Annals of Operations Research 101, 221–253, 2001

One-Parametric Linear-Quadratic Optimization Problems

P. JONKER, G. STILL and F. TWILT
Faculty of Mathematical Sciences, University of Twente, 7500 AE Enschede, The Netherlands

This paper is dedicated to Tony Fiacco, one of the founding fathers of Parametric Optimization

Abstract. We consider families of optimization problems with quadratic object function and affine linear constraints, which depend smoothly on one real parameter. For a generic subclass of such problems only three different types of (generalized) critical points occur, whereas in the general case (of nonlinear one-parameter families of constrained optimization problems on $\mathbb{R}^n$) five types are to be distinguished. We clarify the theoretical background of these phenomena and illustrate the underlying mechanism with simple examples.

Keywords: one-parametric quadratic optimization problems, (generalized) critical points, Whitney regular stratification

AMS subject classification: 90C30, 90C31, 58A35

1. Introduction

One-parametric families of finite, smooth optimization problems in $\mathbb{R}^n$ have been studied intensively during the last decades (see, e.g., [1,3–5,9,10]). Such problems are denoted by $P(t)$, where t stands for a real valued parameter. A basic concept is that of "generalized critical point" (g.c. point). Let us begin with introducing this concept:

We call the pair $(\overline{x}, \overline{t}) \in \mathbb{R}^n \times \mathbb{R}$ *feasible* (for the family $P(\cdot)$), whenever $\overline{x}$ is feasible for $P(\overline{t})$. The feasible pair $(\overline{x}, \overline{t})$ is said to be a *g.c. point* (for $P(\cdot)$) if the gradient at $\overline{x}$ of the objective function for $P(\overline{t})$, together with the gradients at $\overline{x}$ of the constraint functions for $P(\overline{t})$ which are active in $\overline{x}$, form a linear dependent set of vectors in $\mathbb{R}^n$. Note that this concept encompasses all usual notions of "candidate local minimizers" such as "critical point", "stationary point" and "point of Fritz–John type".

Jongen et al. [10] have proved that for a "generic" class of families $P(\cdot)$ the g.c. points classify into precisely *five* characteristic types. A g.c. point $(\overline{x}, \overline{t})$ is said to be of type 1 whenever $P(\overline{t})$ fulfills at $\overline{x}$ the following conditions simultaneously:

1. Linear Independency Constraint Qualification (LICQ);
2. Strict Complementarity condition (SC);
3. Non-Degeneracy (ND) of the restricted Hessian of the Lagrange function (where the restriction is taken to the linearized set of common zeros for the constraint functions, active at $\overline{x}$).

Apart from some technical conditions (cf. [10] and section 2), a g.c. point is called of type 2 when both LICQ and ND hold, but SC not; a g.c. point is said to be of type 3 when LICQ and SC are fulfilled, but ND not. The case where LICQ is violated gives rise to g.c. points of type 4 (when the number of active constraints does not exceed n) or to g.c. points of type 5 (when the number of active constraints equals $n + 1$).

The present paper concerns one-parametric families of C^1-quadratic optimization problems, i.e., problems $P(t)$, where the objective function is quadratic and the constraint functions are linear in the objective variable x, with as coefficients C^1-functions of t. Such problems will be denoted by $PQ(t)$. Recently, Henn et al. (cf. [6]) claimed that – generically – the families $PQ(\cdot)$ admit only g.c. points of the above types 1, 2 and 5. In the present paper we give a full proof as well as a geometrical explanation of this result; in particular, we describe the effects of the non-occurrence of g.c. points of types 3, 4 on the sets of feasible and g.c. points.

Although this point of view is not yet worked out, we believe that our approach is a first step towards a structural analysis of multi- (especially two-) parametric families of smooth optimization problems, and possibly will contribute to a better understanding of Sequential quadratic programming methods.

This work is organized as follows: in section 2 we summarize some generalities on one-parameter (quadratic) optimization problems. A precise formulation of our results is presented in section 3, whereas in section 4 we give some illustrative examples. Finally in section 5 the proofs are to be found.

2. Generalities

Throughout this paper, by x we mean a column vector in $\mathbb{R}^n$, by x^{T} its transpose, and by t a real valued parameter. Moreover, let I and J be two finite (possibly empty) index sets of cardinality m $(< n)$ and s, respectively. For each $t \in \mathbb{R}$ let:

$$\begin{cases} \min\limits_x f(x,t) := \dfrac{1}{2}x^{\mathrm{T}}A(t)x + a^{\mathrm{T}}(t)x & \text{subject to} \\ h_i(x,t) := B_i^{\mathrm{T}}(t)x + b_i(t) = 0, & i \in I, \\ g_j(x,t) := C_j^{\mathrm{T}}(t)x + c_j(t) \leqslant 0, & j \in J, \end{cases} \qquad PQ(t)$$

where (for fixed t): $A(t)$ is a symmetric (n, n)-matrix; $a(t)$, $B_i(t)$ and $C_j(t)$ are column vectors in $\mathbb{R}^n$; $b_i(t)$ and $c_j(t)$ are in $\mathbb{R}$. We assume all entries to be C^1-functions of t.

Let $B(t)$ be the (m, n)-matrix with $B_i^{\mathrm{T}}(t)$ as its i-th row, and put $b(t) = [b_1(t) \ldots b_m(t)]^{\mathrm{T}}$. In a similar way, we define $C(t)$ and $c(t)$ by means of $C_j^{\mathrm{T}}(t)$ and $c_j(t)$, respectively. Then, $PQ(t)$ takes the form

$$\begin{cases} \min\limits_x \dfrac{1}{2}x^{\mathrm{T}}A(t)x + a^{\mathrm{T}}(t)x & \text{subject to} \\ B(t)x + b(t) = 0, \\ C(t)x + c(t) \leqslant 0. \end{cases} \qquad PQ(t)$$

In the sequel, we frequently collect matrices with appropriate sizes into bigger ones. For example, the $(m+s, n)$-matrix $D(t)$ and the $(m+s, 1)$-matrix $d(t)$ are defined by respectively

$$D(t) := \begin{bmatrix} B(t) \\ C(t) \end{bmatrix} \quad \text{and} \quad d(t) := \begin{bmatrix} b(t) \\ c(t) \end{bmatrix}.$$

Given an arbitrary subset J_0 of J with p elements (if $J_0 = \emptyset$ then $p = 0$), we define $D_{J_0}(t)$ as the $(m + p, n)$-submatrix of $D(t)$ generated by all rows of $B(t)$ and all rows of $C(t)$ with indices $j \in J_0$; the column vector $d_{J_0}(t)$ in $\mathbb{R}^{m+p}$ is defined similarly. Throughout this paper, elements of $\mathbb{R}^{n+1} (= \mathbb{R}^n \times \mathbb{R})$ will be partitioned as $z = (x, t)$. The active index set for $PQ(t)$ at z is defined as usual:

$$J_z := \{ j \in J \mid g_j(z) = 0 \}.$$

Let $\overline{z} = (\overline{x}, \overline{t})$ be feasible for $PQ(\cdot)$, thus rank $D_{J_{\overline{z}}}(\overline{t}) = \text{rank}[D_{J_{\overline{z}}}(\overline{t})\ d_{J_{\overline{z}}}(\overline{t})]$, and assume $J_{\overline{z}} = \{1, \ldots, p\}$ if $J_{\overline{z}} \neq \emptyset$. The gradients (with respect to x) of $h_i(x, \overline{t})$, $i \in I$, and $g_j(x, \overline{t})$, $j \in J_{\overline{z}}$, at $\overline{x}$ form a linearly independent set of vectors in $\mathbb{R}^n$ if and only if

$$\textbf{LICQ at } \overline{z} \qquad \text{rank } D_{J_{\overline{z}}}(\overline{t}) = m + p \quad (\text{thus: } m + p \leqslant n).$$

Our target is to investigate the set, say Σ, of all g.c. points for $PQ(\cdot)$. Recall that $\overline{z}$ is a g.c. point if the set of vectors $\{\nabla_x f(\overline{z}),\ \nabla_x h_i(\overline{z}),\ \nabla_x g_j(\overline{z}),\ i = 1, \ldots, m,\ j = 1, \ldots, p\}$ is linearly dependent. Here, $\nabla_x(\cdot)$ stands for gradient with respect to x.

We distinguish between two situations ($\overline{z}$ is feasible):

1. **LICQ at $\overline{z}$ does not hold:** $\overline{z}$ is always a g.c. point for $PQ(\cdot)$.
2. **LICQ at $\overline{z}$ is fulfilled:** $\overline{z}$ is g.c. point for $PQ(\cdot)$ if and only if the following equations

$$\begin{bmatrix} A(t) & D_{J_0}^{\mathrm{T}}(t) \\ D_{J_0}(t) & 0 \end{bmatrix} \begin{bmatrix} x \\ \eta \end{bmatrix} + \begin{bmatrix} a(t) \\ d_{J_0}(t) \end{bmatrix} = 0, \quad J_0 = J_{\overline{z}}, \tag{1}$$

admit a solution $t = \overline{t}, x = \overline{x}, \eta = \overline{\eta}$ $(= (\overline{\lambda}_1, \ldots, \overline{\lambda}_m, \overline{\mu}_1, \ldots, \overline{\mu}_p))$. Here, 0 stands for the $(m + p, m + p)$ null matrix. (Due to LICQ, the "Lagrange vector" $\overline{\eta}$ is unique; put $\overline{\eta} = (\overline{\mu}_1, \ldots, \overline{\mu}_p)$ respectively $\overline{\eta} = (\overline{\lambda}_1, \ldots, \overline{\lambda}_m)$ if $m = 0$, respectively $p = 0$.) Note that in (1), for $t = \overline{t}$, the first n equations yield the "critical point equations for $PQ(\overline{t})$ at $\overline{x}$", whereas the last $m + p$ equations reflect the feasibility of $\overline{x}$ with respect to $PQ(\overline{t})$.

For a g.c. point $\overline{z}$, at which LICQ holds, we introduce the Strict Complementarity (SC) and the Non-Degeneracy (ND) conditions:

SC at $\overline{z}$ $\qquad \overline{\mu}_\ell \neq 0$ for all $\ell \in J_{\overline{z}}$.

ND at $\overline{z}$ $\qquad$ the "restricted" matrix $A(\overline{t})|_{\ker D_{J_{\overline{z}}}(\overline{t})}$ is nonsingular.

By the restricted matrix, we mean any matrix of the form $V^{\mathrm{T}} A(\overline{t}) V$, where V is a matrix with as columns a basis for ker $D_{J_{\overline{z}}}(\overline{t})$; note that the numbers of positive, negative and zero eigenvalues do not depend on the ambiguity in the choice of V, see also [8]. Note: $A(\bar{t})$ = Hessian of the Lagrange function for $PQ(\bar{t})$ at $\bar{x}$.

It is easily shown (see, e.g., [8]) that – under the assumption of LICQ at $\bar{z}$ – the condition ND is equivalent with

$$\widetilde{\mathbf{ND}}\ \textbf{at}\ \bar{z} \qquad \operatorname{rank}\begin{bmatrix} A(\bar{t}) & D^{\mathrm{T}}_{J_{\bar{z}}}(\bar{t}) \\ D_{J_{\bar{z}}}(\bar{t}) & 0 \end{bmatrix} = n + m + p.$$

Apparently, the condition $\widetilde{ND}$ implies LICQ.

Definition 2.1. We call a g.c. point $\bar{z}$ of type 1 if LICQ, SC and ND hold at $\bar{z}$. The set of all g.c. points of type 1 for $PQ(\cdot)$ is denoted by $\Sigma^{(1)}$.

Let $\bar{z}$ be a g.c. point of type 1. Then, the linear index *LI*, the linear co-index *LCI* are defined to be the numbers of negative, positive Lagrange multipliers $\bar{\mu}_\ell$ in SC respectively. The quadratic index *QI*, the quadratic co-index *QCI* are defined as the numbers of negative, positive eigenvalues of $A(\bar{t})|_{\ker D_{J_{\bar{z}}}(\bar{t})}$ (cf. ND) respectively. It can be shown (cf. [8]) that $\bar{x}$ is a local minimizer for $PQ(\bar{t})$ iff $LI + QI = 0$, and a local maximizer iff $LCI + QCI = 0$, whereas in all other cases $\bar{x}$ is a kind of saddle point for $PQ(\bar{t})$.

Lemma 2.1. For any g.c. point $\bar{z}$ of type 1, a neighborhood $\Omega_{\bar{z}}$ of $\bar{z}$ exists, such that the "local critical set" $\Omega_{\bar{z}} \cap \Sigma$ consists of merely type 1 g.c. points, and can be parametrized as a C^1-curve $(x(t), t)$ with $x(\bar{t}) = \bar{x}$ and $|t - \bar{t}| < \varepsilon$, some $\varepsilon > 0$. Moreover, on $\Omega_{\bar{z}} \cap \Sigma$ the index quadruple remains constant (cf. figure 1).

Proof. Follows directly by solving the equation (1), thereby taking LICQ, SC, ND and $\widetilde{ND}$ into account. □

Now, we are going to specify g.c. points of types 2 and 5 (compare section 1).

Definition 2.2. Let $\bar{z}$ be a g.c. point, $J_{\bar{z}} \neq \emptyset$, and put $J_{\bar{z}} = \{1, \ldots, p\}$. Then, $\bar{z}$ is said to be of type 2, whenever LICQ and ND hold at $\bar{z}$, and moreover,

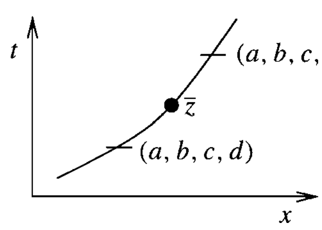

Figure 1. Local structure of Σ around a g.c. point $\bar{z}$ of type 1; the 4-vectors stand for the index quadruple (LI, LCI, QI, QCI); $LI + LCI = p$, $LI + LCI + QI + QCI = n - m$.

(i) exactly one of the Lagrange multipliers in SC, say $\overline{\mu}_p$, vanishes;

(ii) rank $\begin{bmatrix} A(\bar{t}) & D^{\mathrm{T}}_{J_0}(\bar{t}) \\ D_{J_0}(\bar{t}) & 0 \end{bmatrix} = n + m + p - 1,\ J_0 = J_{\bar{z}} \setminus \{p\};$

(iii) $\gamma := \frac{\mathrm{d}}{\mathrm{d}t} g_p(\tilde{x}(t), t)_{t=\bar{t}} \neq 0,$

where $(\tilde{x}(t), t)$ is a parametrization of the local critical set around $\bar{z}$ (in the sense of lemma 2.1) of a family, say $\widetilde{PQ}(\cdot)$, which is obtained from $PQ(\cdot)$ by deleting the constraint g_p. (Note that $\bar{z}$ is g.c. point of type 1 for $\widetilde{PQ}(\cdot)$).

Let $\bar{z}$ be a g.c. point of type 2 for $PQ(\cdot)$. Then, it can be shown (cf. [10]) that a reduced neighborhood of $\bar{z}$, say Ω, exists such that $\Omega \cap \Sigma$ consists of merely type 1 points, and admits the typical *fork structure* as depicted in figure 2 (here the characteristic numbers $\delta \in \{0, 1\}$ are defined as $\delta = \delta_1 - \delta_2$, where δ_1 respectively δ_2 denote the number of negative eigenvalues of $A(\bar{t})|_{\ker D_{J_{\bar{z}} \setminus \{p\}}(\bar{t})})$ respectively $A(\bar{t})|_{\ker D_{J_{\bar{z}}}(\bar{t})})$.

Definition 2.3. Let $\bar{z}$ be a g.c. point of $PQ(\cdot)$, $J_{\bar{z}} \neq \emptyset$, and put $J_{\bar{z}} = \{1, \ldots, p\}$. Then, $\bar{z}$ is said to be of type 5, whenever

(i) $m + p = n + 1$ (thus: LICQ does not hold at $\bar{z}$, and $p \geqslant 2$ (since $m < n$)).

(ii) $\{\nabla h_i(\bar{z}), \nabla g_j(\bar{z}),\ i \in I,\ j \in J_{\bar{z}}\}$ is a set of linear independent vectors in $\mathbb{R}^{n+1}$, where $\nabla(\cdot)$ stands for gradient with respect to z.

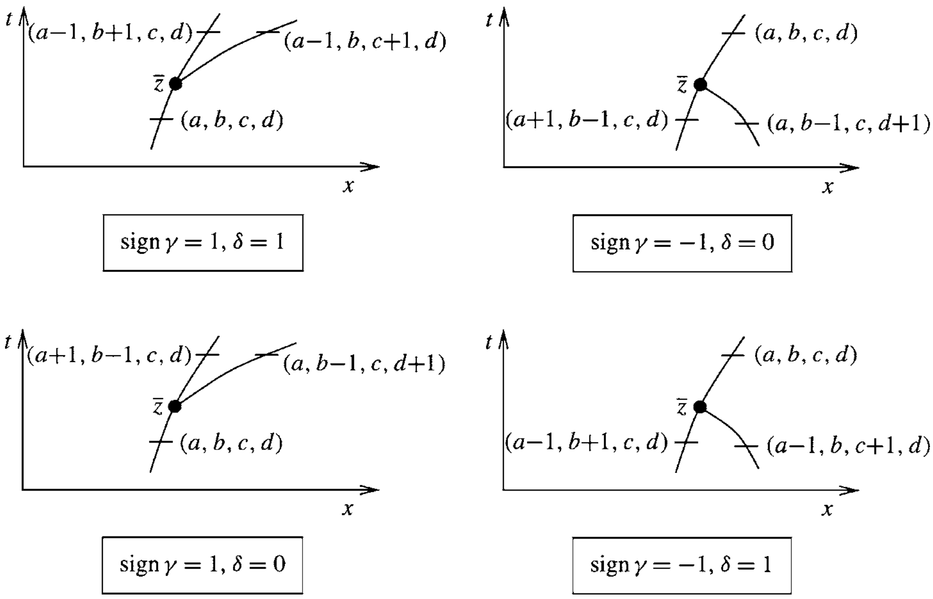

Figure 2. Local structure of Σ around a g.c. point $\bar{z}$ of type 2; the 4-vectors stand for the index quadruple (LI, LCI, QI, QCI); $LI + LCI + QI + QCI = n - m$; $LI + LCI = p$ or $p - 1$.

From (i) and (ii) it follows that there exist $\lambda_i,\ \mu_j,\ i \in I,\ j \in J_{\bar{z}}$, not all vanishing (and unique up to a common multiple) such that

$$\sum_{i=1}^{m} \lambda_i \nabla_x h_i(\bar{z}) + \sum_{j=1}^{p} \mu_j \nabla_x g_j(\bar{z}) = 0. \tag{2}$$

(iii) In (2) we have $\mu_j \neq 0,\ j = 1, \ldots, p$.

From (i) and (ii) we see that unique numbers $\alpha_i, \beta_j,\ i \in I,\ j \in J_{\bar{z}}$ exist, such that

$$\nabla f(z) = \sum_{i=1}^{m} \alpha_i \nabla h_i(\bar{z}) + \sum_{j=1}^{p} \beta_j \nabla g_j(\bar{z}).$$

Put

$$\Delta_{ij} = \beta_i - \beta_j \frac{\mu_i}{\mu_j}, \quad i, j = 1, \ldots, p,$$

and let Δ be the (p, p)-matrix with Δ_{ij} as its (i, j)-element.

(iv) All off-diagonal elements of Δ are unequal to zero.

Let $\bar{z}$ be a g.c. point for $PQ(\cdot)$ of type 5. Then, $\bar{z}$ is a g.c. point for the family $PQ_j(\cdot)$ obtained from $PQ(\cdot)$ by deleting the constraint $g_j,\ j = 1, \ldots, p$. Let $j_0 \in J_{\bar{z}}$ be arbitrary, but fixed. We consider the family $PQ_{j_0}(\cdot)$. The Lagrange multipliers for $PQ_{j_0}(\bar{t})$ corresponding with the active inequalities at $\bar{x}$ are just the numbers $\Delta_{jj_0},\ j \in J_{\bar{z}} \setminus \{j_0\}$. Due to definition 2.3(iv) these multipliers are non-vanishing. From this, we see that $\bar{z}$ is g.c. point for $PQ_{j_0}(\cdot)$ of type 1. Now, it can be shown (cf. [10]) that a reduced neighborhood Ω of $\bar{z}$ exists such that $\Sigma \cap \Omega$ consists of merely g.c. points of type 1 and admits the typical ramification structure as depicted in figure 3 (here, by δ_j we denote the number of negative entries in the j-th column of Δ).

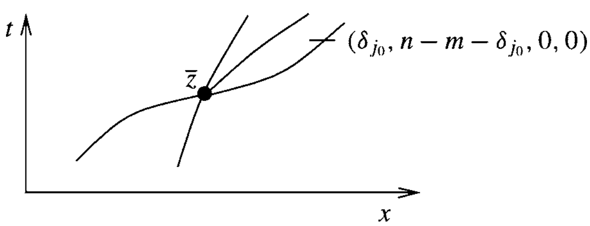

Figure 3. Local structure of Σ around a g.c. point $\bar{z}$ of type 5. At $\bar{z}$, the set Σ ramifies into p C^1-curves, say Σ_j^+, with (constant) index quadruple (LI, LCI, QI, QCI) equal to $(\delta_j, n - m - \delta_j, 0, 0)$, $j = 1, \ldots, p$.

3. Results

We represent $PQ(\cdot)$ by the following mapping Q

$$Q : t \mapsto \begin{bmatrix} A(t) & D^{\mathrm{T}}(t) & a(t) \\ D(t) & 0 & d(t) \end{bmatrix}.$$

Apparently, the map Q may be interpreted as an element of $C^1(\mathbb{R}, \mathbb{R}^N)$, $N = \frac{1}{2}n(n+3) + (m+s)(n+1)$. We endow the space $C^1(\mathbb{R}, \mathbb{R}^N)$ with the so-called strong C^1-topology (cf. [7]), denoted by C^1_s.

Now, we have:

Theorem 3.1 (Theorem of the Three Types). There exists a C^1_s-open and -dense subset, say $\mathcal{O}$, of $C^1(\mathbb{R}, \mathbb{R}^N)$ such that for any $PQ(\cdot)$, represented by $Q \in \mathcal{O}$, each g.c. point is of type 1, 2 or 5.

A sketch of the proof of this theorem (focussing on the non-occurrence of g.c. points of type 3, 4 rather than on the verification of the more technical conditions in definitions 2.2, 2.3) is to be found in [6]. In the present paper we give a full proof.

Theorem 3.1, together with the analysis as given in section 2, yields (see also [10]):

Corollary 3.1. Given any $PQ(\cdot)$, represented by $Q \in \mathcal{O}$ (cf. theorem 3.1), then:

The set $\Sigma^{(1)}$ of g.c. points of type 1 is a C^1-manifold; the g.c. points of types 2 and 5 are isolated points, situated in the topological closure of $\Sigma^{(1)}$.

The changes in the index quadruple (LI, LCI, QI, QCI) when passing along Σ a g.c. point of type 1, 2, 5 are as depicted in figures 1–3, respectively.

Let $\bar{z}$ be a g.c. point of type 1, and denote by $\Sigma^{(1)}_{\bar{z}}$ the connected component of $\Sigma^{(1)}$ which contains $\bar{z}$. In the sequel we always have $|J_{\bar{z}}| = p$. From lemma 2.1 we know that $\Sigma^{(1)}_{\bar{z}}$ is a C^1-curve, parameterized by $(x(t), t)$ with $t \in I_{\bar{z}}$ (=open interval), and $x(\bar{t}) = \bar{x}$. We ask for the evolution of $(x(t), t)$ when t increases. Put $\hat{t} :=$ supremum of t over $I_{\bar{z}}$. If $\hat{t} = \infty$, then the evolution of $(x(t), t)$ is trivial. However, if $\hat{t} < \infty$, then various possibilities may occur:

Theorem 3.2 (Evolution of the critical set). Given $PQ(\cdot)$, represented by $Q \in \mathcal{O}$ (cf. theorem 3.1), then for any g.c. point $\bar{z}$ of type 1 with $\hat{t} < \infty$:

Either $\lim_{t \uparrow \hat{t}} x(t) := \hat{x}$, some $\hat{x} \in \mathbb{R}^n$, (case 1)

or $\lim_{t \uparrow \hat{t}} \|x(t)\| = \infty$, where $\|\cdot\|$ stands for Euclidean norm. (case 2)

In case 1:

Either $(\hat{x}, \hat{t})$ is a g.c. point of type 2 (and then $m + p \leqslant n$),

or $(\hat{x}, \hat{t})$ is a g.c. point of type 5 (and then $m + p = n + 1$).

In case 2: Put

$$M(t) = \begin{bmatrix} A(t) & D^{\mathrm{T}}_{J_{\bar{z}}}(t) \\ D_{J_{\bar{z}}}(t) & 0 \end{bmatrix}, \qquad \overline{M}(t) = \begin{bmatrix} M(t) & \begin{matrix} a(t) \\ d_{J_{\bar{z}}}(t) \end{matrix} \end{bmatrix}. \tag{3}$$

In case 2, only the following two alternatives are possible:

(a) $m + p < n$ (g.c. point at infinity of type 3).
Then, $\operatorname{rank} \overline{M}(\hat{t}) = 1 + \operatorname{rank} M(\hat{t}) = n + m + p$, and $\operatorname{rank} D_{J_{\bar{z}}}(\hat{t}) = m + p$. (Thus there exists a unit vector $(\hat{x}, \hat{\eta}) \in \mathbb{R}^n \times \mathbb{R}^{m+p}$ in $\ker M(\hat{t})$, vector $\hat{x}$ being non-vanishing and uniquely determined up to a sign.)

(b) $m + p = n$ (g.c. point at infinity of type 4).
Then, $\operatorname{rank} \overline{M}(\hat{t}) = 1 + \operatorname{rank} M(\hat{t}) = 2n$, and $\operatorname{rank} D_{J_{\bar{z}}}(\hat{t}) = n - 1$, $\operatorname{rank}[D_{J_{\bar{z}}}(\hat{t}), d_{J_{\bar{z}}}(\hat{t})] = n$. (Thus there exists a unit vector $\hat{x}$ in $\ker D_{J_{\bar{z}}}(\hat{t})$, being determined up to a sign and (of course) non-vanishing.)

In both alternatives we have

$$\lim_{t \uparrow \hat{t}} \frac{x(t)}{\|x(t)\|} = \tilde{x}, \quad \text{where } \tilde{x} = \frac{\hat{x}}{\|\hat{x}\|} \quad \text{or} \quad \tilde{x} = -\frac{\hat{x}}{\|\hat{x}\|}. \tag{4}$$

Interpretation of theorem 3.2 (g.c. points at infinity of type 3 and 4).
The alternatives (a) and (b) in theorem 3.2, case 2, can be interpreted as follows:

Case 2(a) ($m + p < n$). In equation (1), cf. section 2, the feasibility condition holds for $t = \hat{t}$, but the critical point condition not. This fact is reflected in the property that $x(t)$ "tends to infinity", for $t \uparrow \hat{t}$, according to an "asymptotic direction" given by $\tilde{x}$. Recall that $\operatorname{rank} D_{J_{\bar{z}}}(\hat{t}) = m + p$; moreover, in the proof of theorem 3.2 we will see that $\operatorname{corank} A_{J_{\bar{z}}}(\hat{t})_{|\ker D_{J_{\bar{z}}}(\hat{t})} = 1$ and that *all* Lagrange multipliers for $z(t)$ tend to $\pm\infty$ if $t \uparrow \hat{t}$ (in particular they are non-vanishing). Compare also conditions B1–3 for a g.c. point of type 3 in [10]. Therefore, we say that for $t = \hat{t}$ there is a *g.c. point at infinity of type 3*.

Case 2(b) ($m + p = n$). In equation (1), cf. section 2, the feasibility condition is violated for $t = \hat{t}$. This fact is reflected in the property that $x(t)$ "tends to infinity", for $t \uparrow \hat{t}$, according to an "asymptotic direction" given by $\tilde{x}$. Recall that $\operatorname{rank} D_{J_{\bar{z}}}(\hat{t}) = n - 1$, and $m + |J_{\bar{z}}| > 0$; moreover, in the proof of theorem 3.2 we will obtain that *all* Lagrange multipliers for $z(t)$ tend to $\pm\infty$ if $t \uparrow \hat{t}$ (in particular they are non-vanishing). Compare also conditions C1–C4 for a g.c. point of type 4 in [10]. Therefore, we say that for $t = \hat{t}$ there is a *g.c. point at infinity of type 4*. □

In the above theorem, we investigated the evolution of the connected components of $\Sigma^{(1)}$ for increasing values of t. Apparently, similar results can be obtained for decreasing t values.

Theorem 3.3 (Two-sided g.c. points at infinity). Let $PQ(\cdot)$ be represented by the map Q in $\mathcal{O}$, and let $\bar{z}$ be a g.c. point of type 1. Assume that we are in the situation of theorem 3.2, case 2. We distinguish between two cases:

Case $J_{\bar{z}} = J$. Then, there exits a component of $\Sigma^{(1)}$, say $\widetilde{\Sigma}^{(1)}_{\bar{z}}$, with the same set of active constraints ($J_{\bar{z}}$) as $\Sigma^{(1)}_{\bar{z}}$, and parameterized by $(x(t), t)$ with $\hat{t} < t < \hat{t} + \tilde{\varepsilon}$, some $\tilde{\varepsilon} > 0$, such that

$$\lim_{t \downarrow \hat{t}} \frac{x(t)}{\|x(t)\|} = -\tilde{x},$$

where $\tilde{x}$ is the unit vector as introduced in theorem 3.2. Moreover, if (I_1, I_2, I_3, I_4) stands for the index quadruple of g.c. points on $\Sigma^{(1)}_{\bar{z}}$, then the index quadruple of a g.c. point on $\widetilde{\Sigma}^{(1)}_{\bar{z}}$ is given by:

For a g.c. point at infinity of type 3: $(I_2, I_1, I_3 \pm 1, I_4 \mp 1)$;

For a g.c. point at infinity of type 4: $(I_1, I_2, 0, 0)$.

(Note that in the latter case we always have $I_3 = I_4 = 0$ since all points of $\Sigma^{(1)}_{\bar{z}}$ consist of "corner points" $(m + p = n)$.)

Case $J_{\bar{z}} \subsetneqq J$. A component $\widetilde{\Sigma}^{(1)}_{\bar{z}}$ as introduced in case $J_{\bar{z}} = J$ does not occur.

Phenomena, as described in theorem 3.3, case $J_{\bar{z}} = J$, will be referred to as to *two-sided g.c. points at infinity of type 3* (if $m + p < n$) *or type 4* (if $m + p = n$).

Corollary 3.2.

(a) If, in theorem 3.3, we have $J = \emptyset$ (i.e., no inequality constraints) then always $J_{\bar{z}} = J$. Hence, in this case, all g.c. points at infinity are two-sided and of type 3 (due to the overall condition $m < n$).

(b) For $PQ(\cdot)$, $Q \in \mathcal{O}$, two-sided g.c. points at infinity of type 4 are only possible in the case where $J \neq \emptyset$ and all inequality constraints are active.

4. Examples

In this section we give two illustrative examples.

Example 1. We consider the following family of optimization problems ($n = 1$, $m = 0$, $J = \emptyset$):

$$\min \frac{1}{2}(1 - t)x^2 + tx. \qquad PQ(t)$$

The family $PQ(\cdot)$ is represented by the mapping

$$Q : t \longmapsto \overline{M}(t) = \big[(1 - t) \mid t\big].$$

The critical point condition $(1-t)x+t=0$ leads to the critical point set

$$\Sigma = \left\{ \big(x(t), t\big) \mid x(t) = \frac{t}{t-1},\ t \neq 1 \right\}.$$

For $t \neq 1$, the condition $\widetilde{ND}$ $(1-t \neq 0)$ holds. Since there are no constraints, all g.c. points are of type 1, i.e., $\Sigma^{(1)} = \Sigma$ (cf. corollary 3.1). At $\hat{t} = 1$ we have

$$\lim_{t \uparrow \hat{t}} x(t) = -\infty, \quad \lim_{t \downarrow \hat{t}} x(t) = \infty, \quad \text{and} \quad \lim_{t \to \hat{t} \pm 0} \frac{x(t)}{|x(t)|} = \pm 1.$$

Thus, at $t = \hat{t}$ we have given a two-sided g.c. point at infinity of type 3 (cf. theorem 3.3). The problem $Q(\cdot)$ belongs to the generic class $\mathcal{O}$ which is specified in section 5. $\square$

Example 2. We consider the following family of optimization problems ($n = 1$, $m = 0$, $J = \{1, 2\}$):

$$\min \frac{1}{2}x^2 - x \quad \text{subject to} \quad \begin{cases} g_1(x,t) := tx - 2 \leqslant 0, \\ g_2(x,t) := (2t-1)x - 1 \leqslant 0. \end{cases} \qquad PQ(t)$$

Thus,

$$A(t) = 1, \quad a(t) = -1, \quad D(t) = C(t) = \begin{bmatrix} t \\ 2t-1 \end{bmatrix}, \quad d(t) = c(t) = \begin{bmatrix} -2 \\ -1 \end{bmatrix}.$$

Hence, the family $PQ(\cdot)$ is represented by the mapping

$$Q : t \longmapsto \overline{M}(t) = \left[M(t) \begin{matrix} |\ a(t) \\ |\ d(t) \end{matrix} \right] = \begin{bmatrix} 1 & t & 2t-1 & | -1 \\ t & 0 & 0 & | -2 \\ 2t-1 & 0 & 0 & | -1 \end{bmatrix}.$$

Let J_0 be an arbitrary subset of J. We put $\Sigma_{J_0} = \{z \in \Sigma \mid J_z = J_0\}$. We distinguish between four subcases:

Case $J_0 = \emptyset$. At g.c. points where this condition holds, LICQ is trivially fulfilled. Hence, these points follow from (1) together with the conditions $g_1(z) < 0$ and $g_2(z) < 0$:

$$x - 1 = 0, \quad tx - 2 < 0, \quad (2t-1)x - 1 < 0.$$

This yields the following subset $\Sigma_\emptyset$ of Σ:

$$\Sigma_\emptyset = \big\{ \big(x(t), t\big) = (1, t) \mid t < 1 \big\}.$$

Since $A(t) = 1$, the condition $\widetilde{ND}$ holds everywhere on $\Sigma_\emptyset$. Thus, $\Sigma_\emptyset$ consists of merely g.c. points of type 1.

Case $J_0 = \{1, 2\}$. The only feasible point in this case is: $\overline{z} = (3, 2/3)$. The condition LICQ is automatically violated at this point, and thus

$$\Sigma_{\{1,2\}} = \left\{ \left(3, \frac{2}{3} \right) \right\}.$$

By a straightforward verification one checks that this g.c. point is of type 5.

Case $J_0 = \{1\}$. If $t = 0$ then, the condition $g_1 = 0$ is violated (no feasibility). If $t \neq 0$ (and thus LICQ holds), the g.c. points follow from (1) together with the condition $g_2 < 0$. We find:

$$\Sigma_{\{1\}} = \left\{ \left(\frac{2}{t}, t \right) \middle| \, 0 < t < \frac{2}{3} \right\}.$$

Moreover, for the Lagrange multiplier $\mu_1(t)$ we have

$$\mu_1(t) = \frac{t-2}{t^2}, \quad 0 < t < \frac{2}{3}.$$

This multiplier does not vanish (for t values on its domain). Since $A(t) = 1$, the condition $\widetilde{ND}$ holds at all points of $\Sigma_{\{1\}}$, and thus $\Sigma_{\{1\}}$ consists of merely g.c. points of type 1.

Case $J_0 = \{2\}$. If $t = 1/2$ (and thus rank $D_{J_0} = 0$), the condition $g_2 = 0$ is violated (no feasibility). If $t \neq 1/2$ (and thus LICQ holds), the g.c. points follow from (1) together with the condition $g_1 < 0$. We find

$$\Sigma_{\{2\}} = \left\{ \left(\frac{1}{2t-1}, t \right) \, \middle| \, t < \frac{1}{2} \text{ or } t > \frac{2}{3} \right\}.$$

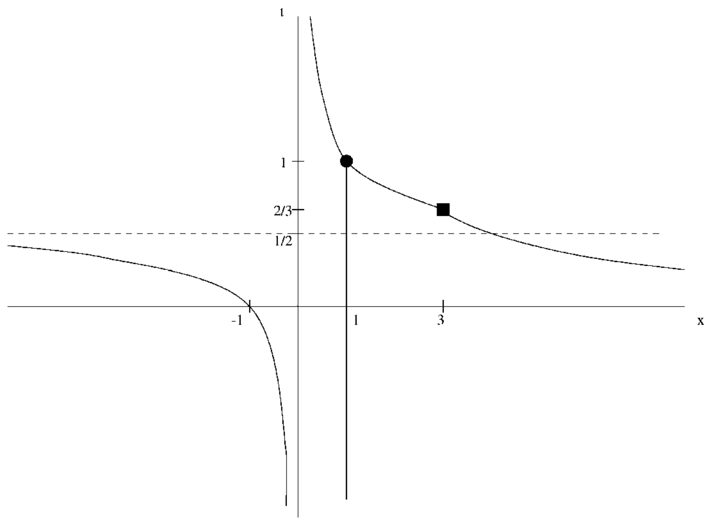

Figure 4. The critical set of $PQ(\cdot)$; $\bullet$ a g.c. point of type 2, $\blacksquare$ a g.c. point of type 5.

For the Lagrange multiplier $\mu_2(t)$ we have

$$\mu_2(t) = \frac{2(t-1)}{(2t-1)^2}, \quad t < \frac{1}{2} \text{ or } t > \frac{2}{3}.$$

This multiplier vanishes if $t = 1$. It is easily verified that $\bar{z} = (1, 1)$ is a g.c. point of type 2. All other points of $\Sigma_{\{2\}}$ are of type 1.

These observations on Σ are summarized in figure 4. Note, that the g.c. points of type 2 and 5 are situated in the closure of the set $\Sigma^{(1)}$ of g.c. points of type 1.

For $t \downarrow 0$ and for $t \uparrow 1/2$ we have a one-sided g.c. point of type 4 (cf. theorems 3.2, 3.3). Note that when the constraint $g_1 \leqslant 0$ is deleted the resulting problem will not admit a g.c. point of type 5 but (for $t = 1/2$) a two-sided g.c. point of type 4 at infinity.

5. Proofs

We begin with the introduction of two classes of special structured matrices. For any integer $q \geqslant 0$, let $\mathcal{M}_{n,q}$ be the set of all real $(n+q, n+q)$-matrices M of the form

$$M = \begin{bmatrix} A & D^{\mathrm{T}} \\ D & 0 \end{bmatrix},$$

where A is a symmetric (n, n)-matrix, D a (q, n)-matrix, and 0 the (q, q) null matrix. (If $q = 0$, then the matrices D and 0 are non-existent.) Moreover, we define $\overline{\mathcal{M}}_{n,q}$ as the class of all matrices $\overline{M} = [Me]$, where $e = [\substack{a \\ d}]$ and $a \in \mathbb{R}^n$, $d \in \mathbb{R}^q$. Apparently, the set $\overline{\mathcal{M}}_{n,q}$ can be identified with $\mathbb{R}^K$, $K = \frac{1}{2}n(n+3) + q(n+1)$. Note that $\overline{\mathcal{M}}_{n,m+s}$ is just the target space of the mappings Q representing the families of optimization problems $PQ(\cdot)$.

Each of the classes $\mathcal{M}_{n,q}$ and $\overline{\mathcal{M}}_{n,q}$ will be partitioned into finitely many mutually disjoint subsets, the so-called *strata* (denoted by $V_{k,\xi}$ and $\overline{V}^{\ell}_{k,\xi,[\tau]}$ respectively).

Specification of $V_{k,\xi}$. For any integer k, with $0 \leqslant k \leqslant \min\{n, q\}$, and any vector $\xi = (\xi^+, \xi^-, \xi^0)$ with non-negative integer components summing up to $(n-k)$, the set $V_{k,\xi}$ is given by:[1]

$$V_{k,\xi} := \left\{ M \in \mathcal{M}_{n,q} \mid \operatorname{rank} D = k,\ \operatorname{In}(A|_{\ker D}) = \xi \right\},$$

where In$(\cdot)$ stands for the *inertia* of a symmetric matrix, i.e. the numbers of respectively positive, negative and zero eigenvalues (counted by multiplicity) of this matrix.

For later purposes we give the so-called *Inertia Theorem* (cf. [11,12]):

For any $M \in V_{k,\xi}$:

$$\operatorname{In}(M) = \operatorname{In}(A|_{\ker D}) + (k, k, q-k). \tag{5}$$

[1] If rank $D = n$, respectively $q = 0$ we define $\operatorname{In}(A|_{\ker D}) = (0, 0, 0)$, respectively $\operatorname{In}(A|_{\ker D}) = \operatorname{In}(A)$.

Table 1
Specification of the $\overline{\mathcal{M}}_{n,q}$-strata $\overline{V}^{\ell}_{k,\xi,[\tau]}$.

ℓ	$\overline{M} \in \overline{V}^{\ell}_{k,\xi,[\tau]}$	iff $M \in V_{k,\xi}$, and moreover:	$\overline{V}^{\ell}_{k,\xi,[\tau]} = \emptyset$ if
1 :		$\operatorname{rank}[Dd] = k + 1$ (thus $d \neq 0$)	$k = q$
2 :	$d \neq 0$	$\operatorname{rank}[Dd] = k$, $\operatorname{rank} \overline{M} = 1 + \operatorname{rank} M$	$k = 0$ or n; or $\xi^0 = 0$
3 :	$d \neq 0$	$\operatorname{rank} \overline{M} = \operatorname{rank} M$; $\tau = \operatorname{sign} \overline{M}$	$k = 0$
4 :	$d = 0$	$\operatorname{rank} \overline{M} = 1 + \operatorname{rank} M$ (thus $a \neq 0$)	$\xi^0 = 0$
5 :	$d = 0,\ a \neq 0$	$\operatorname{rank}[D^{\mathrm{T}}a] = k$ (thus $\operatorname{rank} \overline{M} = \operatorname{rank} M$, $\operatorname{sign} \overline{M} = 0$)	$k = 0$
6 :	$d = 0,\ a \neq 0$	$\operatorname{rank}[D^{\mathrm{T}}a] = k + 1$; $\operatorname{rank} \overline{M} = \operatorname{rank} M,\ \tau = \operatorname{sign} \overline{M}$	$\xi^+ = \xi^- = 0$, or $\xi^- = 0,\ \tau = -1, 0$ or $\xi^+ = 0,\ \tau = 0, 1$
7 :	$d = 0,\ a = 0$	(thus $\operatorname{rank}[Dd] = k$; $\operatorname{rank} \overline{M} = \operatorname{rank} M$)	

In this table, $\operatorname{sign} \overline{M}$ is defined as the signature of $y^{\mathrm{T}}e$, where y is any vector in $\mathbb{R}^{n+q}$ such that $My = e$. (Such y exists iff $\operatorname{rank} M = \operatorname{rank} \overline{M}$; $\operatorname{sign} \overline{M}$ does not depend on the ambiguity of y.) Note that the parameter τ only plays a role if $\ell = 3$ or 6.

Specification of $\overline{V}^{\ell}_{k,\xi,[\tau]}$. Let the integer k and the triple ξ be as above. Then, for the values $\ell = 1, \ldots, 7,\ \tau = -1, 0, 1$, the subsets $\overline{V}^{\ell}_{k,\xi,[\tau]}$ of $\overline{\mathcal{M}}_{n,q}$ are defined according to table 1.

Interpretation of the $\overline{\mathcal{M}}_{n,q}$*-strata.* The conditions specifying the sets $\overline{V}^{\ell}_{k,\xi,[\tau]}$ may look rather fancy. However, on various occasions, an interpretation in terms of quadratic optimization problems is possible.

For example, let us consider a family $PQ(\cdot)$ with J empty (i.e., there are no inequality constraints: $q = m$), represented by the mapping $Q(\cdot)$. Then, we have $Q(t) \in \overline{\mathcal{M}}_{n,m}$ for all t and moreover, for fixed $\bar{t}$:

$PQ(\bar{t})$ does not admit any feasible points if and only if $Q(\bar{t}) \in \overline{V}^{1}_{k,\xi,[\tau]}$.

Now, consider $\tilde{t}$ such that $Q(\tilde{t}) \in \overline{V}^{\ell}_{k,\xi,[\tau]}$ with $\ell \geqslant 2$. Then LICQ holds at all feasible points for $PQ(\tilde{t})$ iff $k = q$ (and none of these feasible points is a g.c. point iff $\ell = 2$ or 4). When $k < q$, each of the feasible points is also a g.c. point.

Using (5) it follows: if $Q(\tilde{t}) \in \overline{V}^{3}_{q,\xi,[\tau]}$ with $\xi^0 = 0$, then $PQ(\tilde{t})$ admits only one g.c. point and at this point both LICQ and ND hold. □

Lemma 5.1. The partitioning of $\overline{\mathcal{M}}_{n,q}$ into $\overline{V}^{\ell}_{k,\xi,[\tau]}$ forms a Whitney-regular stratification.

Proof. See [12]. □

Here, we will not dwell on the precise definition of *Whitney-regular stratification* but merely note that in the context of lemma 5.1 this implies that all strata $\overline{V}^{\ell}_{k,\xi,[\tau]}$ are smooth submanifolds of $\overline{\mathcal{M}}_{n,q}$ and, moreover, neighboring strata stick together in such a regu-

Table 2
Codimensions of non-empty $\overline{\mathcal{M}}_{n,q}$-strata.

ℓ	codim $\overline{V}^{\ell}_{k,\xi,[\tau]}$	ℓ	codim $\overline{V}^{\ell}_{k,\xi,[\tau]}$
1	$(n-k)(q-k)+\frac{1}{2}\xi^0(\xi^0+1)$	4	$q+(n-k)(q-k)+\frac{1}{2}\xi^0(\xi^0+1)$
2	$(n-k+1)(q-k)+\frac{1}{2}\xi^0(\xi^0+1)$	5	$q+(n-k)(q-k+1)+\frac{1}{2}\xi^0(\xi^0+1)$
3	$(n-k+1)(q-k)+\frac{1}{2}\xi^0(\xi^0+3)+1-\tau^2$	6	$q+(n-k)(q-k)+\frac{1}{2}\xi^0(\xi^0+3)+1-\tau^2$
		7	$q+n+(n-k)(q-k)+\frac{1}{2}\xi^0(\xi^0+1)$

lar way that the local topological type of the partitioning remains constant along (the connected components) of each stratum. See [2] for other references on this subject.

Lemma 5.2. The codimensions of $\overline{V}^{\ell}_{k,\xi,[\tau]}$ with respect to $\overline{\mathcal{M}}_{n,q}$ $(=\mathbb{R}^K)$ can be expressed in terms of the parameters $n,\ k,\ q,\ \xi^0$ and τ, according to table 2.

Proof. See [12]. □

For us, the relevance of the latter two lemmas relies upon the possibility of applying Thom's Transversality Theory.

A mapping $Q \in C^1(\mathbb{R}, \overline{\mathcal{M}}_{n,q})$ is said to be *transversal* ($\pitchfork$) to the manifold $\overline{V}^{\ell}_{k,\xi,[\tau]}$ if for any $t_0 \in \mathbb{R}$ either $Q(t_0) \notin \overline{V}^{\ell}_{k,\xi,[\tau]}$, or $Q(t_0) \in \overline{V}^{\ell}_{k,\xi,[\tau]}$ in which case the tangent vector $\frac{\mathrm{d}}{\mathrm{d}t}Q(t_0)$ together with the tangent vectors at $Q(t_0)$ to $\overline{V}^{\ell}_{k,\xi,[\tau]}$, span the whole $\overline{\mathcal{M}}_{n,q}$ (cf. [7; 9, theorem 7.3.2]). We put

$$\pitchfork \overline{\mathcal{M}}_{n,q} := \text{set of all maps in } C^1(\mathbb{R}, \overline{\mathcal{M}}_{n,q}), \text{ transversal to each } \overline{\mathcal{M}}_{n,q}\text{-stratum.}$$

Then, from Thom's transversality theorem (cf. [7,9]) it follows:

Lemma 5.3. The set $\pitchfork \overline{\mathcal{M}}_{n,q}$ is C^1_s-open and -dense in $\mathbb{C}^1(\mathbb{R}, \overline{\mathcal{M}}_{n,q})$ and moreover, for any $Q \in \pitchfork \overline{\mathcal{M}}_{n,q}$, the inverse image $Q^{-1}(\overline{\mathcal{M}}_{n,q})$ is a Whitney-regular stratified subset of $\mathbb{R}$ with strata $Q^{-1}(\overline{V}^{\ell}_{k,\xi,[\tau]})$. Moreover for non-empty strata we have codim $Q^{-1}(\overline{V}^{\ell}_{k,\xi,[\tau]}) =$ codim $\overline{V}^{\ell}_{k,\xi,[\tau]}$.

Corollary 5.1. For any $Q \in \pitchfork \overline{\mathcal{M}}_{n,q}$, the curve $\{Q(t) \mid t \in \mathbb{R}\}$ only intersects strata $\overline{V}^{\ell}_{k,\xi,[\tau]}$ of codimension 0 or 1. In the case of a codimension 1 strata this intersection only occurs for isolated t values.

In view of this corollary, it is interesting to know, for which parameters n, q, k, ξ and τ a stratum of $\overline{\mathcal{M}}_{n,q}$ has codimension 0 or 1. By direct verification of table 2, thereby taking into account that some strata are empty (cf. table 1) the various possibilities are to be found in tables 3 and 4.

Table 3
$\overline{\mathcal{M}}_{n,q}$-strata of codim 0.

	codim $\overline{V}^{\ell}_{k,\xi,[\tau]} = 0$ if	Interpretation
$q \geqslant n+1$	$l = 1,\ k = n\ (\xi^0 = 0)$	no feasible points
$q = n$	$l = 3,\ k = q\ (\xi^0 = 0),\ \tau = \pm 1$	LICQ, ND hold; unique g.c. point
$0 < q < n$	$l = 3,\ k = q,\ \xi^0 = 0,\ \tau = \pm 1$	LICQ, ND hold; unique g.c. point
$q = 0$	$l = 6\ (k = 0),\ \xi^0 = 0,\ \tau = \pm 1$	no constraints, ND holds; unique g.c. point

Table 4
$\overline{\mathcal{M}}_{n,q}$-strata of codim 1.

	codim $\overline{V}^{\ell}_{k,\xi,[\tau]} = 1$ if	Interpretation
$q > n+1$	does not occur	—
$q = n+1$	$\ell = 3,\ k = n\ (\xi^0 = 0),\ \tau = \pm 1$	LICQ violated
$q = n$	$\ell = 1,\ k = n-1\ (= q-1),\ \xi^0 = 0$	no feasible points
	$\ell = 3,\ k = n\ (\xi^0 = 0),\ \tau = 0$	LICQ, ND hold; unique g.c. point
	$\ell = 5,\ k = n = 1\ (\xi^0 = 0)$	LICQ, ND hold; unique g.c. point
$0 < q < n$	$\ell = 2,\ k = q,\ \xi^0 = 1$	LICQ holds; no g.c. points
	$\ell = 3,\ k = q,\ \xi^0 = 0,\ \tau = 0$	LICQ, ND hold; unique g.c. point
	$\ell = 6,\ k = q = 1,\ \xi^0 = 0,\ \tau = \pm 1$	LICQ, ND hold; unique g.c. point
$q = 0$	$\ell = 4\ (k = 0),\ \xi^0 = 1$	no g.c. points
$q = 0\ (n > 1)$	$\ell = 6\ (k = 0),\ \xi^0 = 0,\ \tau = 0$	ND holds; unique g.c. point
$q = 0\ (n = 1)$	$\ell = 7\ (k = 0),\ \xi^0 = 0$	ND holds; unique g.c. point

Lemma 5.4 (Preliminary version of theorem 3.1). There exists a C^1_s-open and -dense subset, say $\mathcal{O}_1$, of $C^1(\mathbb{R}, \mathbb{R}^N)$ such that for any $PQ(\cdot)$, represented by $Q \in \mathcal{O}_1$,

either LICQ and ND hold for all g.c. points,
or LICQ is violated at isolated feasible points (in which case the number of active constraints equals $n+1$).

Proof. Let $J_0 \subset J$ be arbitrary, but fixed and define $Q_{J_0} : \mathbb{R} \to \overline{\mathcal{M}}_{n,m+|J_0|}$ by

$$Q_{J_0}(\cdot) := \begin{bmatrix} A(\cdot) & D^{\mathrm{T}}_{J_0}(\cdot) & a(\cdot) \\ D_{J_0}(\cdot) & 0 & d_{J_0}(\cdot) \end{bmatrix}.$$

Next, we partition the mappings $Q(\cdot)$ in $C^1(\mathbb{R}, \overline{\mathcal{M}}_{n,m+s})$ as follows:

$$Q(\cdot) := \big(Q_{J_0}(\cdot), \big[D_{J\setminus J_0}(\cdot)\ d_{J\setminus J_0}(\cdot)\big]\big) \in C^1\big(\mathbb{R}, \overline{\mathcal{M}}_{n,m+|J_0|}\big) \times C^1\big(\mathbb{R}, \mathbb{R}^{K_{J_0}}\big),$$

where $K_{J_0} = (s - |J_0|)(n+1)$. We put

$$\mathcal{O}_{J_0} := \overline{\pitchfork}\, \overline{\mathcal{M}}_{n,m+|J_0|} \times C^1\big(\mathbb{R}, \mathbb{R}^{K_{J_0}}\big)$$

and

$$\mathcal{O}_1 := \bigcap_{J_0 \subset J} \mathcal{O}_{J_0}.$$

Then, basically due to lemma 5.3, the set $\mathcal{O}_1$ is C^1_s-open and -dense in $C^1(\mathbb{R}, \mathbb{R}^N)$.

Now consider a map $Q(\cdot) \in \mathcal{O}_1$, and let $\bar{z} = (\bar{x}, \bar{t})$ be a g.c. point for the family $PQ(\cdot)$ represented by $Q(\cdot)$. Put $J_0 = J_{\bar{z}}$. Then, the assertion of the lemma follows by straightforward inspection of tables 3, 4 and corollary 5.1. $\square$

We are going to prove theorem 3.1. This will be done by shrinking, in several steps, the set $\mathcal{O}_1$ (see lemma 5.4) until a C^1_s-open and -dense subset $\mathcal{O}$ of $C^1(\mathbb{R}, \mathbb{R}^N)$ is obtained with the property that each g.c. point for $PQ(\cdot)$, $Q(\cdot) \in \mathcal{O}$ is of type 1, 2 or 5 (cf. definitions 2.1, 2.2 and 2.3, respectively).

Idea of the shrinking procedure

1. Put

$$\mathcal{T}_{n,q} := \text{Union of all } \overline{\mathcal{M}}_{n,q}\text{-strata of codim} \geqslant 2.$$

Then, cf. [9, corollary 7.5.3], the set $\mathcal{T}_{n,q}$ is a closed, Whitney regular stratified subset of $\overline{\mathcal{M}}_{n,q}$.

2. Put

$$\overline{\pitchfork}\, \mathcal{T}_{n,q} := \{\varphi \in C^1(\mathbb{R}, \overline{\mathcal{M}}_{n,q}) \mid \varphi \,\overline{\pitchfork}\, \mathcal{T}_{n,q}\},$$

and

$$\mathcal{W}_{n,q} = \overline{\mathcal{M}}_{n,q} \setminus \mathcal{T}_{n,q}.$$

Then, from Thom's transversality theorem, it follows that $\overline{\pitchfork}\, \mathcal{T}_{n,q}$ is C^1_s-open and -dense in $C^1(\mathbb{R}, \overline{\mathcal{M}}_{n,q})$ and moreover, by a dimension argument,

$$\overline{\pitchfork}\, \mathcal{T}_{n,q} = C^1(\mathbb{R}, \mathcal{W}_{n,q}).$$

3. The set $\mathcal{W}_{n,q}$, being open in $\overline{\mathcal{M}}_{n,q}$, is a smooth submanifold of $\overline{\mathcal{M}}_{n,q}$ (of codim 0). Hence, the C^1_s-topology on $C^1(\mathbb{R}, \overline{\mathcal{M}}_{n,q})$ induces a topology (also denoted by C^1_s) on $C^1(\mathbb{R}, \mathcal{W}_{n,q})$. Now, Thom's transversality theorem (for C^1-mappings from $\mathbb{R}$ to $\mathcal{W}_{n,q}$) enables us to select subsets of $\overline{\pitchfork}\, \mathcal{T}_{n,q}$ which are C^1_s-open and -dense in $\overline{\pitchfork}\, \mathcal{T}_{n,q}$, and thus also C^1_s-open and -dense in $C^1(\mathbb{R}, \overline{\mathcal{M}}_{n,q})$. In fact, let T be any closed submanifold of $\mathcal{W}_{n,q}$ of codim 2, and put $\overline{\pitchfork}\, T = \{\varphi \in C^1(\mathbb{R}, \mathcal{W}_{n,q}) \mid \varphi \,\overline{\pitchfork}\, T\}$. Then, $\overline{\pitchfork}\, T$ is C^1_s-open and -dense in $C^1(\mathbb{R}, \overline{\mathcal{M}}_{n,q})$ and $\overline{\pitchfork}\, T = C^1(\mathbb{R}, \mathcal{W}_{n,q} \setminus T)$.

4. Successive application of the procedure in step 3, yields a C^1_s-open and -dense subset, say $\mathcal{S}_{n,q}$, of $C^1(\mathbb{R}, \overline{\mathcal{M}}_{n,q})$.

5. Now, we put for any $J_0 \subset J$:

$$\mathcal{S}_{J_0} := \mathcal{S}_{n,m+|J_0|} \times C^1\big(\mathbb{R}, \mathbb{R}^{K_{J_0}}\big),$$
$$\mathcal{S} := \text{intersection of the sets } \mathcal{S}_{J_0}, \text{ all } J_0 \subset J.$$

   Then, $\mathcal{S}$ is C_s^1-open and -dense in $C^1(\mathbb{R}, \mathbb{R}^N)$.

6. Finally, we shrink $\mathcal{O}_1$ (see lemma 5.4) to a smaller, but still open and dense, subset of $C^1(\mathbb{R}, \mathbb{R}^N)$ by considering the intersection $\mathcal{O}_1 \cap \mathcal{S}$.

In order to apply the above shrinking procedure, six technical lemmas are needed.

Lemma 5.5. Let $0 < q \leqslant n$ and denote by M_ν the $(n+q, n+q)$-matrix obtained from $M \in \mathcal{M}_{n,q}$ after replacing its ν-th column by $\left[\begin{smallmatrix} a \\ d \end{smallmatrix}\right]$. Then, for any j with $0 < j \leqslant q$, the set, say $\mathcal{W}_{n,q}^{n+j}$, of the matrices $\overline{M}$ in $\mathcal{W}_{n,q}$ given by the equations

$$\det M = 0, \qquad \det M_{n+j} = 0$$

is a closed, smooth manifold of codim 2 in $\mathcal{W}_{n,q}$.

Proof. The closedness of $\mathcal{W}_{n,q}^{n+j}$ being trivial, it is sufficient to show that $\mathcal{W}_{n,q}^{n+j}$ is given by two equations which are linearly independent (cf. [9]) in $\mathcal{W}_{n,q}$.

First of all we note that by inspection of tables 1, 3, 4 (and using (5)), for all $\overline{M} \in \mathcal{W}_{n,q}$ we have:

$$\operatorname{rank} \overline{M} = n + q.$$

Furthermore, for all $\overline{M} \in \mathcal{W}_{n,q}$ with $\det M = 0$ it follows:

$$\begin{aligned} &\text{either } \operatorname{rank} D = q,\ q < n, \quad \text{(Case 1)} \\ &\text{or } \operatorname{rank} D = q - 1,\ q = n. \quad \text{(Case 2)} \end{aligned}$$

Moreover, in this case we have:

$$\operatorname{rank} M = n + q - 1$$

and also

$$\left.\begin{array}{l} \text{if all columns of } M \text{ minus the } \nu\text{-th column form a linearly} \\ \text{independent set of vectors in } \mathbb{R}^{n+q}, \text{ then } \det M_\nu \neq 0. \end{array}\right\} \tag{6}$$

Now, let $\overline{M} \in \mathcal{W}_{n,q}^{n+j}$.

Case 1. Since $\operatorname{rank} D = q$, it follows from (6) that $\det M_\nu \neq 0$ for some ν with $1 \leqslant \nu \leqslant n$. Without loss of generality, we assume: $\det M_1 \neq 0$. The first column of M will be denoted by $(a_{11}, \ldots, a_{n1}, d_{11}, \ldots, d_{q1})^{\mathrm{T}}$ and its $(n+j)$-th column by $d_j = (d_{j1}, \ldots, d_{jn}, 0, \ldots, 0)^{\mathrm{T}}$. Taking the symmetry of M into account, we find for the contributions, say $\nabla^{(1)} \det M$, of $a_{11}, \ldots, a_{n1}, d_{11}, \ldots, d_{q1}$ to the gradient of $\det M$:

$$\nabla^{(1)} \det M = 2\Gamma_1 - (\tilde{a}_{11}, 0, \ldots, 0)^{\mathrm{T}},$$

where Γ_1 stands for a vector in $\mathbb{R}^{n+q}$ with as components the cofactors[2] in M for the entries of its first column, and the last vector on the right-hand side is also in $\mathbb{R}^{n+q}$, with $\tilde{a}_{11} =$ (cofactor in M for a_{11}).

Since $\det M_1 \neq 0$, we have $\Gamma_1 \neq 0$.

Now, we note that deleting from M_{n+j} its first column, yields a matrix with the same columns as in the matrix obtained from M_1 by deleting its $(n+j)$-th column. Using this observation, together with the symmetry of A, we find for the contribution of $a_{11}, \ldots, a_{n1}, d_{11}, \ldots, d_{q1}$ to the gradient of $\det(M_{n+j})$:

$$\nabla^{(1)} \det(M_{n+j}) = -\Gamma_2 + \Gamma_3 - \big(\tilde{a}_{11}^{j}, 0, \ldots, 0\big)^{\mathrm{T}},$$

where Γ_2, respectively Γ_3, stands for a vector in $\mathbb{R}^{n+q}$ with as components the cofactors in M_1 for the entries of its $(n+j)$-th column, respectively the cofactors in M_{n+j} for the entries of its first row, and where the last vector on the right-hand side is also in $\mathbb{R}^{n+q}$ with $\tilde{a}_{11}^{j} =$ (cofactor in M_{n+j} for a_{11}). We recall that for quadratic matrices H the formula $H_c^{\mathrm{T}} H = H_c H^{\mathrm{T}} = I \det H$ is valid for the cofactor matrix H_c of H. We now find the relation

$$d_j^{\mathrm{T}} \Gamma_1 = 0, \qquad d_j^{\mathrm{T}} \Gamma_2 \neq 0, \qquad d_j^{\mathrm{T}} \Gamma_3 = 0.$$

In fact, $d_j^{\mathrm{T}} \Gamma_1$ is the $(1, n+j)$-th entry of $M_c^{\mathrm{T}} M$ (which equals the zero-matrix) and $d_j^{\mathrm{T}} \Gamma_2$ is the $(n+j, n+j)$-th entry of $(M_1)_c^{\mathrm{T}} M_1 = \det M_1 I$ ($\det M_1 \neq 0$). To obtain $d_j^{\mathrm{T}} \Gamma_3 = 0$ we have to use that basically due to (6), the cofactor in M_{n+j} for its $(1, n+j)$-th entry vanishes and then $d_j^{\mathrm{T}} \Gamma_3$ equals the $(1, n+j)$-th entry of $(M_{n+j})_c M_{n+j}^{\mathrm{T}}$ and is equal to zero. In view of these relations and $\Gamma_1 \neq 0$ it follows that the vectors Γ_1 and $-\Gamma_2 + \Gamma_3$ are linearly independent. Then also $\nabla^{(1)} \det M$ and $\nabla^{(1)} \det(M_{n+j})$ must be linearly independent and the same is true for the gradients (in $\mathcal{W}_{n,q}$) of $\det M$ and $\det M_{n+j}$. To see that $\nabla^{(1)} \det M$ and $\nabla^{(1)} \det(M_{n+j})$ are linearly independent let us suppose that with c_1, c_2 not both zero we have

$$c_1\big(2\Gamma_1 - (\tilde{a}_{11}, 0, \ldots, 0)^{\mathrm{T}}\big) = c_2\big(-\Gamma_2 + \Gamma_3 - \big(\tilde{a}_{11}^{j}, 0, \ldots, 0\big)^{\mathrm{T}}\big).$$

Since the first component of $-\Gamma_2 + \Gamma_3$ equals $2\tilde{a}_{11}^{j}$ and the first component of Γ_1 equals $\tilde{a}_{11}$, after multiplying the first component of this relation by 2 we would find $c_1 2\Gamma_1 = c_2(-\Gamma_2 + \Gamma_3)$, a contradiction.

Case 2. We have $q = n$, and thus $\det M = (-1)^n \det^2 D$. Hence, the set $\mathcal{W}_{n,n}^{n+j}$ is also given by the equations $\det D = 0$, $\det M_{n+j} = 0$. We are done if we can show that these equations are linearly independent in $\mathcal{W}_{n,n}$. Since, in this case, $\operatorname{rank} D^{\mathrm{T}} = n - 1$, we may assume that the first $n-1$ columns of D^{T} form a linearly independent set of vectors in $\mathbb{R}^n$. We denote the j-th column of D^{T} by $d_j = (d_{j1}, \ldots, d_{jn})$. Then, the contribution

[2] The cofactor in M for its (i, j)-th entry equals $(-1)^{i+j} \det M_{ij}$, where M_{ij} is a matrix obtained from M by deleting its i-th row and j-th column and the matrix with as (i, j)-th entry the element $(-1)^{i+j} \det M_{ij}$ is called the cofactor matrix M_c of M.

of $d_{n1}, \dots, d_{nn}$ to the gradient of $\det D^{\mathrm{T}}$ is a *non-vanishing* vector $\in \mathbb{R}^n$, say Γ_1 (with as entries the cofactors in D^{T} for the entries of $d_n = (d_{n1}, \dots, d_{nn})$).

Moreover, it follows from (6) that $\det M_{n+n} \neq 0$. (In particular, this implies that $0 < j < n$.) We note that, deleting from M_{n+j} the last column yields a matrix with the same columns as in the matrix obtained from M_{n+n} by deleting its $(n + j)$-th column. Using this observation, we find for the contribution of $d_{n1}, \dots, d_{nn}$ to the gradient of $\det M_{n+j}$:

$$\nabla^{(2n)} \det(M_{n+j}) = -\Gamma_2 + \Gamma_3,$$

where Γ_2, respectively Γ_3, stand for vectors in $\mathbb{R}^n$ with as components the cofactors in M_{n+n} for the first n entries in its $(n+j)$-th column, respectively the cofactors in M_{n+j} for the first n entries in its $2n$-th row. Then, using again the properties of cofactor matrices mentioned above (together with the facts that $\det M_{n+n} \neq 0$ and that the cofactor in M_{n+j} for the $2n$-th element of the column $\left[\begin{smallmatrix} a \\ d \end{smallmatrix}\right]$ vanishes due to (6)) it is easily shown that

$$d_j^{\mathrm{T}} \Gamma_1 = 0, \qquad d_j^{\mathrm{T}} \Gamma_2 \neq 0, \qquad d_j^{\mathrm{T}} \Gamma_3 = 0.$$

Now, after a moment of reflection, it will be clear (use also $\Gamma_1 \neq 0$) that Γ_1 and $\nabla^{(2n)} \det(M_{n+j})$ are linearly independent. Hence, this is also true for the gradients (in $\mathcal{W}_{n,n}$) of $\det D$ and $\det M_{n+j}$. □

We define $(0 < q \leqslant n)$:

$$\mathcal{U}_{n,q} := \mathcal{W}_{n,q} \setminus \{\text{union of the sets } \mathcal{W}_{n,q}^{n+j},\ \text{all } j \text{ with } 0 < j \leqslant q\}.$$

Lemma 5.6. Let $1 \leqslant q \leqslant n$, and suppose j, j' are different indices, $1 \leqslant j, j' \leqslant q$. Then, the subset, say $\mathcal{U}_{n,q}^{j,j'}$, of $\mathcal{U}_{n,q}$ given by the equations

$$\det M_{n+j} = 0, \qquad \det M_{n+j'} = 0$$

is a closed, smooth manifold of codim 2 in $\mathcal{U}_{n,q}$.

Proof. The closedness of $\mathcal{U}_{n,q}^{j,j'}$ being trivial, we only prove that the equations $\det M_{n+j} = 0$, $\det M_{n+j'} = 0$ are linearly independent in $\mathcal{U}_{n,q}$. Let $\overline{M} \in \mathcal{U}_{n,q}^{j,j'}$. Then (by construction of $\mathcal{U}_{n,q}$) we have: $\det M \neq 0$. Thus, the contribution of the entries of $\left[\begin{smallmatrix} a \\ d \end{smallmatrix}\right]$ to the gradients of $\det(M_{n+j})$ and $\det(M_{n+j'})$, being equal to the $(n + j)$-th respectively the $(n + j')$-th column of the cofactor matrix of M, are linearly independent. Hence, this is also true for the gradients (in $\mathcal{U}_{n,q}$) of $\det(M_{n+j})$ and $\det(M_{n+j'})$. □

We define $(0 < q \leqslant n)$:

$$\mathcal{V}_{n,q} := \mathcal{U}_{n,q} \setminus \{\text{union of the sets } \mathcal{U}_{n,q}^{j,j'},\ \text{all } j, j',\ 1 \leqslant j, j' \leqslant q,\ j \neq j'\}.$$

Lemma 5.7. Let $0 < j \leqslant q \leqslant n$ and denote by $\widehat{M}^{n+j}$ the matrix obtained from $M \in \mathcal{M}_{n,q}$ after deleting its $(n+j)$-th column and $(n+j)$-th row. Then, the subset, say $\mathcal{V}_{n,q}^{n+j}$, of $\mathcal{V}_{n,q}$ given by the equations

$$\det \widehat{M}^{n+j} = 0, \qquad \det M_{n+j} = 0$$

is a closed, smooth submanifold of codim 2 in $\mathcal{V}_{n,q}$.

Proof. The closedness of $\mathcal{V}_{n,q}^{n+j}$ being trivial, we only prove that the equations $\det \widehat{M}^{n+j} = 0$, $\det M_{n+j} = 0$ are linearly independent in $\mathcal{V}_{n,q}$. For $\overline{M} \in \mathcal{V}_{n,q}^{n+j}$ we distinguish between two possibilities: $q = n$, and $0 < q < n$.

Case $q = n$: Since $\overline{M} \in \mathcal{W}_{n,q}$ it follows from tables 1, 3, 4 that:

$$\begin{array}{lll} \text{Either} & \det M = 0,\ \operatorname{rank} D = n-1 & \big(\text{if } \overline{M} \in \overline{V}^1_{n-1,\xi,[\tau]},\ \xi^0 = 0\big), \\ \text{or} & \det M \neq 0,\ \operatorname{rank} D = n & \big(\text{if } \overline{M} \in \overline{V}^5_{n,\xi,[\tau]},\ n = 1,\ \xi^0 = 0, \text{ or} \\ & & \overline{M} \in \overline{V}^3_{n,\xi,[\tau]},\ \xi^0 = 0,\ \tau = 0, \pm 1\big). \end{array}$$

The first alternative is ruled out since (by construction of $\mathcal{V}_{n,q}$) we have: $\overline{M} \notin \mathcal{W}_{n,q}^{n+j}$, and hence $\det M = 0$, $\det M_{n+j} = 0$ do not hold simultaneously.

The second alternative yields: the contribution of the $\left[\begin{smallmatrix} a \\ d \end{smallmatrix}\right]$-entries to the gradient of $\det M_{n+j}$ (being equal to the array of cofactors in M for the entries in its $(n+j)$-th column) is non-vanishing. On the other hand the contribution of the $\left[\begin{smallmatrix} a \\ d \end{smallmatrix}\right]$-entries to the gradient of $\det \widehat{M}^{n+j}$ are all zero (since $\left[\begin{smallmatrix} a \\ d \end{smallmatrix}\right]$ does not appear in $\widehat{M}^{n+j}$). So, we only have to show that the gradient of $\det \widehat{M}^{n+j}$ is non-vanishing. To this aim we note that, in the second alternative, from $\det M \neq 0$, $\operatorname{rank} D = n$ and $\det \widehat{M}^{n+j} = 0$ it follows: $\operatorname{rank} \widehat{M}^{n+j} = n+q-2$ and $\operatorname{rank} \widehat{D}^j = n-1$ (where $\widehat{D}^j$ is the $(q-1, n)$-matrix obtained from D by deleting its j-th row). So, we may assume that all but the first column of $\widehat{M}^{n+j}$ form a linearly independent set of vectors in $\mathbb{R}^{n+q-1}$. Hence, the contributions of the entries of the first column of $\widehat{M}^{n+j}$ to the gradient of $\det \widehat{M}^{n+j}$ are not all vanishing.

Case $q < n$: Since $\overline{M} \in \mathcal{W}_{n,q}$ it follows from tables 3, 4 that:

$$\begin{array}{lll} \text{Either} & \det M = 0,\ \operatorname{rank} D = q & \big(\text{if } \overline{M} \in \overline{V}^2_{q,\xi,[\tau]},\ \xi^0 = 1\big), \\ \text{or} & \det M \neq 0,\ \operatorname{rank} D = q & \big(\text{if } \overline{M} \in \overline{V}^6_{q,\xi,[\tau]},\ q = 1,\ \xi^0 = 0,\ \tau = \pm 1, \text{ or} \\ & & \overline{M} \in V^3_{q,\xi,[\tau]},\ \xi^0 = 0,\ \tau = 0, \pm 1\big). \end{array}$$

Now, the proof that the gradients in $\mathcal{V}_{n,q}$ of $\det \widehat{M}^{n+j}$ and $\det M_{n+j}$ are linearly independent is the same as in the case where $q = n$, and will be deleted. $\square$

We define $(0 < q \leqslant n)$:

$$\mathcal{R}_{n,q} := \mathcal{V}_{n,q} \setminus \{\text{union of the sets } \mathcal{V}_{n,q}^{n+j},\ \text{all } j = 1, \ldots, q\}.$$

In the proof of theorem 3.1 (see below) we shall use the concept of 1-jet extension. For $\varphi \in C^1(\mathbb{R}, \mathcal{R}_{n,q})$, the reduced 1-jet extension, say $j^1\varphi$ is given by $j^1\varphi(\cdot) = (\varphi(\cdot), \nabla_t\varphi(\cdot))$. Apparently, $j^1\varphi$ is a mapping from $\mathbb{R}$ to $\mathcal{R}_{n,q} \times \mathcal{R}'_{n,q}$ where $\mathcal{R}'_{n,q} = \mathbb{R}^K$. An element of the set $\mathcal{R}_{n,q} \times \mathcal{R}'_{n,q}$ will be denoted by $(\overline{M}, \overline{M}')$; corresponding entries in $\overline{M}$ and $\overline{M}'$ will be distinguished by means of the symbol $'$. For instance, d'_q is the last entry of the last column in matrix $\overline{M}'$.

Next, we consider functions $F^j : \mathcal{R}_{n,q} \times \mathcal{R}'_{n,q} \to \mathbb{R}$ of the form

$$F^j\big(\overline{M}, \overline{M}'\big) = p^j\big(\overline{M}, \overline{M}'\big) + d'_q \cdot \det{}^2\big(\widehat{M}^{n+j}\big),$$

where $\widehat{M}^{n+j}$ is defined as in lemma 5.7, and where p^j is a real polynomial with as variables the entries of $\overline{M}$, $\overline{M}'$, which however does *not* depend on d'_q. The precise specification of F^j will be postponed until the proof of theorem 3.1 (verification of condition (iii) for g.c. points of type 2).

Lemma 5.8. Let $0 < q \leqslant n$ and $0 < j \leqslant q$. Then, the subset of $\mathcal{R}_{n,q} \times \mathcal{R}'_{n,q}$, say $[\mathcal{R}_{n,q} \times \mathcal{R}'_{n,q}]^j$, given by the equations

$$\det M_{n+j} = 0, \qquad F^j\big(\overline{M}, \overline{M}'\big) = 0$$

is a closed, smooth submanifold of codim 2 in $\mathcal{R}_{n,q} \times \mathcal{R}'_{n,q}$.

Proof. We only show that the equations defining $[\mathcal{R}_{n,q} \times \mathcal{R}'_{n,q}]^j$ form a linearly independent pair.

Let $(\overline{M}, \overline{M}')$ be in $[\mathcal{R}_{n,q} \times \mathcal{R}'_{n,q}]^j$. Apparently, the contribution of d'_q to the gradient (with respect to $\mathcal{R}_{n,q} \times \mathcal{R}'_{n,q}$) of $\det M_{n+j}$, respectively F^j, equals 0, respectively $\det^2(\widehat{M}^{n+j})$. From the construction of $\mathcal{R}_{n,q}$ it follows that $\overline{M} \notin \mathcal{V}^{n+j}_{n,q}$, and thus $\det \widehat{M}^{n+j} \neq 0$. On the other hand, we also have: $M \notin \mathcal{W}^{n+j}_{n,q}$ (cf. lemma 5.5), and thus $\det M \neq 0$. Hence, the contribution of the $\left[\begin{smallmatrix} a \\ d \end{smallmatrix}\right]$-entries to the gradient of $\det M_{n+j}$ are not all vanishing.

Altogether, from these observations it follows that the gradients of $\det M_{n+j}$ and $F^j(\overline{M}, \overline{M}')$, both regarded as functions on $\mathcal{R}_{n,q} \times \mathcal{R}'_{n,q}$, form a linearly independent pair of vectors. $\square$

The next two lemmas deal with matrices $\overline{M} = \left[\begin{smallmatrix} A & D^{\mathrm{T}} & a \\ D & 0 & d \end{smallmatrix}\right]$ in $\mathcal{W}_{n,n+1}$. We recall that then D is an $(n+1, n)$-matrix and that by $\widehat{D}^j$ we mean a matrix obtained from D after deleting its j-th row.

Lemma 5.9. Let $0 < j \leqslant n + 1$. Then, the subset, say $\mathcal{W}^j_{n,n+1}$, of $\mathcal{W}_{n,n+1}$ given by

$$\det[Dd] = 0, \qquad \det \widehat{D}^j = 0$$

is a closed, smooth submanifold of codim 2 in $\mathcal{W}_{n,n+1}$.

Proof. We only prove that the defining equations are linearly independent. Note that for any $\overline{M} \in \mathcal{W}_{n,n+1}$ we have:

$$\begin{array}{lll} \text{Either} & \overline{M} \in \overline{V}^1_{n,\xi,[\tau]}, & \xi^0 = 0, \\ \text{or} & \overline{M} \in \overline{V}^3_{n,\xi,[\tau]}, & \xi^0 = 0, \tau = \pm 1. \end{array}$$

We emphasize that in both cases: rank $D = n$. Hence the contributions of the entries of d to the gradient of det$[Dd]$ are not all vanishing. Moreover, rank $\widehat{D}^j = n - 1$ and thus also the contribution of the entries of $\widehat{D}^j$ to the gradient of det $\widehat{D}^j$ are not all vanishing. Since d does not appear in $\widehat{D}^j$, the linear independency of the gradients of det$[Dd]$ and det $\widehat{D}^j$ follows immediately. □

We put

$$\mathcal{U}_{n,n+1} := \mathcal{W}_{n,n+1} \setminus \{\text{union of the sets } \mathcal{W}^j_{n,n+1} \text{ all } j,\ 0 < j \leqslant n+1\}.$$

Lemma 5.10. For any $\overline{M} \in \mathcal{W}_{n,n+1}$ and any pair j, j' with $0 < j \leqslant n+1,\ 0 < j' \leqslant n$, let $M^j_{j'}$ be the matrix obtained from $\overline{M}$ by deleting its $(n+j)$-th column and row, and deleting from the resulting matrix its $(n+j')$-th column. Then the subset, say $\mathcal{U}^{j,j'}_{n,n+1}$, of $\mathcal{U}_{n,n+1}$ given by the equations

$$\det[Dd] = 0, \qquad \det M^j_{j'} = 0$$

is a closed, smooth submanifold of codim 2 in $\mathcal{U}_{n,n+1}$.

Proof. Again we must prove that the defining equations are linearly independent. By construction of $\mathcal{U}_{n,n+1}$, we have for any $\overline{M} \in \mathcal{U}^{j,j'}_{n,n+1}$:

$$\det \begin{bmatrix} A & \widehat{D}^{j^{\mathrm{T}}} \\ \widehat{D}^j & 0 \end{bmatrix} = (-1)^n \det{}^2 \widehat{D}^j \neq 0.$$

From this it follows that the contributions of the entries of $\left[\begin{smallmatrix} a \\ \hat{d}^j \end{smallmatrix}\right]$ to the gradient of det $M^j_{j'}$ are not all vanishing. (Here, $\hat{d}^j$ stands for the vector obtained from d after deleting its j-th component.) On the other hand, again using that det $\widehat{D}^j \neq 0$ for *all* j, the contribution of any entry of d to the gradient of det$[Dd]$ is non-vanishing. Since the j-th component of d does not show up in $M^j_{j'}$, we conclude that the gradients of det$[Dd]$ and det $M^j_{j'}$ are linearly independent. □

Our six technical lemmas being proved we proceed with

Proof of theorem 3.1. We begin with the construction, for any q, of certain C^1_s-open and -dense subsets, say $\mathcal{S}_{n,q}$, of $C^1(\mathbb{R}, \overline{\mathcal{M}}_{n,q})$.

Let $0 < q \leqslant n$. Successive application of the previously described *shrinking procedure* (and using lemmas 5.5, 5.7) yields the set $C^1(\mathbb{R}, \mathcal{R}_{n,q})$, which is C^1_s-open and -dense in $C^1(\mathbb{R}, \overline{\mathcal{M}}_{n,q})$. We define

$$\overline{\pitchfork}\,[\mathcal{R}_{n,q} \times \mathcal{R}'_{n,q}]^j := \bigl\{\varphi \in C^1(\mathbb{R}, \mathcal{R}_{n,q}) \mid j^1\varphi \,\overline{\pitchfork}\,[\mathcal{R}_{n,q} \times \mathcal{R}'_{n,q}]^j\bigr\}.$$

Then, from Thom's transversality theorem (1-jet version) (cf. [7] or [9]), together with lemma 5.8, it follows that $\overline{\pitchfork}\,[\mathcal{R}_{n,q} \times \mathcal{R}'_{n,q}]^j$ is C^1_s-open and -dense in $C^1(\mathbb{R}, \mathcal{R}_{n,q})$ and thus also in $C^1(\mathbb{R}, \overline{\mathcal{M}}_{n,q})$. Moreover, by a dimension argument we have

$$\overline{\pitchfork}\,[\mathcal{R}_{n,q} \times \mathcal{R}'_{n,q}]^j = \bigl\{\varphi \in C^1(\mathbb{R}, \mathcal{R}_{n,q}) \mid j^1\varphi(\mathbb{R}) \cap [\mathcal{R}_{n,q} \times \mathcal{R}'_{n,q}]^j = \emptyset\bigr\}.$$

We put:

$$\mathcal{S}_{n,q} := \bigl\{\text{intersection of the sets } \overline{\pitchfork}\,[\mathcal{R}_{n,q} \times \mathcal{R}'_{n,q}]^j,\ \text{all } j \text{ with } 0 < j \leqslant q\ (\leqslant n)\bigr\}.$$

Apparently, this set $\mathcal{S}_{n,q}$ is C^1_s-open and -dense in $C^1(\mathbb{R}, \mathcal{R}_{n,q})$ and thus also in $C^1(\mathbb{R}, \overline{\mathcal{M}}_{n,q})$; moreover, we have

$$\varphi \in \mathcal{S}_{n,q} \quad \text{iff} \quad j^1\varphi(\mathbb{R}) \cap [\mathcal{R}_{n,q} \times \mathcal{R}'_{n,q}]^j = \emptyset, \quad \text{all } j,\ 0 < j \leqslant q.$$

Next, we turn over to the case where $q = n + 1$. Define: (compare also lemmas 5.9, 5.10)

$$\begin{aligned}
&\mathcal{V}_{n,n+1} := \mathcal{U}_{n,n+1} \setminus \{\text{union of the sets } \mathcal{U}^{j,j'}_{n,n+1},\ \text{all } j, j',\ 1 \leqslant j \leqslant n+1,\ 1 \leqslant j' \leqslant n\},\\
&\mathcal{S}_{n,n+1} := \overline{\pitchfork}\,\mathcal{V}_{n,n+1}\ \bigl(= C^1(\mathbb{R}, \mathcal{V}_{n,n+1})\bigr).
\end{aligned}$$

Then, according to our *shrinking procedure* (and using lemmas 5.9, 5.10) we find that $\mathcal{S}_{n,n+1}$ is C^1_s-open and -dense in $C^1(\mathbb{R}, \overline{\mathcal{M}}_{n,n+1})$.

Finally, if $q = 0$ or $q > n + 1$, we define $\mathcal{S}_{n,q} := C^1(\mathbb{R}, \overline{\mathcal{M}}_{n,q})$.

Now, we are ready to define our set $\mathcal{O}$:

For arbitrary but fixed $J_0 \subset J$, we introduce the following C^1_s-open and -dense subset, say $\mathcal{S}_{J_0}$, of $C^1(\mathbb{R}, \mathbb{R}^N)$:

$$\mathcal{S}_{J_0} := \mathcal{S}_{n,m+|J_0|} \times C^1\bigl(\mathbb{R}, \mathbb{R}^{K_{J_0}}\bigr).$$

We define

$$\mathcal{S} := \bigcap_{J_0 \subset J} \mathcal{S}_{J_0} \quad \text{and} \quad \mathcal{O} := \mathcal{O}_1 \cap \mathcal{S}.$$

(For the meaning of $\mathcal{O}_1$, we refer to the proof of lemma 5.4, where also the integer K_{J_0} is introduced). Apparently, the set $\mathcal{O}$ is C^1_s-open and -dense in $C^1(\mathbb{R}, \mathbb{R}^N)$.

From now on, we assume that $\mathcal{P}Q(\cdot)$ is a 1-parameter family of quadratic optimization problems, represented by a mapping $Q(\cdot) \in \mathcal{O}$ and that $\overline{z}$ is a g.c. point for $\mathcal{P}Q(\cdot)$. Note that by lemma 5.4 we have: $m + |J_{\overline{z}}| \leqslant n + 1$.

Firstly, we consider the case where $m + |J_{\overline{z}}| \leqslant n$ (and thus, due to lemma 5.4 LICQ, ND hold at $\overline{z}$), $J_{\overline{z}} \neq \emptyset$ and at least one of the Lagrange multipliers $\overline{\mu}_j$, $j \in J_{\overline{z}}$, vanishes.

We shall verify that in this case the additional conditions (i)–(iii) of definition 2.2 are fulfilled (and consequently, $\overline{z}$ is a g.c. point of type 2).

Verification of definition 2.2(i). In view of LICQ, the g.c. point $\overline{z}$ follows from equation (1). Due to ND, Cramer's rule applied to (1), yields: $\overline{z}$ admits vanishing Lagrange multipliers for both g_j and $g_{j'}$, $j, j' \in J_{\overline{z}}$, $j \neq j'$, if and only if $\det M_{n+j} = \det M_{n+j'} = 0$, i.e., $Q_{J_{\overline{z}}}(\overline{t}) \in \mathcal{U}^{j,j'}_{n,m+|J_{\overline{z}}|}$ (see also lemma 5.6). This is impossible by the very construction of $\mathcal{S}$.

Verification of definition 2.2(ii). In view of LICQ we have: $m+|J_{\overline{z}}| \leqslant n$. Since $Q \in \mathcal{O}$, and thus in particular $Q \in \mathcal{O}_1$, we know that $Q_{J_{\overline{z}}\setminus\{p\}}(\overline{t})$ is situated in a $\overline{\mathcal{M}_{n,q}}$-stratum of codim 0 or 1, where $q = m + |J_{\overline{z}}| - 1$ $(< n)$. Now, the assertion follows by inspection of tables 3, 4 and the fact that $\overline{z}$ is a g.c. point for the "reduced" family $\widetilde{PQ}(\cdot)$. (Note that under LICQ the condition *ND* and $\widetilde{ND}$ are equivalent.)

Verification of definition 2.2(iii). For γ we find by the Chain rule (D_{m+p} denotes the last row of $D_{J_{\overline{z}}}$):

$$\gamma = D_{m+p}(\overline{t})\frac{\mathrm{d}}{\mathrm{d}t}\tilde{x}(\overline{t}) + \frac{\mathrm{d}}{\mathrm{d}t}\big[D_{m+p}(\overline{t})\big]\tilde{x}(\overline{t}) + \frac{\mathrm{d}}{\mathrm{d}t}d_{m+p}(\overline{t}), \tag{7}$$

where $\tilde{x}(\overline{t})$ can be obtained from: (cf. (1))

$$\begin{bmatrix} \tilde{x}(t) \\ \tilde{\eta}(t) \end{bmatrix} = -\begin{bmatrix} A(t) & D^{\mathrm{T}}_{J_0}(t) \\ D_{J_0}(t) & 0 \end{bmatrix}^{-1} \begin{bmatrix} a(t) \\ d_{J_0}(t) \end{bmatrix}, \quad J_0 = J_{\overline{z}} \setminus \{p\}. \tag{8}$$

(Note that – by the already verified condition (ii) – the above inverse matrix is well defined for $t \approx \overline{t}$.) We put

$$\tilde{\gamma} := \gamma \cdot \det{}^2 \begin{bmatrix} A(\overline{t}) & D^{\mathrm{T}}_{J_0}(\overline{t}) \\ D_{J_0}(\overline{t}) & 0 \end{bmatrix}.$$

By means of (7) and (8) (use Cramer's rule) we can express $\tilde{\gamma}$ in terms of the entries of $A(\cdot)$, $D_{J_{\overline{z}}}(\cdot), a(\cdot), d_{J_{\overline{z}}}(\cdot)$ at $\overline{t}$ as well as the derivatives of these entries at $\overline{t}$. In this way, we obtain for $\tilde{\gamma}$ an expression which is polynomial in the entries as well as their derivatives of the matrices involved . We denote $\tilde{\gamma}$, seen as a function of these entries, by F^q, where F^q is of the form of the function $F^q : \mathcal{R}_{n,q} \times \mathcal{R}'_{n,q} \to \mathbb{R}$ as introduced in the context of lemma 5.8 with $q = m + |J_{\overline{z}}|$, $j = m + p$. (Recall that $|J_{\overline{z}}| = p$.) Apparently, we have $\tilde{\gamma} = F^q(j^1(Q_{J_{\overline{z}}}(\overline{t}))$.

Since $Q(\cdot) \in \mathcal{S}$, we have in particular: $Q(\cdot) \notin \mathcal{V}^q_{n,q}$ (cf. lemma 5.7). On the other hand, by assumption: (use $\overline{\mu}_p = 0$)

$$\det M_{n+m+p} = 0,$$

where M_{n+m+p} stands for the matrix obtained from $\left[\begin{smallmatrix} A(\bar{t}) & D^{\mathrm{T}}_{J_{\bar{z}}}(\bar{t}) \\ D_{J_{\bar{z}}}(\bar{t}) & 0 \end{smallmatrix}\right]$ after replacing its $(n+m+p)$-th column by $\left[\begin{smallmatrix} a(\bar{t}) \\ d_{J_{\bar{z}}}(\bar{t}) \end{smallmatrix}\right]$. Consequently, we find: (use lemma 5.7 and $J_0 = J_{\bar{z}} \setminus \{p\}$)

$$\det \begin{bmatrix} A(\bar{t}) & D^{\mathrm{T}}_{J_0}(\bar{t}) \\ D_{J_0}(\bar{t}) & 0 \end{bmatrix} \neq 0,$$

and thus

$$\gamma \neq 0 \quad \text{iff} \quad \tilde{\gamma} \neq 0.$$

Since, by construction of $\mathcal{S}$, it follows that $Q_{J_{\bar{z}}}(\cdot) \in \mathcal{S}_{n,q}$, $q = m + |J_{\bar{z}}|$, in particular $j^1 Q_{J_{\bar{z}}}(\bar{t}) \notin [\mathcal{R}_{n,q} \times \mathcal{R}'_{n,q}]^q$. Hence, we have $\tilde{\gamma} \neq 0$ (and thus also $\gamma \neq 0$), compare lemma 5.8.

Remark 5.1. It is possible to give an explicit formula for γ (and thus also $\tilde{\gamma}$) in terms of the coefficients, and their derivatives, of the objective/active constraint functions for $PQ(t)$, $t = \bar{t}$ (cf. [10]). However this formula is rather complicated, whereas only its structure is needed (cf. the form of the mappings F^j in lemma 5.8).

We end up with the case where $m + |J_{\bar{z}}| = n + 1$ (and thus LICQ does not hold at $\bar{z}$); compare lemma 5.4. We shall verify that in this case the additional conditions (ii)–(iv) in definition 2.3 are fulfilled (and thus, $\bar{z}$ is a g.c. point of type 5).

Verification of definition 2.3(ii). By construction of $\mathcal{O}$, the curve $\{Q_{J_{\bar{z}}}(t),\ t \in \mathbb{R}\}$ is situated in the (open) set $\mathcal{W}_{n,n+1}$. We recall that this latter set is the union of the $\overline{\mathcal{M}}_{n,n+1}$-strata $\overline{V}^1_{n,\xi,[\tau]}$, $\xi^0 = 0$ (codim 0) and $\overline{V}^3_{n,\xi,[\tau]}$, $\xi^0 = 0$, $\tau = \pm 1$ (codim 1), compare also tables 3, 4. From these tables, together with table 1, it also follows that the latter stratum (as subset of $\mathcal{W}_{n,n+1}$) is given by the equation

$$\det[D\ d] = 0.$$

This equation is a *defining system* (cf. [9]) for $\overline{V}^3_{n,\xi,[\tau]}$, $\xi^0 = 0$, $\tau = \pm 1$ (since $\operatorname{rank} D = n$ on $\mathcal{W}_{n,n+1}$). In view of the feasibility of $\bar{z}$ we have: (use also the very definition of $\mathcal{O}_1$)

$$Q_{J_{\bar{z}}}(t) \text{ intersects } \overline{V}^3_{n,\xi,[\tau]},\ \xi^0 = 0,\ \tau = \pm 1 \textit{ transversally} \text{ for } t = \bar{t}.$$

Put $Q_{J_{\bar{z}}}(t) = (A(t), a(t), D(t), d(t))$ (seen as an element from $\mathbb{R}^K$, $q = n + 1$). We have $\det[D(\bar{t})\ d(\bar{t})] = 0$. Then, the above transversality condition yields:

$$\nabla^{\mathrm{T}} \det\big[D(\bar{t})\ d(\bar{t})\big] \frac{\mathrm{d}}{\mathrm{d}t} Q_{J_{\bar{z}}}(\bar{t}) \neq 0,$$

or equivalently,

$$\widetilde{\nabla}^{\mathrm{T}} \det\big[D(\bar{t})\ d(\bar{t})\big] \frac{\mathrm{d}}{\mathrm{d}t} \big[D(\bar{t})\ d(\bar{t})\big] \neq 0.$$

Here, ∇ stands, as usual, for gradient with respect to the entries of $\overline{M}(\in \mathcal{W}_{n,n+1})$, whereas $\widetilde{\nabla}$ stands for gradient with respect to the entries of $[D\ d]$ $(\in \mathbb{R}^{(n+1)^2})$. By the Chain rule, the latter inequality is equivalent with

$$\frac{\mathrm{d}}{\mathrm{d}t}\det\big[D(t)\ d(t)\big]_{t=\bar{t}} \neq 0.$$

By elementary matrix manipulations, we obtain

$$\det\big[D(t)\ d(t)\big] = \det\begin{bmatrix} D(t) & \begin{matrix} h(\overline{x},t) \\ g(\overline{x},t) \end{matrix} \end{bmatrix},$$

where $h := (h_1, \ldots, h_m)^{\mathrm{T}}$ and $g := (g_1, \ldots, g_p)^{\mathrm{T}}$ stands for respectively the equality and active (at $\overline{z}$) inequality constraints. Taking the feasibility of $\overline{z}$ into account, we find

$$0 \neq \frac{\mathrm{d}}{\mathrm{d}t}\det\big[D(t)\ d(t)\big]_{t=\bar{t}} = \det\begin{bmatrix} D(t) & \begin{matrix} \frac{\mathrm{d}}{\mathrm{d}t}h(\overline{x},t) \\ \frac{\mathrm{d}}{\mathrm{d}t}g(\overline{x},t) \end{matrix} \end{bmatrix}_{t=\bar{t}}.$$

Note that the matrix on the right-hand side has $\nabla^{\mathrm{T}} h_i(\overline{z}),\ i \in I$, and $\nabla^{\mathrm{T}} g_j(\overline{z}),\ j \in J_{\overline{z}}$ as its rows. So, these gradients are linearly independent.

Verification of definition 2.3(iii). Let again $D(\bar{t})$ be the matrix with as rows $\nabla_x^{\mathrm{T}} h_i(\overline{z})$, $i \in I$, and $\nabla_x^{\mathrm{T}} g_j(\overline{z}),\ j \in J_{\overline{z}}$. We assume that our condition is not fulfilled, i.e.,

$$D^{\mathrm{T}}(\bar{t})\eta = 0,$$

for some $\eta \in \mathbb{R}^{n+1},\ \eta \neq 0$, with vanishing last component η_{n+1}. Then we also have:

$$\widehat{D}^{\mathrm{T}}(\bar{t})\hat{\eta} = 0, \tag{9}$$

where $\widehat{D}(\bar{t})$ and $\hat{\eta}$ are obtained from $D(\bar{t})$, respectively η by deleting the last row, respectively the last entry. From the construction of $\mathcal{O}_1$ it follows that $Q_{J_{\overline{z}}\setminus\{p\}}(\bar{t})$, where $p = n - m + 1$, is situated in one of the $\overline{\mathcal{M}}_{n,n}$-strata $\overline{V}^3_{n,\xi,[\tau]}$, $\xi^0 = 0$, $\tau = 0, \pm 1$, $\overline{V}^5_{n,\xi,[\tau]}$ (if $n = 1$), $\overline{V}^1_{n-1,\xi,[\tau]}$, $\xi^0 = 0$. The latter possibility is ruled out since $\overline{z}$ is feasible for $\widetilde{PQ}(\bar{t})$ (= optimization problem obtained from $PQ(\bar{t})$ by deleting the constraint g_p). In the other cases we always have rank $\widehat{D}(\bar{t}) = n$, and thus by (9): $\hat{\eta} = 0$. This leads to a contradiction with our assumption on η.

Verification of definition 2.3(iv). Put $\overline{M} := Q_{J_{\overline{z}}}(\bar{t})$. By construction of $\mathcal{O}$ (especially $S_{n,n+1}$) we have: $\overline{M} \in \mathcal{V}_{n,n+1}$, in particular $\overline{M} \notin \mathcal{U}^{j,j'}_{n,n+1}$, for all pairs $j, j',\ 0 < j \leqslant n+1,\ 0 < j' \leqslant n$. We recall that $\det[D(\bar{t})\ d(\bar{t})] = 0$. Thus $\det M^j_{j'} \neq 0$, where $M^j_{j'}$ is defined as in lemma 5.10. Now, after a moment of reflection it will be clear that the latter condition implies: all off-diagonal elements of matrix Δ in definition 2.3 are non-vanishing.

We proceed to

Proof of theorem 3.2. As stated in the assertion of this theorem, let $PQ(\cdot)$ be represented by $Q \in \mathcal{O}$, and let $\bar{z} = (\bar{x}, \bar{t})$ be a g.c. point of type 1. Moreover, the component $\Sigma^{(1)}_{\bar{z}}$ of $\Sigma^{(1)}$ is parameterized as $z(t) = (x(t), t),\ x(\bar{t}) = \bar{x},\ t \in I_{\bar{z}}$ (= open interval around $\bar{t}$), and $\hat{t}$ (= $\sup_{I_{\bar{z}}} t) < \infty$. Following the curve $(x(t), t)$ for increasing t, there are two mutually exclusive possibilities:

$$\text{Case 1} \quad \lim_{t\uparrow\hat{t}} x(t) = \hat{x},\ \text{some } \hat{x} \in \mathbb{R}^n,\ \text{or}$$

$$\text{Case 2} \quad \lim_{t\uparrow\hat{t}} x(t) \text{ does not exist.}$$

Case 1. A continuity argument yields that $\hat{z} = (\hat{x}, \hat{t})$ is feasible and also $J_{\bar{z}} \subset J_{\hat{z}}$. If LICQ at $\hat{z}$ does not hold, then $\hat{z}$ is (automatically) a g.c. point which by definition of $\hat{t}$ cannot be of type 1, and also not of type 2 (since LICQ is violated). Hence, by theorem 3.1 we find that $\hat{z}$ is g.c. point of type 5.

If LICQ at $\hat{z}$ holds, a continuity argument (use also $J_{\bar{z}} \subset J_{\hat{z}}$) yields that $\hat{z}$ is g.c. point which cannot be of type 1 (by definition of $\hat{t}$) and also not of type 5 (because of LICQ). Hence, by theorem 3.1, we find that $\hat{z}$ is a g.c. point of type 2.

Case 2. We are going to prove that $\lim_{t\uparrow\hat{t}} \|x(t)\| = \infty$ and $\lim_{t\uparrow\hat{t}} x(t)/\|x(t)\| = \tilde{x}$.

As before, we put $|J_{\bar{z}}| = p,\ m + p = q$ and write (cf. (3))

$$M(t) = \begin{bmatrix} A(t) & D^{\mathrm{T}}_{J_{\bar{z}}}(t) \\ D_{J_{\bar{z}}}(t) & 0 \end{bmatrix}, \qquad \overline{M}(t) = Q_{J_{\bar{z}}}(t) = \begin{bmatrix} M(t) & \begin{matrix} a(t) \\ d_{J_{\bar{z}}}(t) \end{matrix} \end{bmatrix}.$$

Since $z(t),\ t \in I_{\bar{z}}$, are g.c. points of type 1, we have:

$$\det M(t) \neq 0, \quad \text{all } t \in I_{\bar{z}}.$$

From this it follows (use equation (1)) that if $\det M(\hat{t}) \neq 0$, then $\lim_{t\uparrow\hat{t}} x(t)$ would exist. However, this is a contradiction to the assumption of case 2. Hence, we have $\det M(\hat{t}) = 0$.

As a further simplification of notations we write:

$$D_{\hat{t}} = D_{J_{\bar{z}}}(\hat{t}), \quad d_{\hat{t}} = d_{J_{\bar{z}}}(\hat{t}), \quad M_{\hat{t}} = M(\hat{t}), \quad \overline{M}_{\hat{t}} = \overline{M}(\hat{t}).$$

By inspection of tables 1, 3, 4 (and using the inertia formula (5) as well as $\det M_{\hat{t}} = 0$) we find the following alternatives (recall that $q \leqslant n$):

Case 2(a). $\overline{M}_{\hat{t}} \in \overline{V}^2_{q,\xi,[\tau]},\ \xi^0 = 1$ (if $0 < q < n$) or $\overline{M}_{\hat{t}} \in \overline{V}^4_{q,\xi,[\tau]},\ \xi^0 = 1$ (if $q = 0$), and thus always $\operatorname{rank} \overline{M}_{\hat{t}} = 1 + \operatorname{rank} M_{\hat{t}}$ $(= n + q)$, $\operatorname{rank}[D_{\hat{t}} d_{\hat{t}}] = q$ $(= m + p)$. Note, that $\xi^0 = 1$ means $\operatorname{corank} A(\hat{t})_{|\ker D_{J_{\bar{z}}}(\bar{t})} = 1$.

Case 2(b). $\overline{M}_{\hat{t}} \in \overline{V}^1_{q-1,\xi,[\tau]},\ \xi^0 = 0$ (if $q = n$), and thus $\operatorname{rank} \overline{M}_{\hat{t}} = 1 + \operatorname{rank} M_{\hat{t}} = n + q$ $(= 2n)$, $\operatorname{rank}[D_{\hat{t}} d_{\hat{t}}] = 1 + \operatorname{rank} D_{\hat{t}} = q$ $(= n)$.

We begin with analyzing case 2(a), $0 < q < n$. Let $y(t) = (x(t), \eta(t)),\ t \in I_{\bar{z}}$, be the solution of equation (1), i.e.,

$$M(t)\begin{bmatrix} x(t) \\ \eta(t) \end{bmatrix} + \begin{bmatrix} a(t) \\ d_{J_{\bar{z}}}(t) \end{bmatrix} = 0. \tag{10}$$

We show that

$$\lim_{t\uparrow\hat{t}} \|y(t)\| = \infty, \quad \text{and} \quad \lim_{t\uparrow\hat{t}} \frac{y(t)}{\|y(t)\|} = \hat{y}, \quad \text{some } \hat{y} \in \ker M_{\hat{t}},\ \|\hat{y}\| = 1.$$

Cramer's rule applied to (10) yields

$$y_\nu(t) = -\frac{\det M_\nu(t)}{\det M(t)}, \quad t \in I_{\bar{z}},\ \nu = 1, \ldots, n+q, \tag{11}$$

where $M_\nu(t)$ is obtained from $M(t)$ by replacing its ν-th column by $\left[\begin{smallmatrix} a(t) \\ d_{J_{\bar{z}}}(t) \end{smallmatrix}\right]$, cf. lemma 5.5. Since $Q(\cdot) \in \mathcal{O}$, we have in particular $\overline{M}_{\hat{t}} \in \mathcal{U}_{n,q},\ 0 < q < n$, and thus (using $\det M_{\hat{t}} = 0$):

$$\det M_{n+j}(\hat{t}\,) \neq 0, \quad \text{all } j = 1, \ldots, q.$$

So, from (11) it follows that $\lim_{t\uparrow\hat{t}} |\eta_j(t)| = \infty,\ j = 1, \ldots, q$, and thus (recall $\eta_j = y_{n+j}$) we have $\lim_{t\uparrow\hat{t}} \|y(t)\| = \infty$. Dividing both sides of (10) by $\|y(t)\|$, yields

$$\lim_{t\uparrow\hat{t}} M(t)\frac{y(t)}{\|y(t)\|} = 0.$$

Since $\operatorname{corank} M_{\hat{t}} = 1$, a unique (up to a sign) solution, say $\hat{y}$, exists for

$$M_{\hat{t}}y = 0, \quad y \in \mathbb{R}^{n+q},\ \|y\| = 1. \tag{12}$$

Hence, the only possible limit points of $y(t)/\|y(t)\|$ for $t \uparrow \hat{t}$ are $\pm\,\hat{y}$. By a continuity argument it follows that not both $\hat{y}$ and $-\hat{y}$ can be limit points. So, we may assume that $\lim_{t\uparrow\hat{t}} y(t)/\|y(t)\| = \hat{y}\ (= (\hat{x}, \hat{\eta}) \in \mathbb{R}^n \times \mathbb{R}^q)$. When $\hat{x}$ would vanish, then by (12): $D_{\hat{t}}^{\mathrm{T}}\hat{\eta} = 0$ and thus $\hat{\eta} = 0$ (since $D_{\hat{t}}$ has full rank). This however is in contradiction to $\|\hat{y}\| = 1$. So, $\hat{x} \neq 0$ and thus $\lim_{t\uparrow\hat{t}} \|x(t)\| = \infty$. Moreover,

$$\lim_{t\uparrow\hat{t}} \frac{x(t)}{\|x(t)\|} = \lim_{t\uparrow\hat{t}} \left[\frac{x(t)}{\|y(t)\|} \cdot \frac{\|y(t)\|}{\|x(t)\|}\right] = \frac{\hat{x}}{\|\hat{x}\|}\ (= \tilde{x}). \tag{13}$$

The proof in case 2(a), $q = 0$, runs essentially along the same lines and will be omitted. (Note that in this subcase: $D_{J_{\bar{z}}}(t)$ and $d_{J_{\bar{z}}}(t)$ are non-existent (thus $M(t) = A(t)$), $\operatorname{rank}[A_{\hat{t}}a_{\hat{t}}] = 1 + \operatorname{rank} A_{\hat{t}} = n$ and $\tilde{x}\ (= \hat{x}$ or $-\hat{x})$ is a unit vector in $\ker A_{\hat{t}}$.)

Finally, we analyze case 2(b) ($q = n$):

As in case 2(a), let $y(t) = (x(t), \eta(t)),\ t \in I_{\bar{z}}$, be a solution of (1). In particular, we then have

$$D_{J_{\bar{z}}}(t)x(t) + d_{J_{\bar{z}}}(t) = 0, \quad t \in I_{\bar{z}}. \tag{14}$$

Cramer's rule yields

$$x_i(t) = (-1)^{n+i+1} \frac{\det[\widehat{D}^i_{J_{\bar{z}}}(t) d_{J_{\bar{z}}}(t)]}{\det D_{J_{\bar{z}}}(t)}, \quad i = 1, \ldots, n, \tag{15}$$

where $\widehat{D}^i_{J_{\bar{z}}}(t)$ is obtained from $D_{J_{\bar{z}}}(t)$ by deleting its i-th column. From rank $D_{\hat{t}} = n-1$ and rank$[D_{\hat{t}}\ d_{\hat{t}}] = n$ it follows that for at least one i, say i_0, we have $\det[\widehat{D}^{i_0}_{\hat{t}}\ d_{\hat{t}}] \neq 0$. So, from (15) we may conclude that $\lim_{t\uparrow\hat{t}} |x_{i_0}(t)| = \infty$, and thus $\lim_{t\uparrow\hat{t}} \|x(t)\| = \infty$. Dividing both sides of (14) by $\|x(t)\|$:

$$\lim_{t\uparrow\hat{t}} D_{J_{\bar{z}}}(t) \frac{x(t)}{\|x(t)\|} = 0.$$

Since corank $D_{\hat{t}} = 1$, from this it follows that $\lim_{t\uparrow\hat{t}} x(t)/\|x(t)\| = \tilde{x}$, with $\tilde{x}$ a unique (up to a sign) unit vector in ker $D_{\hat{t}}$; see case 2(a) for a similar argumentation. □

Proof of theorem 3.3. We distinguish between several cases.

Case $J = J_{\bar{z}}$, $0 < q < n$ (compare theorem 3.2, case 2(a)). By construction of $\mathcal{O}_1$ the curve $Q_{J_{\bar{z}}}(t)$ $(= \overline{M}(t))$ intersects the $\overline{\mathcal{M}}_{n,q}$-stratum $\overline{V}^2_{q,\xi,[\tau]}$, $\xi^0 = 1$ for $t = \hat{t}$ *transversally* at $\overline{M}_{\hat{t}}$. As a subset of the *open* set $\mathcal{W}_{n,q}$, the latter stratum is given by the equation $\det M = 0$ (use table 1, 3, 4 and the inertia formula (5)). This equation yields a defining system for our stratum (since on this stratum corank $M = 1$). Hence, from the above transversality condition it follows

$$\nabla^{\mathrm{T}} \det M|_{\overline{M}_{\hat{t}}} \frac{\mathrm{d}}{\mathrm{d}t} \overline{M}(\hat{t}\,) \neq 0,$$

or equivalently

$$\frac{\mathrm{d}}{\mathrm{d}t} \det M(t)_{t=\hat{t}} = \widetilde{\nabla}^{\mathrm{T}} \det M|_{M_{\hat{t}}} \frac{\mathrm{d}}{\mathrm{d}t} M(\hat{t}\,) \neq 0,$$

where ∇ and $\widetilde{\nabla}$ stands for gradient in $\overline{\mathcal{M}}_{n,q}$ and $\mathcal{M}_{n,q}$, respectively. We conclude that

$$\det M(t) \text{ changes sign at } t = \hat{t}. \tag{16}$$

We consider equation (1), which takes the form: (compare also (10))

$$M(t) \begin{bmatrix} x \\ \eta \end{bmatrix} + \begin{bmatrix} a(t) \\ d_{J_{\bar{z}}}(t) \end{bmatrix} = 0. \tag{17}$$

For $t = \hat{t}$ this equation has no solution (since rank $\overline{M}_{\hat{t}} = 1 + \text{rank}\, M_{\hat{t}}$); for $t \approx \hat{t},\ t \neq \hat{t}$ there is a unique solution, say $y(t) = (x(t), \eta(t))$, because, due to (16), we then have $\det M(t) \neq 0$. Since $J = J_{\bar{z}}$, all inequality constraints for $PQ(t)$ are active at $x(t)$ if $t \approx \hat{t},\ t \neq \hat{t}$. So, for such t values the points $x(t)$ are feasible for $PQ(t)$. In the proof of theorem 3.2 we did already show that $\lim_{t\uparrow\hat{t}} x(t)/\|x(t)\| = \tilde{x}$. In a similar way one proves (use (11) and also (16)) that

$$\lim_{t\downarrow\hat{t}} \frac{x(t)}{\|x(t)\|} = -\tilde{x}. \tag{18}$$

Again by (11) and (16) one finds $\lim_{t\to\hat{t}} |\eta_j(t)| = \infty$, $j = 1,\ldots,q$, all Lagrange multipliers $\eta_j(t)$ changing sign when t passes $\hat{t}$; this explains the interchange of linear indices and co-indices at two sided g.c. points at infinity of type 3. The transition between the quadratic indices and co-indices is obtained by application of the inertia formula (5) in the cases $t < \hat{t}$, $t = \hat{t}$ and $t > \hat{t}$, thereby taking into account (16) as well as corank $M_{\hat{t}} = 1$.

Case $J = J_{\bar{z}}$, $q = n$ (compare also the proof of theorem 3.2, case 2(b)). By construction of $\mathcal{O}_1$, the curve $Q_{J_{\bar{z}}}(t)$ $(= \overline{M}(t))$ intersects the $\overline{\mathcal{M}}_{n,n}$-stratum $\overline{V}^1_{n-1,\xi,[\tau]}$, $\xi^0 = 0$ for $t = \hat{t}$ *transversally* at $\overline{M}_{\hat{t}}$. As a subset of the *open* set $\mathcal{W}_{n,n}$, the latter stratum is given by the equation $0 = \det M$ $(= (-1)^n \det^2 D_{J_{\bar{z}}})$; use tables 1, 3, 4 and the inertia formula (5). The equation $\det D = 0$ is a defining system for our stratum (since on this stratum corank $D = 1$). Hence, from the above transversality condition it follows

$$\nabla^{\mathrm{T}} \det[D_{J_{\bar{z}}}]|_{\overline{M}_{\hat{t}}} \frac{\mathrm{d}}{\mathrm{d}t} \overline{M}(\hat{t}) \neq 0,$$

or equivalently,

$$\frac{\mathrm{d}}{\mathrm{d}t} \det D_{J_{\bar{z}}}(t)|_{t=\hat{t}} = \widetilde{\nabla}^{\mathrm{T}} \det[D_{J_{\bar{z}}}]|_{D_{\hat{t}}} \frac{\mathrm{d}}{\mathrm{d}t} D_{J_{\bar{z}}}(\hat{t}) \neq 0,$$

where ∇ and $\widetilde{\nabla}$ stands for gradient in $\overline{\mathcal{M}}_{n,n}$ and $\mathbb{R}^{n^2}$ respectively. We conclude that

$$\det D_{J_{\bar{z}}}(t) \text{ changes sign at } t = \hat{t}. \tag{19}$$

We consider the *feasibility condition*

$$D_{J_{\bar{z}}}(t)x + d_{J_{\bar{z}}}(t) = 0. \tag{20}$$

For $t = \hat{t}$ there is no solution (since $\operatorname{rank}[D_{\hat{t}} d_{\hat{t}}] = 1 + \operatorname{rank} D_{\hat{t}}$); for $t \approx \hat{t}$, $t \neq \hat{t}$, there is a unique solution, say $x(t)$, because, due to (19), we then have $\det D_{J_{\bar{z}}}(t) \neq 0$. Since $J = J_{\bar{z}}$, all inequality constraints for $PQ(t)$ are active at $x(t)$ if $t \approx \tilde{t}$, $t \neq \tilde{t}$. So, for these t-values the points $x(t)$ are feasible for $PQ(t)$. In the proof of theorem 3.2 we did already show that $\lim_{t\uparrow\hat{t}} x(t)/\|x(t)\| = \tilde{x}$. Similarly, one proves (using (15) and (19)) that

$$\lim_{t\downarrow\hat{t}} \frac{x(t)}{\|x(t)\|} = -\tilde{x}.$$

Recall that

$$\det M(t) = (-1)^n \det{}^2 D_{J_{\bar{z}}}(t). \tag{21}$$

By (19) and (21) there exists on a reduced neighborhood of $\hat{t}$ a solution, say $y(t) = (x(t), \eta(t))$ for (1). By Cramer's rule: (cf. (11))

$$\eta_j(t) = -\frac{\det M_j(t)}{\det M(t)}, \quad t \approx \hat{t},\ t \neq \hat{t},\ j = 1, \dots, q. \tag{22}$$

As in the proof of theorem 3.2 from this it follows that $\lim_{t\to\hat{t}} |\eta_j(t)| = \infty,\ j = 1, \dots, q$. However, by (21) the sign of the Lagrange parameters will not change when t passes $\hat{t}$. This explains the non-changing of the linear (co-)indices in the case of two-sided g.c. points at infinity of type 4.

Remark. Note, that by (15) and $\frac{\mathrm{d}}{\mathrm{d}t} \det D_{J_{\bar{z}}}(\hat{t}) \neq 0$ for each index i with $\det[\widehat{D}^i_{J_{\bar{z}}}(\hat{t})\ d_{J_{\bar{z}}}(\hat{t})] \neq 0$ it follows

$$\lim_{t\to\hat{t}} \big|x_i(t)\big|\big|t - \hat{t}\big| = \sigma_i \tag{23}$$

with some $\sigma_i \neq 0$. In view of (21) and (22) we have (recall $\det M_j(\hat{t}) \neq 0$)

$$\lim_{t\to\hat{t}} \big|\eta_j(t)\big|\big(t - \hat{t}\big)^2 = \kappa_j, \quad j = 1, \dots, q, \tag{24}$$

with some $\kappa_j \neq 0$. This means that (in the case of a g.c. point at infinity of type 4) for $t \to \hat{t}$ the Lagrange multiplier $\eta(t)$ tends to infinity more rapidly than the g.c. point $x(t)$. This effect is illustrated in example 2.

Case $J = J_{\bar{z}}$, $q = 0$. The proof runs along the same lines as in the case $0 < q < n$, and will be deleted. (Compare also the comment in the proof of theorem 3.2, case 2(a), $q = 0$; note that in this case there are no Lagrange multipliers, and thus the linear (co-)indices are zero.)

Case $J_{\bar{z}} \subsetneqq J$. In case of a g.c. point at infinity for $t \uparrow \hat{t}$, the following property holds:

$$\begin{gathered}\text{If } Q(\cdot) \in \mathcal{O} \text{ and } J_{\bar{z}} \subsetneqq J \text{ then, for all indices } j \in J \setminus J_{\bar{z}} \text{ we have} \\ \nabla_x^{\mathrm{T}} g_j\big(x, \hat{t}\big)\hat{x} \neq 0,\end{gathered} \tag{25}$$

where $\hat{x}$ is as introduced in theorem 3.2, case 2. (Note that $\nabla_x^{\mathrm{T}} g_j(x, \hat{t})$ does not depend on x.) In order to prove (25), we suppose that (25) is not true. Then for some $j \in J \setminus J_{\bar{z}}$ ($\neq \emptyset$) we have

$$\nabla_x^{\mathrm{T}} g_j\big(x, \hat{t}\big)\hat{x} = 0.$$

Put $g_j(x, t) = D_j(t)x + d_j(t)$ (thus $D_j(t) = C_j^{\mathrm{T}}(t),\ d_j(t) = c_j(t)$; see the definition of $PQ(t)$ in section 2) and distinguish between the two cases $0 < q < n$ and $q = n$: (recall that $q \leqslant n$; the case $q = 0$ is similar and omitted).

Case $0 < q < n$. Put

$$\overline{M} = \begin{bmatrix} A_{\hat{t}} & D_{\hat{t}}^{\mathrm{T}} & D_j^{\mathrm{T}}(\hat{t}\,) & a_{\hat{t}} \\ D_{\hat{t}} & O & 0 & d_{\hat{t}} \\ D_j(\hat{t}\,) & 0 & 0 & d_j(\hat{t}\,) \end{bmatrix}.$$

Then, using the notations as introduced in lemma 5.7, from the proof of theorem 3.2, case 2(a), cf. (12), it follows: ($\hat{y} = (\hat{x}, \hat{\eta})$)

$$\widehat{M}^{n+q+1}\hat{y} = 0 \quad \text{and} \quad M_{n+q+1}\begin{bmatrix} \hat{y} \\ 0 \end{bmatrix} = 0.$$

Since $\hat{y} \neq 0$, we have

$$\det \widehat{M}^{n+q+1} = \det M_{n+q+1} = 0,$$

which by construction of $\mathcal{O}$ (cf. lemma 5.7) is impossible.

Case $q = n$. Put $D = \begin{bmatrix} D_{\hat{t}} \\ D_j(\hat{t}\,) \end{bmatrix}$ and $d = \begin{bmatrix} d_{\hat{t}} \\ d_j(\hat{t}\,) \end{bmatrix}$. Then, using the notations of lemma 5.9 from the proof of theorem 3.2, case 2(b), it follows

$$\widehat{D}^{n+1}\hat{x} = 0 \quad \text{and} \quad [D\ d]\begin{bmatrix} \hat{x} \\ 0 \end{bmatrix} = 0.$$

Since $\hat{x} \neq 0$, we have

$$\det \widehat{D}^{n+1} = \det[D\ d] = 0,$$

which by construction of $\mathcal{O}$ (cf. lemma 5.9) is impossible. Altogether, we have proved that property (25) holds.

Now we choose some $j_0 \in J \setminus J_{\bar{z}}$. Then, by (25) we have

$$\nabla_x^{\mathrm{T}} g_{j_0}\big(x, \hat{t}\,\big)\hat{x} \,\big(= C_{j_0}^{\mathrm{T}}(\hat{t}\,)\hat{x}\big) \neq 0. \tag{26}$$

As before, let $(x(t), \eta(t))$ be the solution of (10) on a reduced neighborhood of $\hat{t}$. Then, $x(t)$ is feasible for $t < \hat{t}$ and t sufficiently close to $\hat{t}$, i.e., $g_{j_0}(x(t), t) = C_{j_0}^{\mathrm{T}}(t)x(t) + c_{j_0}(t) < 0$. In view of (13) and (26) it follows $C_{j_0}^{\mathrm{T}}(\hat{t}\,)\tilde{x} < 0$. But then $C_{j_0}^{\mathrm{T}}(\hat{t}\,)(-\tilde{x}) > 0$ and in view of (18) for $t > \hat{t}$ and t sufficiently close to $\hat{t}$ it follows $g_{j_0}(x(t), t) = C_{j_0}^{\mathrm{T}}(t)x(t) + c_{j_0}(t) > 0$, i.e., $x(t)$ is not feasible.

Acknowledgement

The authors wish to thank Ruud van Schaik for his comments on a preliminary version of the paper and two referees for their valuable remarks.

References

[1] A.V. Fiacco, *Introduction to Sensitivity and Stability Analysis in Nonlinear Programming* (Academic Press, New York, 1983).

[2] C.G. Gibson, K. Wirthműller, A.A. Du Plessis and E.J.N. Looijenga, *Topological Stability of Smooth Mappings*, Lecture Notes in Math. 552 (Springer, Berlin, 1976).

[3] J. Guddat et al., eds., *Parametric Optimization and Related Topics III*, Series Approximation and Optimization (Peter Lang, Frankfurt, 1993).

[4] J. Guddat et al., eds., *Parametric Optimization and Related Topics IV*, Series Approximation and Optimization (Peter Lang, Frankfurt, 1997).

[5] J. Guddat, F. Guerra and H.Th. Jongen, *Parametric Optimization: Singularities, Pathfollowing and Jumps* (Teubner and John Wiley, Chichester, 1990).

[6] M. Henn, P. Jonker and F. Twilt, *On the Critical Sets of One-Parameter Quadratic Optimization Problems*, eds. Durier and Michelot, Lecture Notes in Economics and Mathematical Systems 429 (Springer, Berlin, 1995).

[7] M.W. Hirsch, *Differential Topology* (Springer, New York, 1976).

[8] H.Th. Jongen, P. Jonker and F. Twilt, *Nonlinear Optimization in $\mathbb{R}^n$, Part I. Morse Theory, Chebyshev Approximation* (Peter Lang, Frankfurt, 1983).

[9] H.Th. Jongen, P. Jonker and F. Twilt, *Nonlinear Optimization in $\mathbb{R}^n$, Part II. Transversality, Flows, Parametric Aspects* (Peter Lang, Frankfurt, 1986).

[10] H.Th. Jongen, P. Jonker and F. Twilt, Critical sets in parametric optimization, Math. Programming 34 (1986) 333–353.

[11] H.Th. Jongen, J. Möbert, J. Rückmann and K. Tammer, On inertia and Schur complement in optimization, Lin. Algebra and Appl. 95 (1987) 97–109.

[12] P. Jonker, G. Still and F. Twilt, On the stratification of a class of special structured matrices, to appear as Memorandum, University of Twente.