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On the nearest neighbor rule for the traveling salesman problem

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Abstract

Rosenkrantz et al. (SIAM J. Comput. 6 (1977) 563) and Johnson and Papadimitriou (in: E.L. Lawler, J.K. Lenstra, A.H.G. Rinnooy Kan, D.B. Shmoys (Eds.), The Traveling Salesman Problem: A Guided Tour of Combinatorial Optimization, Wiley, Chichester, 1985, pp. 145–180, (Chapter 5)) constructed families of TSP instances with n cities for which the nearest neighbor rule yields a tour-length that is a factor $\Omega(\log n)$ above the length of the optimal tour.

We describe two new families of TSP instances, for which the nearest neighbor rule shows the same bad behavior. The instances in the first family are graphical, and the instances in the second family are Euclidean. Our construction and our arguments are extremely simple and suitable for classroom use.

Keywords: Traveling salesman problem; Heuristic; Lower bound

1. Introduction

The *traveling salesman problem* (TSP) is a fundamental and well-known problem in combinatorial optimization; see for instance the book [3] by Lawler et al. An instance of the TSP consists of n cities $1, 2, \ldots, n$ together with the distances $d(i,j)$ for $1 \leqslant i,j \leqslant n$. Throughout this note, we will assume that the distances are symmetric and hence satisfy $d(i,j) = d(j,i)$ for all $1 \leqslant i,j \leqslant n$. Moreover, we will assume that the distances satisfy the triangle inequality $d(i,k) + d(k,j) \geqslant d(i,j)$ for all $1 \leqslant i,j,k \leqslant n$. A *partial tour* is a path that visits each of the cities at most once. A *tour* visits each of the n cities exactly once, and in the end returns to its starting point. The objective in the TSP is to find a tour of minimal length.

The *nearest neighbor rule* (NNR) is a fast and simple heuristic for constructing a TSP tour. NNR starts with an arbitrarily chosen city x_1 as partial tour. Then NNR repeats the following step for $k = 1, \ldots, n-1$: If the current partial tour is $x_1, \ldots, x_k$, then let x_{k+1} be the city closest to x_k subject to the condition that x_{k+1} is not already contained in the partial tour; ties are broken arbitrarily. In the end, the NNR tour returns from city x_n to city x_1. The partial tour constructed after a number of steps of NNR is called a *partial NNR tour*. Subsequently one should note the following: If a partial NNR tour $x_1, \ldots, x_p$ is given and the points $x_{p+1}, x_{p+2}, \ldots, x_q$ are yet unvisited, and if the partial tour $x_p, \ldots, x_q$ can appear as a partial NNR tour, then the partial tour $x_1, \ldots, x_q$ can appear as a partial NNR tour.

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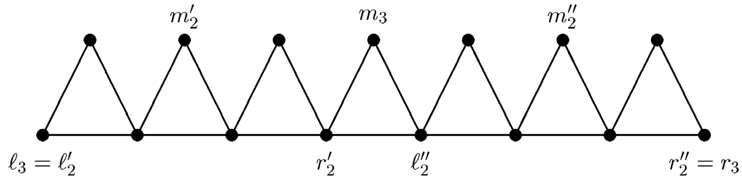

Fig. 1. The graph G_3 with its three special vertices ℓ_3, r_3, and m_3.

Rosenkrantz et al. [4] prove that if the distances $d(i,j)$ are symmetric and satisfy the triangle inequality, then the length of an NNR tour is at most $O(\log n)$ above the length of the optimal tour (all logarithms in this paper are logarithms to the base 2). Moreover [4] exhibit instances for which the length of some NNR tour is a factor $\frac{1}{3}\log n$ above the length of the optimal tour. Johnson and Papadimitriou [2] construct slightly simpler TSP instances that show the same bad lower bound behavior for NNR.

1.1. Contributions of this note

We construct two extremely simple families of TSP instances for which the length of some NNR tour is a factor $\Omega(\log n)$ above the length of the optimal tour. Whereas the arguments in [4,2] are quite involved, our arguments are simple and suitable for classroom use.

The TSP instances in the first family are *graphical*: The distances result from an underlying *undirected* graph $G=(V,E)$ with $V=\{1,2,\ldots,n\}$ such that $d(i,j)$ is the length of the shortest path from vertex i to vertex j in G. All graphical instances satisfy the triangle inequality, but not all instances that satisfy the triangle inequality are graphical. In particular, the instances constructed in [4,2] are non-graphical. The construction for the graphical TSP is given in Section 2. The TSP instances in the second family are *Euclidean*: The cities are points in the Euclidean plane, and the distance $d(i,j)$ between cities i and j is just the Euclidean distance between the corresponding points. The construction for the Euclidean TSP is given in Section 3.

2. The construction for the graphical TSP

For $k \geqslant 1$ we consider the graph $G_k=(V_k,E_k)$ that consists of a chain of 2^k-1 triangles. As an illustrating example, the graph G_3 is depicted in Fig. 1. The graph G_k has 2^k vertices in its lower level, and 2^k-1 vertices in its upper level. The left-most vertex in the lower level is denoted by ℓ_k, the right-most vertex in the lower level is denoted by r_k, and the central vertex in the upper level is denoted by m_k. An equivalent recursive definition of G_k with $k \geqslant 1$ is as follows: The graph G_1 is a triangle on the three vertices ℓ_1, m_1, and r_1. For $k \geqslant 2$ the graph G_k is defined as follows. Take two copies $G'_{k-1}=(V'_k,E'_k)$ and $G''_{k-1}=(V''_k,E''_k)$ of the graph G_{k-1} together with a new vertex m_k. Create an edge between the vertices r'_{k-1} and ℓ''_{k-1}. Create edges from m_k to r'_{k-1} and to ℓ''_{k-1}. Finally, rename vertex ℓ'_{k-1} to ℓ_k, and rename vertex r''_{k-1} to r_k.

Lemma 1. *Let $k \geqslant 1$, and let G be an undirected graph that contains G_k as an induced subgraph such that all edges between G_k and $G-G_k$ are incident either to vertex ℓ_k or to vertex r_k in G_k. Let I_G denote the graphical TSP instance that corresponds to G. Then there exists a partial NNR tour $\mathscr{T}_k$ for the instance I_G:*

(a) *that visits exactly the cities in G_k,*
(b) *that starts in city ℓ_k and ends in city m_k,*
(c) *that has length exactly $(k+3)2^{k-1}-2$.*

Proof. The proof is by induction on k. For $k=1$, we choose the path $\ell_1-r_1-m_1$ of length 2 for $\mathscr{T}_1$. For $k \geqslant 2$, we use the recursive definition of G_k given above that defines G_k in terms of two copies G'_{k-1} and G''_{k-1} of G_{k-1} together with a new vertex m_k.

By the inductive assumption there exists a partial NNR tour $\mathscr{T}'_{k-1}$ of length $(k+2)2^{k-2}-2$ through the subgraph G'_{k-1} that starts in the left city ℓ'_{k-1} $(=\ell_k)$ and ends in the central city m'_{k-1}. Note that city ℓ''_{k-1} is at distance $2^{k-2}+1$ from this central city m'_{k-1}. Since the central city m'_{k-1} is at distance 2^{k-2} from cities ℓ'_{k-1} and r'_{k-1}, all other currently unvisited cities are at distance at least $2^{k-2}+1$ from m'_{k-1}. Therefore it is feasible for NNR to visit ℓ''_{k-1} next after m'_{k-1}.

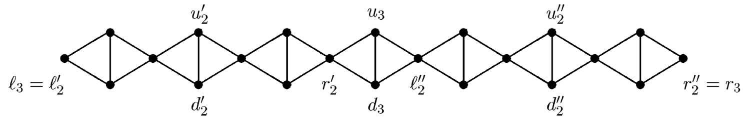

Fig. 2. The point set H_3 with its four special points ℓ_3, r_3, u_3, and d_3.

By the inductive assumption, NNR may then traverse G''_{k-1} from ℓ''_{k-1} to m''_{k-1} according to the partial tour $\mathscr{T}''_{k-1}$ with a total length of $(k+2)2^{k-2} - 2$. Finally, NNR may go the distance $2^{k-2}+1$ from city m''_{k-1} to city m_k since none of the unvisited cities is closer to city m''_{k-1}.

Summarizing, this yields the desired partial tour $\mathscr{T}_k$ from $\ell_k = \ell'_{k-1}$ to m_k through G_k with length $(k+2)2^{k-2} - 2 + (2^{k-2}+1) + (k+2)2^{k-2} - 2 + (2^{k-2}+1) = (k+3)2^{k-1} - 2$. □

Theorem 2. *For every $k \geqslant 1$, there exists a graphical TSP instance with $n = 2^{k+1}$ vertices and an optimal tour of length 2^{k+1}, for which the nearest neighbor rule may yield a tour of length $(k+4)2^{k-1}$.*

In other words, the ratio between the length of this NNR tour and the length of the optimal tour equals $\frac{1}{4}(3 + \log n)$.

Proof. We add a new city v to G_k, and we connect v to ℓ_k and to r_k. Since the resulting graph is Hamiltonian, the corresponding graphical TSP instance has a tour of length $n = 2^{k+1}$. The partial NNR tour $\mathscr{T}_k$ from Lemma 1 together with the distances from m_k to v and from v to ℓ_k yields an NNR tour of length $(k+3)2^{k-1} - 2 + (2^{k-1}+1) + 1 = (k+4)2^{k-1}$. □

3. The construction for the Euclidean TSP

For $k \geqslant 1$ we consider the Euclidean point set H_k that consists of the points in a chain of $2^k - 1$ *diamonds*. As an illustrating example, the point set H_3 is depicted in Fig. 2. All line segments shown in this picture are of unit length. The point set H_k consists of $3 \times 2^k - 2$ points that are arranged in three horizontal layers: The middle layer has 2^k points with coordinates $(j\sqrt{3}; 0)$ for $j = 0, \ldots, 2^k - 1$. The upper layer has $2^k - 1$ points with coordinates $((j+\frac{1}{2})\sqrt{3}; +\frac{1}{2})$ for $j = 0, \ldots, 2^k - 2$, and the lower layer has $2^k - 1$ points with coordinates $((j+\frac{1}{2})\sqrt{3}; -\frac{1}{2})$ for $j = 0, \ldots, 2^k - 2$. Note that in this construction, the sides and the vertical diagonal of each diamond all have length 1. The left-most point of H_k is denoted by ℓ_k, the right most point by r_k, the central point in the upper layer is denoted by u_k, and the central point in the lower layer is denoted by d_k.

An equivalent recursive definition of H_k with $k \geqslant 1$ is as follows: The point set H_1 consists of the four points $\ell_1 = (0; 0)$, $u_1 = (\frac{1}{2}\sqrt{3}; +\frac{1}{2})$, $d_1 = (\frac{1}{2}\sqrt{3}; -\frac{1}{2})$, and $r_1 = (\sqrt{3}; 0)$. For $k \geqslant 2$ the point set H_k is defined as follows. Take two copies H'_{k-1} and H''_{k-1} of the point set H_{k-1}. Keep H'_{k-1} in its original position, and shift H''_{k-1} by $2^{k-1}\sqrt{3}$ units to the right. In the middle between these two copies, add two new points u_k and d_k in the upper and lower layer, respectively. Rename point ℓ'_{k-1} to ℓ_k, and rename point r''_{k-1} to r_k.

Lemma 3. *Let k and t be integers with $1 \leqslant k \leqslant t$. Then the point set H_t contains by definition a copy of H_k as a subset. Let H' be an arbitrary copy of H_k in H_t, and let I_H denote the Euclidean TSP instance that corresponds to H_t. Then there exists a partial NNR tour $\mathscr{T}_k$ for the instance I_H:*

(a) *that visits exactly the cities in the copy H',*
(b) *that starts in city ℓ_k and ends in the upper central city u_k of H',*
(c) *that has length exactly $(4+(k-1)\sqrt{3})2^{k-1} - 1$.*

By symmetry, there also exists a partial NNR tour through H' of the stated length that starts in city ℓ_k and ends in the lower central city d_k.

Proof. The proof is by induction on k. For $k = 1$, we choose the path $\ell_1 - d_1 - r_1 - u_1$ of length 3 for $\mathscr{T}_1$. For $k \geqslant 2$, we use the recursive definition of H_k given above that defines H_k in terms of two copies H'_{k-1} and H''_{k-1} of H_{k-1} together with two new points u_k and d_k.

By the inductive assumption there exists a partial NNR tour $\mathscr{T}'_{k-1}$ of length $(4+(k-2)\sqrt{3})2^{k-2} - 1$

through the subset H'_{k-1} that starts in the left city ℓ'_{k-1} ($=\ell_k$) and ends in the lower central city d'_{k-1}. At that moment, none of the unvisited cities in H is closer to d'_{k-1} than city d_k, and thus we let NNR move on to city d_k. The distance between d'_{k-1} and d_k equals $\sqrt{3}\times 2^{k-2}$. Next, NNR moves the distance 1 from city d_k to city ℓ''_{k-1}. By the inductive assumption, NNR may then traverse H''_{k-1} from ℓ''_{k-1} to u''_{k-1} according to the partial tour $\mathscr{T}''_{k-1}$ with a total length of $(4+(k-2)\sqrt{3})2^{k-2}-1$. Finally, NNR may go the distance $\sqrt{3}\times 2^{k-2}$ from city u''_{k-1} to city u_k since none of the unvisited cities is closer to city u''_{k-1}. Summarizing, this yields a partial tour $\mathscr{T}_k$ from $\ell_k = \ell'_{k-1}$ to u_k through H' of total length

$$\begin{aligned}&(4+(k-2)\sqrt{3})2^{k-2}-1+\sqrt{3}\times 2^{k-2}+1\\&\quad+(4+(k-2)\sqrt{3})2^{k-2}-1+\sqrt{3}\times 2^{k-2}\end{aligned}$$

which equals $(4+(k-1)\sqrt{3})2^{k-1}-1$, exactly as we desired. □

Theorem 4. *For every $k \geqslant 1$, there exists a Euclidean TSP instance with $n = 3\times 2^k - 2$ points and an optimal tour of length $(2+\sqrt{3})2^k - 2\sqrt{3}$, for which the nearest neighbor rule may yield a tour of length at least $(4+k\sqrt{3})2^{k-1}-(\sqrt{3}+1)$.*

Hence, the ratio between the length of this NNR tour and the length of the optimal tour is at least $(\sqrt{3}-\frac{3}{2})(\log n - 2) \approx 0.232(\log n - 2)$.

Proof. We consider the Euclidean TSP instance H_k. Since all points in H_k lie on three parallel lines, an optimal tour can be determined along the arguments of Cutler [1]. The optimal tour is not unique. One optimal tour starts in ℓ_k, then runs through all cities in the upper layer, then moves to r_k, and then makes a zig-zag path back to ℓ_k while alternating between the lower and the middle layer. The length of this optimal tour is $(2+\sqrt{3})\,2^k - 2\sqrt{3}$.

Next, we recall that the partial NNR tour $\mathscr{T}_k$ described in Lemma 3 has a length of exactly $(4+(k-1)\sqrt{3})2^{k-1}-1$. Moreover, the distance for the final step from u_k back to ℓ_k is at least $(2^{k-1}-1)\sqrt{3}$. This yields an NNR tour of length at least $(4+k\sqrt{3})2^{k-1}-(\sqrt{3}+1)$. With this, for $k \geqslant 2$, the ratio between the length of this NNR tour and the length of the optimal tour is at least

$$\begin{aligned}\frac{(4+k\sqrt{3})2^{k-1}-(\sqrt{3}+1)}{(2+\sqrt{3})2^k-2\sqrt{3}} &> \frac{(4+k\sqrt{3})2^{k-1}}{(2+\sqrt{3})2^k}\\&= \frac{4+k\sqrt{3}}{4+2\sqrt{3}}\\&> \frac{k\sqrt{3}}{4+2\sqrt{3}}\\&= \left(\sqrt{3}-\frac{3}{2}\right)k.\end{aligned}$$

Since $k = \log(n+2) - \log 3 \geqslant \log n - 2$, the claimed lower bound on the ratio follows. □

4. Conclusion

We have constructed bad (graphical and Euclidean) instances for the nearest neighbor rule for the TSP. Just as in the instances constructed by Rosenkrantz et al. [4] and by Johnson and Papadimitriou [2], the tie-breakings of NNR are crucial for its bad behavior on our instances. The points in the Euclidean instances from Section 3 can be perturbated by tiny amounts, so that tie-breaking is avoided, whereas the $\Omega(\log n)$ lower bound for NNR remains valid. For the graphical instances from Section 2, however, we do not know how to work around and avoid the tie-breakings. Hence, we currently cannot exclude the possibility that NNR with 'optimal' tie-breakings always yields good approximations for the TSP on graphical instances. We leave this as an open problem.

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