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Synchronization in networks of minimum-phase, non-introspective agents without exchange of controller states: Homogeneous, heterogeneous, and nonlinear ☆

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ABSTRACT

We consider the synchronization problem for a class of directed networks where the agents receive relative output information from their neighbors, but lack independent information about their own state or output (they are *non-introspective*) and are unable to exchange internal controller states with their neighbors. We consider three classes of networks defined by the properties of the agent dynamics: *homogeneous* networks, where the agents are governed by identical linear models; *heterogeneous* networks, where the agents are governed by non-identical linear models; and networks with *nonlinear and time-varying* agent dynamics. In each case, the linear part of the dynamics is assumed to be minimum-phase. Our approach is based on a combination of low-gain and high-gain design techniques.

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1. Introduction

The phenomenon of *synchronization* has attracted a great deal of interest in recent years, due to its ubiquity in nature and potential technological applications in areas such as formation flying, cooperative control, and distributed sensor fusion. Influential work on the study of synchronization criteria was done by Wu and Chua (1995a,b), who used the Kronecker product to analyze systems of coupled oscillators. More recently, synchronization has been widely studied as a control problem, where the goal is to ensure synchronization in a multi-agent system by designing control laws that couple each agent to the system as a whole. The difficulty of this control problem lies in the limited information available to each agent—typically in the form of measurements of its own state or output relative to that of neighboring agents.

Some of the work on synchronization is focused on *state synchronization* based on diffusive *state coupling*, progressing from single- and double-integrator agent dynamics (e.g., Olfati-Saber & Murray, 2003, 2004; Ren & Atkins, 2007) to more general agent dynamics (e.g., Tuna, 2008a; Yang, Roy, Wan, & Saberi, 2011). State synchronization based on diffusive *partial-state coupling* has also been considered by several authors (e.g., Pogromsky & Nijmeijer, 2001; Pogromsky, Santoboni, & Nijmeijer, 2002; Tuna, 2008b). In this context, Li, Duan, Chen, and Huang (2010) introduced a distributed observer that has been expanded upon by several authors (e.g., Yang, Stoorvogel, & Saberi, 2011c). This type of observer makes additional use of the network by allowing the agents to exchange information with their neighbors about their internal estimates, effectively requiring another layer of communication. On the other hand, Seo, Shim, and Back (2009) presented a *low-gain* control design that does not require the exchange of internal states, provided the poles of the agent dynamics are located in the closed left-half complex plane.

The works cited above are concerned with *homogeneous* networks, where the agents are governed by identical dynamical models. A limited amount of work has also been done on *heterogeneous* networks, where the agents are governed by non-identical dynamical models. In a heterogeneous network, the agents' internal states may not be comparable to each other; thus, one often

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aims to achieve *output synchronization*—that is, agreement on some partial-state output.

Some work on heterogeneous networks has focused primarily on synchronization criteria (e.g., Grip, Saberi, & Stoorvogel, 2013a; Xiang & Chen, 2007; Zhao, Hill, & Liu, 2011); other work has been more design-oriented (Bai, Arcak, & Wen, 2011; Chopra & Spong, 2008; Kim, Shim, & Seo, 2011; Wieland, Sepulchre, & Allgöwer, 2011; Yang, Saberi, Stoorvogel, & Grip, 2011). Most designs for heterogeneous networks are based on either modifying the agent dynamics via pre-compensators and local feedbacks, in order to emulate a homogeneous network (Bai et al., 2011; Chopra & Spong, 2008; Yang et al., 2011); or on synchronizing an embedded identical model via the network and then regulating the actual output toward the embedded model output (Kim et al., 2011; Wieland et al., 2011). In either case, the agents are assumed to be *introspective*, meaning that they have access to information about their own state or output in addition to the information received from the network. The authors have recently considered the more challenging case of heterogeneous *non-introspective* agents, and developed a methodology based on a distributed high-gain observer (Grip, Yang, Saberi, & Stoorvogel, 2012). However, like several other designs for heterogeneous networks (Wieland et al., 2011; Yang et al., 2011), it was assumed that the agents can exchange internal controller states with neighboring agents in the network, in the same manner as in Li et al. (2010).

Some authors have also studied synchronization in networks with *nonlinear* agent dynamics (e.g., Arcak, 2007; Chopra & Spong, 2008; Igarashi, Hatanaka, Fujita, & Spong, 2009; Pogromsky & Nijmeijer, 2001; Pogromsky et al., 2002; Xiang & Chen, 2007; Zhao, Hill, & Liu, 2010; Zhao et al., 2011). Explicit control designs for nonlinear networks have largely centered on the relatively strict assumption of *passivity*. Passivity can in some cases be ensured by first applying local pre-feedbacks to the system; however, this requires the system to be introspective.

1.1. Topics of this paper

In this paper, we shall address several combinations of the challenges mentioned above. We start by considering state synchronization in a homogeneous network with partial-state coupling, where the agents are non-introspective and unable to exchange controller states with neighboring agents. This represents a practically significant scenario; for example, one may have multiple vehicles capable of measuring relative distance to their neighbors, but without knowledge of their own absolute position or velocity (i.e., they are non-introspective), and without an additional communication channel for exchanging controller states. Our approach is based on a combination of low- and high-gain design techniques, and solves the synchronization problem subject to the condition that the invariant zeros of the agent dynamics are in the open left-half complex plane. This is in contrast to the pure low-gain approach of Seo et al. (2009), where the same condition was placed on the *poles* of the agent dynamics.

Next, we expand our design to encompass a class of nonlinear time-varying systems that can be transformed to a particular canonical form, where the nonlinearities appear in a lower-triangular pattern. This canonical form does not require the agent dynamics to be passive (or even stable). We discuss in detail when and how a given nonlinear time-varying system can be transformed to this canonical form. Finally, we show how the same design principles can be applied to output synchronization of *heterogeneous* networks without additional assumptions regarding the agent dynamics.

We focus only on single-input single-output (SISO) agent dynamics, while noting that the same principles can be applied to many appropriately chosen classes of multiple-input multiple-output (MIMO) systems. Results from this paper were partially presented at the 2013 *European Control Conference* and the 2014 *American Control Conference* (Grip, Saberi, & Stoorvogel, 2013b, 2014).

1.2. Notation and definitions

For a matrix A, A' denotes its transpose and A^* denotes its conjugate transpose. The Kronecker product between A and B is denoted by $A \otimes B$. We denote by $[X_1; \dots; X_n]$ the vector or matrix obtained by stacking $X_1, \dots, X_n$.

Definition 1. We say that a matrix pair (A, C) *contains* the matrix pair (S, R) if there exists a matrix Π such that $\Pi S = A\Pi$ and $C\Pi = R$.

Remark 1. Definition 1 implies that for any initial condition $\omega(0)$ of the system $\dot{\omega} = S\omega$, $y_r = R\omega$, there exists an initial condition $x(0)$ of the system $\dot{x} = Ax$, $y = Cx$, such that $y(t) = y_r(t)$ for all $t \geq 0$.[1]

2. Network communication

The networks that will be considered in this paper consist of N SISO agents, with the state and output of agent $i \in \{1, \dots, N\}$ denoted by x_i and y_i, respectively. The agents are non-introspective; hence, agent i does not have access to its own state or output. The only information available to each agent is a linear combination of its own output relative to that of the other agents:

$$\zeta_i = \sum_{j=1}^{N} a_{ij}(y_i - y_j),$$

where $a_{ij} \geq 0$ and $a_{ii} := 0$.

The communication topology of the network can be described by a directed graph (digraph) $\mathscr{G}$ with nodes corresponding to the agents in the network and edges given by the coefficients a_{ij}. In particular, $a_{ij} > 0$ implies that an edge exists from agent j to i, in which case j is called a *parent* of agent i and agent i is called a *child* of agent j. The weight of the edge equals the magnitude of a_{ij}. We say that there exists a *directed path* from node i to node j if $\mathscr{G}$ contains a sequence of edges originating at node i and terminating at node j.

We shall make use of the matrix $G = [g_{ij}]$, where $g_{ii} = \sum_{j=1}^{N} a_{ij}$, and $g_{ij} = -a_{ij}$ for $j \neq i$. The matrix G is known as the *Laplacian* of $\mathscr{G}$ and has the property that all the row sums are zero. In terms of the coefficients of G, ζ_i can be rewritten as

$$\zeta_i = \sum_{j=1}^{N} g_{ij} y_j.$$

We shall later refer to the notion of a *directed tree* contained within the network graph $\mathscr{G}$. A directed tree is a subgraph of $\mathscr{G}$ in which every node has exactly one parent, except a single root node with no parents. Moreover, there must be a directed path from the root node to every other node in the tree. A *directed spanning tree* is a directed tree containing all the nodes of the graph.

[1] See Lunze (2011) for a discussion of *system inclusion* and its role in network synchronization.

3. Homogeneous networks of linear agents

We start by considering a homogeneous network of N SISO agents on the form

$$\dot{x}_i = Ax_i + Bu_i, \qquad x_i \in \mathbb{R}^n,\ u_i \in \mathbb{R}, \tag{1a}$$
$$y_i = Cx_i, \qquad y_i \in \mathbb{R}. \tag{1b}$$

Note that no *a priori* couplings exist between the agents. Our goal is to design the input u_i based on available information to achieve state synchronization among the agents, meaning that $\lim_{t\to\infty}(x_i - x_j) = 0$ for all $i, j \in \{1, \ldots, N\}$. We make the following assumption regarding the agent dynamics.

Assumption 1. The triple (A, B, C) is minimum-phase and of relative degree $\rho \geq 1$.

Assumption 1 implies that the triple (A, B, C) is invertible, stabilizable, and detectable (see, e.g., Saberi, Stoorvogel, & Sannuti, 2006, Ch. 3).

Assumption 2. The graph $\mathscr{G}$ contains a directed spanning tree.

Assumption 2 implies that the Laplacian G has a single eigenvalue at the origin and that all the other eigenvalues are located in the open right-half complex plane (Ren & Beard, 2005). For control design, the only information assumed available is a lower bound $\tau > 0$ on the real parts of the non-zero eigenvalues.

3.1. Special coordinate basis

We assume without loss of generality that the triple (A, B, C) is given in the *special coordinate basis* (SCB) (Sannuti & Saberi, 1987). This means that x_i can be decomposed as $x_i = [x_{ia}; x_{id}]$, where $x_{ia} \in \mathbb{R}^{n-\rho}$ and $x_{id} \in \mathbb{R}^{\rho}$, and where

$$\dot{x}_{ia} = A_a x_{ia} + L_{ad} y_i, \tag{2a}$$
$$\dot{x}_{id} = A_d x_{id} + B_d(u_i + E_{da} x_{ia} + E_{dd} x_{id}), \tag{2b}$$
$$y_i = C_d x_{id}. \tag{2c}$$

The matrices $A_d \in \mathbb{R}^{\rho\times\rho}$, $B_d \in \mathbb{R}^{\rho\times 1}$, and $C_d \in \mathbb{R}^{1\times\rho}$ have the special form

$$A_d = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad B_d = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \tag{3}$$
$$C_d = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}.$$

Furthermore, the eigenvalues of A_a are the invariant zeros of (A, B, C).

If the agent dynamics is not in the SCB, then it can be transformed to the SCB via nonsingular state and input transformations. Suppose that the agent dynamics is given by $\dot{\bar{x}}_i = \bar{A}\bar{x}_i + \bar{B}\bar{u}_i$, $y_i = \bar{C}\bar{x}_i$, where $(\bar{A}, \bar{B}, \bar{C})$ satisfies Assumption 1. Then there are nonsingular matrices Γ_x and Γ_u such that, by defining $\bar{x}_i = \Gamma_x x_i$ and $\bar{u}_i = \Gamma_u u_i$, we obtain the system (1) with $A = \Gamma_x^{-1}\bar{A}\Gamma_x$, $B = \Gamma_x^{-1}\bar{B}\Gamma_u$, and $C = \bar{C}\Gamma_x$, where the triple (A, B, C) is in the SCB. The transformations Γ_x and Γ_u can be calculated using available software, either numerically (Liu, Chen, & Lin, 2005) or symbolically (Grip & Saberi, 2010).

3.2. Control design

Let $\delta \in (0, 1]$ and $\varepsilon \in (0, 1]$ denote a low-gain and a high-gain parameter, respectively. It is easy to see that (A_d, B_d, C_d) is controllable and observable. Let therefore K be chosen such that $A_d - KC_d$ is Hurwitz. Furthermore, let $P_\delta = P_\delta' > 0$ be the solution of the algebraic Riccati equation

$$P_\delta A_d + A_d' P_\delta - \tau P_\delta B_d B_d' P_\delta + \delta I = 0, \tag{4}$$

where, as mentioned in Section 3, $\tau > 0$ is a lower bound on the real parts of the eigenvalues of the Laplacian G. Define $F_\delta = -B_d' P_\delta$. Next, define a high-gain scaling matrix

$$S_\varepsilon := \operatorname{diag}(1, \ldots, \varepsilon^{\rho-1}), \tag{5}$$

and define the feedback and output injection matrices

$$F_{\delta\varepsilon} = \varepsilon^{-\rho} F_\delta S_\varepsilon, \qquad K_\varepsilon = \varepsilon^{-1} S_\varepsilon^{-1} K. \tag{6}$$

Now, for each $i \in \{1, \ldots, N\}$, define the following dynamic controller:

$$\dot{\hat{x}}_{ia} = A_a \hat{x}_{ia} + L_{ad} C_d \hat{x}_{id}, \tag{7a}$$
$$\dot{\hat{x}}_{id} = A_d \hat{x}_{id} + B_d(E_{da}\hat{x}_{ia} + E_{dd}\hat{x}_{id}) + K_\varepsilon(\zeta_i - C_d\hat{x}_{id}), \tag{7b}$$
$$u_i = F_{\delta\varepsilon}\hat{x}_{id}. \tag{7c}$$

Remark 2. Note that the internal dynamics of the controller (7) has the form of an observer; however, it is not driven by the output y_i of agent i (which is unavailable), but by $\zeta_i = \sum_{j=1}^N g_{ij} y_j$. The estimate $\hat{x}_i := [\hat{x}_{ia}; \hat{x}_{id}]$ can therefore be interpreted as an estimate of $\sum_{j=1}^N g_{ij} x_j$.

Theorem 1. *Consider the network with agents described by (1) and the dynamic controller described by (7). Under Assumptions 1 and 2 there exists a $\delta^* \in (0, 1]$ such that, for each $\delta \in (0, \delta^*]$, there exists an $\varepsilon^*(\delta) \in (0, 1]$ such that, for all $\varepsilon \in (0, \varepsilon^*(\delta)]$, $\lim_{t\to\infty}(x_i - x_j) = 0$ for all $i, j \in \{1, \ldots, N\}$.*

Proof. For each $i \in \{1, \ldots, N-1\}$, let $\bar{x}_i = [\bar{x}_{ia}; \bar{x}_{id}] := x_N - x_i$ and $\hat{\bar{x}}_i = [\hat{\bar{x}}_{ia}; \hat{\bar{x}}_{id}] := \hat{x}_N - \hat{x}_i$, where $\hat{x}_i = [\hat{x}_{ia}; \hat{x}_{id}]$. The synchronization objective is achieved if $\bar{x}_i \to 0$ for all $i \in \{1, \ldots, N-1\}$. Computing $\dot{\bar{x}}_i$ by subtracting $\dot{x}_i$ from $\dot{x}_N$, we obtain

$$\dot{\bar{x}}_{ia} = A_a \bar{x}_{ia} + L_{ad} C_d \bar{x}_{id},$$
$$\dot{\bar{x}}_{id} = A_d \bar{x}_{id} + B_d(F_{\delta\varepsilon}\hat{\bar{x}}_{id} + E_{da}\bar{x}_{ia} + E_{dd}\bar{x}_{id}).$$

Noting that the row sums of G are zero, we have $\zeta_N - \zeta_i = -\sum_{j=1}^N (g_{ij} - g_{Nj}) y_j = \sum_{j=1}^N (g_{ij} - g_{Nj})(y_N - y_j) = \sum_{j=1}^{N-1} \bar{g}_{ij} C_d \bar{x}_{jd}$, where $\bar{g}_{ij} = g_{ij} - g_{Nj}$, $i, j \in \{1, \ldots, N-1\}$. It follows that we can write

$$\dot{\hat{\bar{x}}}_{ia} = A_a \hat{\bar{x}}_{ia} + L_{ad} C_d \hat{\bar{x}}_{id},$$
$$\dot{\hat{\bar{x}}}_{id} = A_d \hat{\bar{x}}_{id} + B_d(E_{da}\hat{\bar{x}}_{ia} + E_{dd}\hat{\bar{x}}_{id}) + \sum_{j=1}^{N-1} \bar{g}_{ij} K_\varepsilon C_d \bar{x}_{jd} - K_\varepsilon C_d \hat{\bar{x}}_{id}.$$

Next, define $\xi_{ia} = \bar{x}_{ia}$, $\hat{\xi}_{ia} = \hat{\bar{x}}_{ia}$, $\xi_{id} = S_\varepsilon \bar{x}_{id}$, and $\hat{\xi}_{id} = S_\varepsilon \hat{\bar{x}}_{id}$. Then, using the identities $S_\varepsilon A_d S_\varepsilon^{-1} = \varepsilon^{-1} A_d$, $S_\varepsilon B_d = \varepsilon^{\rho-1} B_d$, and $C_d S_\varepsilon^{-1} = C_d$, we have

$$\dot{\xi}_{ia} = A_a \xi_{ia} + V_{iad}\xi_{id}, \qquad \dot{\hat{\xi}}_{ia} = A_a \hat{\xi}_{ia} + \hat{V}_{iad}\hat{\xi}_{id}, \tag{8a}$$
$$\varepsilon\dot{\xi}_{id} = A_d \xi_{id} + B_d F_\delta \hat{\xi}_{id} + V_{ida}^\varepsilon \xi_{ia} + V_{idd}^\varepsilon \xi_{id}, \tag{8b}$$
$$\varepsilon\dot{\hat{\xi}}_{id} = A_d \hat{\xi}_{id} + \hat{V}_{ida}^\varepsilon \hat{\xi}_a + \hat{V}_{idd}^\varepsilon \hat{\xi}_{id} + \sum_{j=1}^{N-1} \bar{g}_{ij} K C_d \xi_{jd} - K C_d \hat{\xi}_{id}, \tag{8c}$$

where $V_{iad} = \hat{V}_{iad} = L_{ad}C_d$, $V^{\varepsilon}_{ida} = \hat{V}^{\varepsilon}_{ida} = \varepsilon^{\rho}B_dE_{da}$, and $V^{\varepsilon}_{idd} = \hat{V}^{\varepsilon}_{idd} = \varepsilon^{\rho}B_dE_{dd}S_{\varepsilon}^{-1}$. Clearly $\|V_{iad}\|$ and $\|\hat{V}_{iad}\|$ are ε-independent, while $\|V^{\varepsilon}_{ida}\|$ and $\|\hat{V}^{\varepsilon}_{ida}\|$ are $O(\varepsilon)$. Moreover, $\|\varepsilon^{\rho}B_dE_{dd}S_{\varepsilon}^{-1}\| \leq \|B_dE_{dd}\|\|\mathrm{diag}(\varepsilon^{\rho},\dots,\varepsilon)\| \leq \varepsilon\|B_dE_{dd}\|$, and hence $\|V^{\varepsilon}_{idd}\|$ and $\|\hat{V}^{\varepsilon}_{idd}\|$ are $O(\varepsilon)$.

Define $\bar{G} = [\bar{g}_{ij}]$, $i,j \in \{1,\dots,N-1\}$. It follows from the proof of Zhang and Tian (2009, Lemma 1) that the eigenvalues of $\bar{G}$ are the nonzero eigenvalues of G. Let $\xi_a = [\xi_{1a};\dots;\xi_{(N-1)a}]$, $\hat{\xi}_a = [\hat{\xi}_{1a};\dots;\hat{\xi}_{(N-1)a}]$, $\xi_d = [\xi_{1d};\dots;\xi_{(N-1)d}]$, and $\hat{\xi}_d = [\hat{\xi}_{1d};\dots;\hat{\xi}_{(N-1)d}]$. Then

$$\dot{\xi}_a = (I_{N-1}\otimes A_a)\xi_a + V_{ad}\xi_d, \qquad \dot{\hat{\xi}}_a = (I_{N-1}\otimes A_a)\hat{\xi}_a + \hat{V}_{ad}\hat{\xi}_d,$$
$$\varepsilon\dot{\xi}_d = (I_{N-1}\otimes A_d)\xi_d + (I_{N-1}\otimes B_dF_\delta)\hat{\xi}_d + V^{\varepsilon}_{da}\xi_a + V^{\varepsilon}_{dd}\xi_d,$$
$$\varepsilon\dot{\hat{\xi}}_d = (I_{N-1}\otimes A_d)\hat{\xi}_d + \hat{V}^{\varepsilon}_{da}\hat{\xi}_a + \hat{V}^{\varepsilon}_{dd}\hat{\xi}_d + (\bar{G}\otimes KC_d)\xi_d - (I_{N-1}\otimes KC_d)\hat{\xi}_d,$$

where $V_{ad} = \mathrm{diag}(V_{1ad},\dots,V_{(N-1)ad})$, and $\hat{V}_{ad}$, V^{ε}_{da}, $\hat{V}^{\varepsilon}_{da}$, V^{ε}_{dd}, and $\hat{V}^{\varepsilon}_{dd}$ are similarly defined. Define U such that $U^{-1}\bar{G}U = J$, where J is the Jordan form of $\bar{G}$, and let $\nu_a = (JU^{-1}\otimes I_{n-\rho})\xi_a$, $\tilde{\nu}_a = \nu_a - (JU^{-1}\otimes I_{n-\rho})\hat{\xi}_a$, $\nu_d = (JU^{-1}\otimes I_\rho)\xi_d$, and $\tilde{\nu}_d = \nu_d - (U^{-1}\otimes I_\rho)\hat{\xi}_d$. Then

$$\dot{\nu}_a = (I_{N-1}\otimes A_a)\nu_a + W_{ad}\nu_d,$$
$$\dot{\tilde{\nu}}_a = (I_{N-1}\otimes A_a)\tilde{\nu}_a + W_{ad}\nu_d - \hat{W}_{ad}(\nu_d - \tilde{\nu}_d),$$
$$\varepsilon\dot{\nu}_d = (I_{N-1}\otimes A_d)\nu_d + (J\otimes B_dF_\delta)(\nu_d - \tilde{\nu}_d) + W^{\varepsilon}_{da}\nu_a + W^{\varepsilon}_{dd}\nu_d,$$
$$\varepsilon\dot{\tilde{\nu}}_d = (I_{N-1}\otimes A_d)\tilde{\nu}_d + (J\otimes B_dF_\delta)(\nu_d - \tilde{\nu}_d) + W^{\varepsilon}_{da}\nu_a - \hat{W}^{\varepsilon}_{da}(\nu_a - \tilde{\nu}_a) + W^{\varepsilon}_{dd}\nu_d - \hat{W}^{\varepsilon}_{dd}(\nu_d - \tilde{\nu}_d) - (I_{N-1}\otimes KC_d)\tilde{\nu}_d,$$

where $W_{ad} = (JU^{-1}\otimes I_{n-\rho})V_{ad}(UJ^{-1}\otimes I_\rho)$, $\hat{W}_{ad} = (JU^{-1}\otimes I_{n-\rho})\hat{V}_{ad}(U\otimes I_\rho)$, $W^{\varepsilon}_{da} = (JU^{-1}\otimes I_\rho)V^{\varepsilon}_{da}(UJ^{-1}\otimes I_{n-\rho})$, $W^{\varepsilon}_{dd} = (JU^{-1}\otimes I_\rho)V^{\varepsilon}_{dd}(UJ^{-1}\otimes I_\rho)$, $\hat{W}^{\varepsilon}_{da} = (U^{-1}\otimes I_\rho)\hat{V}^{\varepsilon}_{da}(UJ^{-1}\otimes I_{n-\rho})$, and $\hat{W}^{\varepsilon}_{dd} = (U^{-1}\otimes I_\rho)\hat{V}^{\varepsilon}_{dd}(U\otimes I_\rho)$. Finally, let N_a and N_d be defined such that $\eta_a := N_a[\nu_a;\tilde{\nu}_a] = [\nu_{1a};\tilde{\nu}_{1a};\dots;\nu_{(N-1)a};\tilde{\nu}_{(N-1)a}]$, and $\eta_d := N_d[\nu_d;\tilde{\nu}_d] = [\nu_{1d};\tilde{\nu}_{1d};\dots;\nu_{(N-1)d};\tilde{\nu}_{(N-1)d}]$. Then

$$\dot{\eta}_a = \tilde{A}_a\eta_a + \tilde{W}_{ad}\eta_d, \tag{9a}$$
$$\varepsilon\dot{\eta}_d = \tilde{A}_\delta\eta_d + \tilde{W}^{\varepsilon}_{da}\eta_a + \tilde{W}^{\varepsilon}_{dd}\eta_d, \tag{9b}$$

where $\tilde{A}_a = (I_{2(N-1)}\otimes A_a)$,

$$\tilde{A}_\delta = I_{N-1}\otimes\begin{bmatrix}A_d & 0\\ 0 & A_d - KC_d\end{bmatrix} + J\otimes\begin{bmatrix}B_dF_\delta & -B_dF_\delta\\ B_dF_\delta & -B_dF_\delta\end{bmatrix},$$

and

$$\tilde{W}_{ad} = N_a\begin{bmatrix}W_{ad} & 0\\ W_{ad}-\hat{W}_{ad} & \hat{W}_{ad}\end{bmatrix}N_d^{-1},$$
$$\tilde{W}^{\varepsilon}_{ds} = N_d\begin{bmatrix}W^{\varepsilon}_{ds} & 0\\ W^{\varepsilon}_{ds}-\hat{W}^{\varepsilon}_{ds} & \hat{W}^{\varepsilon}_{ds}\end{bmatrix}N_s^{-1}, \quad s\in\{a,d\}.$$

Due to its upper block-triangular structure, the eigenvalues of $\tilde{A}_\delta$ are the eigenvalues of the matrices

$$\bar{A}_\delta := \begin{bmatrix}A_d + \lambda B_dF_\delta & -\lambda B_dF_\delta\\ \lambda B_dF_\delta & A_d - KC_d - \lambda B_dF_\delta\end{bmatrix}, \tag{10}$$

for each eigenvalue λ of $\bar{G}$ along the diagonal of J. Following along the lines of Seo et al. (2009), we shall show that $\bar{A}_\delta$ is Hurwitz for all sufficiently small δ. Let $P = P' > 0$ be the solution of the Lyapunov equation $P(A_d - KC_d) + (A_d - KC_d)'P = -I$, and define $\bar{P}_\delta = \mathrm{diag}(P_\delta, \sqrt{\|P_\delta\|}P)$ and $\bar{X}_\delta = \bar{P}_\delta\bar{A}_\delta + \bar{A}^*_\delta\bar{P}_\delta$. We denote by $X_{11} = P_\delta A_d + A_d'P_\delta - 2\mathrm{Re}(\lambda)F_\delta'F_\delta$, $X_{12} = \lambda F_\delta'F_\delta + \lambda^*\sqrt{\|P_\delta\|}F_\delta'B_d'P$, $X_{21} = X_{12}^*$, and $X_{22} = \sqrt{\|P_\delta\|}(P(A_d - KC_d - \lambda B_dF_\delta) + (A_d - KC_d - \lambda B_dF_\delta)^*P)$ the $\rho\times\rho$ blocks of $\bar{X}_\delta$. Using (4), we know that since $\mathrm{Re}(\lambda)\geq\tau$, $X_{11} = -\delta I - (2\mathrm{Re}(\lambda)-\tau)F_\delta'F_\delta \leq -\delta I - \tau F_\delta'F_\delta$, and we also have $X_{22} = -\sqrt{\|P_\delta\|}(I + \lambda PB_dF_\delta + \lambda^*F_\delta'B_d'P) = -\frac{1}{2}\sqrt{\|P_\delta\|}I - \sqrt{\|P_\delta\|}(\frac{1}{2}I + \lambda PB_dF_\delta + \lambda^*F_\delta'B_d'P)$. It follows that

$$\bar{X}_\delta \leq -\begin{bmatrix}\delta I & 0\\ 0 & \dfrac{1}{2}\sqrt{\|P_\delta\|I}\end{bmatrix} - \begin{bmatrix}F_\delta' & 0\\ 0 & I\end{bmatrix}W\begin{bmatrix}F_\delta & 0\\ 0 & I\end{bmatrix},$$

where the blocks of W are given by $W_{11} = \tau$, $W_{12} = -\lambda F_\delta - \lambda^*\sqrt{\|P_\delta\|}B_d'P$, $W_{21} = W_{12}^*$, and $W_{22} = \sqrt{\|P_\delta\|}(\frac{1}{2}I + \lambda PB_dF_\delta + \lambda^*F_\delta'B_d'P)$.

We only need to show that W is positive semidefinite. To this end, let $x = [x_1; x_2]$, $x_1\in\mathbb{C}$, $x_2\in\mathbb{C}^\rho$, be an arbitrary vector. Then we have that x^*Wx is greater than or equal to

$$\begin{bmatrix}|x_1| & \|x_2\|\end{bmatrix}\begin{bmatrix}\tau & -|\lambda|(\|F_\delta\| + \sqrt{\|P_\delta\|}\|PB_d\|)\\ \star & \sqrt{\|P_\delta\|}(\dfrac{1}{2} - 2|\lambda|\|PB_d\|\|F_\delta\|)\end{bmatrix}\begin{bmatrix}|x_1|\\ \|x_2\|\end{bmatrix},$$

where $\star$ denotes a symmetric element. The first-order principal minor of the above matrix is $\tau > 0$. The second-order principal minor is $\frac{1}{2}\tau\sqrt{\|P_\delta\|} - 2\tau\sqrt{\|P_\delta\|}|\lambda|\|PB_d\|\|F_\delta\| - |\lambda|^2(\|F_\delta\| + \sqrt{\|P_\delta\|}\|PB_d\|)^2$. Since all the eigenvalues of A_d are in the closed left-half complex plane, we know by the properties of Riccati-based low-gain design that $P_\delta\to 0$ as $\delta\to 0$ (Lin, 1999, Lemma 2.2.6). Noting that $\|F_\delta\|$ is $O(\|P_\delta\|)$, we see that the second and third term of the above expression are $O(\|P_\delta\|)$, and thus they are dominated by the first term for all sufficiently small δ. It follows that W is positive definite for all sufficiently small δ and $\bar{X}_\delta$ is therefore negative definite. Letting δ be small enough that this holds for all eigenvalues λ of $\bar{G}$, we can therefore conclude that $\tilde{A}_\delta$ is Hurwitz.

Let $\tilde{P}_\delta = \tilde{P}^*_\delta > 0$ be the solution of the Lyapunov equation $\tilde{P}_\delta\tilde{A}_\delta + \tilde{A}^*_\delta\tilde{P}_\delta = -I_{2(N-1)\rho}$, and let $\tilde{P}_a = \tilde{P}_a' > 0$ be the solution of the Lyapunov equation $\tilde{P}_a\tilde{A}_a + \tilde{A}_a'\tilde{P}_a = -I_{2(N-1)(n-\rho)}$, which exists because $\tilde{A}_a$ is Hurwitz. Consider the Lyapunov function $V = \varepsilon\eta_d^*\tilde{P}_\delta\eta_d + \varepsilon\eta_a^*\tilde{P}_a\eta_a$, for which we have

$$\begin{aligned}\dot{V} &= -\|\eta_d\|^2 + 2\mathrm{Re}(\eta_d^*\tilde{P}_\delta\tilde{W}^{\varepsilon}_{da}\eta_a)\\ &\quad + 2\mathrm{Re}(\eta_d^*\tilde{P}_\delta\tilde{W}^{\varepsilon}_{dd}\eta_d) - \varepsilon\|\eta_a\|^2 + 2\varepsilon\mathrm{Re}(\eta_a^*\tilde{P}_a\tilde{W}_{ad}\eta_d)\\ &\leq -(1-2\varepsilon\gamma_1)\|\eta_d\|^2 - \varepsilon\|\eta_a\|^2 + 2\varepsilon\gamma_2\|\eta_d\|\|\eta_a\|,\end{aligned}$$

where $\varepsilon\gamma_1 \geq \|\tilde{P}_\delta\tilde{W}^{\varepsilon}_{dd}\|$ and $\varepsilon\gamma_2 \geq \|\tilde{P}_\delta\tilde{W}^{\varepsilon}_{da}\| + \varepsilon\|\tilde{P}_a\tilde{W}_{ad}\|$. Let ε be chosen small enough that $1 - 2\varepsilon\gamma_1 \geq \frac{1}{2}$. Then

$$\dot{V} \leq -\begin{bmatrix}\|\eta_d\| & \|\eta_a\|\end{bmatrix}\begin{bmatrix}\dfrac{1}{2} & -\varepsilon\gamma_2\\ -\varepsilon\gamma_2 & \varepsilon\end{bmatrix}\begin{bmatrix}\|\eta_d\|\\ \|\eta_a\|\end{bmatrix}.$$

The first-order principal minor of the above matrix is $\frac{1}{2} > 0$. The second-order principal minor is $\frac{1}{2}\varepsilon - \varepsilon^2\gamma_2^2$, which is positive for all $\varepsilon < 1/(2\gamma_2^2)$. It follows that $\eta_a\to 0$ and $\eta_d\to 0$, which implies $\bar{x}_i\to 0$ for all $i\in\{1,\dots,N-1\}$. ■

In addition to selecting a gain matrix K to ensure that $A_d - KC_d$ is Hurwitz, our design methodology requires choosing sufficiently low parameters δ and ε as indicated by Theorem 1. Although it is possible to derive analytical upper bounds on δ and ε, these bounds are likely to be conservative, and the parameters should instead be treated as tuning parameters.

4. Homogeneous networks of nonlinear time-varying agents

In this section we consider nonlinear time-varying agents that can be represented on the following canonical form:

$$\dot{x}_{ia} = A_a x_{ia} + L_{ad} y_i, \tag{11a}$$

$$\dot{x}_{id} = A_d x_{id} + \phi_d(t, x_{ia}, x_{id}) + B_d(u_i + E_{da} x_{ia} + E_{dd} x_{id}), \tag{11b}$$

$$y_i = C_d x_{id}, \tag{11c}$$

where A_a is Hurwitz and A_d, B_d, and C_d have the special form shown in (3). The system (11) differs from (2) only in the presence of a time-varying nonlinearity $\phi_d(t, x_{ia}, x_{id})$. We make the following assumption about this nonlinearity.

Assumption 3. The function $\phi_d(t, x_{ia}, x_{id})$ is continuously differentiable and Lipschitz continuous with respect to (x_{ia}, x_{id}), uniformly in t, and piecewise continuous with respect to t. Moreover, the nonlinearity has the following lower-triangular structure:

$$\frac{\partial \phi_{dj}(t, x_{ia}, x_{id})}{\partial x_{idk}} = 0, \quad \forall k > j, \tag{12}$$

where $\phi_{dj}(t, x_{ia}, x_{id})$ denotes the j'th element of $\phi_d(t, x_{ia}, x_{id})$ and x_{idk} denotes the k'th element of x_{id}.

The canonical form in (11) is similar to various types of chained, lower-triangular canonical forms common in the context of high-gain observer design and output feedback control (see, e.g., Khalil & Praly, 2014). Among the practically relevant types of systems encompassed by this canonical form are mechanical systems with nonlinearities occurring at the acceleration level.

4.1. Control design

Let K_ε and $F_{\delta\varepsilon}$ be designed as in Section 3.2, and define the following dynamic controller:

$$\dot{\hat{x}}_{ia} = A_a \hat{x}_{ia} + L_{ad} C_d \hat{x}_{id}, \tag{13a}$$

$$\dot{\hat{x}}_{id} = A_d \hat{x}_{id} + \phi_d(t, \hat{x}_{ia}, \hat{x}_{id}) + K_\varepsilon(\zeta_i - C_d \hat{x}_{id}) + B_d(E_{da}\hat{x}_{ia} + E_{dd}\hat{x}_{id}), \tag{13b}$$

$$u_i = F_{\delta\varepsilon} \hat{x}_{id}. \tag{13c}$$

Theorem 2. *Consider the network with agents described by (11) and the dynamic controller described by (13). Under Assumptions 2 and 3 there exists a $\delta^* \in (0, 1]$ such that, for each $\delta \in (0, \delta^*]$, there exists an $\varepsilon^*(\delta) \in (0, 1]$ such that, for all $\varepsilon \in (0, \varepsilon^*(\delta)]$, $\lim_{t\to\infty}(x_i - x_j) = 0$ for all $i, j \in \{1, \dots, N\}$.*

Proof. Define $\bar{x}_i$ and $\hat{\bar{x}}_i$ as in the proof of Theorem 1. By Taylor's theorem (see, e.g., Nocedal & Wright, 1999, Theorem 11.1), we can write $\phi_d(t, x_{Na}, x_{Nd}) - \phi_d(t, x_{ia}, x_{id}) = \Phi_{ia}(t)\bar{x}_{ia} + \Phi_{id}(t)\bar{x}_{id}$, where $\Phi_{ia}(t)$ and $\Phi_{id}(t)$ are given by

$$\Phi_{ia}(t) = \int_0^1 \frac{\partial \phi_d}{\partial x_{ia}}(t, x_{ia} + p\bar{x}_{ia}, x_{id} + p\bar{x}_{id})\,\mathrm{d}p,$$

$$\Phi_{id}(t) = \int_0^1 \frac{\partial \phi_d}{\partial x_{id}}(t, x_{ia} + p\bar{x}_{ia}, x_{id} + p\bar{x}_{id})\,\mathrm{d}p.$$

Due to the Lipschitz property of the nonlinearity, the elements of $\Phi_{ia}(t)$ and $\Phi_{id}(t)$ are uniformly bounded, and the lower-triangular structure of the nonlinearity implies that $\Phi_{id}(t)$ is lower-triangular. Similarly, we have $\phi_d(t, \hat{x}_{Na}, \hat{x}_{Nd}) - \phi_d(t, \hat{x}_{ia}, \hat{x}_{id}) = \hat{\Phi}_{ia}(t)\hat{\bar{x}}_{ia} + \hat{\Phi}_{id}(t)\hat{\bar{x}}_{id}$, for matrices $\hat{\Phi}_{ia}(t)$ and $\hat{\Phi}_{id}(t)$ with the same properties. We can now write

$$\dot{\bar{x}}_{ia} = A_a\bar{x}_{ia} + L_{ad}C_d\bar{x}_{id}, \qquad \dot{\hat{\bar{x}}}_{ia} = A_a\hat{\bar{x}}_{ia} + L_{ad}C_d\hat{\bar{x}}_{id},$$

$$\dot{\bar{x}}_{id} = A_d\bar{x}_{id} + \Phi_{ia}(t)\bar{x}_a + \Phi_{id}(t)\bar{x}_d + B_d(F_{\delta\varepsilon}\hat{\bar{x}}_{id} + E_{da}\bar{x}_{ia} + E_{dd}\bar{x}_{id}),$$

$$\dot{\hat{\bar{x}}}_{id} = A_d\hat{\bar{x}}_{id} + \hat{\Phi}_{ia}(t)\hat{\bar{x}}_a + \hat{\Phi}_{id}(t)\hat{\bar{x}}_d + B_d(E_{da}\hat{\bar{x}}_{ia} + E_{dd}\hat{\bar{x}}_{id}) + \sum_{j=1}^{N-1}\bar{g}_{ij}K_\varepsilon C_d\bar{x}_{id} - K_\varepsilon C_d\hat{\bar{x}}_{id}.$$

Next, defining ξ_{ia}, ξ_{id}, $\hat{\xi}_{ia}$, and $\hat{\xi}_{id}$ as in the proof of Theorem 1, we get the same system equations as in (8), but with $V^\varepsilon_{ida} := \varepsilon^\rho B_d E_{da} + \varepsilon S_\varepsilon \Phi_{ia}(t)$, $\hat{V}^\varepsilon_{ida} := \varepsilon^\rho B_d E_{da} + \varepsilon S_\varepsilon \hat{\Phi}_{ia}(t)$, $V^\varepsilon_{idd} := \varepsilon^\rho B_d E_{dd} S_\varepsilon^{-1} + \varepsilon S_\varepsilon \Phi_{id}(t) S_\varepsilon^{-1}$, and $\hat{V}^\varepsilon_{idd} := \varepsilon^\rho B_d E_{dd} S_\varepsilon^{-1} + \varepsilon S_\varepsilon \hat{\Phi}_{id}(t) S_\varepsilon^{-1}$. Clearly $\|V^\varepsilon_{ida}\|$ and $\|\hat{V}^\varepsilon_{ida}\|$ are $O(\varepsilon)$. It is shown in the proof of Theorem 1 that the first term of V^ε_{idd} and $\hat{V}^\varepsilon_{idd}$ is $O(\varepsilon)$. Moreover, the second term is $O(\varepsilon)$ due to the lower-triangular structure of Φ_{id} and $\hat{\Phi}_{id}$ (see, e.g., Grip & Saberi, 2014). The proof can now be completed in the same way as the proof of Theorem 1. ■

4.2. Transforming nonlinear time-varying systems to the canonical form

Our design for nonlinear time-varying agents requires the system to be given in the particular canonical form (11). Given an arbitrary nonlinear time-varying system, one would therefore like to know (i) whether it is possible to transform it to this canonical form; and (ii) how the appropriate transformation can be constructed. If we limit ourselves to *linear* state and input transformations, then both questions are simultaneously answered by the following theorem.

Theorem 3. *Consider the nonlinear time-varying system*

$$\dot{\bar{x}}_i = \bar{A}\bar{x}_i + \bar{B}\bar{u}_i + \bar{\phi}(t, \bar{x}_i), \quad \bar{x}_i \in \mathbb{R}^n, \ \bar{u}_i \in \mathbb{R}, \tag{14a}$$

$$y_i = \bar{C}\bar{x}_i, \quad y_i \in \mathbb{R}, \tag{14b}$$

where $(\bar{A}, \bar{B}, \bar{C})$ is minimum-phase and of relative degree $\rho \geq 1$; and where $\bar{\phi}(t, \bar{x}_i)$ is continuously differentiable and Lipschitz continuous with respect to $\bar{x}_i$, uniformly in t, and piecewise continuous with respect to t. Let $\Gamma_x \in \mathbb{R}^{n\times n}$ and $\Gamma_u \in \mathbb{R}$ be nonsingular state and input transformations such that the triple $(A, B, C) = (\Gamma_x^{-1}\bar{A}\Gamma_x, \Gamma_x^{-1}\bar{B}\Gamma_u, \bar{C}\Gamma_x)$ is in the SCB, and define $\bar{x}_i = \Gamma_x x_i$ and $\bar{u}_i = \Gamma_u u_i$. Then either

- *the system with state x_i, input u_i, and output y_i satisfies the canonical form (11); or*
- *there exists no set of linear, non-singular state and input transformations that take the system to the canonical form.*

Proof. First, note that the linear portion of (11) is in the SCB. Thus, all we have to show is that all transformations that take the linear portion of the system to the SCB are equivalent with respect to satisfying Assumption 3. Consider therefore the system (11) satisfying Assumption 3, and let (A, B, C) denote the corresponding linear triple. Let $\tilde{\Gamma}_x$ and $\tilde{\Gamma}_u$ denote state and input transformations such that $(\tilde{A}, \tilde{B}, \tilde{C}) = (\tilde{\Gamma}_x^{-1}A\tilde{\Gamma}_x, \tilde{\Gamma}_x^{-1}B\tilde{\Gamma}_u, C\tilde{\Gamma}_x)$ is also in the SCB. Define $x_i = \tilde{\Gamma}_x\tilde{x}_i$, and $u_i = \tilde{\Gamma}_u\tilde{u}_i$, and partition $\tilde{x}_i$ as $\tilde{x}_i = [\tilde{x}_{ia}; \tilde{x}_{id}]$, where $\tilde{x}_{ia} \in \mathbb{R}^{n-\rho}$ and $\tilde{x}_{id} \in \mathbb{R}^\rho$. Then we can write

$$\dot{\tilde{x}}_{ia} = \tilde{A}_a\tilde{x}_{ia} + \tilde{L}_{ad}y_i + \tilde{\phi}_a(t, \tilde{x}_{ia}, \tilde{x}_{id}),$$

$$\dot{\tilde{x}}_{id} = A_d\tilde{x}_{id} + \tilde{\phi}_d(t, \tilde{x}_{ia}, \tilde{x}_{id}) + B_d(\tilde{u}_i + \tilde{E}_{da}\tilde{x}_{ia} + \tilde{E}_{dd}\tilde{x}_{id}),$$

$$y_i = C_d\tilde{x}_{id},$$

and we need to show that $\tilde{\phi}_a(t, \tilde{x}_{ia}, \tilde{x}_{id}) = 0$ and that $\tilde{\phi}_d(t, \tilde{x}_{ia}, \tilde{x}_{id})$ satisfies (12).

Let $\tilde{\Gamma}_x = \begin{bmatrix} \Gamma_{xaa} & \Gamma_{xad} \\ \Gamma_{xda} & \Gamma_{xdd} \end{bmatrix}$ be partitioned according to the dimensions of x_{ia} and x_{id}, and define $\mathcal{O}_K(A, C) = [C', \dots, (CA^{K-1})']'$, for matrices A and C of compatible dimensions. Note that $\mathcal{O}_\rho(\tilde{A}, \tilde{C}) = \mathcal{O}_\rho(A, C) = [0, \mathcal{O}_\rho(A_d, C_d)] = [0, I_\rho]$. On the other hand, $\mathcal{O}_\rho(\tilde{A}, \tilde{C}) = \mathcal{O}_\rho(A, C)\tilde{\Gamma}_x = [0, I_\rho]\tilde{\Gamma}_x = [\Gamma_{xda}, \Gamma_{xdd}]$. It follows that $\Gamma_{xda} = 0$ and $\Gamma_{xdd} = I_\rho$, which implies that $\tilde{x}_{id} = x_{id}$.

Next, we have $\tilde{\Gamma}_x\tilde{B} = B\tilde{\Gamma}_u$, which implies $\Gamma_{xad}B_d = 0$, meaning that column ρ of Γ_{xad} is zero. Furthermore, we have $\tilde{\Gamma}_x\tilde{A} = A\tilde{\Gamma}_x$, which implies $\Gamma_{xaa}\tilde{L}_{ad}C_d + \Gamma_{xad}(A_d + B_d\tilde{E}_{dd}) = A_a\Gamma_{xad} + L_{ad}C_d$. It follows that $(\Gamma_{xaa}\tilde{L}_{ad} - L_{ad})C_d = A_a\Gamma_{xad} - \Gamma_{xad}A_d$. Let $1 < k \le \rho$ and note that column k on the left-hand side of the last equation is zero. Suppose that column k of Γ_{xad} is also zero (note that this holds for $k = \rho$) which implies that column k of $A_a\Gamma_{xad}$ is zero. Since column k of $\Gamma_{xad}A_d$ is equal to column $k-1$ of Γ_{xad}, it follows that this column is also zero. By induction, $\Gamma_{xad} = 0$, and hence $x_{ia} = \Gamma_{xaa}\tilde{x}_{ia}$. It now follows that $\tilde{\phi}_a(t, \tilde{x}_{ia}, \tilde{x}_{id}) = 0$ and that $\tilde{\phi}_d(t, \tilde{x}_{ia}, \tilde{x}_{id})$ satisfies (12). ■

5. Heterogeneous networks of linear agents

We now consider *heterogeneous* networks of linear agents, where each agent $i \in \{1, \dots, N\}$ is described by

$$\dot{x}_i = A_i x_i + B_i u_i, \qquad x_i \in \mathbb{R}^{n_i},\ u_i \in \mathbb{R}, \tag{15a}$$
$$y_i = C_i x_i, \qquad y_i \in \mathbb{R}. \tag{15b}$$

We make the following assumption about the agent models.

Assumption 4. For each $i \in \{1, \dots, N\}$, the triple (A_i, B_i, C_i) is minimum-phase and of relative degree $\rho_i \ge 1$.

Unlike the previous sections, our focus here will be on *regulated output* synchronization, where the goal is to ensure synchronization of the outputs toward a reference trajectory generated by an autonomous exosystem

$$\dot{\omega} = S\omega, \qquad \omega \in \mathbb{R}^{n_r} \tag{16a}$$
$$y_r = R\omega, \qquad y_r \in \mathbb{R}. \tag{16b}$$

Because unobservable and asymptotically stable modes play no role asymptotically, we assume without loss of generality that (S, R) is observable and that the eigenvalues of S are in the closed right-half complex plane.

To achieve regulated output synchronization, at least some of the agents must have knowledge of their output relative to that of the exosystem. We therefore assume that each agent has access to the quantity

$$\psi_i = \iota_i(y_i - y_r), \qquad \iota_i = \begin{cases} 1, & i \in \mathcal{I}, \\ 0, & \text{otherwise}, \end{cases}$$

where $\mathcal{I} \subset \{1, \dots, N\}$ represents a subset of the agents. We replace Assumption 2 with the following assumption.

Assumption 5. Every node of $\mathcal{G}$ is a member of a directed tree with its root contained in $\mathcal{I}$.

For the purpose of the derivations in this section we define the matrix $\bar{G} := G + \operatorname{diag}(\iota_1, \dots, \iota_N)$. Note that, according to Lemma 7 of Grip et al. (2012), the eigenvalues of $\bar{G}$ are all in the open right-half complex plane. We shall assume knowledge of a positive lower bound on the real part of the eigenvalues of $\bar{G}$, and for the remainder of the section, $\tau > 0$ represents such a lower bound.

5.1. Special case

We begin by solving the regulated synchronization problem for a special case where

(1) for each $i \in \{1, \dots, N\}$, the pair (A_i, C_i) contains (S, R); and
(2) the triples (A_i, B_i, C_i) for $i \in \{1, \dots, N\}$ are of a common relative degree $\rho \ge 1$.

In Section 5.2 we shall show that our original problem formulation can be transformed to this special case by first augmenting the agents with dynamic pre-compensators.

We can assume without loss of generality that each agent model is given in the SCB, and thus x_i can be partitioned as $x_i = [x_{ia}; x_{id}]$, where $x_{ia} \in \mathbb{R}^{n_i - \rho}$ and $x_{id} \in \mathbb{R}^\rho$, and where

$$\dot{x}_{ia} = A_{ia}x_{ia} + L_{iad}y_i, \tag{17a}$$
$$\dot{x}_{id} = A_d x_{id} + B_d(u_i + E_{ida}x_{ia} + E_{idd}x_{id}), \tag{17b}$$
$$y_i = C_d x_{id}. \tag{17c}$$

The matrices A_d, B_d, and C_d have the special form in (3), and the eigenvalues of A_{ia} are the invariant zeros of the triple (A_i, B_i, C_i), which are all in the open left-half complex plane.

Let K_ε and $F_{\delta\varepsilon}$ be designed as in Section 3.2, and define the following dynamic controller:

$$\dot{\hat{x}}_{id} = A_d\hat{x}_{id} + K_\varepsilon(\zeta_i + \psi_i - C_d\hat{x}_{id}), \tag{18a}$$
$$u_i = F_{\delta\varepsilon}\hat{x}_{id}. \tag{18b}$$

Theorem 4. *Consider the heterogeneous network with agents described by (17) and the dynamic controller described by (18). Suppose that for each $i \in \{1, \dots, N\}$, the pair (A_i, C_i) contains (S, R) and the triple (A_i, B_i, C_i) is of relative degree $\rho \ge 1$. Then, under Assumptions 4 and 5, there exists a constant $\delta^* \in (0, 1]$ such that, for each $\delta \in (0, \delta^*]$, there exists an $\varepsilon^*(\delta) \in (0, 1]$ such that, for all $\varepsilon \in (0, \varepsilon^*(\delta)]$, $\lim_{t\to\infty}(y_i - y_r) = 0$ for all $i \in \{1, \dots, N\}$.*

Proof. For each $i \in \{1, \dots, N\}$, let $\bar{x}_i = x_i - \Pi_i\omega$, where Π_i is such that $\Pi_i S = A_i\Pi_i$, $C_i\Pi_i = R$ in accordance with Definition 1. Then $\dot{\bar{x}}_i = A_i x_i - \Pi_i S\omega + B_i u_i = A_i x_i - A_i\Pi_i\omega + B_i u_i = A_i\bar{x}_i + B_i u_i$. Furthermore, the output synchronization error $e_i = y_i - y_r$ is given by $e_i = C_i x_i - R\omega = C_i x_i - C_i\Pi_i\omega = C_i\bar{x}_i$. Since the dynamics of the $\bar{x}_i$ system with output e_i is governed by the same triple (A_i, B_i, C_i) as the dynamics of agent i, we can decompose it in the same way as in (17), by writing $\bar{x}_i = [\bar{x}_{ia}; \bar{x}_{id}]$, where

$$\dot{\bar{x}}_{ia} = A_{ia}\bar{x}_{ia} + L_{iad}e_i,$$
$$\dot{\bar{x}}_{id} = A_d\bar{x}_{id} + B_d(u_i + E_{ida}\bar{x}_{ia} + E_{idd}\bar{x}_{id}),$$

and $e_i = C_d\bar{x}_{id}$. Define $\xi_{ia} = \bar{x}_{ia}$, $\xi_{id} = S_\varepsilon\bar{x}_{id}$ and $\hat{\xi}_{id} = S_\varepsilon\hat{x}_{id}$. Then it is easy to confirm that we can write

$$\dot{\xi}_{ia} = A_{ia}\xi_{ia} + V_{iad}\xi_{id},$$
$$\varepsilon\dot{\xi}_{id} = A_d\xi_{id} + B_d F_\delta\hat{\xi}_{id} + V^\varepsilon_{ida}\xi_{ia} + V^\varepsilon_{idd}\xi_{id},$$

where $V_{iad} = L_{iad}C_d$, $V^\varepsilon_{ida} = \varepsilon^\rho B_d E_{ida}$, and $V^\varepsilon_{idd} = \varepsilon^\rho E_{idd}S_\varepsilon^{-1}$. We also have $e_i = C_d\xi_{id}$. Furthermore, noting that $\sum_{j=1}^N g_{ij} = 0$, we can write $\zeta_i + \psi_i = \sum_{j=1}^N g_{ij}y_j + \iota_i(y_i - y_r) = \sum_{j=1}^N g_{ij}(y_j - y_r) + \iota_i(y_i - y_r) = \sum_{j=1}^N \bar{g}_{ij}e_j$, where $\bar{g}_{ij}$ represents the coefficients of the matrix $\bar{G} = G + \operatorname{diag}(\iota_1, \dots, \iota_N)$. We therefore have

$$\varepsilon\dot{\hat{\xi}}_{id} = A_d\hat{\xi}_{id} + K\sum_{j=1}^N \bar{g}_{ij}C_d\xi_{jd} - KC_d\hat{\xi}_{id}.$$

Let $\xi_a = [\xi_{1a}; \ldots; \xi_{Na}]$, $\xi_d = [\xi_{1d}; \ldots; \xi_{Nd}]$, and $\hat{\xi}_d = [\hat{\xi}_{1d}; \ldots; \hat{\xi}_{Nd}]$.
Then

$$\dot{\xi}_a = \tilde{A}_a \xi_a + V_{ad}\xi_d,$$
$$\varepsilon\dot{\xi}_d = (I_N \otimes A_d)\xi_d + (I_N \otimes B_d F_\delta)\hat{\xi}_d + V^\varepsilon_{da}\xi_a + V^\varepsilon_{dd}\xi_d,$$
$$\varepsilon\dot{\hat{\xi}}_d = (I_N \otimes A_d)\hat{\xi}_d + (\bar{G} \otimes KC_d)\xi_d - (I_N \otimes KC_d)\hat{\xi}_d,$$

where $\tilde{A}_a = \mathrm{diag}(A_{ia}, \ldots, A_{Na})$, and where $V_{ad}, V^\varepsilon_{da}, V^\varepsilon_{dd}$ are defined in the same way as in the proof of Theorem 1. Note that $\|V_{ad}\|$ is ε-independent, whereas $\|V^\varepsilon_{da}\|$ and $\|V^\varepsilon_{dd}\|$ are $O(\varepsilon)$.

Let U be defined such that $U^{-1}\bar{G}U = J$, where J is the Jordan form of the matrix $\bar{G}$. Define $\nu_a = \xi_a$, $\nu_d = (JU^{-1} \otimes I_\rho)\xi_d$, and $\tilde{\nu}_d = \nu_d - (U^{-1} \otimes I_\rho)\hat{\xi}_d$. Then we have

$$\dot{\nu}_a = \tilde{A}_a \nu_a + W_{ad}\nu_d,$$
$$\varepsilon\dot{\nu}_d = (I_N \otimes A_d)\nu_d + (J \otimes B_d F_\delta)(\nu_d - \tilde{\nu}_d) + W^\varepsilon_{da}\nu_a + W^\varepsilon_{dd}\nu_d,$$
$$\varepsilon\dot{\tilde{\nu}}_d = (I_N \otimes A_d)\tilde{\nu}_d + (J \otimes B_d F_\delta)(\nu_d - \tilde{\nu}_d) + W^\varepsilon_{da}\nu_a + W^\varepsilon_{dd}\nu_d - (I_N \otimes KC_d)\tilde{\nu}_d,$$

where $W_{ad} = V_{ad}(UJ^{-1} \otimes I_\rho)$, $W^\varepsilon_{da} = (JU^{-1} \otimes I_\rho)V^\varepsilon_{da}$, and $W^\varepsilon_{dd} = (JU^{-1} \otimes I_\rho)V^\varepsilon_{dd}(UJ^{-1} \otimes I_\rho)$. Letting $\eta_a = \nu_a$, and letting N_d be defined such that $\eta_d = N_d[\nu_d; \tilde{\nu}_d] := [\nu_{1d}; \tilde{\nu}_{1d}; \ldots; \nu_{Nd}; \tilde{\nu}_{Nd}]$, we obtain dynamics on the same form as (9), where $\tilde{A}_\delta$ is defined as before, and where

$$\tilde{W}_{ad} = \begin{bmatrix} W_{ad} & 0 \end{bmatrix} N_d^{-1}, \qquad \tilde{W}^\varepsilon_{da} = N_d \begin{bmatrix} W^\varepsilon_{da} \\ W^\varepsilon_{da} \end{bmatrix},$$
$$\tilde{W}^\varepsilon_{dd} = N_d \begin{bmatrix} W^\varepsilon_{dd} & 0 \\ W^\varepsilon_{dd} & 0 \end{bmatrix} N_d^{-1}.$$

The remainder of the proof now proceeds in the same way as the proof of Theorem 1, to show that $\bar{x}_i \to 0$, which implies $e_i \to 0$, thus achieving output synchronization. ■

5.2. Recovering the special case via pre-compensators

We now show how to recover the special case specified in the previous section, by augmenting each original agent with two dynamic pre-compensators.

Pre-Compensator 1. The purpose of the first pre-compensator is to add modes from the exosystem to agent i, so that the augmented agent dynamics contains the exosystem. Toward this end, start by constructing a state transformation $\Sigma_i \in \mathbb{R}^{n_i \times n_i}$ taking the pair (A_i, C_i) to the Kalman observable canonical form:

$$\Sigma_i^{-1} A_i \Sigma_i = \begin{bmatrix} A_{i11} & 0 \\ A_{i21} & A_{i22} \end{bmatrix}, \qquad C_i\Sigma_i = \begin{bmatrix} C_{i1} & 0 \end{bmatrix},$$

where $A_{i11} \in \mathbb{R}^{\bar{n}_i \times \bar{n}_i}$ and (A_{i11}, C_{i1}) is observable. Next, let

$$O_i = \begin{bmatrix} C_{i1} & -R \\ \vdots & \vdots \\ C_{i1}A_{i11}^{\bar{n}_i+n_r-1} & -RS^{\bar{n}_i+n_r-1} \end{bmatrix}. \tag{19}$$

Let q_i denote the dimension of the null space of O_i, and define $r_i = n_r - q_i$. Furthermore, let $\Lambda_{iu} \in \mathbb{R}^{\bar{n}_i \times q_i}$ and $\Phi_{iu} \in \mathbb{R}^{n_r \times q_i}$ be chosen such that

$$O_i \begin{bmatrix} \Lambda_{iu} \\ \Phi_{iu} \end{bmatrix} = 0, \qquad \mathrm{rank} \begin{bmatrix} \Lambda_{iu} \\ \Phi_{iu} \end{bmatrix} = q_i.$$

The matrix Φ_{iu} has full column rank because (A_{i11}, C_{i1}) is observable (see Grip et al., 2012, App. D). Let therefore Φ_{io} be chosen such that $\Phi_i := [\Phi_{iu}, \Phi_{io}]$ is nonsingular. We can now state the following lemma, which is proven in the Appendix.

Lemma 1. *We have that*

$$\Phi_i^{-1} S \Phi_i = \begin{bmatrix} S_{i11} & S_{i12} \\ 0 & S_{i22} \end{bmatrix}, \tag{20}$$

where $S_{i11} \in \mathbb{R}^{q_i \times q_i}$, $S_{i12} \in \mathbb{R}^{q_i \times r_i}$, and $S_{i22} \in \mathbb{R}^{r_i \times r_i}$, and where $A_{i11}\Lambda_{iu} = \Lambda_{iu}S_{i11}$. Furthermore, there exists a nonsingular transformation $\Gamma_i \in \mathbb{R}^{r_i \times r_i}$ taking S_{i22} to the companion form

$$\Gamma_i^{-1} S_{i22} \Gamma_i = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -s_{i1} & -s_{i2} & \cdots & -s_{ir_i} \end{bmatrix}.$$

Based on Lemma 1, let A_{ip1} denote the above companion form of S_{i22}, and define $B_{ip1} = [0; \ldots; 0; 1]$ and $C_{ip1} = [1, 0, \ldots, 0]$, so that (A_{ip1}, B_{ip1}) is controllable and (A_{ip1}, C_{ip1}) is observable. We define the following dynamic pre-compensator:

$$\dot{z}_{i1} = A_{ip1}z_{i1} + B_{ip1}\upsilon_i, \tag{21a}$$
$$u_i = C_{ip1}z_{i1}, \tag{21b}$$

where $\upsilon_i \in \mathbb{R}$ is a new input.

Pre-Compensator 2. The purpose of the second pre-compensator is to make the relative degree of the augmented system equal to some fixed ρ, which is chosen such that $\rho \geq \rho_i + r_i$ for all $i \in \{1, \ldots, N\}$, where ρ_i is the relative degree of (A_i, B_i, C_i). Define the matrices

$$A_{ip2} = \begin{bmatrix} 0 & I_{\rho-\rho_i-r_i-1} \\ 0 & 0 \end{bmatrix},$$

$B_{ip2} = [0; \ldots; 0; 1]$, and $C_{ip2} = [1, 0, \ldots, 0]$. Define the following dynamic pre-compensator:

$$\dot{z}_{i2} = A_{ip2}z_{i2} + B_{ip2}v_i, \tag{22a}$$
$$\upsilon_i = C_{ip2}z_{i2}, \tag{22b}$$

where $v_i \in \mathbb{R}$ is a new input.[2]

By stacking the original state and the state of the two pre-compensators as $\chi_i = [x_i; z_{i1}; z_{i2}]$, we obtain the following augmented agent dynamics with input υ_i:

$$\dot{\chi}_i = \mathscr{A}_i\chi_i + \mathscr{B}_i\upsilon_i, \tag{23a}$$
$$y_i = \mathscr{C}_i\chi_i, \tag{23b}$$

where

$$\mathscr{A}_i = \begin{bmatrix} A_i & B_iC_{ip1} & 0 \\ 0 & A_{ip1} & B_{ip1}C_{ip2} \\ 0 & 0 & A_{ip2} \end{bmatrix}, \qquad \mathscr{B}_i = \begin{bmatrix} 0 \\ 0 \\ B_{ip2} \end{bmatrix},$$
$$\mathscr{C}_i = \begin{bmatrix} C_i & 0 & 0 \end{bmatrix}.$$

The following result recovers the result of Theorem 4 for general systems satisfying Assumption 4.

Theorem 5. *The augmented agent dynamics (23) satisfies Assumption 4, and moreover, for each $i \in \{1, \ldots, N\}$, (i) the pair $(\mathscr{A}_i, \mathscr{C}_i)$ contains (S, R); and (ii) the triple $(\mathscr{A}_i, \mathscr{B}_i, \mathscr{C}_i)$ is of relative degree $\rho > 0$.*

Proof. Since the pre-compensators are zero-free and have their poles in the right-half complex plane, no pole-zero cancellations occur in the augmented system, and hence it has the same invariant zeros as the original system and satisfies Assumption 4. The relative degree of the two pre-compensators are r_i and $\rho - \rho_i - r_i$.

[2] In the special case where $\rho - \rho_i - r_i = 0$, the pre-compensator is defined simply as $\upsilon_i = v_i$.

The relative degree of augmented dynamics is therefore $\rho_i + r_i + \rho - \rho_i - r_i = \rho$.

To show that $(\mathscr{A}_i, \mathscr{C}_i)$ contains (S, R), we start by showing that there exists Π_i such that $\Pi_i S = \mathscr{A}_{i1}\Pi_i$, $\mathscr{C}_{i1}\Pi_i = R$, where

$$\mathscr{A}_{i1} = \begin{bmatrix} A_i & B_i C_{ip1} \\ 0 & A_{ip1} \end{bmatrix}, \qquad \mathscr{C}_{i1} = \begin{bmatrix} C_i & 0 \end{bmatrix}.$$

Post-multiplying by Φ_i and defining $\bar{\Pi}_i := \Pi_i \Phi_i$, it can be seen from the proof of Lemma 1 that we get the equivalent expression

$$\begin{bmatrix} \bar{\Pi}_{i11} & \bar{\Pi}_{i12} \\ \bar{\Pi}_{i21} & \bar{\Pi}_{i22} \end{bmatrix} \begin{bmatrix} S_{i11} & S_{i12} \\ 0 & S_{i22} \end{bmatrix} = \begin{bmatrix} A_i & B_i C_{ip1} \\ 0 & A_{ip1} \end{bmatrix} \begin{bmatrix} \bar{\Pi}_{i11} & \bar{\Pi}_{i12} \\ \bar{\Pi}_{i21} & \bar{\Pi}_{i22} \end{bmatrix},$$

$$\begin{bmatrix} C_i & 0 \end{bmatrix} \begin{bmatrix} \bar{\Pi}_{i11} & \bar{\Pi}_{i12} \\ \bar{\Pi}_{i21} & \bar{\Pi}_{i22} \end{bmatrix} = \begin{bmatrix} R\Phi_{iu} & R\Phi_{io} \end{bmatrix}.$$

From Lemma 1 we have $A_{i11}\Lambda_{iu} = \Lambda_{iu} S_{i11}$. As remarked in Section 3, the pair (A_i, C_i) is detectable, and hence the eigenvalues of the matrix A_{i22} are in the open left-half complex plane. Since the eigenvalues of S_{i11} are in the closed right-half complex plane, we can therefore find a solution X_i of the Sylvester equation $X_i S_{i11} = A_{i22} X_i + A_{i21}\Lambda_{iu}$ (see, e.g., Saberi, Stoorvogel, & Sannuti, 2000, App. 2.A). It follows that

$$\begin{bmatrix} \Lambda_{iu} \\ X_i \end{bmatrix} S_{i11} = \begin{bmatrix} A_{i11} & 0 \\ A_{i21} & A_{i22} \end{bmatrix} \begin{bmatrix} \Lambda_{iu} \\ X_i \end{bmatrix}.$$

Letting $\bar{\Pi}_{i11} = \Sigma_i[\Lambda_{iu}; X_i]$, we therefore have $\bar{\Pi}_{i11} S_{i11} = A_i \bar{\Pi}_{i11}$. Furthermore, using the identity $C_{i1}\Lambda_{iu} = R\Phi_{iu}$ from (A.1), we have $C_i\bar{\Pi}_{i11} = [C_{i1}, 0][\Lambda_{iu}; X_i] = C_{i1}\Lambda_{iu} = R\Phi_{iu}$.

Let $\bar{\Pi}_{i21} = 0$. Next, consider the equations $\bar{\Pi}_{i11} S_{i12} + \bar{\Pi}_{i12} S_{i22} = A_i \bar{\Pi}_{i12} + B_i \Xi_i$, $C_i \bar{\Pi}_{i12} = R\Phi_{io}$ with unknowns $\bar{\Pi}_{i12}$ and Ξ_i. This set of *regulator equations* is solvable if the Rosenbrock system matrix $\begin{bmatrix} A_i - \lambda I & B_i \\ C_i & 0 \end{bmatrix}$ has rank $n_i + 1$ for each λ that is an eigenvalue of S_{i22} (Saberi et al., 2000, Corollary 2.5.1). The normal rank of this matrix is $n_i + 1$, because the system is right-invertible (Saberi, Sannuti, & Chen, 1995, Proposition 3.1.6). The matrix retains its normal rank for each λ that is an eigenvalue of S_{i22}, since these are all in the closed right-half complex plane while the invariant zeros of (A_i, B_i, C_i) are all in the open left-half complex plane. Finally, consider the equations $\bar{\Pi}_{i22} S_{i22} = A_{ip1} \bar{\Pi}_{i22}$, $C_{ip1}\bar{\Pi}_{i22} = \Xi_i$ with unknown $\bar{\Pi}_{i22}$. To see that these can be solved, note that we can equivalently write $\tilde{\Pi}_{i22} S_{i22} = S_{i22}\tilde{\Pi}_{i22}$, $C_{ip1}\Gamma_i^{-1}\tilde{\Pi}_{i22} = \Xi_i$, where $\tilde{\Pi}_{i22} = \Gamma_i \bar{\Pi}_{i22}$. Letting $\bar{\mathrm{O}}_i$ denote the observability matrix of the pair $(\mathrm{diag}(S_{i22}, S_{i22}), [C_{ip1}\Gamma_i^{-1}, -\Xi_i])$, it follows from the Cayley–Hamilton theorem that

$$\operatorname{rank} \bar{\mathrm{O}}_i = \operatorname{rank} \begin{bmatrix} C_{ip1}\Gamma_i^{-1} & -\Xi \\ \vdots & \vdots \\ C_{ip1}\Gamma_i^{-1} S_{i22}^{r_i - 1} & -\Xi_i S_{i22}^{r_i - 1} \end{bmatrix} \le r_i.$$

The first r_i columns of the above matrix constitute the observability matrix of the observable pair $(S_{i22}, C_{ip1}\Gamma_i^{-1})$, and it follows that $\tilde{\Pi}_{i22}$ can be chosen such that $\bar{\mathrm{O}}_i[\tilde{\Pi}_{i22}; I] = 0$; that is, $[\tilde{\Pi}_{i22}; I]$ spans the unobservable subspace of $(\mathrm{diag}(S_{i22}, S_{i22}), [C_{ip1}\Gamma_i^{-1}, -\Xi_i])$. Then $C_{ip1}\Gamma_i^{-1}\tilde{\Pi}_{i22} = \Xi_i$ and

$$\begin{bmatrix} S_{i22} & 0 \\ 0 & S_{i22} \end{bmatrix} \begin{bmatrix} \tilde{\Pi}_{i22} \\ I \end{bmatrix} = \begin{bmatrix} \tilde{\Pi}_{i22} \\ I \end{bmatrix} S_{i22},$$

which implies $S_{i22}\tilde{\Pi}_{i22} = \tilde{\Pi}_{i22} S_{i22}$. Combining the above expressions, we have

$$\begin{bmatrix} \bar{\Pi}_{i11} & \bar{\Pi}_{i12} \\ \bar{\Pi}_{i21} & \bar{\Pi}_{i22} \end{bmatrix} \begin{bmatrix} S_{i11} & S_{i12} \\ 0 & S_{i22} \end{bmatrix} = \begin{bmatrix} \bar{\Pi}_{i11} S_{i11} & \bar{\Pi}_{i11} S_{i12} + \bar{\Pi}_{i12} S_{i22} \\ 0 & \bar{\Pi}_{i22} S_{i22} \end{bmatrix}$$

$$= \begin{bmatrix} A_i \bar{\Pi}_{i11} & A_i \bar{\Pi}_{i12} + B_i \Xi_i \\ 0 & A_{ip1}\bar{\Pi}_{i22} \end{bmatrix} = \begin{bmatrix} A_i & B_i C_{ip1} \\ 0 & A_{ip1} \end{bmatrix} \begin{bmatrix} \bar{\Pi}_{i11} & \bar{\Pi}_{i12} \\ \bar{\Pi}_{i21} & \bar{\Pi}_{i22} \end{bmatrix}$$

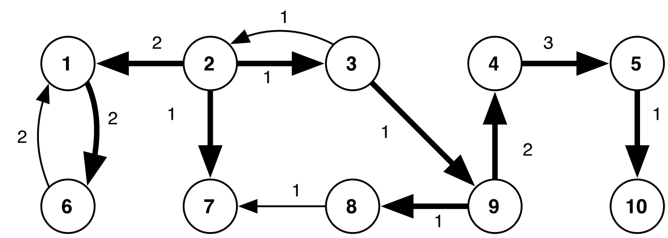


Fig. 1. Example network graph, with a directed spanning tree rooted at node 2 illustrated with bold arrows.

and

$$\begin{bmatrix} C_i & 0 \end{bmatrix} \begin{bmatrix} \bar{\Pi}_{i11} & \bar{\Pi}_{i12} \\ \bar{\Pi}_{i21} & \bar{\Pi}_{i22} \end{bmatrix} = \begin{bmatrix} C_i \bar{\Pi}_{i11} & C_i \bar{\Pi}_{i12} \end{bmatrix} = \begin{bmatrix} R\Phi_{iu} & R\Phi_{io} \end{bmatrix}.$$

Defining $\mathscr{B}_{i1} = [0; B_{ip1}]$, we can write

$$\mathscr{A}_i = \begin{bmatrix} \mathscr{A}_{i1} & \mathscr{B}_{i1} C_{ip2} \\ 0 & A_{ip2} \end{bmatrix}, \qquad \mathscr{C}_i = \begin{bmatrix} \mathscr{C}_{i1} & 0 \end{bmatrix}.$$

It is now straightforward to see that the matrix $\Pi_i^* := [\Pi_i; 0]$ verifies that the pair $(\mathscr{A}_i, \mathscr{C}_i)$ contains (S, R). ■

6. Example

Consider a network of $N = 10$ agents, illustrated in Fig. 1. This network has a directed spanning tree rooted at node 2, and thus it satisfies Assumption 2. The real part of the non-zero eigenvalues are bounded below by approximately 0.76. For design purposes, we assume that a lower bound $\tau = 0.6$ is known. We shall first consider a homogeneous linear example and then a homogeneous nonlinear and time-varying example. A heterogeneous example is given in a conference paper by Grip et al. (2013b).

Consider the linear agent model described by the matrices

$$A = \begin{bmatrix} -1 & 1 & -1 \\ -0.1 & 0 & 1.1 \\ -0.1 & 1 & -0.9 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}.$$

This model is of relative degree $\rho = 2$ and has an invariant zero at -2, and hence it satisfies Assumption 1. It also has a pole at $\lambda \approx 0.66$, and thus it is exponentially unstable. By using the state and input transformations

$$\Gamma_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \qquad \Gamma_u = 1,$$

the agent dynamics is transformed to the SCB (2):

$$\dot{x}_{ia} = -2x_{ia} + y_i,$$

$$\dot{x}_{id} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_{id} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \left(u_i + x_{ia} + \begin{bmatrix} 0 & 0.1 \end{bmatrix} x_{id} \right),$$

$$y_i = \begin{bmatrix} 1 & 0 \end{bmatrix} x_{id}.$$

For the remainder of the example, we shall work only with this representation. We start the design by selecting $K = [3; 2]$, to place the poles of $A_d - KC_d$ at -1 and -2. Next, we find the solution of the algebraic Riccati equation (4) for a given $\delta \in (0, 1]$, and we compute $K_\varepsilon = \varepsilon^{-1} S_\varepsilon^{-1} K$ and $F_{\delta\varepsilon} = \varepsilon^{-2} F_\delta S_\varepsilon$ for a given $\varepsilon \in (0, 1]$. Finally, we implement the controller (7) with the computed values. After some trial and error, we find that $\delta = 10^{-5}$ and $\varepsilon = 0.07$ ensures synchronization, which yields $K_\varepsilon \approx [42.86; 408.15]$ and $F_{\delta\varepsilon} = [-0.83, \ -1.67]$. Fig. 2 shows the synchronization error $y_i - y_{10}$ in the output channel for agents $1, \ldots, 9$.

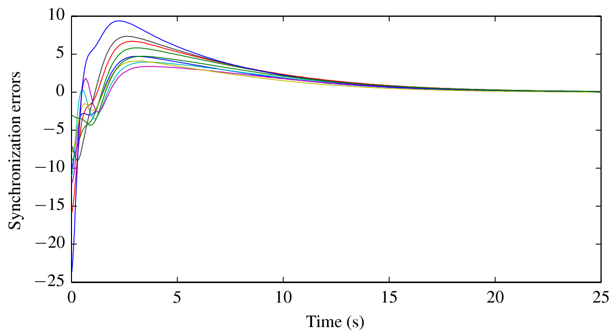


Fig. 2. Synchronization errors for linear example.

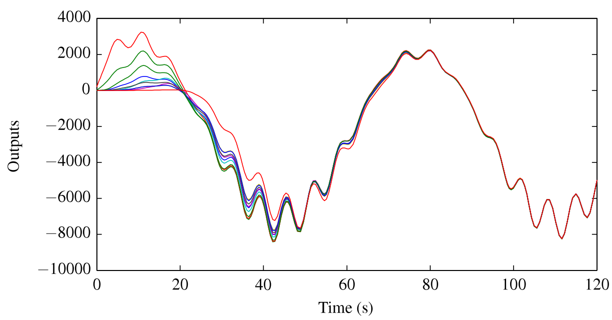


Fig. 3. Agent outputs for nonlinear example.

Next, we consider the agent model

$$\dot{x}_{ia} = -2x_{ia} + y_i,$$
$$\dot{x}_{id} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_{id} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (u_i + \phi_d(t, x_{ia}, x_{id})),$$
$$y_i = \begin{bmatrix} 1 & 0 \end{bmatrix} x_{id},$$

where $\phi_d(t, x_{ia}, x_{id}) = 0.3\sin(t)x_{ia} + \sin(0.1x_{id1}) + 2\ln(1 + x_{id2}^2)$. It is easy to see that this model is in the canonical form (11) and that the nonlinearity satisfies Assumption 3. We follow the same procedure for finding K_ε and $F_{\delta\varepsilon}$, and implement the controller (13). We find that $\delta = 10^{-5}$ and $\varepsilon = 1$ ensures synchronization, which yields $K_\varepsilon = [3; 2]$ and $F_{\delta\varepsilon} \approx [-0.0041, -0.1167]$. Fig. 3 shows the simulated agent outputs $y_1, \ldots, y_{10}$, which clearly synchronize.[3]

7. Concluding remarks

In this paper we have addressed the output synchronization problem for classes of directed networks that present a number of challenges, including unstable, nonlinear time-varying, and heterogeneous agent dynamics; agents' lack of information about their own state or output; and agents' inability to exchange controller states.

In each of the designs, an observer gain K and a control gain F_δ are designed based on the matrices A_d, B_d, and C_d alone, which are uniquely defined by the agents' relative degree. Consequently, the controllers do not make full use of the available information about the agent dynamics, and they may dominate intrinsically stabilizing dynamics through the use of unnecessarily low or high gain. Design methodologies that better exploit the dynamics of the agents are likely to lead to improved performance and are an interesting topic for future research. Generalizations to broad classes of MIMO systems is another area that will be addressed in future work.

[3] The simulations were carried out using the Dormand–Prince variable-step integration method implemented in Matlab/Simulink.

Appendix. Proof of Lemma 1

The columns of $[\Lambda_{iu}; \Phi_{iu}]$ span the unobservable subspace of the pair $(\text{diag}(A_{i11}, S), [C_{i1}, -R])$, which is $\text{diag}(A_{i11}, S)$-invariant, and hence there exists a $S_{i11} \in \mathbb{R}^{q_i \times q_i}$ such that

$$\begin{bmatrix} A_{i11} & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} \Lambda_{iu} \\ \Phi_{iu} \end{bmatrix} = \begin{bmatrix} \Lambda_{iu} \\ \Phi_{iu} \end{bmatrix} S_{i11}, \qquad \begin{bmatrix} C_{i1} & -R \end{bmatrix} \begin{bmatrix} \Lambda_{iu} \\ \Phi_{iu} \end{bmatrix} = 0. \quad \text{(A.1)}$$

It follows that $A_{i11}\Lambda_{iu} = \Lambda_{iu}S_{i11}$. Moreover, $S\Phi_{iu} = \Phi_{iu}S_{i11}$, which means that

$$S\begin{bmatrix} \Phi_{iu} & \Phi_{io} \end{bmatrix} = \begin{bmatrix} \Phi_{iu} & \Phi_{io} \end{bmatrix} \begin{bmatrix} S_{i11} & S_{i12} \\ 0 & S_{i22} \end{bmatrix},$$

for some matrices S_{i12} and S_{i22}. This, in turn, implies (20). Next, because (S, R) is observable, we have $\text{rank}\begin{bmatrix} S-\lambda I \\ R \end{bmatrix} = n$ for all eigenvalues λ of S, which implies that $\text{rank}(S - \lambda I) = n - 1$ for all eigenvalues of S. Due to the triangular form obtained via the similarity transform in (20), we therefore have $\text{rank}(S_{i22} - \lambda I) = r_i - 1$ for all eigenvalues λ of S_{i22}. It follows from this that S_{i22} is a *non-derogatory* matrix that can be transformed to the companion form (Golub & van Loan, 1996, Section 7.4.6).

References

Arcak, M. (2007). Passivity as a design tool for group coordination. *IEEE Transactions on Automatic Control*, *52*(8), 1380–1390.

Bai, H., Arcak, M., & Wen, J. (2011). Cooperative control design: A systematic, passivity-based approach. In *Communications and control engineering*. Springer.

Chopra, N., & Spong, M. (2008). Output synchronization of nonlinear systems with relative degree one. In V. Blondel, S. Boyd, & H. Kimura (Eds.), *Lecture notes in control and information sciences*: *Vol. 371*. *Recent advances in learning and control* (pp. 51–64). Springer-Verlag.

Golub, G. H., & van Loan, C. F. (1996). *Matrix computations* (3rd ed.). Baltimore, MD: Johns Hopkins University Press.

Grip, H. F., & Saberi, A. (2010). Structural decomposition of linear multivariable systems using symbolic computations. *International Journal of Control*, *83*(7), 1414–1426.

Grip, H. F., & Saberi, A. (2014). High-gain observer design for multi-output systems: Transformation to a canonical form by dynamic output shaping. *International Journal of Robust and Nonlinear Control, Special Issue on High Gain Observers in Nonlinear Output Feedback Control*, *24*(6), 1016–1042.

Grip, H. F., Saberi, A., & Stoorvogel, A. A. (2013a). On the existence of virtual exosystems for synchronized linear networks. *Automatica*, *49*(10), 3145–3148.

Grip, H. F., Saberi, A., & Stoorvogel, A. A. (2013b). Regulated output synchronization for heterogeneous networks of non-introspective, minimum-phase SISO agents without exchange of controller states. In: Proc. European Contr. Conf. Zurich, Switzerland.

Grip, H. F., Saberi, A., & Stoorvogel, A. A. (2014). Synchronization in networks of nonlinear, non-introspective, minimum-phase agents without exchange of controller states. In: Proc. American Contr. Conf. Portland, OR.

Grip, H. F., Yang, T., Saberi, A., & Stoorvogel, A. A. (2012). Output synchronization for heterogeneous networks of non-introspective agents. *Automatica*, *48*(10), 2444–2453.

Igarashi, Y., Hatanaka, T., Fujita, M., & Spong, M. W. (2009). Passivity-based attitude synchronization in *SE*(3). *IEEE Transactions on Control Systems Technology*, *17*(5), 1119–1133.

Khalil, H. K., & Praly, L. (2014). High-gain observers in nonlinear feedback control. *International Journal of Robust and Nonlinear Control, Special Issue on High Gain Observers in Nonlinear Output Feedback Control*, *24*(6), 993–1015.

Kim, H., Shim, H., & Seo, J. H. (2011). Output consensus of heterogeneous uncertain linear multi-agent systems. *IEEE Transactions on Automatic Control*, *56*(1), 200–206.

Li, Z., Duan, Z., Chen, G., & Huang, L. (2010). Consensus of multiagent systems and synchronization of complex networks: A unified viewpoint. *IEEE Transactions on Circuits and Systems I: Regular Papers*, *57*(1), 213–224.

Lin, Z. (1999). *Lecture notes in contr. inform. sci.*: *Vol. 240*. *Low gain feedback*. Berlin: Springer.

Liu, X., Chen, B. M., & Lin, Z. (2005). Linear systems toolkit in Matlab: Structural decompositions and their applications. *Journal of Control Theory and Applications*, *3*(3), 287–294.

Lunze, J. (2011). An internal-model principle for the synchronisation of autonomous agents with individual dynamics. In: Proc. IEEE Conf. Dec. Contr. Orlando, FL, pp. 2106–2111.

Nocedal, J., & Wright, S. J. (1999). *Numerical optimization*. New York: Springer-Verlag.

Olfati-Saber, R., & Murray, R. M. (2003). Agreement problems in networks with directed graphs and switching topology. In: Proc. IEEE Conf. Dec. Contr. Maui, HI, pp. 4126–4132.

Olfati-Saber, R., & Murray, R. M. (2004). Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on Automatic Control*, *49*(9), 1520–1533.

Pogromsky, A., & Nijmeijer, H. (2001). Cooperative oscillatory behavior of mutually coupled dynamical systems. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, *48*(2), 152–162.

Pogromsky, A., Santoboni, G., & Nijmeijer, H. (2002). Partial synchronization: from symmetry towards stability. *Physica D*, *172*(4–5), 65–87.

Ren, W., & Atkins, E. (2007). Distributed multi-vehicle coordinated control via local information exchange. *International Journal of Robust and Nonlinear Control*, *17*(10–11), 1002–1033.

Ren, W., & Beard, R. W. (2005). Consensus seeking in multiagent systems under dynamically changing interaction topologies. *IEEE Transactions on Automatic Control*, *50*(5), 655–661.

Saberi, A., Sannuti, P., & Chen, B. M. (1995). *H_2 Optimal control*. London: Prentice-Hall.

Saberi, A., Stoorvogel, A. A., & Sannuti, P. (2000). Control of linear systems with regulation and input constraints. In *Communication and control engineering*. Springer.

Saberi, A., Stoorvogel, A. A., & Sannuti, P. (2006). *Filtering theory*. Boston: Birkhäuser.

Sannuti, P., & Saberi, A. (1987). Special coordinate basis for multivariable linear systems—Finite and infinite zero structure, squaring down and decoupling. *International Journal of Control*, *45*(5), 1655–1704.

Seo, J. H., Shim, H., & Back, J. (2009). Consensus of higher-order linear systems using dynamic output feedback compensator: Low gain approach. *Automatica*, *45*(11), 2659–2664.

Tuna, S. E. (2008a). LQR-based coupling gain for synchronization of linear systems [online], available arXiv:0801.3390.

Tuna, S. E. (2008b). Synchronizing linear systems via partial-state coupling. *Automatica*, *44*(8), 2179–2184.

Wieland, P., Sepulchre, R., & Allgöwer, F. (2011). An internal model principle is necessary and sufficient for linear output synchronization. *Automatica*, *47*(5), 1068–1074.

Wu, C. W., & Chua, L. O. (1995a). Application of graph theory to the synchronization in an array of coupled nonlinear oscillators. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, *42*(8), 494–497.

Wu, C. W., & Chua, L. O. (1995b). Application of Kronecker products to the analysis of systems with uniform linear coupling. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, *42*(10), 775–778.

Xiang, J., & Chen, G. (2007). On the V-stability of complex dynamical networks. *Automatica*, *43*(6), 1049–1057.

Yang, T., Roy, S., Wan, Y., & Saberi, A. (2011). Constructing consensus controllers for networks with identical general linear agents. *International Journal of Robust and Nonlinear Control*, *21*(11), 1237–1256.

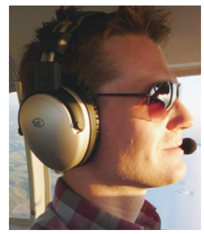
Yang, T., Saberi, A., Stoorvogel, A. A., & Grip, H. F. (2011). Output consensus for networks of non-identical introspective agents. In: Proc. IEEE Conf. Dec. Contr. Orlando, FL, pp. 1286–1292.

Yang, T., Stoorvogel, A.A., & Saberi, A. (2011c). Consensus for multi-agent systems—synchronization and regulation for complex networks. In: Proc. American Contr. Conf. San Francisco, CA, pp. 5312–5317.

Zhang, Y., & Tian, Y.-P. (2009). Consentability and protocol design of multi-agent systems with stochastic switching topology. *Automatica*, *45*, 1195–1201.

Zhao, J., Hill, D.J., & Liu, T. (2010). Passivity-based output synchronization of dynamical networks with non-identical nodes. In: Proc. IEEE Conf. Dec. Contr. Atlanta, GA, pp. 7351–7356.

Zhao, J., Hill, D. J., & Liu, T. (2011). Synchronization of dynamical networks with nonidentical nodes: Criteria and control. *IEEE Transactions on Circuits and Systems I: Regular papers*, *58*(3), 584–594.



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