

Best Monotone Degree Conditions for Graph Properties: A Survey

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Abstract We survey sufficient degree conditions, for a variety of graph properties, that are best possible in the same sense that Chvátal’s well-known degree condition for hamiltonicity is best possible.

Keywords Best monotone degree conditions · Hamiltonicity · Connectivity · Toughness · k -factor · Binding number

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1 Introduction

We consider only finite graphs without loops or multiple edges. Our terminology and notation are standard except as indicated. A good reference for undefined terms is [28]. We mention only that given graphs G, H with disjoint vertex sets, we will denote their disjoint union by $G \cup H$, and their join by $G + H$.

We generally use the standard abbreviation for integer sequences; e.g., $(4, 4, 4, 4, 4, 5, 5, 6)$ will be denoted $4^5 5^2 6^1$. An integer sequence $\pi = (d_1 \leq \dots \leq d_n)$ is called *graphical* if there exists a graph G having π as its vertex degree sequence; in that case, G is called a *realization* of π . If P is a graph property, such as ‘hamiltonian’ or ‘ k -connected’, we call a graphical sequence π *forcibly* P if every realization of π has property P . If $\pi = (d_1 \leq \dots \leq d_n)$ and $\pi' = (d'_1 \leq \dots \leq d'_n)$ are integer sequences, we say π' *majorizes* π , denoted $\pi' \geq \pi$, if $d'_i \geq d_i$ for $1 \leq i \leq n$. There is an analogous definition and notation for π' *minorizes* π .

Historically, the vertex degrees of a graph have been used to provide sufficient conditions for the graph to have certain properties, such as hamiltonicity or k -connectedness. In particular, sufficient conditions for π to be forcibly hamiltonian were given by several authors in [21, 33, 49], culminating in the following theorem of Chvátal [30].

Theorem 1.1 (Chvátal [30]) *Let $\pi = (d_1 \leq \dots \leq d_n)$ be a graphical sequence, with $n \geq 3$. If*

$$d_i \leq i \Rightarrow d_{n-i} \geq n - i, \quad \text{for } 1 \leq i \leq \frac{1}{2}(n - 1), \quad (1)$$

then π is forcibly hamiltonian.

Unlike its predecessors, Theorem 1.1 has the property that if a sequence π fails to satisfy condition (1) for some index i , then π is majorized by the sequence $\pi' = i^i (n-i-1)^{n-2i} (n-1)^i$, with nonhamiltonian realization $G' = K_i + (\overline{K}_i \cup K_{n-2i})$. As we will see in Sect. 2, this key property implies that condition (1) in Theorem 1.1 is the best of an entire important class of degree conditions for π to be forcibly hamiltonian.

Sufficient conditions for π to be forcibly k -connected have been given by several authors in [25, 26], culminating in the following theorem of Bondy [21] (though the form in which we present it is due to Boesch [20]).

Theorem 1.2 (Bondy [21]) *Let $\pi = (d_1 \leq \dots \leq d_n)$ be a graphical sequence, with $n \geq 2$ and $1 \leq k \leq n - 1$. If*

$$d_i \leq i + k - 2 \Rightarrow d_{n-k+1} \geq n - i, \quad \text{for } 1 \leq i \leq \frac{1}{2}(n - k + 1), \quad (2)$$

then π is forcibly k -connected.

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If the sequence π fails to satisfy (2) for some index i , then π is majorized by $\pi' = (i + k - 2)^i (n - i - 1)^{n-k-i+1} (n - 1)^{k-1}$, with not- k -connected realization $G' = K_{k-1} + (K_i \cup K_{n-k-i+1})$. Thus (2) is the best condition for π to be forcibly k -connected in precisely the same way (1) is the best condition for π to be forcibly hamiltonian.

Our goal in this paper is to survey degree conditions, for a wide variety of graph properties, that are best in precisely the same sense as conditions (1) and (2) above. In Sect. 2, we present a formal framework which allows us—at least in principle—to identify and construct such degree conditions, and to evaluate their inherent complexity. In Sect. 3, we apply this framework, and consider best degree conditions for the following graph properties and parameters: edge-connectivity (Sect. 3.1), binding number (Sect. 3.2), toughness (Sect. 3.3), existence of factors (Sect. 3.4), existence of paths and cycles (Sect. 3.5), and independence number, clique number, chromatic number, and vertex arboricity (Sect. 3.6). In Sect. 4, we consider best degree conditions for π to be forcibly $P_1 \Rightarrow P_2$. Such conditions represent the least amount of degree strength that needs to be added to P_1 to get a sufficient condition for P_2 , and are especially interesting when P_1 is a necessary condition for P_2 . We will focus in Sect. 4 on the situation where P_2 is ‘hamiltonian’, and P_1 belongs to the set { ‘traceable’, ‘2-connected’, ‘1-binding’, ‘contains a 2-factor’, ‘1-tough’ } of prominent necessary conditions for hamiltonicity. In Sect. 5, we consider situations where P_1 does not imply P_2 , but the best degree condition for P_1 implies the best degree condition for P_2 . Such results may be considered improvements in a degree sense over what is true structurally.

2 Framework for Best Monotone Degree Conditions

A graph property P is *increasing (decreasing)* if whenever a graph G has P , so does every edge-augmented supergraph (edge-deleted subgraph) of G . Thus ‘hamiltonian’ and ‘ k -connected’ are increasing properties, while ‘ k -colorable’ is a decreasing property. In the rest of this section, we assume P is an increasing graph property; a completely analogous development can be given for decreasing graph properties.

Given a graph property P , consider a theorem T which declares certain graphical sequences forcibly P , rendering no decision on the remaining graphical sequences. Such a theorem T is called a (*forcibly*) P -theorem. Thus Theorem 1.1 is a hamiltonian theorem.

A P -theorem T is *monotone* if whenever T declares a graphical sequence π forcibly P , T declares every graphical $\pi' \geq \pi$ forcibly P .

A P -theorem T is *P -optimal* (or *optimal*, if P is understood) if every graphical sequence which T does not declare forcibly P is not forcibly P . Note that Theorem 1.1 is not optimal in this sense; e.g., Theorem 1.1 does not declare $\pi = (2k)^{4k+1}$ forcibly hamiltonian for any $k \geq 1$, but all such π are forcibly hamiltonian [48].

A P -theorem T is *P -weakly-optimal* (or *weakly optimal*, if P is understood) if every graphical sequence which T does not declare forcibly P is majorized by a graphical sequence that is not forcibly P . As noted in the previous section, Theorem 1.1 is weakly optimal in this sense.

A monotone P -theorem that is also P -weakly-optimal is a ‘best’ monotone P -theorem in the following sense.

Theorem 2.1 *Let T, T_0 be monotone P -theorems, and let T_0 be P -weakly-optimal. Then any graphical sequence declared forcibly P by T is also declared forcibly P by T_0 .*

Proof Suppose to the contrary that T declares some graphical sequence π forcibly P , but T_0 does not. Since T_0 is P -weakly-optimal, there exists a graphical $\pi' \geq \pi$ which is not forcibly P . But since T is monotone and declares π forcibly P , T must declare $\pi' \geq \pi$ forcibly P , a contradiction. \square

Since Theorem 1.1 is monotone and weakly optimal, Theorem 1.1 is a best monotone hamiltonian theorem. Similarly, Theorem 1.2 is a best monotone k -connected theorem.

By Theorem 2.1, all best monotone P -theorems declare the same set of graphical sequences forcibly P ; we denote this set of graphical sequences by $\text{BM}(P)$. So in terms of their effect, all best monotone P -theorems are equivalent.

In the following three paragraphs, we describe a generic way to construct—at least in principle—best monotone P -theorems. Consider the partially-ordered set G_n whose elements are the graphical sequences of length n , and whose partial-order relation is degree majorization. The graphical sequences of length n that are not forcibly P induce a subposet of G_n , denoted \overline{P}_n . A maximal element in \overline{P}_n is called a (P, n) -sink. The set of all (P, n) -sinks will be denoted $S(P, n)$.

Given a graphical sequence $\pi = (a_1 \leq \dots \leq a_n)$, note that π fails to satisfy the degree condition $C(\pi)$ defined by

$$C(\pi) : d_1 \geq a_1 + 1 \vee \dots \vee d_n \geq a_n + 1;$$

indeed, $C(\pi)$ is the weakest monotone degree condition which ‘blocks’ π (i.e., which π fails to satisfy). We call $C(\pi)$ the *Chvátal-type condition* for π . In the sequel, we will usually write $C(\pi)$ in the more traditional form

$$d_1 \leq a_1 \wedge \dots \wedge d_{j-1} \leq a_{j-1} \Rightarrow d_j \geq a_j + 1 \vee \dots \vee d_n \geq a_n + 1,$$

for some $j < n$.

If $\pi \in \overline{P}_n$, then by definition there exists $\pi' \in S(P, n)$ majorizing π , and thus π fails to satisfy $C(\pi')$. Put differently, if a graphical n -sequence π satisfies the degree condition $\bigwedge_{\pi \in S(P, n)} C(\pi)$, then π is forcibly P ; i.e., the theorem T with degree condition $\bigwedge_{\pi \in S(P, n)} C(\pi)$ is a forcibly P -theorem. But certainly T is monotone, and T is also P -weakly-optimal (if π fails to satisfy the degree condition of T , then π is majorized by some $\pi' \in S(P, n) \subseteq \overline{P}_n$ which is not forcibly P). Thus T is a best monotone P -theorem.

In practical terms, it may be almost impossible to identify the precise set of sinks $S(P, n)$. Fortunately, it is not necessary to make this precise identification to get a best monotone P -theorem: If one can merely identify a set A of non- P graphs on n vertices whose set of degree sequences $\prod(A)$ contains all of $S(P, n)$ (so $S(P, n) \subseteq \prod(A) \subseteq \overline{P}_n$), then—as above—the theorem with degree condition $\bigwedge_{\pi \in \prod(A)} C(\pi)$ will also be a best monotone P -theorem. Although it is typically difficult to find even

such a set A , we will see in the following sections that this is possible for a remarkable number of graph properties.

Finally, we note that $|S(P, n)|$ may be considered the ‘inherent complexity’ of a best monotone theorem on n vertices. More precisely, we have the following.

Theorem 2.2 *When the degree condition of a best monotone P -theorem on n vertices is expressed as a conjunction $\bigwedge C(\pi)$ of P -weakly-optimal Chvátal-type conditions, the conjunction must contain at least $|S(P, n)|$ such conditions.*

Proof It suffices to show that any $\pi \in S(P, n)$ satisfies $\bigwedge_{\pi' \in S(P, n) - \{\pi\}} C(\pi')$; for then the conjunction must contain each Chvátal-type condition $C(\pi)$, as π ranges over $S(P, n)$. Suppose to the contrary that some sink $\pi_a = (a_1 \leq \dots \leq a_n)$ violates $C(\pi_b)$, where $\pi_b = (b_1 \leq \dots \leq b_n)$ is another sink. Then $a_i \leq b_i$, for $1 \leq i \leq n$, and so $\pi_a \leq \pi_b$, contradicting the assumption that π_a is a sink. \square

3 Best Monotone Conditions for Graph Properties P

3.1 Edge-Connectivity

We noted in Sect. 1 that Bondy [21] (see also Boesch [20]) gave a best monotone condition for k -vertex-connectedness (Theorem 1.2). While Theorem 1.2 is also a sufficient condition for k -edge-connectedness, it is not a best monotone condition when $k \geq 2$.

A best monotone condition for 2-edge-connectedness was given in [6].

Theorem 3.1 (Bauer et al. [6]) *Let $\pi = (d_1 \leq \dots \leq d_n)$ be a graphical sequence. If*

$$d_1 \geq 2; \tag{3a}$$

$$d_{i-1} \leq i - 1 \wedge d_i \leq i \Rightarrow d_{n-1} \geq n - i \vee d_n \geq n - i + 1, \text{ for } 3 \leq i < \frac{1}{2}n; \tag{3b}$$

$$d_{n/2} \leq \frac{1}{2}n - 1 \Rightarrow d_{n-2} \geq \frac{1}{2}n \vee d_n \geq \frac{1}{2}n + 1, \text{ if } n \text{ is even,} \tag{3c}$$

then π is forcibly 2-edge-connected.

For the weak optimality of Theorem 3.1, let $G(n, i)$, $i \geq 1$, denote disjoint cliques K_i and K_{n-i} joined by a single edge. If π fails to satisfy (3a), then π is majorized by the degrees of $G(n, 1)$. If π fails to satisfy (3b) for some i , then π is majorized by the degrees of $G(n, i)$. If π fails to satisfy (3c), then π is majorized by the degrees of $G(n, n/2)$. Since none of these graphs is 2-edge-connected, Theorem 3.1 is weakly optimal.

Kriesell [42] and Yin and Guo [62] independently established a best monotone condition for 3-edge-connectedness, which had been conjectured in [6].

Theorem 3.2 (Kriesell [42], Yin and Guo [62]) *Let $\pi = (d_1 \leq \dots \leq d_n)$ be a graphical sequence. If*

$$d_1 \geq 3; \tag{4a}$$

$$d_{i-2} \leq i-1 \wedge d_i \leq i \Rightarrow d_{n-2} \geq n-i \vee d_n \geq n-i+1, \quad \text{for } 4 \leq i < \frac{1}{2}n; \quad (4b)$$

$$d_{i-1} \leq i-1 \wedge d_i \leq i+1 \Rightarrow d_{n-2} \geq n-i \vee d_n \geq n-i+1, \quad \text{for } 4 \leq i < \frac{1}{2}(n-1); \quad (4c)$$

$$d_{i-2} \leq i-1 \wedge d_i \leq i \Rightarrow d_{n-1} \geq n-i \vee d_n \geq n-i+2, \quad \text{for } 4 \leq i < \frac{1}{2}n; \quad (4d)$$

$$d_{n/2} \leq \frac{1}{2}n-1 \Rightarrow d_{n-4} \geq \frac{1}{2}n \vee d_n \geq \frac{1}{2}n+1, \quad \text{if } n \text{ is even}; \quad (4e)$$

$$d_{(n-3)/2} \leq \frac{1}{2}(n-3) \Rightarrow d_{n-3} \geq \frac{1}{2}(n+1) \vee d_n \geq \frac{1}{2}(n+3), \quad \text{if } n \text{ is odd}; \quad (4f)$$

$$d_{n/2} \leq \frac{1}{2}n-1 \Rightarrow d_{n-3} \geq \frac{1}{2}n \vee d_{n-1} \geq \frac{1}{2}n+1 \vee d_n \geq \frac{1}{2}n+2, \quad \text{if } n \text{ is even}; \quad (4g)$$

then π is forcibly 3-edge-connected.

The increase in the number of conditions in Theorem 3.2 when $k = 3$, compared to Theorem 3.1 when $k = 2$, is notable. Indeed, we now prove that the number of weakly optimal Chvátal-type conditions in a best monotone condition for k -edge-connectedness grows superpolynomially in k , for n sufficiently large. A more involved proof of this was given previously by Kriesell [42].

By Theorem 2.2, it suffices to prove the following.

Theorem 3.3 *Let $k \geq 2$, and let $n \geq 4k - 2$ be an even integer. Then there are at least $p(k-1)$ k -edge-connected sinks in G_n , where p denotes the integer partition function, so that $p(r) \sim \frac{1}{4\sqrt{3}r} e^{\pi\sqrt{\frac{2r}{3}}}$ [37].*

Proof Construct a family of $p(k-1)$ edge-maximal not k -edge-connected graphs on n vertices as follows: Begin with disjoint copies X, Y of $K_{n/2}$. Let $a_1 + a_2 + \dots + a_j$ be any partition of $k-1$, and choose vertices $x_1, x_2, \dots, x_j \in X$. Add $k-1$ edges between X and Y so that a_i of these edges are incident at $x_i \in X$, for $1 \leq i \leq j$, and the edges are incident to $k-1$ distinct vertices in Y . Call the resulting graph $G(a_1, \dots, a_j)$, noting that it has minimum degree $\delta(G(a_1, \dots, a_j)) = \frac{1}{2}n - 1$.

To complete the proof, it suffices to show

Claim. $\pi(G(a_1, \dots, a_j))$ is a k -edge-connected sink in G_n .

Proof of the Claim. Let $G = G(a_1, \dots, a_j)$. Suppose to the contrary that $\pi(G)$ is majorized by $\pi(H) \neq \pi(G)$, where H is an edge-maximal not k -edge-connected graph, necessarily consisting of two disjoint cliques X, Y such that $|X| + |Y| = n$ and $|E(X, Y)| = k-1$. We may assume $|X| > |Y|$, so that $|Y| < \frac{1}{2}n$ (if $|X| = |Y|$, then $\pi(G)$ and $\pi(H)$ have the same degree sum and $\pi(H)$ could not majorize $\pi(G)$).

We consider two cases.

Case 1. $|Y| \geq k$

Since $|Y| > k-1 = |E(X, Y)|$, some vertex $y \in Y$ is not incident to any edge in $E(X, Y)$. So $d_H(y) \leq |Y| - 1 < \frac{1}{2}n - 1 = \delta(G)$, and $\pi(H)$ would not majorize $\pi(G)$.

Case 2. $|Y| \leq k-1$

Then any $y \in Y$ satisfies $d_H(y) \leq |E(X, Y)| + (|Y| - 1) \leq (k-1) + (k-2) = 2k-3 = \frac{1}{2}(4k) - 3 \leq \frac{1}{2}(n+2) - 3 = \frac{1}{2}n - 2 < \frac{1}{2}n - 1 = \delta(G)$. Again, π would not majorize $\pi(G)$.

This proves the Claim, and completes the proof of Theorem 3.3. \square

In light of Theorem 3.3, it would be desirable to have a simple, though not best monotone, condition for k -edge-connectedness that is at least better than Theorem 1.2 as a sufficient condition for k -edge-connectedness. The following such condition was given in [6].

Theorem 3.4 (Bauer et al. [6]) *Let $\pi = (d_1 \leq \dots \leq d_n)$ be a graphical sequence and let $k \geq 1$ be an integer. If*

$$d_1 \geq k; \tag{5a}$$

$$d_{i-k+1} \leq i - 1 \wedge d_i \leq i+k - 2 \Rightarrow d_n \geq n - i + k - 1, \text{ for } k + 1 \leq i \leq \lfloor \frac{1}{2}n \rfloor, \tag{5b}$$

then π is forcibly k -edge-connected.

A sufficient condition for k -edge-connectedness stronger than Theorem 3.4 was given by Yin and Guo [62], though their degree condition is substantially more involved than conditions (5a) and (5b). We refer the reader to [62] for details.

3.2 Binding Number

The concept of the binding number of a graph was first used by Anderson [2, p. 185], and then given its present definition by Woodall [59].

Given $S \subseteq V(G)$, let $N(S) \subseteq V(G)$ denote the neighbor set of S . Let

$$\mathcal{S} = \{ S \subseteq V(G) \mid S \neq \emptyset \text{ and } N(S) \neq V(G) \}.$$

The *binding number* of G , denoted $\text{bind}(G)$, is defined by

$$\text{bind}(G) = \min_{S \in \mathcal{S}} \frac{|N(S)|}{|S|}.$$

In particular, $\text{bind}(K_n) = n - 1$, for $n \geq 1$. A set $S \in \mathcal{S}$ for which the above minimum is attained is called a *binding set* of G . For $b \geq 0$, we call G *b -binding* if $\text{bind}(G) \geq b$. Cunningham [32] has shown that computing $\text{bind}(G)$ is tractable.

A number of theorems in the literature guarantee that a graph G has a certain property if $\text{bind}(G)$ is appropriately bounded from below. The following three theorems exhibit the best possible lower bound on $\text{bind}(G)$ to guarantee that G has the indicated property.

Theorem 3.5 (Anderson [2]) *If $|V(G)|$ is even and $\text{bind}(G) \geq 4/3$, then G contains a 1-factor.*

Theorem 3.6 (Woodall [59,60]) *If $\text{bind}(G) \geq 3/2$, then G is hamiltonian.*

Theorem 3.7 (Shi [52]) *If $\text{bind}(G) \geq 3/2$, then G contains a cycle of length l , for $3 \leq l \leq |V(G)|$.*

Other graph properties which are guaranteed by lower bounds on binding number include the existence of an f -factor [34, 41, 61], the existence of a k -clique [40, 46], and k -extendability [29, 51].

In [10], a best monotone condition was given for a graph to be b -binding, first for $0 < b \leq 1$ and then for $b \geq 1$.

Theorem 3.8 (Bauer et al. [10]) *Let $0 < b \leq 1$ and let $\pi = (d_1 \leq \dots \leq d_n)$ be a graphical sequence, with $n \geq \lceil b + 1 \rceil = 2$. If*

$$d_i \leq \lceil bi \rceil - 1 \Rightarrow d_{n-\lceil bi \rceil+1} \geq n - i, \quad \text{for } 1 \leq i \leq \left\lfloor \frac{n}{b+1} \right\rfloor; \quad (6a)$$

$$d_{\lfloor n/(b+1) \rfloor+1} \geq n - \left\lfloor \frac{n}{b+1} \right\rfloor, \quad (6b)$$

then π is forcibly b -binding.

If π fails to satisfy (6a) for some i , then π is majorized by the degrees of $K_{\lceil bi \rceil-1} + (K_{n-i-\lceil bi \rceil+1} \cup \overline{K_i})$. If π fails to satisfy (6b), then π is majorized by the degrees of $K_{n-\lfloor n/(b+1) \rfloor-1} + \overline{K_{\lfloor n/(b+1) \rfloor+1}}$. Since neither graph is b -binding, Theorem 3.8 is weakly optimal.

Theorem 3.9 (Bauer et al. [10]) *Let $b \geq 1$, and let $\pi = (d_1 \leq \dots \leq d_n)$ be a graphical sequence, with $n \geq \lceil b + 1 \rceil$. If*

$$d_i \leq n - \left\lfloor \frac{n-i}{b} \right\rfloor - 1 \Rightarrow d_{\lfloor (n-i)/b \rfloor+1} \geq n - i, \quad \text{for } 1 \leq i \leq \left\lfloor \frac{n}{b+1} \right\rfloor; \quad (7a)$$

$$d_{\lfloor n/(b+1) \rfloor+1} \geq n - \left\lfloor \frac{n}{b+1} \right\rfloor, \quad (7b)$$

then π is forcibly b -binding.

If π fails to satisfy (7a) for some i , then π is majorized by the degrees of $K_{n-\lfloor (n-i)/b \rfloor-1} + (K_{\lfloor (n-i)/b \rfloor-i+1} \cup \overline{K_i})$. If π fails to satisfy (7b) (which is the same as (6b)), then π is majorized by the degrees of $K_{n-\lfloor n/(b+1) \rfloor-1} + \overline{K_{\lfloor n/(b+1) \rfloor+1}}$. Since neither graph is b -binding, Theorem 3.9 is weakly optimal.

3.3 Toughness

The concept of toughness in graphs was introduced by Chvátal in [31]. Let $\omega(G)$ denote the number of components in a graph G . For $t \geq 0$, we call G t -tough if $t \cdot \omega(G - X) \leq |X|$, for every $X \subseteq V(G)$ with $\omega(G - X) \geq 2$. The toughness of G , denoted $\tau(G)$, is the maximum $t \geq 0$ such that G is t -tough (taking $\tau(K_n) = n - 1$, for $n \geq 1$). Thus if G is not complete, then

$$\tau(G) = \min \left\{ \frac{|X|}{\omega(G - X)} \mid X \subseteq V(G) \text{ with } \omega(G - X) \geq 2 \right\}.$$

In [7], it was shown that computing $\tau(G)$ is NP-hard.

Toughness has been especially prominent in connection with the existence of long cycles in graphs. Indeed, it was a longstanding conjecture that every 2-tough graph is hamiltonian. But Bauer, Broersma, and Veldman [5] disproved this conjecture by constructing $(\frac{9}{4} - \epsilon)$ -tough nonhamiltonian graphs. Unfortunately, the methods in [5] do not extend to higher levels of toughness, and it remains an open question whether there exists a constant $t_0 \geq 9/4$ such that every t_0 -tough graph is hamiltonian.

In [4], a best monotone condition for a graph to be t -tough was given for $t \geq 1$.

Theorem 3.10 (Bauer et al. [4]) *Let $t \geq 1$, $n \geq \lceil t \rceil + 2$, and $\pi = (d_1 \leq \dots \leq d_n)$ be a graphical sequence. If*

$$d_{\lfloor i/t \rfloor} \leq i \Rightarrow d_{n-i} \geq n - \lfloor i/t \rfloor, \text{ for } t \leq i < \frac{tn}{t+1}, \tag{8}$$

then π is forcibly t -tough.

If π fails to satisfy (8) for some i , then π is majorized by the degrees of $K_i + (K_{\lfloor i/t \rfloor} \cup K_{n-i-\lfloor i/t \rfloor})$ which is not t -tough. Thus Theorem 3.10 is weakly optimal. Note also that condition (8) of Theorem 3.10 reduces to condition (1) of Theorem 1.1 when $t = 1$.

By Theorem 3.10, a best monotone t -tough condition on degree sequences of length n requires fewer than n weakly optimal Chvátal-type conditions, for $t \geq 1$. But this changes markedly as $t \rightarrow 0$. In particular, for any integer $k \geq 2$, a best monotone $\frac{1}{k}$ -tough condition on degree sequences of length n requires at least $f(k)n$ weakly optimal, Chvátal-type conditions, where $f(k)$ grows superpolynomially as $k \rightarrow \infty$. This is implied by Theorem 2.2 and the following result [4, Lemma 4.2].

Theorem 3.11 (Bauer et al. [4]) *Let $n = m(k + 1)$, where $k \geq 2$ and $m \geq 9$ are integers. Then the number of $\frac{1}{k}$ -tough sinks in G_n is at least $\frac{p(k-1)}{5(k+1)}n$, where p is the integer partition function.*

The superpolynomial growth in the complexity of a best monotone t -tough theorem as $t \rightarrow 0$ suggests the desirability of having a simple t -tough theorem, for $0 < t < 1$. The following was given in [4].

Theorem 3.12 (Bauer et al. [4]) *Let $0 < t < 1$, $n \geq \lfloor 1/t \rfloor + 2$, and $\pi = (d_1 \leq \dots \leq d_n)$ be a graphical sequence. If*

$$d_i \leq i - \lfloor 1/t \rfloor + 1 \Rightarrow d_{n-i+\lfloor 1/t \rfloor-1} \geq n - i, \text{ for } \lfloor 1/t \rfloor \leq i < \frac{1}{2}(n + \lfloor 1/t \rfloor - 1); \tag{9a}$$

$$d_i \leq i - 1 \Rightarrow d_n \geq n - i, \text{ for } 1 \leq i \leq \frac{1}{2}n, \tag{9b}$$

then π is forcibly t -tough.

3.4 Factors

The *deficiency* of a graph G , denoted $\text{def}(G)$, is the number of vertices unmatched under a maximum matching in G . In particular, G has a 1-factor if and only if $\text{def}(G) = 0$. We call G β -deficient if $\text{def}(G) \leq \beta$.

In [44] (see also [22]), a best monotone condition was given for a graph to be β -deficient.

Theorem 3.13 (Las Vergnas [44]) *Let $\pi = (d_1 \leq \dots \leq d_n)$ be a graphical sequence, and let $0 \leq \beta \leq n$ with $n \equiv \beta \pmod{2}$. If*

$$d_{i+1} \leq i - \beta \Rightarrow d_{n+\beta-i} \geq n - i - 1, \quad \text{for } 1 \leq i \leq \frac{1}{2}(n + \beta - 2), \quad (10)$$

then π is forcibly β -deficient.

If π fails to satisfy (10) for some i , then π is majorized by the degrees of $K_{i-\beta} + (\overline{K_{i+1}} \cup K_{n-2i+\beta-1})$, which is not β -deficient. Thus Theorem 3.13 is weakly optimal.

Taking $\beta = 0$ in Theorem 3.13, we obtain a best monotone condition for a graph to contain a 1-factor.

In [3], a best monotone condition was given for a graph to contain a 2-factor. We now give a slightly better form of the condition.

Theorem 3.14 (Bauer et al. [3]) *Let $\pi = (d_1 \leq \dots \leq d_n)$ be a graphical sequence, with $n \geq 3$. If (setting $d_0 = 0$)*

$$d_{(n+1)/2} \geq \frac{1}{2}(n + 1), \quad \text{if } n \text{ is odd}; \quad (11a)$$

$$d_{n/2-1} \geq \frac{1}{2}n \vee d_{n/2+1} \geq \frac{1}{2}n + 1, \quad \text{if } n \text{ is even}; \quad (11b)$$

$$d_i \leq i \wedge d_{i+1} \leq i + 1 \Rightarrow d_{n-i-1} \geq n - i - 1 \vee d_{n-i} \geq n - i, \\ \text{for } 0 \leq i \leq \frac{1}{2}(n - 3); \quad (11c)$$

$$d_{i-1} \leq i \wedge d_{i+2} \leq i + 1 \Rightarrow d_{n-i-3} \geq n - i - 2 \vee d_{n-i} \geq n - i - 1, \\ \text{for } 1 \leq i \leq \frac{1}{2}(n - 5), \quad (11d)$$

then π forcibly contains a 2-factor.

If π fails to satisfy (11a), then π is majorized by the degrees of $K_{(n-1)/2} + \overline{K_{(n+1)/2}}$. If π fails to satisfy (11b), then π is majorized by the degrees of $K_{(n-2)/2} + (\overline{K_{(n-2)/2}} \cup K_2)$. If π fails to satisfy (11c) for some i , then π is majorized by the degrees of $K_i + (\overline{K_{i+1}} \cup K_{n-2i-1})$ with an edge added joining $\overline{K_{i+1}}$ and K_{n-2i-1} . If π fails to satisfy (11d) for some i , then π is majorized by the degrees of $K_i + (\overline{K_{i+2}} \cup K_{n-2i-2})$ with three independent edges joining $\overline{K_{i+2}}$ and K_{n-2i-2} . Since none of these graphs contains a 2-factor, Theorem 3.14 is weakly optimal.

We conjecture that the number of weakly optimal Chvátal-type conditions in a best monotone condition for a graph to contain a k -factor grows rapidly with k . More precisely, we put forth the following (cf. Theorem 3.3 and Theorem 3.11).

Conjecture 3.15 *Let $f(k, n)$ denote the number of k -factor sinks in G_n . Then there exist $a, b > 0$ such that if $n \geq ak + b$, then $f(k, n)$ grows superpolynomially in k .*

3.5 Paths and Cycles

In Sect. 1, we noted that Theorem 1.1 gives a best monotone condition for hamiltonicity.

A graph G is k -hamiltonian if for all $X \subseteq V(G)$ with $|X| \leq k$, the induced subgraph $\langle V(G) - X \rangle$ is hamiltonian. Thus ‘0-hamiltonian’ is the same as ‘hamiltonian’. A best monotone condition for k -hamiltonicity was first given in [30] (although the form in which we present it is from [22] and [45]). Of course, Theorem 1.1 is the special case $k = 0$.

Theorem 3.16 (Chvátal [30]) *Let $\pi = (d_1 \leq \dots \leq d_n)$ be a graphical sequence with $n \geq 3$, and let $0 \leq k \leq n - 3$. If*

$$d_i \leq i + k \Rightarrow d_{n-i-k} \geq n - i, \text{ for } 1 \leq i < \frac{1}{2}(n - k), \tag{12}$$

then π is forcibly k -hamiltonian.

If π fails to satisfy (12) for some i , then π is majorized by the degrees of $K_{i+k} + (\overline{K_i} \cup K_{n-2i-k})$, which is not k -hamiltonian. Thus Theorem 3.16 is weakly optimal.

A graph is *traceable* if it contains a hamiltonian path. A best monotone condition for traceability was given in [30]. More generally, G is k -path-coverable if $V(G)$ can be covered by k or fewer vertex-disjoint paths. In particular, ‘1-path-coverable’ is the same as ‘traceable’. A best monotone condition for k -path-coverability was obtained independently in [22] and [45].

Theorem 3.17 (Bondy and Chvátal [22], Lesniak [45]) *Let $\pi = (d_1 \leq \dots \leq d_n)$ be a graphical sequence and let $k \geq 1$. If*

$$d_{i+k} \leq i \Rightarrow d_{n-i} \geq n - i - k, \text{ for } 1 \leq i < \frac{1}{2}(n - k), \tag{13}$$

then π is forcibly k -path-coverable.

If π fails to satisfy (13) for some i , then π is majorized by the degrees of $K_i + (\overline{K_{i+k}} \cup K_{n-2i-k})$, which is not k -path-coverable (adding k complete vertices to a graph which is k -path coverable results in a hamiltonian graph, while adding k complete vertices to the above graph results in a graph which is not even 1-tough). Thus Theorem 3.17 is weakly optimal.

A graph is *hamiltonian-connected* if every pair of vertices is joined by a hamiltonian path. A best monotone condition for hamiltonian-connectedness was given in [19, Chapter 10, Theorem 12].

Theorem 3.18 (Berge [19]) *Let $\pi = (d_1 \leq \dots \leq d_n)$ be a graphical sequence with $n \geq 4$. If*

$$d_{i-1} \leq i \Rightarrow d_{n-i} \geq n - i + 1, \text{ for } 2 \leq i < \frac{1}{2}(n + 1), \tag{14}$$

then π is forcibly hamiltonian-connected.

If π fails to satisfy (14) for some i , then π is majorized by the degrees of $K_i + (\overline{K_{i-1}} \cup K_{n-2i+1})$, which is not hamiltonian-connected (there is no hamiltonian path joining two vertices in K_i). Thus Theorem 3.18 is weakly optimal.

A graph G is k -edge-hamiltonian if any collection of vertex-disjoint paths with at most k edges altogether belong to a hamiltonian cycle in G . A best monotone condition for k -edge-hamiltonicity was given in [43] (see also [19, Chapter 10, Theorem 8]).

Theorem 3.19 (Kronk [43]) *Let $\pi = (d_1 \leq \dots \leq d_n)$ be a graphical sequence with $n \geq 3$, and let $0 \leq k \leq n - 3$. If*

$$d_{i-k} \leq i \Rightarrow d_{n-i} \geq n - i + k, \text{ for } k + 1 \leq i < \frac{1}{2}(n + k), \tag{15}$$

then π is forcibly k -edge-hamiltonian.

If π fails to satisfy (15) for some i , then π is majorized by the degrees of $K_i + (\overline{K_{i-k}} \cup K_{n-2i+k})$, which is not k -edge-hamiltonian (consider a path in K_i with k edges). Thus Theorem 3.19 is weakly optimal.

A graph G is *pancyclic* if it contains an l -cycle for any l such that $3 \leq l \leq |V(G)|$. We have the following best monotone condition for π to be forcibly pancyclic.

Theorem 3.20 *Let $\pi = (d_1 \leq \dots \leq d_n)$ be a graphical sequence with $n \geq 3$. If*

$$d_i \leq i \Rightarrow d_{n-i} \geq n - i, \text{ for } 1 \leq i < \frac{1}{2}n; \tag{16a}$$

$$d_n \geq \frac{1}{2}n + 1, \text{ if } n \text{ is even,} \tag{16b}$$

then π is forcibly pancyclic.

Proof In [17], it is shown that if π satisfies (16a), then G is pancyclic or bipartite. But if G is bipartite, then, since G is hamiltonian by (16a) and Theorem 1.1, n is even and both bipartition sets have $\frac{1}{2}n$ vertices. Thus $d_n \leq \frac{1}{2}n$, which contradicts (16b). \square

If π fails to satisfy (16a) for some i , then π is majorized by the degrees of $K_i + (\overline{K_i} \cup K_{n-2i})$, which has no n -cycle. If π fails to satisfy (16b), then π is majorized by the degrees of $K_{n/2, n/2}$, which has no odd length cycles. Thus Theorem 3.20 is weakly optimal.

3.6 Independence Number, Clique Number, Chromatic Number, and Vertex Arboricity

We consider best monotone conditions for a graphical sequence π to be forcibly $p(G) \leq k$ or forcibly $p(G) \geq k$, where p denotes any of the graph parameters α (independence number), ω (clique number), χ (chromatic number), or a (vertex arboricity). Note that if $p \in \{\omega, \chi, a\}$, then $p(G) \leq k$ is a decreasing property and $p(G) \geq k$ is an increasing property (so that we seek upper bounds on π in the first case and lower bounds in the second); while if $p = \alpha$ then it is the other way around.

We begin with best monotone conditions for upper bounds $p(G) \leq k$. We consider first best monotone conditions for $\alpha(G) \leq k$ and $\omega(G) \leq k$. Since $\omega(G) \leq k \Leftrightarrow \alpha(\overline{G}) \leq k$, the development is analogous for α and ω ; we do only α .

Theorem 3.21 *Let $\pi = (d_1 \leq \dots \leq d_n)$ be a graphical sequence, and $k \geq 1$ an integer. If*

$$d_{k+1} \geq n - k, \tag{17}$$

then π is forcibly $\alpha(G) \leq k$.

Proof Suppose π satisfies (17), but has a realization G with $\alpha(G) \geq k + 1$. If $S \subseteq V(G)$ is an independent set with $|S| \geq k + 1$, then each vertex in S has degree at most $n - k - 1$, and thus $d_{k+1} \leq n - k - 1$, contradicting (17). \square

If π fails to satisfy (17), then π is majorized by the degrees of $G' = \overline{K_{k+1}} + K_{n-k-1}$, with $\alpha(G') = k + 1 > k$. Thus Theorem 3.21 is weakly optimal.

We also note that the optimal condition for $\alpha(G) \leq k$ is tractable. We begin with the following result of Rao [50].

Theorem 3.22 (Rao [50]) *A graphical sequence π has a realization G with $\alpha(G) \geq k$ if and only if π has a realization in which vertices with the k smallest degrees form an independent set.*

Using Theorem 3.22, it is easy to determine whether or not $\pi = (d_1 \leq \dots \leq d_n)$ is forcibly $\alpha(G) \leq k$: Iteratively consider $k = 1, 2, \dots, n - 1$. To decide if π has a realization with $k + 1$ independent vertices, form the graph $H = \overline{K_{k+1}} + K_{n-k-1}$, letting v_1, \dots, v_{k+1} (resp., v_{k+2}, \dots, v_n) denote the vertices of $\overline{K_{k+1}}$ (resp., K_{n-k-1}). Assign degree d_i to v_i for $1 \leq i \leq n$, and determine if H contains a subgraph H' with the assigned degrees. If so, then π has a realization G with $\alpha(G) \geq k + 1$, and π is not forcibly $\alpha(G) \leq k$. Otherwise, by Theorem 3.22, π is forcibly $\alpha(G) \leq k$. Tutte [54] proved the existence of H' is equivalent to the existence of a 1-factor in a graph that can be efficiently constructed from H and d_1, \dots, d_n .

Structural conditions guaranteeing $\chi(G) \leq k$ (that G is k -colorable) have a long and rich history [23,35,53,58]. Regarding degree conditions, we first note the trivial bound $\chi(G) \leq \Delta(G) + 1$, and thus

Theorem 3.23 *The graphical sequence $\pi = (d_1 \leq \dots \leq d_n)$ is forcibly $\chi(G) \leq d_n + 1$.*

A best monotone condition for $\chi(G) \leq k$ was given by Welsh and Powell [56].

Theorem 3.24 (Welsh and Powell [56]) *Let $\pi = (d_1 \leq \dots \leq d_n)$ be a graphical sequence. Then π is forcibly*

$$\chi(G) \leq \max_{1 \leq j \leq n} \min\{n - j + 1, d_j + 1\}.$$

Reexpressing Theorem 3.24 with an equivalent Chvátal-type degree condition, we have the following.

Theorem 3.25 *Let $\pi = (d_1 \leq \dots \leq d_n)$ be a graphical sequence, and let $1 \leq k \leq n$. If (setting $d_0 = 0$)*

$$d_{n-k} \leq k - 1, \tag{18}$$

then π is forcibly $\chi(G) \leq k$.

If π fails to satisfy (18), then π is minorized by the vertex degrees of $G = K_{k+1} \cup \overline{K_{n-k-1}}$, with $\chi(G) = k + 1 > k$. Thus Theorem 3.25 is weakly optimal.

Analogous to the bound $\chi(G) \leq \Delta(G) + 1$, we have $a(G) \leq \lfloor \frac{1}{2}\Delta(G) \rfloor + 1$ [27], and thus we get the following.

Theorem 3.26 *Let $\pi = (d_1 \leq \dots \leq d_n)$ be a graphical sequence. Then π is forcibly $a(G) \leq \lfloor \frac{1}{2}d_n \rfloor + 1$.*

A best monotone condition for $a(G) \leq k$ was given in [36]; it is analogous to Theorem 3.24.

Theorem 3.27 (Hakimi and Schmeichel [36]) *Let $\pi = (d_1 \leq \dots \leq d_n)$ be a graphical sequence. Then π is forcibly*

$$a(G) \leq \max_{1 \leq j \leq n} \min \left\{ \left\lceil \frac{1}{2}(n - j + 1) \right\rceil, \left\lceil \frac{1}{2}(d_j + 1) \right\rceil \right\}.$$

Reexpressing Theorem 3.27 with an equivalent Chvátal-type degree condition, we have the following.

Theorem 3.28 *Let $\pi = (d_1 \leq \dots \leq d_n)$ be a graphical sequence, and let $1 \leq k \leq \frac{1}{2}n$. If*

$$d_{n-2k} \leq 2k - 1, \tag{19}$$

then π is forcibly $a(G) \leq k$.

If π fails to satisfy (19), then π is minorized by the vertex degrees of $G = K_{2k+1} \cup \overline{K_{n-2k-1}}$, with $a(G) = k + 1 > k$. Thus Theorem 3.28 is weakly optimal.

We turn next to best monotone conditions for lower bounds $p(G) \geq k$. The most prominent degree condition for $\alpha(G) \geq k$, although not best monotone, is independently due to Caro [24] and Wei [55]. An elegant probabilistic proof appears in [1, p. 81] (see also [57, p. 428]).

Theorem 3.29 (Caro [24], Wei [55]) *Let $\pi = (d_1 \leq \dots \leq d_n)$ be a graphical sequence. Then π is forcibly $\alpha(G) \geq \sum_{j=1}^n \frac{1}{d_j + 1}$.*

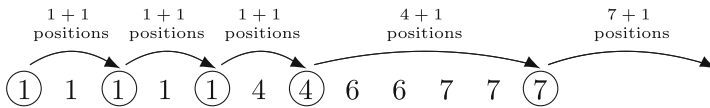
A best monotone condition for $\alpha(G) \geq k$ was given by Murphy [47]. Let $\pi = (d_1 \leq \dots \leq d_n)$ be a graphical sequence. Define $f : \mathbb{Z}^+ \rightarrow \{d_1, d_2, \dots, d_n, \infty\}$ recursively as follows: Set $f(1) = d_1$. If $f(i) = d_j$, then set

$$f(i + 1) = \begin{cases} d_{j+f(i)+1}, & \text{if } j + f(i) + 1 \leq n; \\ \infty, & \text{otherwise;} \end{cases}$$

while if $f(i) = \infty$, then $f(i + 1) = \infty$.

Murphy’s condition is the following.

Theorem 3.30 (Murphy [47]) *Let $\pi = (d_1 \leq \dots \leq d_n)$ be a graphical sequence. Then π is forcibly $\alpha(G) \geq \max\{i \in \mathbb{Z}^+ \mid f(i) < \infty\}$.*



Example If $\pi = 1^5 4^2 6^2 7^3$, then $f(1) = d_1 = 1, f(2) = d_3 = 1, f(3) = d_5 = 1, f(4) = d_7 = 4, f(5) = d_{12} = 7$, and $f(6) = f(7) = \dots = \infty$. The calculation of the $f(i)$ can be nicely visualized, as shown above.

So by Theorem 3.30, π is forcibly $\alpha(G) \geq 5$ (the number of circled vertices). Δ

By comparison, Theorem 3.29 guarantees only that π in the above example is forcibly $\alpha(G) \geq 4$. Indeed, Theorem 3.30 can be arbitrarily better than Theorem 3.29: For the graphical sequence $\pi = 1^{l+1} 2^{l+1} 3^{l+1} \dots d^d$ with $d \equiv 0 \pmod{4}$, Theorem 3.29 (resp., Theorem 3.30) guarantees that π is forcibly $\alpha(G) \geq \sum_{i=1}^d (1 - \frac{1}{i+1}) \sim d - \ln d$ (resp., $\alpha(G) \geq d$).

To see that Theorem 3.30 is weakly optimal for $\alpha(G) \geq k$, suppose Theorem 3.30 fails to guarantee that π is forcibly $\alpha(G) \geq k$. Consider the degree sequence $\pi' = f(1)^{f(1)+1} f(2)^{f(2)+1} \dots f(k-2)^{f(k-2)+1} l^{l+1}$, where l denotes the number of degrees in π with index greater than $f(k-1)$. Note that $l \leq f(k-1)$, since $l \geq f(k-1) + 1$ implies $f(k) < \infty$, contradicting that Theorem 3.30 does not declare π forcibly $\alpha(G) \geq k$. Thus π' minorizes π . But π' has realization $G' = K_{f(1)+1} \cup \dots \cup K_{f(k-2)+1} \cup K_{l+1}$ consisting of $k-1$ disjoint cliques, with $\alpha(G') = k-1 < k$. Thus Theorem 3.30 is weakly optimal for $\alpha(G) \geq k$.

Using Theorem 3.30, we can easily obtain best monotone conditions for $\omega(G) \geq k$ and $\chi(G) \geq k$. Let $\pi = (d_1 \leq \dots \leq d_n)$ be a graphical sequence, and define $g(\pi) = \max\{i \in \mathbb{Z}^+ \mid f(i) < \infty\}$ as above (so that π is forcibly $\alpha(G) \geq g(\pi)$). Define $h : \{\text{Graphical Sequences}\} \rightarrow \mathbb{Z}^+$ by $h(\pi) = g(\bar{\pi})$, where $\bar{\pi} = ((n-1) - d_n \leq \dots \leq (n-1) - d_1)$ is the degree sequence complementary to π . If G, \bar{G} are arbitrary realizations of $\pi, \bar{\pi}$, then $h(\pi) = g(\bar{\pi}) \leq \alpha(\bar{G}) = \omega(G) \leq \chi(G)$. Since g is monotone decreasing, h is monotone increasing. So to prove that $\omega(G) \geq h(\pi)$ and $\chi(G) \geq h(\pi)$ are best monotone lower bounds, it suffices to show these lower bounds are weakly optimal. But if $h(\pi) = g(\bar{\pi}) \leq k-1$, then as above there exists a $\bar{\pi}'$ minorizing $\bar{\pi}$ with a realization \bar{G}' consisting of $k-1$ disjoint cliques. So $\pi' \geq \pi$ has a realization $G' = \bar{G}'$ that is a complete $(k-1)$ -partite graph. Thus $\omega(G'), \chi(G') = k-1 < k$, and the above lower bounds for ω and χ are weakly optimal.

Finally, if P is the property $a(G) \geq k$, it was proved in [15] that $|S(P, n)|$ grows superpolynomially in n . Indeed, $|S(P, n)| \geq D\left(\lfloor \frac{n}{k-1} \rfloor\right)$ if $k-1$ divides n , where $D(m)$ denotes the number of different degree sequences of unlabeled m -vertex trees. We refer the reader to [15] for details. But $D(m) = p(m-2)$ [19, Chapter 6, Theorem 8] grows superpolynomially in m , where p is the integer partition function. Thus, any best monotone condition for P will be inherently complex.

4 Best Monotone Degree Conditions for Implications $P_1 \Rightarrow P_2$

In this section, we consider best monotone degree conditions for $P_1 \Rightarrow P_2$, where P_1, P_2 are monotone increasing graph properties. Conditions of this type were first

considered in [16]. A framework for such considerations is given in Sect. 2 upon substituting $P_1 \Rightarrow P_2$ for P throughout. Best monotone $P_1 \Rightarrow P_2$ conditions are particularly interesting when P_1 is a necessary condition for P_2 , since they provide the minimum degree strength which needs to be added to P_1 to get a sufficient condition for P_2 .

In this section, we will focus on best monotone degree conditions for $P_1 \Rightarrow P_2$ —some proved, some conjectured—when P_2 is ‘hamiltonian’, and P_1 belongs to the set {‘traceable’, ‘2-connected’, ‘1-binding’, ‘contains a 2-factor’, ‘1-tough’} of well-known necessary conditions for hamiltonicity.

We begin with two results which are essentially immediate corollaries of Theorem 1.1.

Theorem 4.1 *Let $\pi = (d_1 \leq \dots \leq d_n)$ be a graphical sequence, with $n \geq 3$. If*

$$d_i \leq i \Rightarrow d_{n-i} \geq n - i, \quad \text{for } 1 \leq i \leq \frac{1}{2}(n - 1), \quad (20)$$

then every traceable realization of π is hamiltonian.

Note that (20) is the same degree condition as in Theorem 1.1. If π fails to satisfy (20) for some i , then π is majorized by the degrees of $K_i + (\overline{K}_i \cup K_{n-2i})$, which is traceable and nonhamiltonian. Thus Theorem 4.1 is weakly optimal for traceable \Rightarrow hamiltonian.

Theorem 4.2 *Let $\pi = (d_1 \leq \dots \leq d_n)$ be a graphical sequence, with $n \geq 3$. If*

$$d_i \leq i \Rightarrow d_{n-i} \geq n - i, \quad \text{for } 2 \leq i \leq \frac{1}{2}(n - 1), \quad (21)$$

then every 2-connected realization of π is hamiltonian.

If π fails to satisfy (21) for some i , then π is majorized by the degree sequence of $K_i + (\overline{K}_i \cup K_{n-2i})$, which is 2-connected (since $i \geq 2$) and nonhamiltonian. Thus Theorem 4.2 is weakly optimal for 2-connected \Rightarrow hamiltonian.

We have the following best monotone condition for 1-binding \Rightarrow hamiltonian [16].

Theorem 4.3 (Bauer et al. [16]) *Let $\pi = (d_1 \leq \dots \leq d_n)$ be a graphical sequence, with $n \geq 3$. If (setting $d_0 = 0$)*

$$d_i \leq i \Rightarrow d_{n-i} \geq n - i, \quad \text{for } 1 \leq i \leq \frac{1}{2}(n - 2); \quad (22a)$$

$$d_{i-1} \leq i \Rightarrow d_{n-i} \geq \frac{1}{2}(n + 1), \quad \text{for } 1 \leq i \leq \frac{1}{2}(n - 3), \text{ if } n \text{ is odd,} \quad (22b)$$

then every 1-binding realization of π is hamiltonian.

If π fails to satisfy (22a) (resp., (22b)) for some i , then π is majorized by the degrees of $K_i + (\overline{K}_i \cup K_{n-2i})$ (resp., $K_i + (\overline{K}_{i-1} \cup 2K_{(n+1)/2-i})$), which are each 1-binding and nonhamiltonian. Thus Theorem 4.3 is weakly optimal for 1-binding \Rightarrow hamiltonian.

Since 1-binding is also a necessary condition for a graph to contain a 1-factor, the following is of some interest [14].

Theorem 4.4 (Bauer et al. [14]) *Let $\pi = (d_1 \leq \dots \leq d_n)$ be a graphical sequence, with n even. If (setting $d_0 = 0$)*

$$d_i \leq i \wedge d_{i+2j+1} \leq i + 2j \Rightarrow d_{n-i} \geq n - (i + 2j + 1),$$

$$\text{for } 1 \leq i \leq \frac{1}{2}(n - 6) \text{ and } 1 \leq j \leq \frac{1}{4}(n - 2i - 2); \tag{23a}$$

$$d_{n/2-5} \geq \frac{1}{2}n - 3 \vee d_{n/2+4} \geq \frac{1}{2}n - 1, \text{ if } n \geq 10, \tag{23b}$$

then every 1-binding realization of π contains a 1-factor.

If π fails to satisfy (23a) for some i, j (resp., (23b)), then π is majorized by the degree sequence of $K_i + (\overline{K_i} \cup K_{2j+1} \cup K_{n-2i-2j-1})$ (resp., $K_{n/2-4} + (\overline{K_{n/2-5}} \cup 3K_3)$) which are each 1-binding without a 1-factor. Thus Theorem 4.4 is weakly optimal for 1-binding \Rightarrow 1-factor.

For $1 < b < 3/2$, a best monotone degree condition for b -binding \Rightarrow hamiltonian is not currently known. An asymptotically best minimum degree condition for b -binding \Rightarrow hamiltonian when $1 < b < 3/2$, namely $\delta(G) \geq \left(\frac{2-b}{3-b}\right)n$, was established in [18]. A somewhat involved best monotone degree condition for b -binding \Rightarrow 1-tough was given in [16], where it was conjectured that this condition is also a best monotone degree condition for b -binding \Rightarrow hamiltonian. We refer the reader to [16] for details.

A best monotone degree condition for 2-factor \Rightarrow hamiltonian is also not currently known. As the graph $K_1 + 2K_{(n-1)/2}$ shows, a best minimum degree condition for 2-factor \Rightarrow hamiltonian is Dirac’s hamiltonian condition $\delta(G) \geq \frac{1}{2}n$. On the other hand, we have the following best monotone degree condition for 2-factor \Rightarrow 1-tough [13].

Theorem 4.5 (Bauer et al. [13]) *Let $\pi = (d_1 \leq \dots \leq d_n)$ be a graphical sequence, with $n \geq 3$. If (setting $d_0 = 0$)*

$$d_i \leq i \Rightarrow d_{n-i} \geq n - i, \text{ for } 1 \leq i \leq \frac{1}{2}(n - 3); \tag{24a}$$

$$d_{i-1} \leq i \Rightarrow d_{n-i} \geq \frac{1}{2}(n + 1), \text{ for } 1 \leq i \leq \frac{1}{2}(n - 5), \text{ if } n \text{ is odd}; \tag{24b}$$

$$d_{i-1} \leq i \Rightarrow d_{n/2-1} \geq \frac{1}{2}n \vee d_{n-i} \geq \frac{1}{2}n + 1,$$

$$\text{for } 1 \leq i \leq \frac{1}{2}(n - 4), \text{ if } n \text{ is even}, \tag{24c}$$

then every realization of π with a 2-factor is 1-tough.

If π fails to satisfy (24a), (24b), or (24c), resp., for some i , then π is majorized by the degree sequence of $K_i + (\overline{K_i} \cup K_{n-2i})$, $K_i + (\overline{K_{i-1}} \cup 2K_{(n+1)/2-i})$, or $K_i + (\overline{K_{i-1}} \cup K_{n/2-i} \cup K_{n/2+1-i})$, resp., where each graph contains a 2-factor, but is not 1-tough. Thus Theorem 4.5 is weakly optimal for 2-factor \Rightarrow 1-tough.

We put forth the following conjecture.

Conjecture 4.6 *The degree condition in Theorem 4.5 is a best monotone degree condition for 2-factor \Rightarrow hamiltonian.*

A best monotone condition for 1-tough \Rightarrow hamiltonian is again not currently known. However, a best minimum degree condition for 1-tough \Rightarrow hamiltonian was given by Jung [39] (see also [11]).

Theorem 4.7 (Jung [39]) *Let G be a 1-tough graph on $n \geq 11$ vertices. If $\delta(G) \geq \frac{1}{2}n - 2$, then G is hamiltonian.*

In [38], Hoàng gave the following simple, but not best monotone, degree condition for 1-tough \Rightarrow hamiltonian, and noted the difficulty of determining the 1-tough \Rightarrow hamiltonian sinks.

Theorem 4.8 (Hoàng [38]) *Let $\pi = (d_1 \leq \dots \leq d_n)$ be a graphical sequence, with $n \geq 3$. If*

$$d_i \leq i \wedge d_{n-i+1} \leq n - i - 1 \Rightarrow d_j + d_{n-j+1} \geq n, \text{ for } 1 \leq i < j \leq \lceil \frac{1}{2}n \rceil,$$

then every 1-tough realization of π is hamiltonian.

Corollary 4.9 *Let $\pi = (d_1 \leq \dots \leq d_n)$ be a graphical sequence, with $n \geq 3$. If*

$$d_i \leq i \Rightarrow d_{n-i+1} \geq n - i, \text{ for } 1 \leq i \leq \frac{1}{2}(n - 1),$$

then every 1-tough realization of π is hamiltonian.

A best monotone condition for 1-tough \Rightarrow 2-factor was given in [12].

Theorem 4.10 (Bauer et al. [12]) *Let $\pi = (d_1 \leq \dots \leq d_n)$ be a graphical sequence, with $n \geq 3$. If (setting $d_0 = 0$)*

$$\begin{aligned} & d_i \leq i + j \wedge d_{i+2j+1} \leq i + j + 1 \\ & \Rightarrow d_{n-i-3j-1} \geq n - i - 2j - 1 \vee d_{n-i-j} \geq n - i - 2j, \\ & \text{for } 0 \leq i \leq \frac{1}{2}(n - 7) \text{ and } 1 \leq j \leq \frac{1}{5}(n - 2i - 2); \end{aligned} \tag{25a}$$

$$\begin{aligned} & d_i \leq i + 2 \wedge d_{i+4} \leq i + 3 \Rightarrow d_{n-i-6} \geq \frac{1}{2}n - 1 \vee d_{n-i-2} \geq \frac{1}{2}n, \\ & \text{for } 0 \leq i \leq \frac{1}{2}(n - 18), \text{ if } n \geq 18 \text{ is even}; \end{aligned} \tag{25b}$$

$$\begin{aligned} & d_i \leq i + 1 \wedge d_{i+2} \leq i + 2 \wedge d_{i+3} \leq i + 3 \Rightarrow d_{n-i-5} \geq \frac{1}{2}n - 1 \\ & \vee d_{n-i-1} \geq \frac{1}{2}n, \text{ for } 0 \leq i \leq \frac{1}{2}(n - 16), \text{ if } n \geq 16 \text{ is even}; \end{aligned} \tag{25c}$$

$$d_{n/2-5} \geq \frac{1}{2}n - 2 \vee d_{n/2} \geq \frac{1}{2}n - 1 \vee d_{n/2+3} \geq \frac{1}{2}n + 1, \text{ if } n \geq 10 \text{ is even}, \tag{25d}$$

then every 1-tough realization of π contains a 2-factor.

If π fails to satisfy (25a) for some i, j , then π is majorized by the degrees of $K_{i+j} + (\overline{K_{i+2j+1}} \cup K_{n-2i-3j-1})$ with $\overline{K_{i+2j+1}}$ and $K_{n-2i-3j-1}$ joined by $2j + 1$

independent edges. If π fails to satisfy (25b) for some i , then π is majorized by the degrees of $K_{i+2} + (\overline{K_{i+4}} \cup 2K_{n/2-i-3})$ with $\overline{K_{i+4}}$ joined to the two copies of $K_{n/2-i-3}$ by four independent edges, three of the edges to one copy, and one to the other. If π fails to satisfy (25c) for some i , then π is majorized by the degrees of $K_{i+1} + (\overline{K_{i+3}} \cup 2K_{n/2-i-2})$ with $x, y, z \in V(\overline{K_{i+3}})$ joined by three independent edges to one copy of $K_{n/2-i-2}$, and x joined by one edge to the other copy. If π fails to satisfy (25d), then π is majorized by the degree sequence of $K_{n/2-3} + (\overline{K_{n/2-1}} \cup K_3 \cup K_1)$ with $\overline{K_{n/2-1}}$ joined by four independent edges to $K_3 \cup K_1$. Each of these graphs is 1-tough and does not contain a 2-factor. Thus Theorem 4.10 is weakly optimal for 1-tough \Rightarrow 2-factor.

We conclude this section with the following question.

Query *Is the degree condition in Theorem 4.10 also a best monotone degree condition for 1-tough \Rightarrow hamiltonian?*

5 Improving Structural Results in a Best Monotone Sense

Recall from Sect. 2 that if P is a graph property and π is a graphical sequence, then $\pi \in \text{BM}(P)$ if and only if every graphical sequence $\pi' \geq \pi$ is forcibly P . Let P_1, P_2 be graph properties. If $P_1 \Rightarrow P_2$ and $\pi \in \text{BM}(P_1)$, then π is forcibly P_2 . But more is true [8].

Theorem 5.1 (Bauer et al. [8]) *Let P_1, P_2 be graph properties such that $P_1 \Rightarrow P_2$. Then $\pi \in \text{BM}(P_1) \Rightarrow \pi \in \text{BM}(P_2)$.*

In the remainder of this section we will abbreviate $\pi \in \text{BM}(P_1) \Rightarrow \pi \in \text{BM}(P_2)$ (equivalent to $\text{BM}(P_1) \subseteq \text{BM}(P_2)$) by $\text{BM}(P_1) \Rightarrow \text{BM}(P_2)$.

For example, since 3/2-binding \Rightarrow hamiltonian by Theorem 3.6, we have

$$\text{BM}(3/2\text{-binding}) \Rightarrow \text{BM}(\text{hamiltonian}) \tag{26}$$

by Theorem 5.1. We may think of (26) as a best monotone analogue of the structural implication 3/2-binding \Rightarrow hamiltonian.

Our interest in this section will be in implications of the form $\text{BM}(P_1) \Rightarrow \text{BM}(P_2)$ when the analogous structural implication $P_1 \Rightarrow P_2$ does not hold. In that case, we will call $\text{BM}(P_1) \Rightarrow \text{BM}(P_2)$ an *improvement in a best monotone sense* of the structural $P_1 \not\Rightarrow P_2$. In the remainder of this section, we illustrate the possibility of obtaining such improvements with several examples.

1. Although hamiltonian \Rightarrow 1-tough, the converse 1-tough \Rightarrow hamiltonian fails to hold. But the converse does hold in a best monotone sense, i.e., $\text{BM}(1\text{-tough}) \Rightarrow \text{BM}(\text{hamiltonian})$, by Theorem 3.10 ($t = 1$) and Theorem 1.1.
2. Although $a(G) \leq k \Rightarrow \chi(G) \leq 2k$, the converse $\chi(G) \leq 2k \Rightarrow a(G) \leq k$ is not true. But the converse is true in a best monotone sense, i.e., $\text{BM}(\chi(G) \leq 2k) \Rightarrow \text{BM}(a(G) \leq k)$, by Theorem 3.25 and Theorem 3.28.
3. As noted in Sect. 3.2, the bound $\text{bind}(G) \geq 3/2$ in Theorem 3.6 is best possible, and thus there is no structural implication of the form $b\text{-binding} \Rightarrow \text{hamiltonian}$, for any $b < 3/2$. But this implication does hold in a best monotone sense for $b > 1$ [10].

Theorem 5.2 (Bauer et al. [10]) *If $b > 1$, then $\text{BM}(b\text{-binding}) \Rightarrow \text{BM}(\text{hamiltonian})$.*

The hypothesis $b > 1$ in Theorem 5.2 is best possible: If

$$\pi = \left(\lfloor \frac{1}{2}n \rfloor - 1 \right)^{\lfloor n/2 \rfloor - 1} \left(n - \lfloor \frac{1}{2}n \rfloor \right)^{n - 2\lfloor n/2 \rfloor + 2} (n - 1)^{\lfloor n/2 \rfloor - 1},$$

then $\pi \in \text{BM}(1\text{-binding})$ by Theorem 3.9, but $\pi \notin \text{BM}(\text{hamiltonian})$, since π fails to satisfy Theorem 1.1 for $i = \lfloor \frac{1}{2}n \rfloor - 1$.

4. The following was proved in [9].

Theorem 5.3 (Bauer et al. [9]) *Let G be a graph with $\text{bind}(G) \geq 2$. Then*

$$\tau(G) \geq \begin{cases} 3/2, & \text{if } \text{bind}(G) = 2; \\ 2, & \text{if } \text{bind}(G) = 9/4, \text{ or } \text{bind}(G) = 2 + 1/(2m - 1), \\ & \text{for some } m \geq 2; \\ 2 + 1/m, & \text{if } \text{bind}(G) = 2 + 2/(2m - 1), \text{ for some } m \geq 2; \\ \text{bind}(G), & \text{otherwise.} \end{cases}$$

Moreover, these bounds are best possible for every value of $\text{bind}(G) \geq 2$.

Thus, the structural implication $b\text{-binding} \Rightarrow b\text{-tough}$ fails to hold for infinitely many $b \geq 2$. But this implication is true in a best monotone sense for all $b \geq 2$ [8].

Theorem 5.4 (Bauer et al. [8]) *If $b \geq 2$, then $\text{BM}(b\text{-binding}) \Rightarrow \text{BM}(b\text{-tough})$.*

The hypothesis $b \geq 2$ in Theorem 5.4 is best possible: If $m \geq 2$ and $\pi = (2m - 3)^{m-2} (2m - 2)^2 (3m - 4)^{2m-3}$, then taking $b = 2 - 1/m$, we have $\pi \in \text{BM}(b\text{-binding})$ by Theorem 3.9, but $\pi \notin \text{BM}(b\text{-tough})$, since π fails to satisfy Theorem 3.10 for $i = 2m - 3$.

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