L^2 Sampled Signal Reconstruction With Causality Constraints—Part II: Theory

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***Abstract*—This paper provides the theoretic foundation for the design of L^2 optimal reconstructors (also known as interpolators/holds) with a prescribed degree of causality. A compact frequency-domain solution is derived that mimics known interpolation techniques for ordinary transfer functions. In parallel, an extensive state space solution is documented. It complements the frequency-domain solution in that it constructively proves the various claims, and it also makes the solution concrete. The state space solution requires the solution of one Riccati and one Lyapunov matrix equation.**

***Index Terms*—Causality constraints, consistent reconstruction, hybrid model matching, lifting, L^2 optimization.**

I. Introduction

We study the problem of reconstructing an analog signal from its sampled measurements with a prescribed degree of causality. In the first part [1], the problem was cast as a hybrid L^2 model matching of the form depicted in Fig. 1. Here, analog shift-invariant, but not necessarily stable, signal generators $\mathcal{G}_v$ and $\mathcal{G}_y$, the covariance of the measurement noise $\Sigma \geq 0$, and the ideal sampler $\mathcal{S}$ are given and the reconstructor, consisting of a pure discrete part $\bar{\mathcal{F}}$ and an D/A converter $\mathcal{H}$, is to be designed. The given components, those in the gray box, driven by a normalized white input w, shape properties and dependences of the analog signal to be reconstructed v and the discrete measurement signal $\bar{y}$. The reconstructor generates the analog reconstruction u of v according to

$$u(t) = \sum_{i\in\mathbb{Z}} \phi(t-ih)\bar{y}[i], \quad t\in\mathbb{R} \tag{1}$$

where $\phi(t)$ is an interpolation kernel or hold function to be found and h is the sampling period. The causality constraint that we impose is that

$$\phi(t) = 0, \quad \text{whenever } t < -lh, \tag{2}$$

for a given $l \in \mathbb{Z}^+$ called the *smoothing lag*. The design is then formulated as the problem of stabilizing the error system $\mathcal{G}_\mathrm{e}$ and minimizing its L^2 norm under constraint (2).

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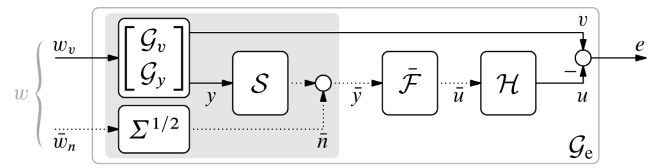

Fig. 1. The problem setup.

While the first part focuses on the solution of the problem, its interpretations and applications, in this paper we develop the theory behind the solution. This theory hinges on three main themes.

1) *Lifting*, as a framework within which all components of the hybrid system in Fig. 1 can be addressed in a unified manner.
2) *Coprime factorization*, as a systematic way of resolving stabilization constraints. This is an established tool in feedback control theory, especially if the system to be controlled is unstable [2]. In signal processing applications, feedback and the problem of stabilization is less of an issue and this may be the reason that coprime factorization is not widely used. One of our aims in this paper is to demonstrate the usefulness of this tool.
3) *State-space representation*, as an efficient computational tool. This topic hardly needs an introduction. However, there is a peculiar twist which has to do with the fact that our systems are a mixture of analog and discrete elements. What we need in the solution to our design problem are state representations of transfer functions. This shows up naturally in lifting because Fourier transformation is performed only with respect to discrete time (multiples of the sampling period) and so the Fourier transform still depends on intersample time.

Although our main goal is to provide a proof for the solution presented in [1], the results of this part are of independent interest and can be used in other sampling and reconstruction applications.

The paper is organized as follows. In Section II, we summarize lifting and systems norms in the lifted domain and then reformulate the reconstruction problem in the lifted frequency domain. To gain a preliminary insight into the proposed solution procedure, we solve a simple hold design problem for an integrator in Section III, and then in Section IV, we formulate and prove the general frequency domain solution of the problem. In Section V, we use this solution to address the consistency of the optimal reconstruction. In Section VI, which is rather technical, we set up a state-space equivalent of the frequency domain solution. Concluding remarks are provided in Section VII.

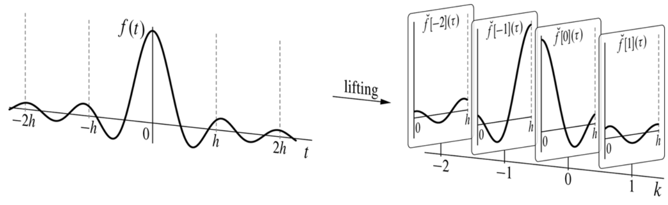

Fig. 2. Lifting analog signals (with $f(t) = \mathrm{sinc}_{0.44h}(t)$).

Notation

We follow the notation conventions of the first part [1], so below we outline only the most frequently used nonstandard definitions. Analog systems that are linear and invariant with respect to any time shift are said to be linear continuous time invariant (LCTI) systems. Systems that are linear and invariant with respect to integer multiples of the sampling period h are said to be linear discrete time invariant (LDTI). The space $L^2[0, h)$ is denoted by $\mathbb{L}$.

II. Reformulation in the Lifted Domain

In this section, we show how our (hybrid and LDTI) reconstruction problem can be converted to a pure discrete shift-invariant model-matching problem in the lifted domain. To this end, we review some material from [3], to which the reader is referred for further details and proofs.

A. Lifting and the Setup in the Lifted Domain

To deal with analog and discrete signals in a unified way, we represent all analog signals as discrete signals while preserving their analog, intersample, behavior. This process is called *lifting* [4], and it is reminiscent of the polyphase decomposition [5]. Fig. 2 explains the idea on real-valued signals. For an arbitrary $f : \mathbb{R} \to \mathbb{C}^{n_f}$, its lifting $\breve{f} : \mathbb{Z} \to \{[0, h) \to \mathbb{C}^{n_f}\}$ is defined as

$$\breve{f}[k](\tau) := f(kh + \tau), \quad k \in \mathbb{Z}$$

which is a sequence of functions. The lifted z-transform of $\breve{f}$ is defined as

$$\breve{f}(z; \tau) := \sum_{k \in \mathbb{Z}} \breve{f}[k](\tau) z^{-k}$$

for all $z \in \mathbb{C}$ for which the series converges. With z replaced by $\mathrm{e}^{\mathrm{j}\theta}$ for $\theta \in [-\pi, \pi]$ we obtain the lifted Fourier transform $\breve{f}(\mathrm{e}^{\mathrm{j}\theta}; \tau)$. Normally, we suppress the intersample time τ and simply write $\breve{f}[k]$, $\breve{f}(z)$, and $\breve{f}(\mathrm{e}^{\mathrm{j}\theta})$.

By lifting all analog signals in Fig. 1 we convert this hybrid system to a pure discrete one depicted in Fig. 3 without loosing the intersample information. The lifted signal generators $\breve{\mathcal{G}}_v$ and $\breve{\mathcal{G}}_y$ are merely the original analog generators $\mathcal{G}_v$ and $\mathcal{G}_y$ viewed as the mappings between the lifted versions of w_v and v and y. Likewise, the lifted ideal sampler $\acute{\mathcal{S}}$ connects the lifting $\breve{y}$ of y with its sampled version (according to $(\acute{\mathcal{S}}\breve{y})[k] = \breve{y}[k](0)$, as a matter of fact) and the lifted D/A converter $\grave{\mathcal{H}}$ transforms $\bar{u}$ to the lifting $\breve{u}$ of u.

Remark 2.1: The accents above the lifted systems serve a purpose in that they keep track of the dimensionality of the domain and range of the mapping. The breve accent, like $\breve{\mathcal{G}}$, indicates that this system maps a sequence of functions (of intersample time) to another sequence of functions. The acute accent, like $\acute{\mathcal{S}}$ on the lifted sampler, indicates that this system maps a sequence of functions to a sequence of numbers, and the grave accent, like $\grave{\mathcal{H}}$ on a hold, indicates that it maps a sequence of numbers to that of functions. ■

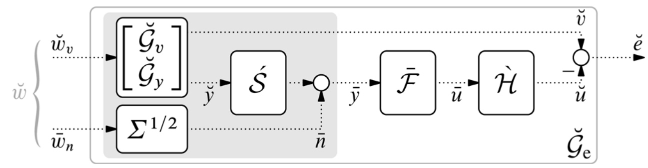

Fig. 3. The problem setup in the lifted domain.

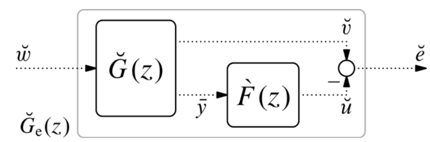

Fig. 4. The compact problem setup in the lifted frequency domain.

Because all blocks of our lifted setup are discrete systems, we no longer need to distinguish between them according to the kind of signals with which they operate. The only distinction making sense in the problem statement stage is that between given and to be designed parts. Moreover, as all blocks in Fig. 3 are now shift invariant [3], we may treat them in the z-domain. These observations lead us to the equivalent setup in Fig. 4, where the transfer function of the signal generator is

$$\breve{G}(z) = \begin{bmatrix} \breve{G}_v(z) & 0 \\ \acute{G}_y(z) & \Sigma^{1/2} \end{bmatrix} \tag{3}$$

where $\breve{G}_v(z)$ and $\acute{G}_y(z)$ are the transfer functions of $\breve{\mathcal{G}}_v$ and $\acute{\mathcal{G}}_y := \acute{\mathcal{S}}\breve{\mathcal{G}}_y$, respectively, and $\grave{F}(z)$ is the transfer function of the lifted reconstructor $\grave{\mathcal{F}} := \grave{\mathcal{H}}\bar{\mathcal{F}}$.

The term transfer *function* may be somewhat confusing when applied to the blocks in Fig. 4 because these "functions" are operators over $\mathbb{L} := L^2[0, h)$. Specifically, let

$$\begin{bmatrix} v(t) \\ y(t) \end{bmatrix} = \int_{-\infty}^{\infty} \begin{bmatrix} g_v(t - s) \\ g_y(t - s) \end{bmatrix} w_v(s)\mathrm{d}s$$

for some impulse responses g_v and g_y and the reconstructor act according to (1). Then, at almost every $z \in \mathbb{C}$:

- $\breve{G}_v(z)$ is an integral operator $\mathbb{L} \to \mathbb{L}$, for which the relation $\breve{v}(z) = \breve{G}_v(z)\breve{w}_v(z)$ reads
$$\breve{v}(z; \tau) = \int_0^h \breve{g}_v(z; \tau - \sigma)\breve{w}_v(z; \sigma)\mathrm{d}\sigma; \tag{4a}$$
- $\acute{G}_y(z)$ is an integral operator $\mathbb{L} \to \mathbb{C}^{n_{\bar{y}}}$, for which the relation $\bar{y}(z) = \acute{G}_y(z)\breve{w}_v(z)$ reads
$$\bar{y}(z) = \int_0^h \breve{g}_y(z; -\sigma)\breve{w}_v(z; \sigma)\mathrm{d}\sigma; \tag{4b}$$
- $\grave{F}(z)$ is a multiplication operator $\mathbb{C}^{n_{\bar{y}}} \to \mathbb{L}$, for which the relation $\breve{u}(z) = \grave{F}(z)\bar{y}(z)$ reads
$$\breve{u}(z; \tau) = \grave{\phi}(z; \tau)\bar{y}(z). \tag{4c}$$

Still, the calculus of these "transfer operators" is very similar to that of ordinary transfer functions, so we proceed with this terminology.

B. Stability and Causality in the Lifted Frequency Domain

The problem formulation in [1, Sec. II] requires the stability of both the error system $\mathcal{G}_e$ and the reconstructor $\mathcal{H}\bar{\mathcal{F}}$. By stability in both these cases, we understand the boundedness of the corresponding operators:

$$\sup_{\|w_v\|_2=1} \|\mathcal{G}_e w_v\|_2 < \infty \quad \text{and} \quad \sup_{\|\bar{y}\|_2=1} \|\mathcal{H}\bar{\mathcal{F}}\bar{y}\|_2 < \infty$$

where $\|\cdot\|_2$ stands for either $L^2(\mathbb{R})$ or $\ell^2(\mathbb{Z})$ signal norm. These requirements translate to the lifted frequency domain as the requirements that the frequency responses $\breve{G}_e(e^{j\theta})$ and $\grave{F}(e^{j\theta})$ are bounded at almost all θ.

In addition, we require that the reconstructor is l-causal in the sense (2). Combined with stability, the causality requirement has an elegant lifted frequency domain characterization. To this end, define the Hardy space H^∞ as the set of transfer functions $\grave{F}(z) : \mathbb{C}^{n_{\bar{y}}} \to \mathbb{L}$, which are analytic in $|z| > 1$ and satisfy

$$\operatorname{ess\,sup}_{|z|>1} \sigma_{\max}[\grave{F}(z)] < \infty$$

where $\sigma_{\max}[\cdot]$ stands for the operator maximal singular value, which equals the $\mathbb{C}^{n_{\bar{y}}} \to \mathbb{L}$ induced norm of $\grave{F}(z)$. Furthermore, let $z^l H^\infty$ be the space of operators $\grave{F}(z)$ such that $z^{-l}\grave{F} \in H^\infty$. Then, [3, Theorem 6.2] for every $l \in \mathbb{Z}^+$,

$$\mathcal{H}\bar{\mathcal{F}} \text{ is stable and } l\text{-causal} \iff \grave{F} \in z^l H^\infty.$$

It is worth emphasizing that when causality constraints are imposed, we have to analyze lifted transfer functions in the whole exterior of the unit disk. This is in contrast to the noncausal case studied in [6], where only the behavior on the unit circle $\mathbb{T}$ matters.

C. Performance in the Lifted Frequency Domain

As discussed in [1, Sec. II-C], we consider the familiar mean square performance measure. In the lifted frequency domain we minimize the L^2-norm of the error system, which is

$$\|\breve{G}_e\|_2 := \left(\frac{1}{2\pi h}\int_{-\pi}^{\pi} \|\breve{G}_e(e^{j\theta})\|_{HS}^2 d\theta\right)^{1/2}$$

where $\|\cdot\|_{HS}$ is the Hilbert–Schmidt norm [7], which is an operator version of the Frobenius (trace) matrix norm. The space of all transfer functions that are well defined on the unit circle $\mathbb{T}$ and have finite L^2-norm is denoted $L^2(\mathbb{T})$, or simply L^2 when the context is clear. We use this notation not only for operators $\mathbb{L} \to \mathbb{L}$, but also for operators $\mathbb{C} \to \mathbb{L}$ and so on. In all cases, $L^2(\mathbb{T})$ is a Hilbert space with an inner product, of which we only need to know that it exists and that it has the trace-like property:

$$\langle \tilde{A}, \tilde{B}\tilde{X}\rangle_2 = \langle \tilde{A}\tilde{X}^\sim, \tilde{B}\rangle_2 \tag{5}$$

where $\tilde{X}^\sim(z) := [\tilde{X}(1/\bar{z})]^*$ is the *conjugate* transfer function.

The treatment of L^2 optimization problems with stability requirements might be hampered by the fact that stability and performance are expressed in terms of different spaces, relations between which are not always clear. In our case the analysis is greatly simplified by the fact [3, Prop. 5.3] that

$$z^l H^\infty \subset L^2(\mathbb{T}) \tag{6}$$

whenever they are considered over *finite-rank* operators. This, in particular, is always true for the space of reconstructors.

D. Problem Formulation

We are now in the position to reformulate $\mathbf{RP}_l$ from [1] in the lifted frequency domain. Consider the reconstruction setup in Fig. 4. Then, we have

$\breve{\mathbf{RP}}_l$ Given a causal $\breve{G}$ as in (3) and $l \in \mathbb{Z}_0^+$, find $\grave{F}_l \in z^l H^\infty$ stabilizing the error system $\breve{G}_e$ and minimizing

$$\mathcal{J}_l := \|\breve{G}_e\|_2^2$$

over all reconstructors $\grave{F} \in z^l H^\infty$.

The shift invariance of all blocks in Fig. 4 facilitates the use of frequency-domain methods. In particular, we adapt the approach of [8]. This adaptation is not straightforward as the extension of many standard methods, well known for transfer functions over finite-dimensional input and output spaces, to lifted transfer function is quite nontrivial. Moreover, some of these methods are not well exposed in the signal processing literature. For these reasons, we start with a simple particular case of $\breve{\mathbf{RP}}_l$, which motivates the main steps of the theory to be developed later on.

III. A Motivating Example

To motivate the various steps of the general solution, we solve in this section the reconstruction problem for the case that the discrete noise $\bar{n} = \Sigma^{1/2}\bar{w}_n$ is absent and the signal generators $\mathcal{G}_v, \mathcal{G}_y$ are integrators,

$$G_v(s) = G_y(s) = 1/s.$$

In this case, the error system reduces to

$$\breve{G}_e = \breve{G}_v - \grave{F}\acute{G}_y \quad (\text{where } \acute{G}_y = \acute{S}\breve{G}_v).$$

The integrator, with its pole at the origin, enforces that any admissible reconstructor $\grave{F}$ will have to cancel the pole at the origin, meaning, as we shall soon see, that the hold recovers constant signals error free. In systems theory this is a common technique to endow admissible controllers (holds in this case) with desirable frequency-dependent properties.

Let us derive the lifted transfer functions $\breve{G}_v(z)$ and $\acute{G}_y(z)$. The lifting of the impulse response of $\mathcal{G}_v$, $g_v(t) = \mathbb{1}(t)$, is

$$\begin{aligned}\breve{g}_v(z;\tau-\sigma) &= \sum_{k\in\mathbb{Z}} \mathbb{1}(kh+\tau-\sigma)z^{-k} \\ &= \mathbb{1}(\tau-\sigma) + \sum_{k\in\mathbb{N}} z^{-k} \\ &= \mathbb{1}(\tau-\sigma) + 1/(z-1).\end{aligned}$$

Thus, by (4a), $\breve{v}(z) = \breve{G}_v(z)\breve{w}(z)$ is defined by the relation

$$\breve{v}(z;\tau) = \int_0^\tau \breve{w}(z;\sigma)d\sigma + \int_0^h \frac{1}{z-1}\breve{w}(z;\sigma)d\sigma. \tag{7}$$

The output of the sampler $\acute{\mathcal{G}}_y = \acute{\mathcal{S}}\breve{\mathcal{G}}_v$ equals $\breve{v}(z;0)$ and, hence, $\bar{y}(z) = \acute{G}_y(z)\breve{w}(z)$ follows directly by substituting $\tau = 0$ in (7),

$$\bar{y}(z) = \int_0^h \frac{1}{z-1}\breve{w}(z;\sigma)\mathrm{d}\sigma.$$

This equals the second integral of (7). The $\breve{G}_v(z)$ can thus be rewritten as

$$\breve{G}_v(z) = \breve{N}_v(z) + \grave{H}_{\mathrm{ZOH}}(z)\acute{G}_y(z)$$

where $\breve{N}_v$ is defined by the first term in the right-hand side of (7), which is the integrator with reset at every $t = kh$, and $\grave{H}_{\mathrm{ZOH}}$ is (the lifted transfer function of) the standard zero-order hold, whose interpolation kernel $\phi_{\mathrm{ZOH}}(t) = \mathbb{1}_{[0,h)}(t)$ has z-transform $\breve{\phi}_{\mathrm{ZOH}}(z;\tau) = 1$ for all $\tau \in [0,h)$ and all $z \in \mathbb{C}$.

As both $\breve{N}_v$ and $\grave{H}_{\mathrm{ZOH}}$ are *static* lifted systems (their transfer functions are constant in z), they are stable. Instabilities in the estimation channel $\breve{G}_v$ are thus of the same form as in the measurement channel $\acute{G}_y$. This, in particular, implies that the error system is stabilizable. Indeed, the trivial pick $\grave{F} = \grave{H}_{\mathrm{ZOH}}$ produces the stable error system $\breve{G}_{\mathrm{e}} = \breve{N}_v$. This can be intuitively explained: if v is asymptotically constant, a piecewise-constant reconstruction of its sampled noise-free measurements yields asymptotically perfect reconstruction.

a) Parameterization of All Stabilizing Holds: Although the zero-order hold stabilizes the error system, it is not necessarily optimal. This particular stabilizing solution, however, can be used to generate all other stabilizing solutions. To see this, consider the error transfer function $\breve{G}_{\mathrm{e}}(z)$, defined by the relation $\breve{e}(z) = (\breve{G}_v(z) - \grave{F}(z)\acute{G}_y(z))\breve{w}(z)$, that is

$$\breve{e}(z;\tau) = \int_0^\tau \breve{w}(z;\sigma)\mathrm{d}\sigma + \int_0^h \frac{1-\breve{\phi}(z;\tau)}{z-1}\breve{w}(z;\sigma)\mathrm{d}\sigma$$

where $\breve{\phi}(z;\tau)$ is the z-transform the interpolation kernel of $\grave{F}$. Whilst the first of the two integrals defines a stable system (it equals $\breve{N}_v\breve{w}$), the second integral contains a singularity on the unit circle $\mathbb{T}$, at $z = 1$. Every stabilizing hold must therefore cancel this singularity and this is the only requirement on stabilizing holds (apart from introducing no new instabilities, of course). Thus, the requirement that the hold be stabilizing can be cast as the following interpolation constraint on its z-transformed interpolation kernel,

$$\breve{\phi}(1;\tau) \equiv 1 \quad \forall\tau \in [0,h). \tag{8}$$

Clearly, the zero order hold, $\grave{H}_{\mathrm{ZOH}}$, satisfies this constraint as $\breve{\phi}_{\mathrm{ZOH}}(z;\tau) = 1$ for all z. Standard interpolation arguments [9, Theorem 10.18] yield then that all l-causal holds satisfying (8) are parameterized as

$$\grave{F}(z) = \grave{H}_{\mathrm{ZOH}}(z) + \grave{Q}(z)\bar{M}_y(z) \tag{9}$$

where

$$\bar{M}_y(z) = \frac{z-1}{a_1 z + a_0}$$

for any fixed $|a_0| < |a_1|$, and $\grave{Q} \in z^l H^\infty$ but otherwise arbitrary. In other words, all l-causal stabilizing holds are the parallel interconnection of a particular solution ($\grave{H}_{\mathrm{ZOH}}$) and the cascade of a discrete stable and proper transfer function having its zero at the interpolation point ($\bar{M}_y$) and an arbitrary $\grave{Q} \in z^l H^\infty$. The freedom in a_1 and a_0 (which does not affect $\grave{H}$ as the term $a_1 z + a_0$ can always be canceled by $\grave{Q}$), will be exploited later. With this parameterization, the error systems becomes

$$\breve{G}_{\mathrm{e}}(z) = \breve{N}_v(z) - \grave{Q}(z)\acute{N}_y(z), \tag{10}$$

with $\acute{N}_y := \bar{M}_y\acute{G}_y$ verifying

$$\acute{N}_y(z)\breve{w}(z) = \int_0^h \frac{1}{a_1 z + a_0}\breve{w}(z;\sigma)\mathrm{d}\sigma. \tag{11}$$

This $\acute{N}_y$ is causal and stable (i.e., $\acute{N}_y \in H^\infty$).

b) The Optimal Hold: Now that the stability issue is resolved, the solution of $\breve{\mathbf{RP}}_{\boldsymbol{l}}$ follows from a standard projection argument. By (6), our design parameter $\grave{Q} \in z^l H^\infty$ resides in a subspace of L^2. It is further easy to see that $\breve{N}_v$ and $\acute{N}_y$ are both stable and have finite L^2 norms.

By the Projection Theorem (orthogonality principle [5]) a hold $\grave{Q}_l \in z^l H^\infty$ minimizes the L^2-norm of (10) if

$$\langle \breve{N}_v - \grave{Q}_l\acute{N}_y, \grave{Q}\acute{N}_y\rangle_2 = 0, \quad \forall\grave{Q} \in z^l H^\infty.$$

In other words, $\grave{Q}_l$ solves $\breve{\mathbf{RP}}_{\boldsymbol{l}}$ if

$$\grave{V} - \grave{Q}_l\acute{N}_y\acute{N}_y^\sim \perp z^l H^\infty \tag{12}$$

where $\grave{V} := \breve{N}_v\acute{N}_y^\sim$. The orthogonality here is satisfied if the impulse response of $\grave{V} - \grave{Q}_l\acute{N}_y\acute{N}_y^\sim$ is zero at all $k < -l$. This condition might not be easy to enforce for an arbitrary $\acute{N}_y$ of the form (11), because $\acute{N}_y\acute{N}_y^\sim$ is in general noncausal, so that the l-causality of $\grave{Q}_l$ is not preserved in $\grave{Q}_l\acute{N}_y\acute{N}_y^\sim$. Indeed, by the results of [3, Sec. V-B], the conjugate is

$$\acute{N}_y^\sim(z) = \frac{1}{a_1 z^{-1} + a_0}$$

(the zero-order hold preceded by a discrete filter), so that

$$\acute{N}_y(z)\acute{N}_y^\sim(z) = \frac{h}{(a_1 z + a_0)(a_1 z^{-1} + a_0)}$$

which is noncausal for all $a_0 \neq 0$. Yet if $a_0 = 0$, we have that $\acute{N}_y\acute{N}_y^\sim \equiv h/a_1^2$, which is static and therefore causal (and causally invertible). It is convenient to normalize this static system by choosing $a_1 = \sqrt{h}$. In this case, the orthogonality condition (12) reads

$$\grave{V} - \grave{Q}_l \perp z^l H^\infty.$$

This is trivially achieved by taking the orthogonal projection

$$\grave{Q}_l = \mathrm{proj}_{z^l H^\infty}\grave{V}$$

which, hence, is the solution that we seek. This projection is merely the truncation of the impulse response of $\grave{V}$ to $\mathbb{Z}_l^+$.

Given our choice of $a_1 = \sqrt{h}$ and $a_0 = 0$, we have

$$\grave{V}(z) = \breve{N}_v(z)\acute{N}_y^{\sim}(z) = \int_0^{\tau} \frac{z}{\sqrt{h}}\mathrm{d}\sigma = \frac{\tau}{\sqrt{h}}\, z,$$

and therefore the solution $\grave{Q}_l$ to our problem is

$$\grave{Q}_l(z) = \begin{cases} 0 & \text{if } l = 0 \\ \frac{\tau}{\sqrt{h}}\, z & \text{if } l \geq 1. \end{cases}$$

The optimal hold, finally, is obtained by substituting this $\grave{Q}_l$ into (9). For $l = 0$ (no preview), the optimal hold is the *zero-order hold*. For $l \geq 1$ (finite preview), the optimal hold turns out to be the *first-order hold*:

$$\grave{F}_l(z) = \grave{H}_{\mathrm{ZOH}}(z) + \frac{\tau z}{\sqrt{h}} \frac{z-1}{\sqrt{h}z} = \frac{\tau}{h}\, z + \frac{h-\tau}{h} = \grave{H}_{\mathrm{FOH}}(z)$$

(see [3, Example 4.4]).

Remark 3.1: It is worth emphasizing that the optimal causal reconstructor, $\mathcal{H}_{\mathrm{ZOH}}$, is *not* a truncated version of the optimal noncausal reconstructor, $\mathcal{H}_{\mathrm{FOH}}$. The truncation is involved in the optimal solution, yet in an intermediate stage only. ∎

Remark 3.2: Quite interesting is that the optimal reconstructor in this case, as well as in all cases where $\mathcal{G} = \mathcal{G}_y$ are first-order systems, exploits only one preview step. Even if we allow a wider preview window ($l > 1$), the optimal solution is 1-causal. This property, however, is not generic in the L^2-optimization, see the discussion in [8, Sec. IV-C]. In general, the optimal reconstructor exploits all preview available and the larger the preview length is, the better reconstruction performance is achieved, see the examples in [1]. ∎

To complete the solution, we need to calculate the achieved optimal reconstruction performance. By orthogonality,

$$\begin{aligned} \|\breve{G}_{\mathrm{e}}\|_2^2 &= \|\breve{N}_v - \grave{Q}_l\acute{N}_y\|_2^2 = \|\breve{N}_v\|_2^2 - \|\grave{Q}_l\acute{N}_y\|_2^2 \\ &= \|\breve{N}_v\|_2^2 - \|\grave{Q}_l\|_2^2 \end{aligned}$$

where the fact that $\acute{N}_y\acute{N}_y^{\sim} = I$ was used. Routine calculation yields that

$$\|\breve{N}_v\|_2^2 = \frac{1}{2\pi h}\int_{-\pi}^{\pi}\int_0^h\int_0^h [\mathbb{1}(\tau-\sigma)]^2 \mathrm{d}\tau\mathrm{d}\sigma\mathrm{d}\theta = \frac{h}{2}.$$

Finally, if $l = 0$, we clearly have $\|\grave{Q}_l\|_2^2 = 0$ and if $l \geq 1$, then $\grave{Q}_l^{\sim}\grave{Q}_l \equiv h^2/3$, so that $\|\grave{Q}_l\|_2^2 = h/3$. Thus, the optimal performance index is

$$\|\breve{G}_{\mathrm{e}}\|_2^2 = \begin{cases} h/2 & \text{if } l = 0 \\ h/6 & \text{if } l \geq 1. \end{cases}$$

It shows that the availability of preview improves the reconstruction performance by a factor of 3 in this case. Also, for all preview lengths, $\lim_{h\to 0}\|\breve{G}_{\mathrm{e}}\|_2 = 0$, which agrees with our intuition that this signal can be perfectly reconstructed from its analog noise-free measurements.

We are now in the position to describe the general solution procedure. The solution follows the same lines as the example of this section.

IV. Frequency-Domain Solution

Stability of the error system is the first issue to be addressed when solving $\mathbf{RP}_{\boldsymbol{l}}$. As we saw in the previous section, stabilization amounts to canceling the instabilities of the signal generators by the reconstructor $\grave{F}$. For the simple system considered in the previous section stabilization is fairly straightforward. For the general case, when

$$\breve{G}_{\mathrm{e}} = [\, I \quad -\grave{F}\,]\breve{G},$$

we use the coprime factorization approach [2]. This offers an elegant formalism to parameterize all stabilizing holds as we shall soon see.

We say that H^∞ functions $\tilde{M}(z)$ and $\tilde{N}(z)$ are *left coprime over* H^∞ if there exist compatibly dimensioned H^∞ functions $\tilde{X}(z)$ and $\tilde{Y}(z)$ such that

$$\tilde{M}\tilde{X} + \tilde{N}\tilde{Y} = [\,\tilde{M} \quad \tilde{N}\,]\begin{bmatrix}\tilde{X}\\ \tilde{Y}\end{bmatrix} = I.$$

This equation is called the Bézout equation and the corresponding $\tilde{X}$ and $\tilde{Y}$ are the Bézout factors of $\tilde{M}$ and $\tilde{N}$. Left coprimeness effectively says that $\tilde{M}$ and $\tilde{N}$ have no common unstable (i.e., in $\mathbb{C}\setminus\mathbb{D}$) zeros, including their multiplicity and output directions. Another way to say this is that $[\,\tilde{M} \quad \tilde{N}\,]$ is right invertible in H^∞. Consequently, if $\tilde{M}$ and $\tilde{N}$ are left coprime, then $\tilde{T}[\,\tilde{M} \quad \tilde{N}\,] \in H^\infty$ necessarily implies that $\tilde{T} \in H^\infty$ as well.

A. Stabilization

We start with the following result, which states that the stabilizability is equivalent to the existence of a special upper triangular coprime factorization.

Proposition 4.1: There is $\grave{F} \in H^\infty$ rendering G_{e} stable iff

$$\breve{G} = \begin{bmatrix} I & \grave{M}_v \\ 0 & \bar{M}_y\end{bmatrix}^{-1}\begin{bmatrix}\breve{N}_v \\ \acute{N}_y\end{bmatrix} \tag{13}$$

for some left coprime $\bar{M}_y, \acute{N}_y \in H^\infty$ and some $\grave{M}_v, \breve{N}_v \in H^\infty$. In this case the right-hand side of (13) is a left coprime factorization.

Proof (Essentially From [8]): Let $\bar{M}_y^{-1}\acute{N}_y$ be a left coprime factorization of $[\,\acute{G}_y \quad \Sigma^{1/2}\,]$. This exists because $\acute{G}_y$ is rational and proper and Σ is stable (in fact, we construct one in Section VI). If in addition stable $\grave{M}_v$ and $\breve{N}_v$ can be found to satisfy (13), then $\grave{F} = -\grave{M}_v$ is a stable and stabilizing hold because it gives the stable $\breve{G}_{\mathrm{e}} = \breve{N}_v$. Conversely, if $\grave{F}$ and $\breve{G}_{\mathrm{e}}$ are both stable, then (13) holds for $\grave{M}_v = -\grave{F}$ and $\breve{N}_v = \breve{G}_{\mathrm{e}}$.

Once $\bar{M}_y^{-1}\acute{N}_y$ is coprime, the coprimeness of the right-hand side of (13) follows from the Bézout identity

$$\begin{bmatrix} I & \grave{M}_v \\ 0 & \bar{M}_y\end{bmatrix}\begin{bmatrix} I & -\grave{M}_v\bar{X}_y - \breve{N}_v\grave{Y}_y \\ 0 & \bar{X}_y\end{bmatrix} + \begin{bmatrix}\breve{N}_v \\ \acute{N}_y\end{bmatrix}[\,0 \quad \grave{Y}_y\,] = I$$

where $\bar{X}_y$ and $\grave{Y}_y$ are the Bézout factors of $\bar{M}_y$ and $\acute{N}_y$. ∎

Remark 4.1: Note that Proposition 4.1 considers $\grave{F} \in H^\infty$, which might appear to be more restrictive than what we need ($\grave{F} \in z^l H^\infty$). It can be shown, however, that if $\breve{G}_y(z)$ is proper (i.e., bounded in $|z| > \rho$ for sufficiently large ρ), the preview has no effect on the stabilization. This is because the relaxation of the causality constraints does not relax the requirement that $\grave{F}(z)$ is analytic in $\mathbb{C} \setminus \bar{\mathbb{D}}$. ∎

Factorization (13) facilitates the parameterization of the set of all stabilizing reconstructors and corresponding error systems. The following result is essentially a systematic generalization of (9) and (10).

Proposition 4.2: Suppose (13) is a coprime factorization. Then $\grave{F} \in z^l H^\infty$ stabilizes $\breve{G}_e$ iff

$$\grave{F} = -[I \quad -\grave{Q}] \begin{bmatrix} \grave{M}_v \\ \bar{M}_y \end{bmatrix} = -\grave{M}_v + \grave{Q}\bar{M}_y \tag{14a}$$

for some $\grave{Q} \in z^l H^\infty$. In this case,

$$\breve{G}_e = [I \quad -\grave{Q}] \begin{bmatrix} \breve{N}_v \\ \acute{N}_y \end{bmatrix} = \breve{N}_v - \grave{Q}\acute{N}_y \tag{14b}$$

parameterizes the set of all stable error transfer functions.

Proof (Essentially From [8]): If $\grave{Q} \in z^l H^\infty$ and we take $\grave{F}$ as defined in (14a), then it follows by direct substitutions that (14b) holds, which is stable. To prove the converse, let $\grave{F} \in z^l H^\infty$ stabilize the error system. Then, $\grave{Q} := (\grave{M}_v + \grave{F})\bar{M}_y^{-1}$ solved from (14a) satisfies

$$[I \quad -\grave{Q}] \begin{bmatrix} I & \grave{M}_v & \breve{N}_v \\ 0 & \bar{M}_y & \acute{N}_y \end{bmatrix} = [I \quad -\grave{F} \quad \breve{G}_e] \in z^l H^\infty.$$

Therefore,

$$z^{-l}[I \quad -\grave{Q}] \begin{bmatrix} I & \grave{M}_v & \breve{N}_v \\ 0 & \bar{M}_y & \acute{N}_y \end{bmatrix} \in H^\infty.$$

As (13) is a left coprime factorization, the right-most 2×3 partitioned matrix has a right inverse in H^∞. Hence, we have $z^{-l}\grave{Q} \in H^\infty$, i.e., $Q \in z^l H^\infty$. ∎

B. Normalization and Orthogonalization

The choice of coprime factors in (13) is not unique. Indeed, given any particular solution, we may redefine the coprime factors using appropriately dimensioned $\grave{U} \in H^\infty$ and $\bar{T}, \bar{T}^{-1} \in H^\infty$ as

$$\begin{bmatrix} I & \grave{M}_v & \breve{N}_v \\ 0 & \bar{M}_y & \acute{N}_y \end{bmatrix} \to \begin{bmatrix} I & -\grave{U}\bar{T} \\ 0 & \bar{T} \end{bmatrix} \begin{bmatrix} I & \grave{M}_v & \breve{N}_v \\ 0 & \bar{M}_y & \acute{N}_y \end{bmatrix}$$

because the 2×2 matrix in the middle is bistable and cancels in the factorization (13). We exploit this freedom to supplement the factors in (13) with desirable properties facilitating the L^2 performance analysis.

First, motivated by the analysis in Section III, we choose $\bar{T}$ so that the factor in (14b) that depends on the design parameter $\grave{Q}$, is co-inner in the redefined coprime factors, i.e., such that

$$\bar{T}\acute{N}_y\acute{N}_y^\sim\bar{T}^\sim = I \tag{15}$$

(normalization). Since on the unit disk the conjugate transfer function is the adjoint, the (rational and *matrix-valued*) transfer function

$$\bar{\Phi}_y(z) := \acute{N}_y(z)\acute{N}_y^\sim(z) \tag{16}$$

is self-adjoint for $z \in \mathbb{T}$. Equation (15) rewrites then as

$$\bar{\Phi}_y = \bar{T}^{-1}(\bar{T}^\sim)^{-1}. \tag{17}$$

This shows that the required $\bar{T}$, if it exists, is merely the inverse of the *spectral factor* [10] of $\bar{\Phi}_y$. The existence of this spectral factor is equivalent to the nonsingularity of $\bar{\Phi}_y(z)$ on the unit disk. This condition is also the standard nonsingularity condition [11] for the estimation problem associated with (14b): if it does not hold, the optimal $\grave{Q}$ might not belong to H^∞, albeit can be arbitrarily closely approximated by a stable $\grave{Q}$. To rule out such situations, we assume hereafter that

$\breve{\mathcal{A}}_1$: $\bar{\Phi}_y(e^{j\theta}) > 0$ for all $\theta \in [-\pi, \pi]$.

It is worth emphasizing that this condition does not depend on the particular choice of coprime factorization in (13). Indeed, $\acute{N}_y$ (as well as $\bar{M}_y$) is unique [2] modulo the left multiplication by a bi-stable $\bar{T}(z)$ (i.e., $\bar{T}, \bar{T}^{-1} \in H^\infty$), which is well defined and nonsingular on $z \in \mathbb{T}$.

Consider now the transfer function, obtained from the first term in (14b) times the conjugate of second term of (14b), without $\grave{Q}$, for the redefined coprime factors and with $\bar{T}$ satisfying (17),

$$\grave{V} := \breve{N}_v\acute{N}_y^\sim = \breve{N}_v\acute{N}_y^\sim\bar{T}^\sim - \grave{U}. \tag{18}$$

Since $\breve{N}_v, \acute{N}_y, \bar{T} \in H^\infty$, the first term in (18) is stable but most probably not causal. This transfer function can always be decomposed into causal and strictly anticausal parts. Denoting the former by $(\cdot)_+$, the choice

$$\grave{U} = (\breve{N}_v\acute{N}_y^\sim\bar{T}^\sim)_+ \tag{19}$$

yields a strictly anticausal $\grave{V}$ (that is, $\grave{V}^\sim \in z^{-1}H^\infty$). We thus just proved the following result:

Proposition 4.3: Let $\breve{\mathcal{A}}_1$ hold and $\breve{G}$ admit a left coprime factorization as in (13). Then the factors can be chosen so that

$$\begin{bmatrix} \breve{N}_v \\ \acute{N}_y \end{bmatrix} \acute{N}_y^\sim = \begin{bmatrix} \grave{V} \\ I \end{bmatrix} \tag{20}$$

with $\grave{V}^\sim \in z^{-1}H^\infty$.

C. L^2 Optimization

With the left coprime factorization we turned the design of stabilizing $\grave{F} \in z^l H^\infty$ into that of a $\grave{Q} \in z^l H^\infty$. The minimization of the L^2-norm of $\breve{G}_e$ follows the standard Hilbert space optimization arguments presented in Section III. To apply these arguments, we have to assume that

$\breve{\mathcal{A}}_2$: $\breve{N}_v \in L^2$.

This assumption does not depend on the choice of coprime factorization because it is in fact equivalent to the assumption that $\mathbf{RP}_l$ admits a solution with finite cost $\mathcal{J}_l$. Indeed, as the term $\grave{Q}\acute{N}_y$ in (14b) has finite rank at each frequency, it is in L^2 for every $\grave{Q} \in z^l H^\infty$. Hence, $\breve{G}_e$ is in L^2 iff the first term, $\breve{N}_v$, of (14b) is in L^2.

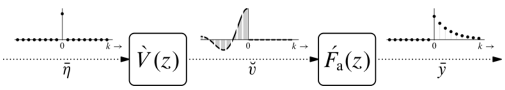

Fig. 5. Impulse response pattern of $\acute{\mathcal{F}}_{\rm a}\grave{\mathcal{V}}$.

By the Projection Theorem, $\grave{Q}_l$ is optimal if and only if the error system is orthogonal to all possible $\grave{Q}\acute{N}_y$ i.e.,

$$\breve{N}_v - \grave{Q}_l\acute{N}_y \perp \grave{Q}\acute{N}_y$$

for all $\grave{Q} \in z^l H^\infty$. By (5) and (20) this condition holds iff

$$\grave{V} - \grave{Q}_l \perp z^l H^\infty$$

[mind (6)]. This, in turn, is achieved by taking

$$\grave{Q}_l = \mathrm{proj}_{z^l H^\infty} \grave{V}. \tag{21}$$

The required projection amounts to truncation of the interpolation kernel of $\grave{V}$, which has support in strictly negative time $\mathbb{Z}_0^-$, to $\mathbb{Z}_{-l}^-$. The result is an FIR hold $\grave{Q}_l$. This optimal FIR $\grave{Q}_l$, substituted in (14a), determines the optimal hold $\grave{F}_l$. Typically, this $\grave{F}_l$ is not FIR.

Finally, by Pythagoras, the optimal performance level can be expressed as

$$\mathcal{J}_l = \|\breve{N}_v\|_2^2 - \|\grave{Q}_l\|_2^2.$$

Because $\grave{Q}_0 = 0$, the quantity $\|\grave{Q}_l\|_2^2$ expresses the improvement of the achievable performance level due to the preview.

V. Consistency Analysis

A widely used approach in the design of hold devices, is one based on the notion of *consistency* [12, p. 2918]. Loosely speaking, a hold is said to be consistent if its output, when again sampled, recovers the samples that where injected into the hold. A precise definition follows shortly.

Interestingly, consistency and L^2-optimality are often not conflicting criteria. To make the link, we assume in this section that the discrete noise $\bar{n}_w = \Sigma^{1/2}\bar{w}_n$ is absent and that $\mathcal{G}_y$ is a filtered version of $\mathcal{G}_v$, that is,

$$\breve{G}_y = \breve{F}_{\rm a}\breve{G}_v$$

for some LCTI $\breve{F}_{\rm a}$. The filter $\breve{F}_{\rm a}$ in this context is known as an antialiasing filter. The series $\acute{S}\breve{F}_{\rm a}$ is a generalized sampler and we denote it as $\acute{F}_{\rm a}$. Now the signal $\bar{y}$ that is injected into the hold is in the image of $\acute{F}_{\rm a}\breve{G}_v$ and the consistency property in this situation is then that the hold has the property

$$\acute{F}_{\rm a}\grave{F}\acute{F}_{\rm a}\breve{G}_v = \acute{F}_{\rm a}\breve{G}_v \iff (I - \acute{F}_{\rm a}\grave{F})\acute{F}_{\rm a}\breve{G}_v = 0.$$

This is implied by the identity

$$\acute{F}_{\rm a}\grave{F} = I. \tag{22}$$

In fact, in most of the cases, $\acute{F}_{\rm a}\breve{G}_v$ is right-invertible, so consistency is then equivalent to (22).

In Subsections V-A and V-B, we prove that L^2-optimal holds are always consistent if no causality constraint are imposed ($l = \infty$) and that consistency at any positive finite preview, $l > 0$, is also guaranteed provided that the impulse response of the antialiasing filter has support on $[0, h)$.

A. Noncausal Reconstruction ($l = \infty$)

Take $l = \infty$. In the absence of Σ we have $\acute{F}_{\rm a}\breve{G}_v = \acute{G}_y = \bar{M}_y^{-1}\acute{N}_y$, and therefore

$$\acute{F}_{\rm a}\breve{G}_v\breve{G}_v^\sim\acute{F}_{\rm a}^\sim = \bar{M}_y^{-1}\acute{N}_y\acute{N}_y^\sim\left(M_y^{-1}\right)^\sim = (\bar{M}_y^\sim\bar{M}_y)^{-1}$$

and

$$\begin{aligned}\grave{V} &= \breve{N}_v\acute{N}_y^\sim = (I + \grave{M}_v\acute{F}_{\rm a})\breve{G}_v(\bar{M}_y\acute{F}_{\rm a}\breve{G}_v)^\sim \\ &= (I + \grave{M}_v\acute{F}_{\rm a})\breve{G}_v\breve{G}_v^\sim\acute{F}_{\rm a}^\sim\bar{M}_y^\sim.\end{aligned}$$

Now, using the fact that the optimal $\grave{Q}$ equals $\grave{V}$, we get for the optimal hold

$$\begin{aligned}\grave{F}_\infty &= -\grave{M}_v + \grave{V}\bar{M}_y = -\grave{M}_v + (I + \grave{M}_v\acute{F}_{\rm a})\breve{G}_v\breve{G}_v^\sim\acute{F}_{\rm a}^\sim\bar{M}_y^\sim \\ &= \breve{G}_v\breve{G}_v^\sim\acute{F}_{\rm a}^\sim(\acute{F}_{\rm a}\breve{G}_v\breve{G}_v^\sim\acute{F}_{\rm a}^\sim)^{-1}.\end{aligned}$$

It is now trivial to see that this hold satisfies the consistency criterion (22). In other words, noncausal L^2 optimal holds are always consistent if $\Sigma = 0$ and $\mathcal{G}_y = \mathcal{F}_{\rm a}\mathcal{G}_v$.

B. l-Causal Reconstruction for FIR Antialiasing Filters

Since consistency is guaranteed for infinite preview, $\grave{Q} = \grave{V}$, it makes sense to write the finite preview case in terms of the infinite preview case. So express the optimal $\grave{Q}_l$ as

$$\grave{Q}_l = \grave{V} - \grave{V}_{\rm tail}$$

where

$$\grave{V}_{\rm tail} := \grave{V} - \mathrm{proj}_{z^l H^\infty}(\grave{V}) = \cdots + z^{l+2}\grave{V}_{-l-2} + z^{l+1}\grave{V}_{-l-1}$$

for some static holds $\grave{V}_i$. Using the consistency of infinite preview we now have

$$\acute{F}_{\rm a}\grave{F}_l = \acute{F}_{\rm a}(-\grave{M}_v + \grave{V}\bar{M}_y - \grave{V}_{\rm tail}\bar{M}_y) = I - \acute{F}_{\rm a}\grave{V}_{\rm tail}\bar{M}_y.$$

Thus, the optimal hold is consistent if $\acute{F}_{\rm a}\grave{V}_{\rm tail}\bar{M}_y = 0$, which is the same as

$$\acute{F}_{\rm a}\grave{V}_{\rm tail} = 0 \tag{23}$$

because $\bar{M}_y$ is nonsingular.

A key observation, which we shall use in the analysis, is that while $\grave{V}$ is anticausal, the series

$$\begin{aligned}\acute{F}_{\rm a}\grave{V} &= \acute{F}(I + \grave{M}_v\acute{F}_{\rm a})\breve{G}_v\breve{G}_v^\sim\acute{F}^\sim\bar{M}_y^\sim \\ &= (\acute{F}_{\rm a}\breve{G}_v\breve{G}_v^\sim\acute{F}_{\rm a}^\sim)\bar{M}_y^\sim + \acute{F}_{\rm a}\grave{M}_v(\acute{F}_{\rm a}\breve{G}_v\breve{G}_v^\sim\acute{F}_{\rm a}^\sim)\bar{M}_y^\sim \\ &= \bar{M}_y^{-1} + \acute{F}_{\rm a}\grave{M}_v\bar{M}_y^{-1} = (I + \acute{F}_{\rm a}\grave{M}_v)\bar{M}_y^{-1}\end{aligned}$$

is *causal* because all its factors are. In other words, we have a causal system as the series interconnection of an anticausal and a causal system. Fig. 5 illustrates this situation in terms of its impulse response.

Now assume that the antialiasing filter $\mathcal{F}_{\rm a}$ is an LCTI system with the impulse response having support in $[0, h)$. This is the case that $\acute{\mathcal{F}}_{\rm a}$ is a *zero-order generalized sampler*, acting as

$$\bar{y}[k] = \int_0^h f_{\rm a}(\tau)v(kh - \tau)\mathrm{d}\tau, \tag{24}$$

with $f_{\rm a}(\tau)$ the impulse response of $\mathcal{F}_{\rm a}$. This includes the ideal sampler (if $f_{\rm a}(\tau) = \delta(\tau)$) and the averaging sampler (if $f_{\rm a}(\tau) = \frac{1}{h}\mathbb{1}_{[0,h)}(t)$).

In the lifted domain, (24) reads $\bar{y}[k] = \acute{F}_1\breve{v}[k-1]$, where $\acute{F}_1 : \mathbb{L} \to \mathbb{R}^{n_{\bar{y}}}$ is the integral operator with the kernel $f_{\rm a}$. This means that in this case $\acute{F}_{\rm a}(z) = z^{-1}\acute{F}_1$ and

$$\begin{aligned}\acute{F}_{\rm a}(z)\grave{V}(z) &= z^{-1}\acute{F}_1(\cdots + z^3\grave{V}_{-3} + z^2\grave{V}_{-2} + z\grave{V}_{-1})\\ &= \cdots + z^2\acute{F}_1\grave{V}_{-3} + z\acute{F}_1\grave{V}_{-2} + \acute{F}_1\grave{V}_{-1}.\end{aligned}$$

Since this system is causal, only the final term can remain, so all others are necessarily zero, $\acute{F}_1\grave{V}_i = 0$ for all $i \in \mathbb{Z}^-_{-1}$. This implies that (23) holds for all $l \in \mathbb{N}$. Thus, l-causal L^2-optimal reconstruction always produces consistent solutions if $l > 0$ and the antialiasing filter $\mathcal{F}_{\rm a}$ has an impulse response with support on $[0, h)$.

C. General Antialiasing Filters

It is as yet not clear what category of antialiasing filters result in consistent holds. Apart from the two cases considered above, we showed in [13] that if the output of the antialiasing filter is its state, then again consistency follows for every $l \in \mathbb{N}$. Extensive numerical testing bears out that in about all other cases L^2-optimal holds are not consistent; see [13] for an example.

VI. State-Space Solution

The frequency domain solution of Section IV may not yet be regarded as explicit, since it is formulated in terms of operator-valued lifted transfer functions. Every step of this solution, however, can be spelled out in time domain (peeling-off) and this leads to an implementable form of the optimal reconstructor and a calculable expression for the optimal performance. That is the topic of this section. A certain level of technicality cannot be avoided.

We argue that it is advantageous to carry out the peeling-off procedure in terms of state-space realizations. State-space methods are rigorous, equally suit for SISO and MIMO systems, and results in efficient computational algorithms. We, therefore, bring in a minimal state-space realization of the combined $\mathcal{G}_v$ and $\mathcal{G}_y$ as described in [1, Sec. III]:

$$G(s) = \begin{bmatrix} G_v(s) \\ G_y(s)\end{bmatrix} = \begin{bmatrix} C_v \\ C_y\end{bmatrix}(sI - A)^{-1}B \qquad (25)$$

where $(C_y, {\rm e}^{Ah})$ is detectable and $[\,C_y \quad \Sigma\,]$ has full row rank.

Before we proceed with the algorithm, we need to review some aspects of the state-space theory for lifted transfer functions. This is the subject of the rest of this section (for more details, the reader is referred to [4]).

A. Preliminaries: State Space in the Lifted Domain

Based on (25), we develop state-space realizations of $\breve{G}_v(z)$ and $\acute{G}_y(z)$. As the systems are assumed causal, the impulse response of $\mathcal{G}_v$ in terms of its state-space realization is

$$g_v(t) = C_v{\rm e}^{At}B\,\mathbb{1}(t).$$

Its lifted z-transform, for $\tau, \sigma \in [0, h)$ thus is

$$\begin{aligned}\breve{g}_v(z; \tau - \sigma) &= \sum_{k\in\mathbb{Z}} g_v(kh + \tau - \sigma)z^{-k}\\ &= C_v{\rm e}^{A(\tau-\sigma)}B\,\mathbb{1}(\tau - \sigma)\\ &\quad + \sum_{k=1}^{\infty} C_v{\rm e}^{A(kh+\tau-\sigma)}Bz^{-k}\\ &= C_v{\rm e}^{A(\tau-\sigma)}B\,\mathbb{1}(\tau - \sigma)\\ &\quad + C_v{\rm e}^{A\tau}(zI - {\rm e}^{Ah})^{-1}{\rm e}^{A(h-\sigma)}B.\end{aligned}$$

Taking into account (4a), we have

$$\breve{G}_v(z) = \breve{D}_v + \grave{C}_v(zI - \bar{A})^{-1}\acute{B}, \qquad (26)$$

where [with n denoting the state dimension in (25)]

$$\bar{A} : \mathbb{R}^n \to \mathbb{R}^n \quad \bar{\xi} \mapsto {\rm e}^{Ah}\bar{\xi}, \qquad (27a)$$

$$\acute{B} : \mathbb{L} \to \mathbb{R}^n \quad \breve{v} \mapsto \int_0^h {\rm e}^{A(h-\sigma)}B\breve{v}(\sigma){\rm d}\sigma, \qquad (27b)$$

$$\grave{C}_v : \mathbb{R}^n \to \mathbb{L} \quad \bar{\xi} \mapsto C_v{\rm e}^{A\tau}\bar{\xi}, \qquad (27c)$$

$$\breve{D}_v : \mathbb{L} \to \mathbb{L} \quad \breve{v} \mapsto C_v\int_0^\tau {\rm e}^{A(\tau-\sigma)}B\breve{v}(\sigma){\rm d}\sigma. \qquad (27d)$$

As we can see, (26) has the form of a discrete state-space realization. The only difference from the "conventional" form is that the "B," "C," and "D" parameters of (26) are operators from or/and to infinite-dimensional space, $\mathbb{L}$, rather than plain matrices. This difference, however, is not crucial.

Eventually, we shall see that all lifted systems we face in the development of the solution of $\breve{\mathbf{RP}}_{\boldsymbol{l}}$ either have transfer functions of the form

$$\tilde{G}(z) = \tilde{D} + \tilde{C}(zI - \bar{A})^{-1}\tilde{B} \qquad (28)$$

or are conjugate of such transfer functions. Here, we use the tilde accent to indicate that the corresponding operator, say $\tilde{O}$, might be either $\bar{O}$ or $\acute{O}$ or $\grave{O}$ or $\breve{O}$. In all cases we consider, the parameters of $\tilde{G}(z)$ are *bounded* operators. For example, the lifted transfer function of $\mathcal{G}_y$ is

$$\acute{G}_y(z) = \bar{C}_y(zI - \bar{A})^{-1}\acute{B}$$

where $\bar{A}$ and $\acute{B}$ are as in (27a) and (27b), respectively, and $\bar{C}_y = C_y$ (just take $\tau = 0$ in (27) and replace C_v with C_y).

Using the definition of the conjugate transfer function in Section II-C, it can be shown that

$$\tilde{G}^{\sim}(z) = \tilde{D}^* + \tilde{B}^*(z^{-1}I - \bar{A}')^{-1}\tilde{C}^*.$$

This implies that we shall need to calculate the adjoints of the parameters of lifted state-space realizations. This can be done by the use of the very definition of the adjoint operator. For example, to calculate the adjoint of $\acute{B}$ in (27b), write the definition $\langle\acute{B}\breve{w}, \bar{\xi}\rangle_{\mathbb{R}^n} = \langle\breve{w}, \acute{B}^*\bar{\xi}\rangle_{\mathbb{L}}$ as

$$\bar{\xi}'\int_0^h {\rm e}^{A(h-\sigma)}Bw(\sigma){\rm d}\sigma = \int_0^h (B'{\rm e}^{A'(h-\sigma)}\bar{\xi})'\breve{w}(\sigma){\rm d}\sigma.$$

This yields

$$\acute{B}^* : \mathbb{R}^n \to \mathbb{L} \quad \bar{\xi} \mapsto B' \mathrm{e}^{A'(h-\tau)} \bar{\xi}. \tag{29b}$$

Analogously, it is straightforward to show that

$$\grave{C}_v^* : \mathbb{L} \to \mathbb{R}^n \quad \breve{v} \mapsto \int_0^h \mathrm{e}^{A\sigma} C_v' \breve{v}(\sigma) \mathrm{d}\sigma. \tag{29c}$$

We shall use these formulas in Sections VI-C and VI-D.

Note that the "A" part in (28) is always finite dimensional. This is a fundamental property of lifted state-space realizations associated with finite-dimensional analog systems. It plays an important role in our developments. The first consequence of this fact is that the stability of (operator-valued) transfer function (28) can be verified in terms of eigenvalues of a matrix, exactly like in the case of matrix-valued transfer functions. We have the following.

Proposition 6.1: Let $\tilde{G}(z)$ be as in (28). Then $\tilde{G} \in H^\infty$ if $\bar{A}$ is Schur (i.e., with all eigenvalues in $\mathbb{D}$).

Proof: If $\bar{A}$ is Schur, $zI - \bar{A}$ is invertible for all $z \in \mathbb{C} \setminus \mathbb{D}$. Hence, $\tilde{G}(z)$ is analytic and bounded in $\mathbb{C} \setminus \mathbb{D}$. ■

Like in the matrix-valued case, the impulse response of a stable causal system having the transfer function (28) is

$$\tilde{G}[k] = \begin{cases} \tilde{C}\bar{A}^{k-1}\tilde{B} & \text{if } k > 0 \\ \tilde{D} & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Using this formula, the following results can be proved:

Proposition 6.2: Let $\tilde{G}(z)$ given by (28) be the transfer function of a causal system and let $\bar{A} \in \mathbb{R}^{n \times n}$ be Schur. Then, $\tilde{G} \in L^2$ iff $\tilde{D}$ is a Hilbert–Schmidt operator and in this case

$$\|\tilde{G}\|_2^2 = \frac{1}{h}\|\tilde{D}\|_{\mathrm{HS}}^2 + \frac{1}{h}\mathrm{tr}(\tilde{C}^*\tilde{C}W_{\mathrm{c}}) \tag{30a}$$

$$= \frac{1}{h}\|\tilde{D}\|_{\mathrm{HS}}^2 + \frac{1}{h}\mathrm{tr}(W_{\mathrm{o}}\tilde{B}\tilde{B}^*) \tag{30b}$$

where $W_{\mathrm{c}}, W_{\mathrm{o}} \in \mathbb{R}^{n \times n}$ are the controllability and observability Gramians of (28), respectively, which are the solutions of the Lyapunov equations

$$W_{\mathrm{c}} = \bar{A}W_{\mathrm{c}}\bar{A}' + \tilde{B}\tilde{B}^* \quad \text{and} \quad W_{\mathrm{o}} = \bar{A}'W_{\mathrm{o}}\bar{A} + \tilde{C}^*\tilde{C}.$$

Proof: Because $(\mathrm{e}^{\mathrm{j}\theta}I - \bar{A})^{-1} \in \mathbb{C}^{n \times n}$ is bounded at each $\theta \in [-\pi, \pi]$, $\tilde{G}(\mathrm{e}^{\mathrm{j}\theta})$ is a bounded finite-rank perturbation of $\tilde{D}$ for all possible i/o spaces. This proves the first statement. To calculate the norm, we use [3, eq. (33)]

$$\begin{aligned} h\|\tilde{G}\|_2^2 &= \|\tilde{D}\|_{\mathrm{HS}}^2 + \sum_{i \in \mathbb{N}} \|\tilde{C}\bar{A}^{\,i-1}\tilde{B}\|_{\mathrm{HS}}^2 \\ &= \|\tilde{D}\|_{\mathrm{HS}}^2 + \sum_{i \in \mathbb{N}} \mathrm{tr}(\tilde{C}\bar{A}^{\,i-1}\tilde{B}\tilde{B}^*(\bar{A}')^{i-1}\tilde{C}^*) \\ &= \|\breve{D}\|_{\mathrm{HS}}^2 + \mathrm{tr}\left(\tilde{C}^*\tilde{C}\sum_{i \in \mathbb{N}} \tilde{A}^{\,i-1}\tilde{B}\tilde{B}^*(\tilde{A}')^{i-1}\right) \\ &= \|\breve{D}\|_{\mathrm{HS}}^2 + \mathrm{tr}\left(\sum_{i \in \mathbb{N}} (\tilde{A}')^{i-1}\tilde{C}^*\tilde{C}\tilde{A}^{\,i-1} \cdot \tilde{B}\tilde{B}^*\right). \end{aligned}$$

The result follows by the fact that the last two sums equal W_{c} and W_{o}, respectively. ■

It is readily seen that both $\tilde{B}\tilde{B}^*$ and $\tilde{C}^*\tilde{C}$ are $n \times n$ matrices, so that the second terms in the right-hand sides of (30) are the plain matrix traces. As we shall see in Section VI-D (Lemma 6.3), the evaluation of the Hilbert–Schmidt norm of $\tilde{D}$ also reduces to a matrix trace calculation.

We are now in the position to peel off the lifted solution of Section IV. To simplify the exposition, we first assume that $\Sigma = 0$ (no measurement noise). At the end of this section, we explain how the formulas should be adjusted when $\Sigma \neq 0$.

B. Constructing Coprime Factors (Proposition 4.1)

Now, define $\bar{A}_1 := \mathrm{e}^{Ah} + LC_y$ for some matrix L such that $\bar{A}_1$ is Schur (exists by detectability) and consider the transfer function

$$\bar{M}_y(z) = \Xi(I + C_y(zI - \bar{A}_1)^{-1}L) \in H^\infty \tag{31a}$$

where Ξ is some square nonsingular matrix which we determine later. It can be verified that in this case $\bar{M}_y(z)C_y(zI - \mathrm{e}^{Ah})^{-1} = \Xi C_y(zI - \bar{A}_1)^{-1}$, so

$$\acute{N}_y(z) := \bar{M}_y(z)\acute{G}_y(z) = \Xi C_y(zI - \bar{A}_1)^{-1}\acute{B} \in H^\infty \tag{31b}$$

where $\acute{B}$ is defined by (27b). By construction, $\acute{G}_y = \bar{M}_y^{-1}\acute{N}_y$. Moreover, as shown in Lemma A.1, these factors are coprime in H^∞. Thus, for any stabilizing L and nonsingular Ξ, (31a) and (31b) define coprime factors of $\acute{G}_y$.

As a candidate for $\grave{M}_v$ consider the transfer function

$$\grave{M}_v(z) = z\,\grave{C}_v\mathrm{e}^{-Ah}(zI - \bar{A}_1)^{-1}L \in H^\infty \tag{31c}$$

where $\grave{C}_v$ is as in (27c). This is more than an educated guess but its derivation would lead too far (it somehow follows from [14]). We can, however, verify that this guess works, which is all we need here. In this case

$$\begin{aligned} \grave{C}_v(zI - \mathrm{e}^{Ah})^{-1} + \grave{M}_v(z)C_y(zI - \mathrm{e}^{Ah})^{-1} \\ = \grave{C}_v\mathrm{e}^{-Ah}\bar{A}_1(zI - \bar{A}_1)^{-1} \end{aligned}$$

so that $\breve{N}_v := \breve{G}_v + \grave{M}_v\acute{G}_y$ verifies

$$\breve{N}_v(z) = \breve{D}_v + \grave{C}_v\mathrm{e}^{-Ah}\bar{A}_1(zI - \bar{A}_1)^{-1}\acute{B} \tag{31d}$$

and is indeed stable (belongs to H^∞).

Thus, the construction of a coprime factorization of $\breve{G}$ as in (13) amounts to the choice of L such that $\mathrm{e}^{Ah} + LC_y$ is Schur. The factors are then explicitly given by (31). This proves, by construction, that assumption $\boldsymbol{\mathcal{A}_1}$ from [1] is sufficient for the stabilizability of $\mathcal{G}_{\mathrm{e}}$.

C. Normalization (Proposition 4.3)

The freedom we have in the choice of L and Ξ will be used to normalize the factorization as in (20) with $\grave{V}^{\sim} \in z^{-1}H^\infty$. The conjugate of $\acute{N}_y$ defined by (31b) is

$$\acute{N}_y^{\sim}(z) = \acute{B}^*(z^{-1}I - \bar{A}_1')^{-1}C_y'\Xi' \tag{32}$$

where $\acute{B}^*$ is given in (29b). It is readily seen that

$$\acute{B}\acute{B}^* = \int_0^h \mathrm{e}^{A\tau}BB'\mathrm{e}^{A'\tau}\mathrm{d}\tau =: \Gamma_w(h) > 0$$

(the positive definiteness of $\Gamma_w(h)$ for all $h > 0$ follows from the controllability of (A, B)). Hence, $\bar{\Phi}_y$ from (16) reads

$$\bar{\Phi}_y(z) = \Xi C_y(zI - \bar{A}_1)^{-1}\Gamma_w(h)(z^{-1}I - \bar{A}_1')^{-1}C_y'\Xi'. \quad (33)$$

The nonsingularity of $\Gamma_w(h)$, $\mathrm{e}^{\mathrm{j}\theta}I - \bar{A}_1$ ($\bar{A}_1$ is Schur), and Ξ yields then that $\breve{\boldsymbol{\mathcal{A}}}_1$ is equivalent to the full row rank of C_y, which, in turn, is exactly $\boldsymbol{\mathcal{A}}_2$ from [1] if $\Sigma = 0$.

We will now exploit the freedom in L and Ξ to render $\Phi(z) = I$. As $\bar{A}_1$ is Schur and $\Gamma_w(h) > 0$, the Lyapunov equation

$$Y = \bar{A}_1 Y \bar{A}_1' + \Gamma_w(h) \quad (34)$$

is solvable by $Y > 0$. This allows us to split $\Phi(z)$ into causal and anticausal parts. To this end, we first split

$$\begin{aligned}(zI - \bar{A}_1)^{-1}&\Gamma_w(h)(z^{-1}I - \bar{A}_1')^{-1} \\ &= (zI - \bar{A}_1)^{-1}(Y - \bar{A}_1 Y \bar{A}_1')(z^{-1}I - \bar{A}_1')^{-1} \\ &= Y + (zI - \bar{A}_1)^{-1}\bar{A}_1 Y + Y\bar{A}_1'(z^{-1} - \bar{A}_1')^{-1}.\end{aligned}$$

Substituting this split into (33) gives

$$\bar{\Phi}_y(z) = \Xi(\bar{\Phi}_0 + \bar{\Phi}_{\mathrm{c}}(z) + \bar{\Phi}_{\mathrm{c}}^{\sim}(z))\Xi' \quad (35)$$

with $\bar{\Phi}_0 := C_y Y C_y'$ and $\bar{\Phi}_{\mathrm{c}}(z) := C_y(zI - \bar{A}_1)^{-1}\bar{A}_1 Y C_y'$. To render $\bar{\Phi}_y(z)$ static, we now choose L such that $\bar{\Phi}_c(z) = 0$. This is guaranteed if

$$0 = \bar{A}_1 Y C_y' = (\mathrm{e}^{Ah} + LC_y)YC_y'. \quad (36)$$

Any such L (assuming it exists and is stabilizing) yields $Y > 0$ and so by the full row rank of C_y the matrix $C_y Y C_y'$ is nonsingular. Consequently, (36) has a unique solution

$$L = -\mathrm{e}^{Ah}YC_y'(C_y Y C_y')^{-1}. \quad (37)$$

It is not yet clear that this gain is stabilizing. Substituting (37) into (34), we end up with the following equation for Y:

$$Y = \mathrm{e}^{Ah}(Y - YC_y(C_y Y C_y')^{-1}C_y'Y)\mathrm{e}^{A'h} + \Gamma_w(h). \quad (38)$$

This is a standard discrete algebraic Riccati equation (DARE) [11], [15]. The detectability of (C_y, e^{Ah}) and the nonsingularity of Γ_w (which, together with the full rank of C_y, implies that

$$\begin{bmatrix} \mathrm{e}^{Ah} - \mathrm{e}^{\mathrm{j}\theta}I & \Gamma_w^{1/2}(h) \\ C_y & 0 \end{bmatrix}$$

is right invertible for all $\theta \in [-\pi, \pi]$) guarantee that this DARE admits a stabilizing solution $Y \geq 0$ such that $\bar{A}_1$ is Schur and $C_y Y C_y'$ is nonsingular (in fact, $Y > 0$).

Thus, by solving the DARE (38) we obtain the static $\bar{\Phi}_y(z) = \Xi\bar{\Phi}_0\Xi'$. To render it identity, we just choose Ξ as an arbitrary square matrix that satisfies

$$\Xi'\Xi = (C_y Y C_y')^{-1} \quad (39)$$

(e.g., Ξ' may be the Cholesky factor of $(C_y Y C_y')^{-1}$).

It is time to check the condition on $\grave{V}$ in (20). Using (31d) and (32), we obtain

$$\begin{aligned}\grave{V} = &\breve{D}_v\acute{B}^*(z^{-1}I - \bar{A}_1')^{-1}C_y'\Xi' \\ &+ \grave{C}_v\mathrm{e}^{-Ah}\bar{A}_1(zI - \bar{A}_1)^{-1}\Gamma_w(h)(z^{-1}I - \bar{A}_1')^{-1}C_y'\Xi'.\end{aligned}$$

By (34), we have that

$$\begin{aligned}\bar{A}_1(zI - \bar{A}_1)^{-1}&\Gamma_w(h)(z^{-1}I - \bar{A}_1')^{-1}C_y' \\ &= \bar{A}_1(Y + (zI - \bar{A}_1)^{-1}\bar{A}_1 Y + Y\bar{A}_1'(z^{-1} - \bar{A}_1')^{-1})C_y' \\ &= \bar{A}_1 Y \bar{A}_1'(z^{-1} - \bar{A}_1')^{-1}C_y'\end{aligned}$$

where we used (36) to obtain the last equality. Thus, we end up with

$$\grave{V}(z) = \grave{C}_V(z^{-1}I - \bar{A}_1')^{-1}C_y'\Xi' \quad (40)$$

where

$$\grave{C}_V := \breve{D}_v\acute{B}^* + \grave{C}_v\mathrm{e}^{-Ah}\,\bar{A}_1 Y \bar{A}_1'.$$

This $\grave{V}$ is indeed the conjugate of a $z^{-1}H^\infty$ system. So we need not adjust the coprime factors by choice of $\acute{U}$ as we did in the frequency domain solution (19). Thus, the choices of L (unique) and Ξ according to (37) and (39), respectively, where Y is the stabilizing solution of (38), renders the factors in (31) satisfying (20).

We conclude this section with spelling out $\grave{C}_V$ and its adjoint. Using (27d), (29b), (27c), and then (34), we obtain

$$\begin{aligned}\grave{C}_V\bar{\xi} &= C_v\left(\int_0^\tau \mathrm{e}^{A(\tau-\sigma)}BB'\mathrm{e}^{A'(h-\sigma)}\mathrm{d}\sigma\right. \\ &\qquad\left. + \mathrm{e}^{A(\tau-h)}\bar{A}_1 Y \bar{A}_1'\right)\bar{\xi} \\ &= C_v\mathrm{e}^{A(\tau-h)}(Y - \Gamma_w(h-\tau))\bar{\xi}. \quad (41)\end{aligned}$$

The adjoint of this operator, $\grave{C}_V^* : \mathbb{L} \to \mathbb{R}^{n_v}$, transforms

$$\breve{v} \mapsto \int_0^h (Y - \Gamma_w(h-\tau))\mathrm{e}^{A'(\tau-h)}C_v'\breve{v}(\tau)\mathrm{d}\tau \quad (42)$$

which can be verified by the direct use of the definition.

D. Assumption $\breve{\boldsymbol{\mathcal{A}}}_2$

Here we establish that $\breve{\boldsymbol{\mathcal{A}}}_2$ always holds in our case and quantify the norm of $\breve{N}_v$. To this end, define

$$\begin{aligned}\Gamma_v &:= \int_0^h \mathrm{e}^{-A'\tau}C_v'C_v\mathrm{e}^{-A\tau}\mathrm{d}\tau \\ \Gamma_{vw} &:= \int_0^h\int_0^\tau B'\mathrm{e}^{A'\sigma}C_v'C_v\mathrm{e}^{A\sigma}B\mathrm{d}\sigma\mathrm{d}\tau.\end{aligned}$$

Then, the following result can be formulated:

Lemma 6.3: $\breve{N}_v \in L^2$ and

$$\|\breve{N}_v\|_2^2 = \frac{1}{h}\mathrm{tr}(\Gamma_{vw}) + \frac{1}{h}\mathrm{tr}(\Gamma_v(Y - \Gamma_w(h))).$$

Proof: It is known [16, Theorem 8.8] that $\breve{D}_v$ defined by (27d) is a Hilbert–Schmidt operator. Then, the first statement follows by Proposition 6.2.

To compute the norm, we use (30a). First, it is a known fact [4, Example 12.2.2] that $\|\breve{D}_v\|_{\mathrm{HS}}^2 = \mathrm{tr}(\Gamma_{vw})$. Now, it follows from (34) and the equality $\breve{B}\breve{B}^* = \Gamma_w(h)$ that Y is actually the controllability Gramian of the realization (31d) of $\breve{N}_v$. Thus, the second term in the right-hand side of (30a) is

$$\mathrm{tr}(\bar{A}_1'\mathrm{e}^{-A'h}\grave{C}_v^*\grave{C}_v\mathrm{e}^{-Ah}\bar{A}_1 Y) = \mathrm{tr}(\mathrm{e}^{-A'h}\grave{C}_v^*\grave{C}_v\mathrm{e}^{-Ah}\bar{A}_1 Y\bar{A}_1').$$

The result then follows by the facts that $\mathrm{e}^{-A'h}\grave{C}_v^*\grave{C}_v\mathrm{e}^{-Ah} = \Gamma_v$ [just combine (29c) and (27c)] and $\bar{A}_1 Y\bar{A}_1' = Y - \Gamma_w(h)$, see (34). ■

Remark 6.1: The strict properness of $G_v(s)$ in (25) is necessary for establishing that $\breve{N}_v \in L^2$. Indeed, if $G_v(s) = D_v + C_v(sI - A)^{-1}B$ for some $D_v \neq 0$, the only change in $\breve{N}_v$ is its feedthrough $\breve{D}_v$ term, which in this case would transform $\breve{v} \mapsto D_v v(\tau) + C_v\int_0^\tau \mathrm{e}^{A(\tau-\sigma)}B\breve{v}(\sigma)\mathrm{d}\sigma$. This $\breve{D}_v$ is not compact, and thus not a Hilbert–Schmidt operator [16, Theorem 8.7]. ■

E. Projection: The Optimal $\grave{Q}_l$ by (21) and Its Norm

Now, consider $\grave{V}(z)$ from (40). Because $\bar{A}_1$ is Schur, the power series expansion $\grave{V}(z) = \sum_{i\in\mathbb{N}}\grave{C}_V\bar{A}_{i-1}'C_y'\Xi' z^i$ is well defined, where, with some abuse of notation,

$$\bar{A}_i := \bar{A}_1^i.$$

The coefficients of z^i are the impulse response of $\grave{V}$ at the time instance $-i$. By (21), the optimal $\grave{Q}$, denoted by $\grave{Q}_l$, is then the (FIR) truncation of this series to its first l terms:

$$\grave{Q}_l(z) = \grave{C}_V\sum_{i=1}^{l}\bar{A}_{i-1}'C_y'\Xi' z^i. \tag{43}$$

Denote

$$\begin{aligned}\grave{Q}_{l,\mathrm{tail}}(z) &:= z^{-l}(\grave{V}(z) - \grave{Q}_l(z)) = \grave{C}_V\sum_{i\in\mathbb{N}}\bar{A}_{i+l-1}'C_y'\Xi' z^i \\ &= \grave{C}_V\bar{A}_l'(z^{-1}I - \bar{A}_1')^{-1}C_y'\Xi'.\end{aligned}$$

We thus may also write $\grave{Q}_l = \grave{V} - \grave{Q}_{l,\mathrm{tail}}$, which is a useful form to carry out state-space calculations involving $\grave{Q}_l$.

Our next step is to calculate the L^2-norm of $\grave{Q}_l$. By [3, eq. (33)], it can be obtained directly from (43) as follows:

$$\begin{aligned}\|\grave{Q}_l\|_2^2 &= \frac{1}{h}\sum_{i=1}^{l}\mathrm{tr}(\Xi C_y\bar{A}_{i-1}\grave{C}_V^*\grave{C}_V\bar{A}_{i-1}'C_y'\Xi') \\ &= \frac{1}{h}\mathrm{tr}\left(\grave{C}_V^*\grave{C}_V\sum_{i=0}^{l-1}\bar{A}_i'C_y'\Xi'\Xi C_y\bar{A}_i\right).\end{aligned}$$

It follows from (41) and (42) that $\grave{C}_V^*\grave{C}_V = \Gamma_V$, where

$$\Gamma_V := \int_0^h (Y - \Gamma_w(\tau))\mathrm{e}^{-A'\tau}C_v'C_v\mathrm{e}^{-A\tau}(Y - \Gamma_w(\tau))\mathrm{d}\tau. \tag{44}$$

Standard Lyapunov arguments yield that

$$\sum_{i=0}^{l-1}\bar{A}_i'C_y'\Xi'\Xi C_y\bar{A}_i = X - \bar{A}_l'X\bar{A}_l$$

where $X \geq 0$ is the solution of the Lyapunov equation

$$X = \bar{A}_1'X\bar{A}_1 + C_y'\Xi'\Xi C_y. \tag{45}$$

Thus, we just proved the following result:

Lemma 6.4: $\|\grave{Q}_l\|_2^2 = \frac{1}{h}\mathrm{tr}((X - \bar{A}_l'X\bar{A}_l)\Gamma_V)$.

F. When Σ Is Nonzero

The derivation for nonzero Σ is similar to that for $\Sigma = 0$. The only difference is that the intermediate steps are now quite a bit longer. In essence, however, the derivation is equally involved. We briefly outline the modifications to the formulas.

A key observation is that a coprime factorization of the form (13) for a nonzero constant Σ is readily derived from that for zero Σ. Indeed, it can be verified that the factorization

$$\begin{bmatrix}\breve{G}_v & 0 \\ \acute{G}_y & \Sigma^{1/2}\end{bmatrix} = \begin{bmatrix} I & \grave{M}_v \\ 0 & \bar{M}_y\end{bmatrix}^{-1}\begin{bmatrix}\breve{N}_v & \grave{M}_v\Sigma^{1/2} \\ \acute{N}_y & \bar{M}_y\Sigma^{1/2}\end{bmatrix}$$

is coprime provided so is its first column. This means that the addition of Σ amounts to the replacement of $\acute{N}_y$ and $\breve{N}_v$ with $[\,\acute{N}_y \quad \bar{M}_y\Sigma^{1/2}\,]$ and $[\,\breve{N}_v \quad \grave{M}_v\Sigma^{1/2}\,]$, respectively. These replacements affect then the choice of the matrices L and Ξ in (31), assumptions $\breve{\boldsymbol{A}}_{\mathbf{1},\mathbf{2}}$, and the optimal performance.

Let us start with $\breve{\boldsymbol{A}}_{\mathbf{1}}$. Because now $\bar{\Phi}_y = \acute{N}_y\acute{N}_y^{\sim} + \bar{M}_y\Sigma\bar{M}_y^{\sim}$, this assumption fails iff $\exists\eta \neq 0$ such that

$$\eta'\acute{N}_y(\mathrm{e}^{\mathrm{j}\theta}) = 0 \quad \text{and} \quad \eta'\bar{M}_y(\mathrm{e}^{\mathrm{j}\theta})\Sigma^{1/2} = 0 \tag{46}$$

for some $\theta \in [-\pi, \pi]$. We already saw in Section VI-C that the first condition above is equivalent to $\eta'\Xi C_y = 0$. For every such η, we have that $\eta'\bar{M}_y(z) = \eta'\Xi$, which implies that (46) holds iff $\eta'\Xi[\,C_y \quad \Sigma\,] = 0$. Therefore, $\breve{\boldsymbol{A}}_{\mathbf{1}}$ is equivalent to assumption $\breve{\boldsymbol{A}}_{\mathbf{2}}$ from [1].

To normalize $\bar{\Phi}_y$, use (31a) and (31b) to obtain

$$\begin{aligned}\Xi^{-1}\bar{\Phi}_y(z)\Xi^{-\prime} &= \Sigma + C_y(zI - \bar{A}_1)^{-1}L\Sigma \\ &\quad + \Sigma L'(z^{-1}I - \bar{A}_1')^{-1}C_y' + C_y(zI - \bar{A}_1)^{-1} \\ &\quad \times (\Gamma_w(h) + L\Sigma L')(z^{-1}I - \bar{A}_1')^{-1}C_y'.\end{aligned} \tag{33$'$}$$

To split this transfer function to causal and anticausal parts, we need to replace the Lyapunov equation (34) with

$$Y = \bar{A}_1 Y\bar{A}_1' + \Gamma_w(h) + L\Sigma L' \tag{34$'$}$$

for which we still have that $Y > 0$ because $\Gamma_w(h) + L\Sigma L' > 0$. Following the steps in Section VI-C, we end up with the split as in (35), but now with $\bar{\Phi}_0 = \Sigma + C_y Y C_y'$ and

$$\Phi_{\mathrm{c}}(z) = C_y(zI - \bar{A}_1)^{-1}(\bar{A}_1 Y C_y' + L\Sigma).$$

Thus, (36) is replaced with $\bar{A}_1 Y C_y' + L\Sigma = 0$, from which

$$L = -\mathrm{e}^{Ah}YC_y'(\Sigma + C_y Y C_y')^{-1} \tag{37$'$}$$

where the invertibility is guaranteed by the nonsingularity of Y and $\breve{\mathcal{A}}_1$. This yields

$$Y = e^{Ah}(Y - YC_y(\Sigma + C_yYC_y')^{-1}C_y'Y)e^{A'h} + \Gamma_w(h) \tag{38$'$}$$

and

$$\Xi'\Xi = (\Sigma + C_yYC_y')^{-1} \tag{39$'$}$$

instead of (38) and (39), respectively. It is now a matter of a straightforward algebra to verify that the other formulas of Section VI-C remain unchanged, including the fact that $\grave{V} = \breve{N}_v\breve{N}_y^\sim + \grave{M}_v\Sigma\bar{M}_y^\sim$ is still strictly anticausal and given by (40).

Now, the presence of Σ changes $\breve{N}_v$ by adding to it a *finite-rank* column $\grave{M}_v$. This implies that assumption $\breve{\mathcal{A}}_2$ is still valid. To compute the norm in Lemma 6.3, note that

$$\|[\,\breve{N}_v \quad \grave{M}_v\Sigma^{1/2}\,]\|_2 = \|[\,\breve{N}_v \quad z^{-1}\grave{M}_v\Sigma^{1/2}\,]\|_2$$

because the L^2-norm is computed column-wise and z^{-1} is inner and, hence, does not affect the norm. Thus, we need to calculate the L^2-norm of

$$[\,\breve{N}_v \quad z^{-1}\grave{M}_v\Sigma^{1/2}\,] = [\,\breve{D}_v \quad 0\,] + \grave{C}_v(zI - \bar{A}_1)^{-1}[\,\acute{B} \quad L\Sigma^{1/2}\,]$$

[obtained by (31c) and (31d)]. The feedthrough term of this realization has obviously the same norm as that of $\breve{N}_v$. Also, the controllability Gramian is still Y, which can be seen from (34$'$) and the fact that $\acute{B}\acute{B}^* = \Gamma_w$. These arguments show that the result of Lemma 6.3 remains unchanged.

G. Formulating the Solution in the Form of [1, Theorem 3.1]

In this subsection, we show that the solution derived so far is exactly the solution of [1, Theorem 3.1].

We start with expressing matrix exponentials and their integrals used in this section in terms of[1] $\Lambda(t)$ and Δ defined in [1, eq. (6) and (7)]. Because $e^{Ah} = \Lambda_{11}$ and $\Gamma_w(h) = \Lambda_{12}\Lambda_{11}'$ (follows by the Van Loan formulas; see [1, Lemma A.1]), the matrix $\bar{A}_1$, (38$'$) and (45) equal their counterparts defined by [1, eq. (8)–(10)]. Further use of the Van Loan formulas yields that $\Gamma_v = \Delta_{21}\Lambda_{11}^{-1}$ and

$$e^{-A\tau}(Y - \Gamma_w(\tau)) = [\,I \quad 0\,]\Lambda(-\tau)\begin{bmatrix} Y \\ I \end{bmatrix}. \tag{47}$$

The latter, in turn, allows us to apply the Van Loan formulas to (44), resulting in $\Gamma_V = [\,-I \quad Y\,]\Delta\Lambda^{-1}\begin{bmatrix} Y \\ I \end{bmatrix}$. Finally, it is known [17, Lemma 5.5] that $\operatorname{tr}(\Gamma_{vw}) = \operatorname{tr}(\Delta_{22}\Lambda_{11}')$. It then follows that the norms calculated in Lemmas 6.3 and 6.4 add up to the optimal performance of [1, Theorem 3.1].

Now, denote

$$\bar{F}_c(z) := -z\,(zI - \bar{A}_1)^{-1}L$$
$$\bar{F}_{\bar{c},l}(z) := \sum_{i=1}^{l} \bar{A}_{i-1}'C_y'\Xi'\Xi\,z^i.$$

[1]Following [1], the notation Λ (without the argument) indicates $\Lambda(h)$.

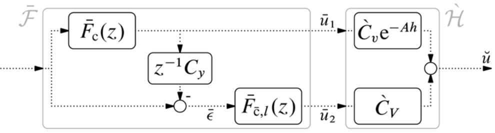

Fig. 6. The optimal l-causal reconstructor $\grave{\mathcal{F}}_l$ in the lifted domain.

It is readily verified that these transfer functions are equivalent to their namesakes defined in [1, Theorem 3.1]. By formulas (31c), (31a), and (43), we have that $\grave{M}_v(z) = -\grave{C}_v e^{-Ah}\bar{F}_c(z)$, $\bar{M}_y(z) = \Xi(I - z^{-1}C_y\bar{F}_c(z))$, and $\grave{Q}_l(z) = \grave{C}_V\bar{F}_{\bar{c},l}(z)\Xi^{-1}$. It then follows from Proposition 4.2 that the optimal (i.e., that with $\grave{Q} = \grave{Q}_l$) reconstructor $\grave{\mathcal{F}}_l$ can be presented in the form depicted in Fig. 6. The discrete part of this block-diagram, $\bar{\mathcal{F}}$, is equivalent to the discrete part of the optimal reconstructor in [1, Fig. 3]. The same is true regarding the D/A converter $\mathcal{H}$, which can be seen from (27c), (41), and (47) (just substitute $\tau \to h - \tau$).

VII. Concluding Remarks

In this second part, we have addressed the L^2 optimal design of D/A converters (reconstructors) with causality constraints imposed on them. Optimal solutions have been derived in both frequency domain and time domain (state-space) representations of the signal generators. The frequency domain solution revolves around coprime and spectral factorizations and the state space solution around Riccati and Lyapunov equations. The state-space machinery facilitates both computational and efficiency of implementation of the optimal reconstructors.

Although our main objective of this part was to provide proofs for the solution presented in [1], the presented results are of independent interest and can be useful in other sampling and reconstruction applications. For example, the presented state-space machinery plays a key role in the solution of the L^2 reconstruction problem with FIR constraints in [18]. We also expect that the factorization formulas could be used in the solution of the L^∞ (minmax) version of the problem.

Appendix

Lemma A.1: The factors $\bar{M}_y$ and $\acute{N}_y$ defined by (31a) and (31b) are coprime in H^∞.

Proof: Let F be any matrix such that $A_F := A + BF$ is Hurwitz (this is always possible because the pair (A, B) is controllable). Consider then the following candidates Bézout factors:

$$\bar{X}_y(z) = (I - C_y(zI - e^{A_F h})^{-1}L)\Xi^{-1}$$
$$\grave{Y}_y(z) = \grave{C}_F(zI - e^{A_F h})^{-1}L\Xi^{-1}$$

where $\grave{C}_F$ verifies $\grave{C}_F\bar{\xi} = Fe^{A_F\tau}\bar{\xi}$. Then,

$$\begin{aligned}\bar{M}_y\bar{X}_y &= \Xi(I + C_y(zI - \bar{A}_1)^{-1}L) \\ &\quad\times (I - C_y(zI - e^{A_F h})^{-1}L)\Xi^{-1} \\ &= I + \Xi C_y(zI - \bar{A}_1)^{-1}(zI - e^{A_F h} \\ &\quad - zI + \bar{A}_1 - LC_y)(zI - e^{A_F h})^{-1}L\Xi^{-1} \\ &= I + \Xi C_y(zI - \bar{A}_1)^{-1} \\ &\quad\times (e^{Ah} - e^{A_F h})(zI - e^{A_F h})^{-1}L\Xi^{-1}.\end{aligned}$$

Also,

$$\acute{N}_y \grave{Y}_y = \Xi C_y (zI - \bar{A}_1)^{-1} \times \int_0^h \mathrm{e}^{A(h-\sigma)} BF \mathrm{e}^{A_F \sigma} \mathrm{d}\sigma (zI - \mathrm{e}^{A_F h})^{-1} L \Xi^{-1}.$$

The integral in the last expression can be interpreted as the response, at the time instance $t = h$, of the continuous-time system $G_1 := (sI - A)^{-1}B$ to the input $F\mathrm{e}^{A_F t}$, which, in turn, is the impulse response of the system $G_2 := F(sI - A_F)^{-1}$. Thus, the integral can be interpreted as the impulse response of the system $G_1 G_2$ taken at the time instance $t = h$. The cascade $G_1 G_2$ can be also represented as a parallel interconnection:

$$\begin{aligned} G_1 G_2 &= (sI - A)^{-1} BF (sI - A_F)^{-1} \\ &= (sI - A_F)^{-1} - (sI - A)^{-1}. \end{aligned}$$

Hence, the impulse response of $G_1 G_2$ is the difference of the impulse responses of $(sI - A_F)^{-1}$ and $(sI - A)^{-1}$:

$$\int_0^h \mathrm{e}^{A(h-\sigma)} BF \mathrm{e}^{A_F \sigma} \mathrm{d}\sigma = \mathrm{e}^{A_F h} - \mathrm{e}^{Ah}$$

so that

$$\begin{aligned} \acute{N}_y \grave{Y}_y &= \Xi C_y (zI - \bar{A}_1)^{-1} (\mathrm{e}^{A_F h} - \mathrm{e}^{Ah})(zI - \mathrm{e}^{A_F h})^{-1} L \Xi^{-1} \\ &= I - \bar{M}_y \bar{X}_y. \end{aligned}$$

Thus, $\bar{X}_y$ and $\grave{Y}_y$ are Bézout factors of $\bar{M}_y$ and $\acute{N}_y$, which proves the statement. ■

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