1(1)$ will violate the constraints. Therefore constrained stabilization is not possible.

However, for the constraint set defined by

$$\mathcal{S} = \{z : |z_1| \leqslant 1, |z_2| \leqslant 1\} \quad \text{and} \quad \mathcal{T} = \mathbb{R}^2$$

it is easily seen that the feedback $u = 0$ achieves constrained stabilization.

The shape dependence on the constraint sets causes trouble in developing solvability conditions for non-right invertible systems. Hence, it is meaningful to ask under what conditions the solvability conditions for the non-right invertible constraints will not depend on the specific shape of constraint sets. The following theorem provides an answer to this question.

Theorem 5.3
Consider the system (1). The following two statements are equivalent:

(i) The global or semi-global constrained stabilization is possible for all constraint sets $\mathcal{S}$ and $\mathcal{T}$ satisfying Assumption 2.1.
(ii) The constraints of system Σ are at most weakly non-minimum phase and of type one. Moreover, the subsystem $\tilde{\Sigma}_1$ defined in (23) takes the following form:

$$\begin{pmatrix} x_a(k+1) \\ x_b(k+1) \end{pmatrix} = \begin{pmatrix} A_{aa} & A_{ab} \\ 0 & \alpha I \end{pmatrix} \begin{pmatrix} x_a(k) \\ x_b(k) \end{pmatrix} + \begin{pmatrix} \bar{K}_a \\ 0 \end{pmatrix} \bar{z}(k)$$

$$z_b(k) = \begin{pmatrix} 0 & C_{zb} \end{pmatrix} \begin{pmatrix} x_a(k) \\ x_b(k) \end{pmatrix} \tag{27}$$

where $\bar{z}^{\mathrm{T}} = (z_0^{\mathrm{T}}, z_d^{\mathrm{T}})$, the matrix C_{zb} is injective, and $\alpha \in [0, 1)$.

Proof
The proof of (ii) $\Rightarrow$ (i) is obvious. It remains to prove (i) $\Rightarrow$ (ii).

We decompose $K_a = (K_{a0}\ K_{ab}\ K_{ad})$ and $K_b = (K_{b0}\ K_{bb}\ K_{bd})$. Then we rewrite the subsystem $\tilde{\Sigma}_1$ defined in (23) as

$$\begin{pmatrix} x_a(k+1) \\ x_b(k+1) \end{pmatrix} = \begin{pmatrix} A_{aa} & A_{ab} \\ 0 & \bar{A}_{bb} \end{pmatrix} \begin{pmatrix} x_a(k) \\ x_b(k) \end{pmatrix} + \begin{pmatrix} \bar{K}_a \\ \bar{K}_b \end{pmatrix} \bar{z}(k)$$

$$z_b(k) = (0 \quad C_{zb}) \begin{pmatrix} x_a(k) \\ x_b(k) \end{pmatrix} \tag{28}$$

where $A_{ab} = K_{ab}C_{zb}$, $\bar{A}_{bb} = A_{bb} + K_{bb}C_{zb}$, $\bar{K}_a = (K_{a0}\ K_{ad})$, $\bar{K}_a = (K_{a0}\ K_{ad})$, and the partial state x_a represents the zero dynamics. Viewing $\bar{z}$ as input to the zero dynamics and noting that z_b is constrained, the necessary condition for global or semi-global constrained stabilization as stated in condition (v) of Theorem 5.1 requires that

$$\ker C_{zb} \subset \ker C_{zb}\bar{A}_{bb}$$

This means that $\ker C_{zb}$ is part of the zero dynamics. But all of the zero dynamics of the original system has been included in the dynamics of x_a. Hence, $\ker C_{zb} = \{0\}$, i.e. C_{zb} is injective.

Knowing that C_{zb} is injective, we can choose a constraint set on z_b so that x_b is constrained to be arbitrarily small. However, $\bar{z}(0)$ can be anywhere in the constraint set for $\bar{z}$ which can be arbitrarily large. If $K_b \neq 0$ then we cannot guarantee that x_b is small enough to be in its constraint set and we get a constraint violation. Hence, we must have $K_b = 0$.

With $K_b = 0$, the subsystem of x_b becomes completely uncontrollable. For asymptotic stabilization of the whole system we need $|\alpha| < 1$. However, if the constraint set on x_b is not symmetric, to avoid constraint violation we must have $\alpha \in [0, 1)$. □

6. CONCLUSION

This paper has considered the semi-global and global stabilization problems of discrete-time linear systems in the presence of magnitude and rate constraints on both state and input variables. It turns out that the solvability conditions are largely dependent on the structural properties of linear plants such as the right invertibility, the location of the constraint invariant zeros, and the order of infinite zeros (or relative degree). New notions like constraint invariant zeros, constraint infinite zeros, right invertible constraints, and non-right invertible constraints have been introduced to characterize the conditions under which the global and semi-global stabilization problems are solvable. The general results presented here include the stabilization problems of linear systems with input constraints as a special case.

APPENDIX A: GLOBAL AND SEMI-GLOBAL STABILIZATION WITH AMPLITUDE AND RATE CONSTRAINTS AND ℓ_2 DISTURBANCE

In this section we first develop a nonlinear control law satisfying the amplitude and rate constraints that achieves globally asymptotic stabilization for an asymptotically null controllable system without disturbance, meanwhile it achieves global attractivity of the origin when an ℓ_2 disturbance is in presence. Then we develop a linear control for the semi-global case which achieves a similar result.

Theorem A1
Consider the system

$$x(k+1) = Ax(k) + Bu(k) + Bw(k) \tag{A1}$$

with input subject to the amplitude and rate constraints:

$$\|u(k)\|_\infty \leqslant \alpha, \quad \|u(k+1) - u(k)\|_\infty \leqslant \beta, \quad \forall k \geqslant 0 \tag{A2}$$

for some $\alpha > 0$ and $\beta > 0$. The sequence $w(k)$ is any disturbance in ℓ_2. Assume that (A, B) is stabilizable with all eigenvalues of A in the closed unit disc. Then, there exists a static nonlinear state feedback which has the following properties:

- The constraints in (A2) are not violated.
- In the absence of disturbance the equilibrium point $x = 0$ of the closed-loop system is globally asymptotically stable and locally exponentially stable.
- In the presence of any ℓ_2 disturbance the state $x = 0$ remains globally attractive.

Proof
We first recall the following fact. Let $Q(\varepsilon)$ be any parameterized positive definite matrix satisfying: $Q(\varepsilon) > 0$ for $\varepsilon > 0$, $Q(\varepsilon) \to 0$ as $\varepsilon \to 0$, and $(\mathrm{d}/\mathrm{d}\varepsilon)Q(\varepsilon) > 0$. Then the discrete-time algebraic Riccati equation (DARE)

$$A^{\mathrm{T}}PA - P - A^{\mathrm{T}}PB(B^{\mathrm{T}}PB + I)^{-1}B^{\mathrm{T}}PA + Q(\varepsilon) = 0$$

has a unique positive definite solution $P(\varepsilon)$ for any $\varepsilon \in (0, 1]$. Moreover, this positive definite solution $P(\varepsilon)$ has the following properties:

(i) The matrix $[A - B(B^{\mathrm{T}}PB + I)^{-1}B^{\mathrm{T}}P(\varepsilon)A]$ is Schur-stable for all $\varepsilon > 0$.
(ii) $\lim_{\varepsilon \to 0} P(\varepsilon) = 0$.
(iii) $P(\varepsilon)$ is continuously differentiable with $\mathrm{d}P(\varepsilon)/\mathrm{d}\varepsilon > 0$ for any $\varepsilon \in (0, 1]$.
(iv) There exists a constant $M > 0$ such that

$$\|P^{1/2}(\varepsilon)AP^{-1/2}(\varepsilon)\| \leqslant M \tag{A3}$$

for any $\varepsilon \in (0, 1]$.

For simplicity, we choose $Q(\varepsilon) = \varepsilon I$.

The idea of scheduling is to choose the parameter ε to be state dependent, that is, we define

$$\varepsilon(x(k)) = \max\{\varepsilon \in (0, 1]\colon\ x^{\mathrm{T}}(k)P(\varepsilon)x(k)\mathrm{tr}[P(\varepsilon)] \leqslant \delta^{*2}\} \tag{A4}$$

so that when $Q(\varepsilon)$ in the above Riccati equation is replaced by $Q(\varepsilon(x(k)))$, it yields a unique solution $P(\varepsilon(x(k)))$. Note that, because of the properties possessed by $P(\varepsilon)$, $\varepsilon(x(k))$ is well defined. To simplify notation, we denote $\varepsilon(k) = \varepsilon(x(k))$, $Q_k = Q(\varepsilon(x(k))) = \varepsilon(k)I$ and $P_k = P(\varepsilon(x(k)))$. Following this, we define a nonlinear static control law as

$$u(k) = -(B^{\mathrm{T}}P_kB + I)^{-1}B^{\mathrm{T}}P_kAx(k) \tag{A5}$$

and show that there exists a sufficiently small $\delta^* > 0$ such that the control law satisfies the amplitude and rate constraints (A2) and achieves global asymptotic stabilization of system (A1) when $w = 0$.

Let $\rho = \min\{\alpha, \ \beta/2\}$ and choose $\delta^* > 0$ small enough so that

$$M^2 \lambda_{\max}(BB^{\mathrm{T}})\delta^{*2} \leqslant \rho^2$$

where M is the constant defined in (A3). Then,

$$\begin{aligned} \|u(k)\|^2 &= x^{\mathrm{T}}(k)A^{\mathrm{T}}P_k B(B^{\mathrm{T}}P_k B + I)^{-2}B^{\mathrm{T}}P_k A x(k) \\ &\leqslant \|P_k^{-1/2}AP_k^{1/2}\|\lambda_{\max}(BB^{\mathrm{T}})\{x^{\mathrm{T}}(k)P_k x(k)\,\mathrm{tr}\,P_k\} \\ &\leqslant M^2\lambda_{\max}(BB^{\mathrm{T}})\delta^{*2} \\ &\leqslant \rho^2 \leqslant \alpha^2 \end{aligned}$$

which implies that $\|u(k)\|_\infty \leqslant \|u(k)\| \leqslant \alpha$ for all k, i.e. the control law (A5) does not violate the amplitude constraint. On the other hand, the above also yields that $\|u(k)\|_\infty \leqslant \beta/2$ for all k. Hence,

$$\|u(k+1) - u(k)\|_\infty \leqslant \|u(k+1)\|_\infty + \|u(k)\|_\infty \leqslant \beta$$

for all k which shows that the control law also does not violate the rate constraint.

Next we show that the closed-loop system is globally asymptotically stable when $w \equiv 0$. Choose a Lyapunov function

$$V(k) := V(x(k)) = x^{\mathrm{T}}(k)P_k x(k)$$

The variation of $V(k)$ along the state trajectory of the closed-loop system is

$$\begin{aligned} &V(k+1) - V(k) \\ &\quad = x^{\mathrm{T}}(k+1)[P_{k+1} - P_k]x(k+1) - \varepsilon(k)x^{\mathrm{T}}(k)x(k) - u^{\mathrm{T}}(k)u(k) \\ &\qquad + w^{\mathrm{T}}(k)B^{\mathrm{T}}P_k B w(k) - 2u^{\mathrm{T}}(k)w(k) \end{aligned} \tag{A6}$$

$$= x^{\mathrm{T}}(k+1)[P_{k+1} - P_k]x(k+1) - \varepsilon(k)\|x(k)\|^2 - \|u(k) + w(k)\|^2 \tag{A7}$$

$$+ w^{\mathrm{T}}(k)(B^{\mathrm{T}}P_k B + I)w(k) \tag{A8}$$

When $w \equiv 0$, we get

$$V(x(k+1)) - V(x(k)) \leqslant -\varepsilon(k)\|x(k)\|^2 + x^{\mathrm{T}}(k+1)[P_{k+1} - P_k]x(k+1) \tag{A9}$$

Consider the following two cases:

Case 1: If $\varepsilon(k+1) \leqslant \varepsilon(k)$, we find by the monotonicity of $P(\varepsilon)$ that $P_{k+1} \leqslant P_k$ and using (A9) that $V(x(k+1)) - V(x(k)) < 0$ for $x(k) \neq 0$.

Case 2: If $1 \geqslant \varepsilon(k+1) > \varepsilon(k)$, then $P_{k+1} > P_k$ and

$$V(x(k))\,\mathrm{tr}P_k = \delta^{*2} \geqslant V(x(k+1))\,\mathrm{tr}\,P_{k+1}$$

which yields $V(x(k+1)) - V(x(k)) < 0$ for $x(k) \neq 0$.

In conclusion, the control law (A5) guarantees that $V(x(k+1)) - V(x(k)) < 0$ for $x(k) \neq 0$, which implies the global asymptotic stability. The local exponential stability follows easily by noting that $\varepsilon(x(k)) \equiv 1$ if the system starts sufficiently close to the origin and the control law is linear and the input saturation is never overloaded.

The global attractivity of the origin in the presence of ℓ_2 disturbance follows from the following argument. From (A8) we claim that if $V(k+1) \geqslant V(k)$ then

$$V(k+1) - V(k) \leqslant -\varepsilon(k)\|x(k)\|^2 + \eta\|w(k)\|^2 \tag{A10}$$

where $\eta = \lambda_{\max}[B^{\mathrm{T}}P(1)B + I]$. This is because the scheduling defined in (A4) guarantees that $P_{k+1} \leqslant P_k$ whenever $V(k+1) \geqslant V(k)$. This yields

$$V(k+1) - V(k) \leqslant \eta\|w(k)\|^2 \tag{A11}$$

for all $k \geqslant 0$. This inequality guarantees that, given an ℓ_2 disturbance w, the state starting from anywhere in $\mathbb{R}^n$ is bounded. This implies that $\varepsilon(k)$ has a lower bound $\varepsilon_{\min} > 0$. It remains to show that the state x starting from any point in $\mathbb{R}^n$ is also in ℓ_2, hence it approaches to the origin.

First, note that if the initial state is sufficiently close to the origin, say $\|x(0)\| \leqslant r_0$ for some $r_0 > 0$ small enough, and the disturbance is bounded by $\|w(k)\| \leqslant d_0$, then for sufficiently small d_0 the amplitude and rate constraints (A2) will not be violated, and the closed-loop system is linear and exponentially stable. Hence, $x(k) \in \ell_2$.

Now, let $d_0^2 < \varepsilon_{\min} r_0/\eta$. We show that for any initial state $x(0) \in \mathbb{R}^n$ and any disturbance $w \in \ell_2$ there exists $K > 0$ such that $\|x(K)\| \leqslant r_0$ and $\|w(k)\| \leqslant d_0$ for all $k > K$. Since w is vanishing, there exists $K_1 > 0$ such that $\|w(k)\| \leqslant d_0$ for all $k \geqslant K_1$. On the other hand, if $\|x(k)\| > r_0$ and $V \times (k+1) \geqslant V(k)$ for some $k \geqslant K_1$ then from (A10) we have

$$V(k+1) - V(k) \leqslant -\varepsilon(k)\|x(k)\|^2 + \eta\|w(k)\|^2 < -\varepsilon_{\min} r_0 + \eta d_0 < 0$$

for $k \geqslant K_1$. This contradiction yields that either $\|x(k)\| \leqslant r_0$ or $V(k+1) < V(k)$. For the former case, we are done. For the later case, there exists $K > K_1$ such that $\|x(K)\| \leqslant r_0$. In conclusion, there exists $K > 0$ such that $\|x(K)\| \leqslant r_0$. This shows the global attractivity. □

Theorem A2

Consider the system (A1) with input subject to the amplitude and rate constraints (A2). Assume the same condition as stated in Theorem A1. Then, given any compact set $\mathscr{K}$ in the state space and any $D > 0$ there exists a linear state feedback which has the following properties:

- The constraints in (A2) are not violated.
- In the absence of disturbance the equilibrium point $x = 0$ of the closed-loop system is asymptotically stable with $\mathscr{K}$ contained in the region of attraction.
- In the presence of any ℓ_2 disturbance satisfying $\|w\|_{\ell_2} \leqslant D$ the state $x = 0$ remains attractive.

Proof

The proof of this theorem is easily adapted from the proof of Theorem A1. Since we are dealing with semi-global stabilization, we can fix ε to be a constant, instead of being state dependent. Let $V(x) = x^{\mathrm{T}}P(\varepsilon)x$ be the Lyapunov function and $L_V(c) := \{x : x^{\mathrm{T}}P(\varepsilon)x \leqslant c\}$ be the level set. Choose a sufficiently small $\varepsilon \in (0, 1]$ so that

$$2\eta D^2 \operatorname{tr} P(\varepsilon) \leqslant \delta^{*2} \quad \text{and} \quad \mathscr{K} \subset L_V(\eta D^2)$$

where the constants η and δ^* are defined in the proof of Theorem A1. Following this choice of ε, we claim that the level set $L_V(2\eta D^2)$ is an invariant set for trajectories starting from any point in $\mathscr{K}$ and any disturbance w satisfying $\|w\|_{\ell_2} \leqslant D$. This claim follows easily from the inequality (A10) which holds for all $k \geqslant 0$ when ε is fixed. The rest of the proof follows similarly as the global case. □

APPENDIX B: COMPLETION TO THE PROOF OF THEOREM 4.1

Lemma B1

Consider the following system

$$x(k+1) = Ax(k) + \lambda^k Gx(k)$$

where A is Schur stable and $|\lambda| < 1$. Then, for all $x(0) \in \mathbb{R}^n$ there exists $\kappa > 0$ such that

$$\|x(k)\| \leqslant \kappa \|x(0)\|$$

for all $k \geqslant 0$.

Proof

Since A is Schur stable, there exists a positive definite matrix $P > 0$ such that $A^{\mathrm{T}}PA - P = -I$. Let $V(x) = x^{\mathrm{T}}Px$ and denote $V_k = V(x(k))$. Then

$$\begin{aligned} V_{k+1} - V_k &= -x^{\mathrm{T}}(k)x(k) + 2\lambda^k x^{\mathrm{T}}(k)G^{\mathrm{T}}PAx(k) + \lambda^{2k}x^{\mathrm{T}}(k)G^{\mathrm{T}}PGx(k) \\ &\leqslant 2|\lambda|^k (x^{\mathrm{T}}(k)A^{\mathrm{T}}PAx(k))^{1/2}(x^{\mathrm{T}}(k)G^{\mathrm{T}}PGx(k))^{1/2} + |\lambda|^{2k}x^{\mathrm{T}}(k)G^{\mathrm{T}}PGx(k) \\ &\leqslant (2\beta^{1/2}|\lambda|^k + \beta|\lambda|^{2k})V_k \\ &\leqslant c_0|\lambda|^k V_k \end{aligned}$$

where we have used $x^{\mathrm{T}}(k)A^{\mathrm{T}}PAx(k) \leqslant x^{\mathrm{T}}(k)Px(k)$, $\beta = \lambda_{\max}(G^{\mathrm{T}}PG)/\lambda_{\min}(P)$, and $c_0 = 2\beta^{1/2} + \beta$. It follows that

$$V_{k+1} \leqslant (1 + c_0|\lambda|^k)V_k \leqslant \exp\{c_0|\lambda|^k\} V_k$$

Thus,

$$\prod_{i=0}^{k-1} \frac{V_{i+1}}{V_i} \leqslant \exp\left\{\sum_{k=0}^{\infty} c_0|\lambda|^k\right\}$$

i.e.

$$V_k \leqslant \exp\left\{\sum_{k=0}^{\infty} c_0|\lambda|^k\right\} V_0$$

for all $k \geqslant 0$. Hence the lemma follows. □

Completion to the Proof of Theorem 4.1

Note that the feedback $f(x_a)$ as constructed in Appendix A is globally Lipschitz and locally linear. Let $r > 0$ be sufficiently small and let $f(x_a) = -F_a x_a$ for $\|x_a\| \leqslant r$. As we shall see later, we can choose the initial conditions sufficiently small to guarantee that the trajectory of x_a remains in this ball. The construction of $f(x_a)$ guarantees that $\tilde{A}_{aa} := A_{aa} - K_a F_a$ is Schur stable. We decompose $F_a = (F_{a0}^{\mathrm{T}},\ F_{ad}^{\mathrm{T}})^{\mathrm{T}}$ and continue by writing out the first subsystem

$$\begin{aligned} x_a(k+1) &= A_{aa}x_a(k) + K_a f(x_a(k)) + K_a v(k) \\ &= (A_{aa} - K_a F_a)x_a(k) + K_a \begin{pmatrix} v_0 \\ v_d \end{pmatrix} \\ &= \tilde{A}_{aa}x_a(k) + K_a \begin{pmatrix} 0 \\ \lambda^k x_d(0) - \lambda^k f_d(x_a(k)) \end{pmatrix} \\ &= \tilde{A}_{aa}x_a(k) + \lambda^k K_a \begin{pmatrix} 0 \\ x_d(0) \end{pmatrix} + \lambda^k K_a \begin{pmatrix} 0 \\ F_{ad} \end{pmatrix} x_a(k) \end{aligned}$$

This system is equivalent to the following dynamics

$$\begin{pmatrix} x_a(k+1) \\ \xi(k+1) \end{pmatrix} = \begin{pmatrix} \tilde{A}_{aa} & I \\ 0 & \lambda I \end{pmatrix} \begin{pmatrix} x_a(k) \\ \xi(k) \end{pmatrix} + \lambda^k G \begin{pmatrix} x_a(k) \\ \xi(k) \end{pmatrix}$$

where

$$\xi(0) = K_a \begin{pmatrix} 0 \\ x_d(0) \end{pmatrix} \quad \text{and} \quad G = \begin{pmatrix} K_a \begin{pmatrix} 0 \\ F_{ad} \end{pmatrix} & 0 \\ 0 & 0 \end{pmatrix}$$

Applying Lemma B1, there exist $\kappa > 0$ and $\kappa_1 > 0$ such that

$$\|x_a(k)\| \leqslant \|(x_a(k)^{\mathrm{T}} \quad \xi(k)^{\mathrm{T}})^{\mathrm{T}}\| \leqslant \kappa \|(x_a(0)^{\mathrm{T}} \quad \xi(0)^{\mathrm{T}})^{\mathrm{T}}\| \leqslant \kappa_1 (\|x_a(0)\| + \|x_d(0)\|) \qquad \square$$

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