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An Analytic Description of the Vector Constrained KP Hierarchy

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Abstract: In this paper we give a geometric description in terms of the Grassmann manifold of Segal and Wilson, of the reduction of the KP hierarchy known as the vector k-constrained KP hierarchy. We also show in a geometric way that these hierarchies are equivalent to Krichever's general rational reductions of the KP hierarchy.

1. Introduction

In recent years (vector) constrained KP hierarchies have attracted considerable attention both from the mathematical and the physical community [2–27, 29, 31, 32]. Many interesting integrable systems like the AKNS, Yajima–Oikawa and Melnikov hierarchies appear amongst these constrained families. In the physics literature they are studied in connection with multi-matrix models.

The (vector) constrained KP hierarchies were introduced as reductions of the KP hierarchy

$$\frac{\partial L}{\partial t_n} = [(L^n)_+, L], \quad n \geq 1,$$

for the first order pseudodifferential operator $L = \partial + \sum_{j<0} \ell_j \partial^j$. This reduction consists of assuming that

$$(L^k)_- = \sum_{j=1}^{m} q_j \partial^{-1} r_j,$$

such that the following conditions on the functions q_j and r_j hold:

$$\frac{\partial q_j}{\partial t_n} = (L^n)_+(q_j) \quad \text{and} \quad \frac{\partial r_j}{\partial t_n} = -(L^n)_+^*(r_j) \quad \text{for all} \quad n \geq 1.$$

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In this way it generalizes the well-known Gelfand–Dickey hierarchies ($(L^k)_- = 0$).

Much is known about these constrained hierarchies and many well-known features are investigated, e.g. it was shown that they possess a bi-Hamiltonian structure [9, 20, 24, 29, 32], a bilinear representation [13], [21], [22], [32] and Bäcklund-Darboux and Miura transformations [2, 4–7, 10, 23]. However, until recently, the geometry remained unclear. It is well-known that one can associate to a point in an infinite Grassmannian a solution L of the KP hierarchy [28, 30]. In this paper we consider the Segal–Wilson Grassmannian. Let H be the Hilbert space of all square integrable functions on the circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, which decomposes in a natural way as the direct sum of two infinite dimensional orthogonal closed subspaces $H_+ = \{\sum_{n\geq 0} a_n z^n \in H\}$ and $H_- = \{\sum_{n<0} a_n z^n \in H\}$. The Segal–Wilson Grassmannian $Gr(H)$ consists of all closed subspaces $W \subset H$ such that the orthogonal projection on H_- is a Hilbert-Schmidt operator. In this setting, the k^{th} Gelfand–Dickey hierarchy has the following simple geometrical interpretation. The KP operator L belongs to the k^{th} Gelfand–Dickey hierarchy if and only if the corresponding $W \in Gr(H)$ satisfies $z^k W \subset W$. One of the authors gave in [19] (see also [18]) a simple interpretation of the constrained KP hierarchy for the case of polynomial tau-functions, viz L belongs to the m-vector k-constrained KP hierarchy if and only if the corresponding $W \in Gr(H)$ has a subspace W' of codimension m such that $z^k(W') \subset W$. We show in this paper that the same interpretation also holds in the Segal–Wilson case. Using this geometrical interpretation, we prove in Sect. 5 that the vector constrained KP hierarchy describes the same reduction of KP as the general rational reductions of Krichever [17] (see also [15]). Our geometrical interpretation is also useful to give solutions of these hierarchies (see e.g. [19]).

2. The KP Hierarchy Revisited

In this section we recall some results for the KP-hierarchy that we will need in this paper. The KP hierarchy starts with a commutative ring R and a privileged derivation ∂ of R. In order to be able to take roots of differential operators in ∂ with coefficients form R, one extends this ring $R[\partial]$ to the ring $R[\partial, \partial^{-1})$ of pseudodifferential operators with coefficients in R. It consists of all expressions

$$\sum_{i=-\infty}^{N} a_i \partial^i \quad , \quad a_i \in R \quad \text{for all} \quad i,$$

that are added in an obvious way and multiplied according to

$$\partial^j \circ a\partial^i = \sum_{k=0}^{\infty} \binom{j}{k} \partial^k(a)\partial^{i+j-k}.$$

Each operator $P = \sum p_j \partial^j$ decomposes as $P = P_+ + P_-$ with $P_+ = \sum_{j\geq 0} p_j \partial^j$ its differential operator part and $P_- = \sum_{j<0} p_j \partial^j$ its integral operator part. We denote by $Res_\partial P = p_{-1}$ the *residue* of P. On $R[\partial, \partial^{-1})$ we have an anti-algebra morphism called *taking the adjoint*. The adjoint of $P = \sum p_i \partial^i$ is given by

$$P^* = \sum_i (-\partial)^i p_i.$$

Further one has a set of derivations $\{\partial_n \mid n \geq 1\}$ of R that commute with ∂. The equations of the hierarchy can be formulated in a compact way in a set of relations for a so-called *Lax operator* in $R[\partial, \partial^{-1})$, i.e. an operator of the form

$$L = \partial + \sum_{j<0} \ell_j \partial^j \quad , \ell_j \in R \quad \text{for all } \ j < 0. \tag{2.1}$$

These equations are

$$\partial_n(L) = \sum_{j<0} \partial_n(\ell_j)\partial^j = [(L^n)_+, L], \quad n \geq 1. \tag{2.2}$$

Since this equation for $n = 1$ boils down to $\partial_1(\ell_j) = \partial(\ell_j)$ for all j, we assume from now on that $\partial = \partial_1$. Equation (2.2) has at least the trivial solution $L = \partial$ and can be seen as the compatibility equation of the linear system

$$L\psi = z\psi \quad \text{and} \quad \partial_n(\psi) = (L^n)_+(\psi). \tag{2.3}$$

One needs a context in which the actions of (2.3) make sense and that allows you to derive (2.2) from (2.3). For the trivial solution (2.3) becomes

$$\partial\psi = z\psi \quad \text{and} \quad \partial_n\psi = z^n\psi \quad \text{for all } \ n \geq 1.$$

Hence if one takes $\partial_n = \frac{\partial}{\partial t_n}$ then the function $\gamma(z) = \exp(\sum_{i\geq 1} t_i z^i)$ is a solution. The space M of the so-called *oscillating functions* for which we make sense of (2.3) can be seen as a collection of perturbations of this solution. It is defined as

$$M = \{(\sum_{i\leq N} a_i z^i) e^{\sum t_i z^i} \mid a_i \in R, \ \text{ for all } \ i\}.$$

The space M becomes a $R[\partial, \partial^{-1})$-module by the natural extension of the actions

$$\begin{aligned} b\{(\textstyle\sum_j a_j z^j) e^{\sum t_i z^i}\} &= (\textstyle\sum_j b a_j z^j) e^{\sum t_i z^i}, \\ \partial\{(\textstyle\sum_j a_j z^j) e^{\sum t_i z^i}\} &= (\textstyle\sum_j \partial(a_j) z^j + \sum_j a_j z^{j+1}) e^{\sum t_i z^i}. \end{aligned}$$

It is even a free $R[\partial, \partial^{-1})$-module, since we have

$$(\sum p_j \partial^j) e^{\sum t_i z^i} = (\sum p_j z^j) e^{\sum t_i z^i}.$$

An element ψ in M is called an *oscillating function of type* z^ℓ, if it has the form

$$\psi(z) = \{z^\ell + \sum_{j<\ell} \alpha_j z^j\} e^{\sum t_i z^i}.$$

The fact that M is a free $R[\partial, \partial^{-1})$-module permits you to show that each oscillating function of type z^ℓ that satisfies (2.3) gives you a solution of (2.2). This function is then called a *wavefunction* of the KP-hierarchy.

Segal and Wilson give in [30] an analytic approach to construct wavefunctions of the KP-hierarchy. They considered the Hilbert space

$$H = \{\sum_{n\in\mathbb{Z}} a_n z^n \mid a_n \in \mathbb{C}, \sum_{n\in\mathbb{Z}} \mid a_n \mid^2 < \infty\},$$

with decomposition $H = H_+ \oplus H_-$, where

$$H_+ = \{\sum_{n\geq 0} a_n z^n \in H\} \quad \text{and} \quad H_- = \{\sum_{n<0} a_n z^n \in H\}$$

and inner product $< \cdot \mid \cdot >$ given by

$$< \sum_{n\in\mathbb{Z}} a_n z^n \mid \sum_{m\in\mathbb{Z}} b_m z^m >= \sum_{n\in\mathbb{Z}} a_n \overline{b_n}.$$

To this decomposition is associated the Grassmannian $Gr(H)$ consisting of all closed subspaces W of H such that the orthogonal projection $p_+ : W \to H_+$ is Fredholm and the orthogonal projection $p_- : W \to H_-$ is Hilbert-Schmidt. The connected components of $Gr(H)$ are given by

$$Gr^{(\ell)}(H) = \left\{W \in Gr(H) | \ p_+ : z^\ell W \to H_+ \ \text{ has index zero}\right\}.$$

On each of these components we have a natural action by multiplication of the group of commuting flows

$$\Gamma_+ = \{\exp(\sum_{i\geq 1} t_i z^i) \mid t_i \in \mathbb{C}, \ \sum \mid t_i \mid (1+\epsilon)^i < \infty \quad \text{for some } \ \epsilon > 0\}.$$

Now we take for R the ring of meromorphic functions on Γ_+ and for ∂_n the partial derivative w.r.t. t_n. Then there exists for each W in $Gr^{(-\ell)}(H)$ a wavefunction ψ_W of type z^ℓ that is defined on a dense open subset of Γ_+ and that takes values in W. Moreover, it is known that the range of ψ_W spans a dense subspace of W. Hence, if we write $\psi_W = P_W \cdot e^{\sum t_i z^i}$ with $P_W \in R[\partial, \partial^{-1})$, then $L_W = P_W \partial P_W^{-1}$ is a solution of the KP-hierarchy. Each component of $Gr(H)$ generates in this way the same set of solutions of the KP-hierarchy, so it would suffice, as is done in [30], to consider only $Gr^{(0)}(H)$. However, it is more convenient here to consider all components.

A subsystem of the KP-hierarchy consists of all solutions L that are the k^{th} root of a differential operator. This gives you solutions of the KP-hierarchy that do not depend on the $\{t_{kn}, \text{ with } \ n \geq 1\}$. Those operators satisfy the condition $L^k = (L^k)_+$. The set of equations corresponding to this condition is called the k^{th} Gelfand–Dickey hierarchy. Now it has been shown that, among the solutions coming from the Segal–Wilson Grassmannian, the ones that satisfy the k^{th} Gelfand–Dickey hierarchy are exactly characterized by $z^k W \subset W$. In the next section we consider a generalization of this condition.

3. An Extension of the Condition $z^k W \subset W$

In this section we consider, for each k and m in $\mathbb{N} = \{0, 1, 2, \dots\}$, $k \neq 0$ subspaces W in $Gr(H)$ that possess the m**-Vector** k**-Constrained** $(mVkC)$**-Condition**:

$$\textit{There is a subspace } W' \textit{ of } W \textit{ of codimension } m \textit{ such that } z^k(W') \subset W. \tag{3.1}$$

This is a natural generalization of the condition that describes inside $Gr(H)$ the solutions of the k^{th} Gelfand–Dickey hierarchy. We will show here in a geometric way how you can associate to each W, satisfying the $mVkC$-condition, $2m$ functions $\{q_j \mid 1 \leq j \leq m\}$ and $\{r_j \mid 1 \leq j \leq m\}$ for which the following equations hold:

$$\partial_n(q_j) = (L_W^n)_+(q_j) \quad \text{for all} \ \ n \geq 1, \tag{3.2}$$

$$\partial_n(r_j) = -(L_W^n)_+^*(r_j) \quad \text{for all} \ \ n \geq 1. \tag{3.3}$$

Here A^* denotes the adjoint of A in $R[\partial, \partial^{-1})$. Moreover L_W satisfies

$$L_W^k = (L_W^k)_+ + \sum_{j=1}^{m} q_j \partial^{-1} r_j. \tag{3.4}$$

At the same time we will give links with the paper of Zhang [31].

Take any W in $Gr^{(-\ell)}(H)$ that satisfies the $mVkC$-condition. It is no restriction to assume that the m occurring in (3.1) is optimal, i.e. there is an orthonormal basis $\{u_1, \ldots, u_m\}$ of the orthocomplement of W' in W such that

$$(\text{Span}\{z^k u_1, \ldots, z^k u_m\}) \cap W = \{0\}.$$

Since multiplication with z is unitary, the vectors $\{z^k(u_1), \ldots, z^k(u_m)\}$ are an orthonormal basis of the orthocomplement of W in $z^k W + W$. To the space W we associate the subspaces

$$W_j = W \oplus \mathbb{C} z^k u_j, 1 \leq j \leq m.$$

Clearly the W_j all belong to $Gr^{(-\ell+1)}(H)$ and hence, they have wavefunctions ψ_{W_j} of type $z^{\ell-1}$, i.e.

$$\psi_{W_j} = \psi_{W_j}(t,z) = \{z^{\ell-1} + \sum_{s \geq 1} a_{js}(t) z^{\ell-1-s}\} e^{\sum t_i z^i}. \tag{3.5}$$

Recall that $\psi_{W_j}(t,z)$ is well-defined for all t belonging to the open dense subset

$$\Gamma_+^{W_j} = \{\gamma(z) = \exp(\sum t_i z^i) \in \Gamma_+ | \gamma^{-1} W_j \ \text{ is transverse to } \ z^{\ell-1} H_+\}.$$

On $\Gamma_+^{W_j}$ we consider the function

$$s_j(t) = < \psi_{W_j}(t,z) \mid z^k u_j > . \tag{3.6}$$

Since the vectors $\{\psi_{W_j}(t,z) \mid t \in \Gamma_+^{W_j}\}$ are lying dense in W_j and m was assumed to be optimal, the functions $\{s_j\}$ do not vanish. Hence, on a dense open subset of Γ_+, there is defined the function

$$\varphi_j = \frac{1}{s_j} \psi_{W_j} := r_j \psi_{W_j}. \tag{3.7}$$

It takes values in W_j and has moreover the following useful property

$$\varphi_j(t) - z^k u_j \in W, \tag{3.8}$$

for all t in a dense open subset of Γ_+. This property is a consequence of the facts that $\varphi_j(t) - z^k u_j$ is by construction orthogonal to $z^k u_j$ and that W is the orthocomplement of $\mathbb{C} z^k u_j$ inside W_j. In [31], similar functions $\{\varphi_j\}$ are introduced, only not using the geometry, but as solutions of a certain system of differential equations. In particular, we can dispose of the condition (a) in the proposition of [31]. Thus we have obtained m functions $\{r_j\}$.

To define the $\{q_j\}$ we consider

$$z^k\psi_W - (L_W^k)_+(\psi_W) = (L_W^k)_-(\psi_W) = \{\sum_{s\geq 0} b_s(t)z^{\ell-1-s}\}e^{\sum t_iz^i}. \qquad (3.9)$$

For each $j, 1 \leq j \leq m$, we have a function q_j on $\Gamma_+^{W_j}$,

$$\begin{aligned} q_j(t) &= < z^k\psi_W(t,z) - (L_W^k)_+\psi_W(t,z) \mid z^k u^j > \\ &= < z^k\psi_W(t,z) \mid z^k u_j > \\ &= < \psi_W(t,z) \mid u_j > . \end{aligned}$$

Because m is optimal, the functions $\{q_j\}$ are non-zero on an open dense subset of Γ_+. Since u_j does not depend on t and since $\frac{\partial}{\partial t_n}\psi_W = (L_W^n)_+(\psi_W)$, we get directly for q_j,

$$\begin{aligned} \frac{\partial q_j}{\partial t_n} &= < \frac{\partial}{\partial t_n}(\psi_W)(t,z) \mid u_j > = < (L_W^n)_+(\psi_W(t,z)) \mid u_j > \\ &= (L_W^n)_+(< \psi_W \mid u_j >) = (L_W^n)_+(q_j). \end{aligned} \qquad (3.10)$$

Thus Eqs. (3.2) for the derivatives of the $\{q_j\}$ are clear. Those for the $\{r_j\}$ require more work.

First we derive an expression for $(L_W^k)_-(\psi_W)$. Thereto we consider

$$\Phi(t) = z^k\psi_W - (L_W^k)_+(\psi_W) - \sum_{j=1}^m q_j\varphi_j. \qquad (3.11)$$

Since φ_j takes values in W_j, the function $(L_W^k)_+(\psi_W)$ does so in the space W and $z^k\psi_W$ in z^kW. Hence we have that $\Phi(t)$ belongs to $W + z^kW$ for all relevant t. By construction we have that for all $j, 1 \leq j \leq m$, $\Phi(t)$ is orthogonal to $z^k u_j$, hence $\Phi(t)$ even belongs to W. From the form of the φ_j, we see that on an open dense set of Γ_+ one has

$$\Phi(t) = \{\sum_{s\geq 0} c_s z^{\ell-1-s}\}e^{\sum t_iz^i}.$$

By construction, there holds

$$W \cap (z^\ell H_+)^\perp \gamma(z) = \{0\},$$

so that we arrive at

$$z^k\psi_W - (L_W^k)_+(\psi_W) = \sum_{j=1}^m q_j\varphi_j. \qquad (3.12)$$

This equation is part of the system of differential equations for the φ_j as used in [31]. Recall that φ_j has the form

$$\varphi_j = \{r_j z^{\ell-1} + \text{ lower order terms in } z\}e^{\sum t_iz^i}.$$

Hence,

$$\frac{\partial\varphi_j}{\partial x} = \frac{\partial\varphi_j}{\partial t_1} = \{r_j z^\ell + \text{ lower order terms }\}e^{\sum t_i z^i}.$$

On the other hand we know that $\varphi_j(t) - z^k u_j$ belongs to W for all t. Thus also $\frac{\partial\varphi_j}{\partial x}(t)$ belongs to W. In W we have that

$$\frac{\partial\varphi_j}{\partial x} - r_j\psi_W = \{\sum_{s\geq 0}\alpha_s z^{\ell-1-s}\}e^{\sum t_i z^i} \in (z^\ell H_+)^\perp\gamma,$$

and this has to be zero. By definition we have $\varphi_j = r_j\psi_{W_j}$ and differentiation w.r.t. x gives

$$\psi_W = \frac{1}{r_j}\partial(r_j\psi_{W_j}) = (r_j^{-1}\partial r_j)(\psi_{W_j}). \tag{3.13}$$

Consequently, we have for φ_j,

$$\varphi_j = r_j\psi_{W_j} = r_j(r_j^{-1}\partial^{-1}r_j)\psi_W = \partial^{-1}r_j\psi_W.$$

Now we substitute this in Eq. (3.12) and obtain

$$(L_W^k)_-(\psi_W) = \{\sum_{j=1}^m q_j\partial^{-1}r_j\}\psi_W. \tag{3.14}$$

Since the pseudodifferential operators act freely on wavefunctions, we see that L_W and the functions $\{q_j\}$ and $\{r_j\}$ are exactly connected by Eq. (3.4)

$$(L_W^k)_- = \sum_{j=1}^m q_j\partial^{-1}r_j.$$

What remains to be shown, is the differential Eq. (3.3) for the r_j. As $\varphi_j(t) - z^k u_j$ belongs to W, it follows that for all $n \geq 1$, $\frac{\partial\varphi_j}{\partial t_n}(t)$ lies in W. Recall that

$$\varphi_j = \{r_j z^{\ell-1} + \text{ lower order terms in } z\}e^{\sum t_i z^i}.$$

Then we have

$$\begin{aligned}\frac{\partial\varphi_j}{\partial t_n} &= \{r_j z^{n+\ell-1} + \text{ lower order terms}\}e^{\sum t_i z^i}\\ &= \{r_j\partial^{n-1}\}\psi_W + \{\textstyle\sum_{s\geq 0}\alpha_s z^{n-1+\ell-s}\}e^{\sum t_i z^i}\\ &= A_{nj}(\psi_W) + \{\textstyle\sum_{s\geq 0}\beta_s z^{\ell-1-s}\}e^{\sum t_i z^i},\end{aligned}$$

with A_{nj} a uniquely determined differential operator in ∂ of order $n-1$ and with leading coefficient r_j. Since both $\frac{\partial\varphi_j}{\partial t_n}$ as $A_{nj}(\psi_W)$ are lying in W, we get

$$\frac{\partial\varphi_j}{\partial t_n} - A_{nj}(\psi_W) = 0 = W \cap (z^\ell H_+)^\perp\gamma(z).$$

On the other hand we know that $\varphi_j = \partial^{-1}r_j\psi_W$ and this leads to

$$A_{nj}(\psi_W) = \partial^{-1}\frac{\partial r_j}{\partial t_n}\psi_W + \partial^{-1}r_j(L_W^n)_+(\psi_W). \tag{3.15}$$

This gives you an expression for A_{nj} in L_W and r_j,

$$A_{nj} = \partial^{-1}(\frac{\partial r_j}{\partial t_n} + r_j(L_W^n)_+).$$

By taking the residue in ∂ of the operators in this equation, we see that

$$\mathrm{Res}_\partial(A_{nj}) = 0 = \frac{\partial r_j}{\partial t_n} + \mathrm{Res}_\partial(\partial^{-1} r_j(L_W^n)_+) = \frac{\partial r_j}{\partial t_n} + (L_W^n)_+^*(r_j).$$

The last equality is a direct consequence of the following property of residues of pseudodifferential operators.

Lemma 3.1. *In the ring $R[\partial, \partial^{-1})$ of pseudodifferential operators with coefficients in R, we have for each f in R and $P = \sum_{j \le N} p_j \partial^j$ in $R[\partial, \partial^{-1})$,*

$$Res_\partial(\partial^{-1} f P) = (P^*)_+(f),$$

where $(P^*)_+ = \sum_{0 \le j \le N} (-\partial)^j p_j$ is the differential operator part of the adjoint of P.

Proof. First we recall that Res_∂ behaves as follows w.r.t. to taking the adjoint $P^* = \sum_{j \le N} (-\partial)^j p_j$ of P,

$$\mathrm{Res}_\partial(P^*) = -\mathrm{Res}_\partial P.$$

This is easily reduced to operators of the form $a\partial^n, n \in \mathbb{Z}$. Next one notices that it suffices to prove the equality in the lemma for differential operators. The left-hand side for such a P transforms as

$$\mathrm{Res}_\partial(\partial^{-1} f P) = -\mathrm{Res}_\partial(P^* f(-\partial)^{-1}) = \mathrm{Res}_\partial(P^* f \partial^{-1}).$$

As $P^* f$ is a differential operator with constant term $P^*(f)$, this gives the proof of the lemma. □

So we have shown that each r_j satisfies Eq. (3.3):

$$\frac{\partial r_j}{\partial t_n} = -(L_W^n)_+^*(r_j),$$

and we can conclude that L_W, the $\{q_j\}$ and the $\{r_j\}$ form a solution of the m-vector k-constrained KP-hierarchy.

4. The Main Theorem

In this subsection we will prove the converse of the result from the foregoing subsection and thus come to the main theorem. So we start with a W in $Gr^{(-\ell)}(H)$ and functions $\{q_j\}$ and $\{r_j\}$, all defined on a dense open subset of Γ_+, such that the Eqs. (3.2) , (3.3) and (3.4) are satisfied. We will show that such a W fulfills the $mVkC$-condition from Sect. 3. Recall that there is a unique pseudodifferential operator P_W such that $\psi_W = P_W(e^{\sum t_i z^i})$. It has the form

$$P_W = \partial^\ell + \sum_{j<\ell} p_j \partial^j = \{1 + \sum_{s<0} p_{\ell+s} \partial^s\} \partial^\ell. \tag{4.1}$$

It is not difficult to see that the fact that ψ_W is a wavefunction is equivalent to P_W satisfying the Sato-Wilson equations

$$\frac{\partial P_W}{\partial t_n} P_W^{-1} = -(P_W \partial^n P_W^{-1})_-, \tag{4.2}$$

where P_- denotes the integral operator part $\sum_{i<0} p_i \partial^i$ of the element $P = \sum p_j \partial^j$ in $R[\partial, \partial^{-1})$. Next we consider for each $j, 1 \leq j \leq m$, the operators Q_j and R_j defined by

$$Q_j := q_j \partial q_j^{-1} P_W \quad \text{and} \quad R_j = r_j^{-1} \partial^{-1} r_j P_W. \tag{4.3}$$

We want to show that the Q_j and the R_j also satisfy the Sato-Wilson equations. To do so, we need some properties of the ring $R[\partial, \partial^{-1})$ of pseudodifferential operators with coefficients from R. We resume them in a lemma

Lemma 4.1. *If f belongs to R and Q to $R[\partial, \partial^{-1})$, then the following identities hold:*

(a) $(Qf)_- = Q_- f$,
(b) $(fQ)_- = fQ_-$,
(c) $Res_\partial(Qf) = Res_\partial(fQ) = f\, Res_\partial(Q)$,
(d) $(\partial Q)_- = \partial Q_- - Res_\partial(Q)$,
(e) $(Q\partial)_- = Q_- \partial - Res_\partial(Q)$,
(f) $(Q\partial^{-1})_- = Q_- \partial^{-1} + Res_\partial(Q\partial^{-1})\partial^{-1}$,
(g) $(\partial^{-1} Q)_- = \partial^{-1} Q_- + \partial^{-1}\, Res_\partial(Q^* \partial^{-1})$.

Since the proof of this lemma consists of straightforward calculations, we leave this to the reader. Now we can show

Proposition 4.1. *The operators Q_j and $R_j, 1 \leq j \leq m$, satisfy the Sato-Wilson equations.*

Proof. If we denote $\frac{\partial}{\partial t_n}$ by ∂_n, then we get for $Q_j = q_j \partial q_j^{-1} P_W$ that

$$\begin{aligned} \partial_n(Q_j)Q_j^{-1} &= \partial_n(q_j \partial q_j^{-1}) q_j \partial_j^{-1} q_j^{-1} + q_j \partial q_j^{-1} \partial_n(P_W) P_W^{-1} q_j \partial^{-1} q_j^{-1} \\ &= -q_j \partial q_j^{-1} (L_W^n)_- q_j \partial^{-1} q_j^{-1} + \partial_n(q_j \partial q_j^{-1}) q_j \partial^{-1} q_j^{-1}. \end{aligned}$$

Now we apply successively the identities from Lemma 4.1 to the first operator of the right-hand side

$$\begin{array}{lll} q_j \partial q_j^{-1} (L_W^n)_- q_j \partial^{-1} q_j^{-1} & = q_j \partial (q_j^{-1} L_W^n q_j)_- \partial^{-1} q_j^{-1} & = \\ q_j \partial (q_j^{-1} L_W^n q_j \partial^{-1})_- q_j^{-1} & - q_j \partial \mathrm{Res}_\partial (q_j^{-1} L_W^n q_j \partial^{-1}) \partial^{-1} q_j^{-1} & = \\ q_j (\partial q_j^{-1} L_W^n q_j \partial^{-1})_- q_j^{-1} & + q_j \mathrm{Res}_\partial (q_j^{-1} L_W^n q_j \partial^{-1}) q_j^{-1} & - \\ q_j \partial \mathrm{Res}_\partial (q_j^{-1} L_W^n q_j \partial^{-1}) \partial^{-1} q_j^{-1} & = (q_j \partial q_j^{-1} L_W^n q_j \partial^{-1} q_j^{-1})_- & + \\ q_j^{-1} \mathrm{Res}_\partial (L_W^n q_j \partial^{-1}) & - q_j \partial q_j^{-1} \mathrm{Res}_\partial (L_W^n q_j \partial^{-1}) \partial^{-1} q_j^{-1}. & \end{array}$$

By applying Lemma 3.1 to these last two residues we get

$$(q_j \partial q_j^{-1} L_W^n q_j \partial^{-1} q_j^{-1})_- + (L_W^n)_+(q_j) q_j^{-1} - q_j \partial q_j^{-1} (L_W^n)_+(q_j) \partial^{-1} q_j^{-1}.$$

On the other hand

$$\partial_n(q_j\partial q_j^{-1})q_j\partial^{-1}q_j^{-1} = \partial_n(q_j)q_j^{-1} - q_j\partial q_j^{-2}\partial_n(q_j)q_j\partial^{-1}q_j^{-1}.$$

Thus we see that, if $\partial_n(q_j) = (L_W^n)_+(q_j)$, the operator Q_j satisfies the Sato-Wilson equation

$$\partial_n(Q_j)Q_j^{-1} = -(Q_j\partial^n Q_j^{-1})_-. \tag{4.4}$$

For R_j, we proceed in a similar fashion

$$\begin{aligned}\partial_n(R_j)R_j^{-1} &= -r_j^{-1}\partial^{-1}r_j(L_W^n)_- r_j\partial r_j + \partial_n(r_j^{-1}\partial^{-1}r_j)r_j^{-1}\partial r_j\\ &= -r_j^{-1}\partial^{-1}(r_jL_W^n r_j^{-1})_-\partial r_j + -\partial_n(r_j)r_j^{-1} + r_j^{-1}\partial^{-1}(\partial_n(r_j)r_j^{-1})\partial r_j.\end{aligned}$$

Now we successively apply Lemma 4.1 (g) and (c) and (4.2) to the first term of the right-hand side of this equation

$$\begin{aligned}&- r_j^{-1}\partial^{-1}(r_jL_W^n r_j^{-1})_-\partial r_j = -r_j^{-1}\{(\partial^{-1}r_jL_W^n r_j^{-1})_-\\ &\qquad\qquad - \partial^{-1}\mathrm{Res}_\partial(r_j^{-1}(L_W^n)_+^* r_j\partial^{-1})\}\partial r_j\\ &= -r_j^{-1}(\partial^{-1}r_jL_W^n r_j^{-1})_-\partial r_j + r_j^{-1}\partial^{-1}r_j^{-1}(L_W^n)_+^*(r_j)\partial r_j\\ &= -r_j^{-1}\{(\partial^{-1}r_jL_W^n r_j^{-1}\partial)_- + \mathrm{Res}_\partial(\partial^{-1}r_jL_W^n r_j^{-1})\}r_j + r_j^{-1}\partial^{-1}r_j^{-1}(L_W^n)_+^*(r_j)\partial r_j\\ &= -(r_j^{-1}\partial^{-1}r_jL_W^n r_j^{-1}\partial r_j)_- - r_j^{-1}(L_W^n)_+^*(r_j) + r_j^{-1}\partial^{-1}r_j^{-1}(L_W^n)_+^*(r_j)\partial r_j.\end{aligned}$$

Since $\partial_n(r_j) = -(L_W^n)^*(r_j)$, we see that the last two terms cancel $\partial_n(r_j^{-1}\partial r_j)r_j^{-1}\partial r_j$ and thus we have obtained the Sato-Wilson equation for R_j,

$$\partial_n(R_j)R_j = -(R_j\partial^n R_j^{-1})_-. \tag{4.5}$$

This concludes the proof of Proposition 4.1. □

This proposition has some important consequences. Since the $\{r_j\}$ and the $\{q_j\}$ are non-zero on a dense open subset of Γ_+, we define on such a subset of Γ_+ oscillating functions ψ_{Q_j} and ψ_{R_j} of type $z^{\ell+1}$ resp. $z^{\ell-1}$ by

$$\psi_{Q_j} = q_j\partial q_j^{-1}\cdot\psi_W \quad\text{and}\quad \psi_{R_j} = r_j^{-1}\partial^{-1}r_j\cdot\psi_W. \tag{4.6}$$

In fact Q_j and R_j are Bäcklund–Darboux transformations of the KP hierarchy. To be more precise, we conclude from Proposition 4.1.

Corollary 4.1. *The functions ψ_{Q_j} and ψ_{R_j} are wavefunctions of planes W_{Q_j} and W_{R_j}. Moreover we have the following codimension 1 inclusions:*

$$W_{Q_j}\subset W \quad\textit{and}\quad W\subset W_{R_j}.$$

Proof. From the Sato-Wilson equations one deduces directly that for all $n\geq 1$,

$$\partial_n\psi_{Q_j} = (Q_j\partial^n Q_j^{-1})_+\psi_{Q_j} \quad\text{and}\quad \partial_n\psi_{R_j} = (R_j\partial^n R_j^{-1})_+\psi_{R_j}.$$

This shows the first part of the claim. Consider the following subspace in $Gr(H)$:

$$W_{Q_j} = \text{the closure of Span}\{\psi_{Q_j}(t,z)\}.$$

The inclusions between the spaces W and W_{Q_j} follows from the first relation of (4.6) and the fact that the values of a wavefunction corresponding to an element of $Gr(H)$

are lying dense in that space. Since for a suitable γ in Γ_+ the orthogonal projections of $\gamma^{-1}W_{R_j}$ on $z^\ell H_+$ resp. $\gamma^{-1}W$ on $z^{\ell+1}H_+$ have a one dimensional kernel, one obtains the codimension one result. For the inclusions between the spaces W and W_{R_j} we consider the adjoint wavefunctions $\psi^*_W = P_W^{*-1}e^{-\sum t_i z^i}$ and $\psi^*_{R_j} = -r\partial r^{-1}\psi^*_W$. Since the complex conjugate $\overline{z\psi^*_W(t,z)}$ of $z\psi^*_W(t,z)$ corresponds to the space $W^\perp$, the same argument as before shows the codimension 1 inclusion:

$$W_j := \text{the closure of Span}\{\overline{z\psi^*_{R_j}(t,z)}\} \subset W^\perp.$$

Hence $\psi_{R_j}(t,z)$ corresponds to $W_j^\perp$, which must be W_{R_j} = the closure of Span $\{\psi_{R_j}(t,z)\}$. This concludes the proof of the corollary. $\square$

Now we can formulate the main results of this paper.

Theorem 4.1. *Let W be a plane in $Gr(H)$ and let L_W be the corresponding solution of the KP-hierarchy. Then for $m, k \in \mathbb{N}$, $k \neq 0$, the following 2 conditions are equivalent:*

(a) *The space W satisfies the $mVkC$-condition.*
(b) *There exist functions $\{q_j \mid 1 \leq j \leq m\}$ and $\{r_j \mid 1 \leq j \leq m\}$ defined on an open dense subset of Γ_+ such that the following conditions are fulfilled:*
   (i) $\partial_n(q_j) = (L_W^n)_+(q_j)$ *for all $n \geq 1$,*
   (ii) $\partial_n(r_j) = -(L_W^n)_+^*(r_j)$ *for all $n \geq 1$,*
   (iii) $L_W^k = (L_W^k)_+ + \sum_{j=1}^m q_j\partial^{-1}r_j$.

Proof. In Sect. 2 it has been shown that (a) implies (b). So we assume from now on (b). The relation (b) (iii) leads to

$$\begin{aligned} L_W^k(\psi_W) &= z^k\psi_W \\ &= (L_W^k)_+(\psi_W) + \sum_{j=1}^m q_j\partial^{-1}r_j\psi_W \\ &= (L_W^k)_+(\psi_W) + \sum_{\substack{j \\ r_j\neq 0}} q_j r_j r_j^{-1}\partial^{-1}r_j\psi_W \\ &= (L_W^k)_+(\psi_W) + \sum_{\substack{j \\ r_j\neq 0}} q_j r_j\psi_{R_j}. \end{aligned}$$

Thus we see with the usual density argument that

$$z^kW \subset W + \sum_j W_{R_j} = \sum_j W_{R_j} = \tilde{W}.$$

Since each W has codimension one in W_{R_j}, we see that the codimension of W in $\tilde{W}$ is $\leq m$. Let W_1 be the orthocomplement of W in $\tilde{W}$ and $p_1 : H \to W_1$ the orthogonal projection on W_1. Inside W we consider

$$W^1 = \{w \in W \mid p_1(z^kw) = 0\}.$$

Since $\dim(W_1) \leq m$, we see that W^1 is a subspace of W of codimension $\leq m$ and by construction $z^kW^1 \subset W$. This completes the proof of the theorem. $\square$

5. General Rational Reductions of the KP Hierarchy

We are now going to connect the vector constrained KP hierarchy to reductions of the KP hierarchy introduced by Krichever [17]. For that purpose we assume that W is a plane in $Gr(H)$ that satisfies the $mVkC$-condition, where we choose m to be as minimal as is possible for that plane. Let $L_W = P_W \partial P_W^{-1}$, with P_W of the form (4.1), be the corresponding solution of the KP hierarchy and let $W^1 \subset W$ be the subspace of codimension m such that $W_1 = z^k W^1 \subset W$. Notice first that W_1 is a subspace of W and $z^k W$ of codimension $k+m$ and m, respectively. Hence there exist differential operators L_1 and L_2 of order $k+m$ and m, respectively, such that

$$L_1 \psi_W = \psi_{W_1}, \quad L_2 z^k \psi_W = \psi_{W_1} \tag{5.1}$$

and that ψ_{W_1} is again a wavefunction. From (5.1) one immediately deduces that

$$L_W^k = L_2^{-1} L_1. \tag{5.2}$$

We first prove the following lemma.

Lemma 5.1. *Let $L = P\partial^k P^{-1}$ be a pseudodifferential operator of order k and let L_1 and L_2 be differential operators of order $k+m$ and m, respectively, such that $L = L_2^{-1} L_1$. Then one has the following identities:*

$$L_1 (L_2^{-1} L_1)^{i/k} = (L_1 L_2^{-1})^{i/k} L_1, \quad L_2 (L_2^{-1} L_1)^{i/k} = (L_1 L_2^{-1})^{i/k} L_2.$$

Proof. Since $L_1 P = L_2 P \partial^k$, one can find a pseudodifferential operator Q of the same order as P such that $L_1 = Q\partial^{k+m} P^{-1}$, $L_2 = Q\partial^m P^{-1}$, and thus $L_1 L_2^{-1} = Q\partial^k Q^{-1}$. Since also $L_2^{-1} L_1 = P\partial^k P^{-1}$, one finds that their k^{th} roots satisfy

$$(L_2^{-1} L_1)^{1/k} = P\partial P^{-1}, \quad (L_1 L_2^{-1})^{1/k} = Q\partial Q^{-1}.$$

Using this, one easily verifies the identities of the Lemma. □

Since both ψ_W and ψ_{W_1} are wavefunctions that are connected by Eqs. (5.1), we find, using (5.2) and Lemma 5.1, that

$$L_W = (L_2^{-1} L_1)^{1/k} \quad \text{and} \quad L_{W_1} = L_1 (L_2^{-1} L_1)^{1/k} L_1^{-1} = (L_1 L_2^{-1})^{1/k}. \tag{5.3}$$

Hence

$$\partial_i \psi_{W_1} = ((L_1 L_2^{-1})^{i/k})_+ \psi_{W_1} = ((L_1 L_2^{-1})^{i/k})_+ L_1 \psi_W,$$

and on the other hand is also equal to

$$\partial_i (L_1 \psi_W) = \partial_i (L_1) \psi_W + L_1 ((L_2^{-1} L_1)^{i/k})_+ \psi_W,$$

from which one deduces that

$$\partial_i L_1 = ((L_1 L_2^{-1})^{i/k})_+ L_1 - L_1 ((L_2^{-1} L_1)^{i/k})_+. \tag{5.4}$$

In a similar way one obtains from the other identity of (5.1) that

$$\partial_i L_2 = ((L_1 L_2^{-1})^{i/k})_+ L_2 - L_2 ((L_2^{-1} L_1)^{i/k})_+. \tag{5.5}$$

Notice that in this way we have exactly obtained Krichever's general rational reductions of the KP hierarchy [17]. Krichever considers KP pseudodifferential operators L of the form (2.1), such that $L^k = L_2^{-1}L_1$, where L_1 and L_2 are coprime differential operators of order $k + m$ and m, respectively. It can be shown that Eqs. (5.4) and (5.5) for L_1 and L_2 are equivalent to the KP Lax equations for L. It is not difficult to see that our operators must be coprime, since we have chosen our m to be minimal. We will now prove that the converse also holds, i.e, that the following theorem holds.

Theorem 5.1. *Let W be a plane in $Gr(H)$ and let L_W be the corresponding solution of the KP-hierarchy. Then for $m, k \in \mathbb{N}$, $k \neq 0$, the following 2 conditions are equivalent:*

(a) *The space W satisfies the $mVkC$-condition, with m as minimal as possible.*
(b) *There exist coprime differential operators L_1 and L_2 of order $k+m$ and m, respectively, such that the following conditions are fulfilled:*
  (i) $L_W^k = L_2^{-1}L_1$,
  (ii) $\partial_i L_1 = ((L_1L_2^{-1})^{i/k})_+L_1 - L_1((L_2^{-1}L_1)^{i/k})_+$,
  (iii) $\partial_i L_2 = ((L_1L_2^{-1})^{i/k})_+L_2 - L_2((L_2^{-1}L_1)^{i/k})_+$.

Proof. We have already shown that (a) implies (b). So we assume from now on (b). Let ψ_1 be the oscillating function $L_1\psi_W$, then by using Lemma 5.1:

$$(L_1L_2^{-1})^{1/k}\psi_1 = (L_1L_2^{-1})^{1/k}L_1\psi_W = L_1(L_2^{-1}L_1)^{1/k}\psi_W = zL_1\psi_W = z\psi_1.$$

Now consider

$$\begin{aligned}\partial_i\psi_1 &= \partial_i(L_1)\psi_W + L_1\partial_i\psi_W\\ &= (((L_1L_2^{-1})^{i/k})_+L_1 - L_1((L_2^{-1}L_1)^{i/k})_+ + L_1((L_2^{-1}L_1)^{i/k})_+)\psi_W\\ &= ((L_1L_2^{-1})^{i/k})_+L_1\psi_W\\ &= ((L_1L_2^{-1})^{i/k})_+\psi_1.\end{aligned}$$

Hence ψ_1 is again a wavefunction of the KP hierarchy. If we let W_1 be the closure of the span of the $\psi_1(t,z)$, then $\psi_{W_1} = \psi_1$. Since $z^k\psi_W$ is also a wavefunction,

$$L_2z^k\psi_W = \psi_{W_1}.$$

Thus we see with the usual density argument that

$$\begin{aligned}&W_1 \subset z^kW \quad \text{of codimension } m,\\ &W_1 \subset W \quad \text{of codimension } k+m.\end{aligned} \tag{5.6}$$

Hence $W^1 = z^{-k}W_1$ is a subset of W of codimension m such that $z^kW^1 \subset W$. Since our differential operators are coprime, one cannot find lower order operators M_1 and M_2 such that $L_W = M_2^{-1}M_1$. Hence there is no smaller subspace W_1 and no smaller m such that (5.6) is satisfied. □

As a consequence of this, we obtain that in the Segal–Wilson setting, the vector constrained KP hierarchy and Krichever's general rational reduction define the same reduction of the KP hierarchy.

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