

Local Nonlinear Model Matching: From Linearity to Nonlinearity*

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Under the assumption that the model is decouplable and under generic assumptions on the plant, the nonlinear model matching problem is locally solvable if and only if it is solvable for the linearization of plant and model around an equilibrium point

Key Words—Nonlinear control systems; control systems synthesis; linearization techniques.

Abstract—The model matching problem consists of designing a compensator for a given system, called the plant, in such a way that the resulting input-output behaviour matches that of a prespecified model. In this paper a local solution of the nonlinear model matching problem is given for the case that the model is decouplable by static state feedback. The main theorem states that under generic conditions on the plant the problem is solvable around an equilibrium point if and only if it is solvable for the linearization of plant and model. The generic conditions are identified. They naturally appear in the solution of the dynamic input-output decoupling problem for the plant. The theory is illustrated by means of two examples.

1. INTRODUCTION

IN THIS paper we study the so called *model matching problem* (MMP) for nonlinear systems. That is, given a nonlinear control system, to be referred to as the plant P , together with another nonlinear system, to be called the model M , the question is whether or not it is possible to design a suitable precompensator for the plant such that the input-output behaviour of the precompensated plant matches that of the given model M . The idea is that in case a positive answer to this question exists, one is able to follow all the desired input-output trajectories of the model with those of the plant with precompensator.

The model matching problem has received a lot of attention during the last decades and considerable progress towards a characterization of necessary and sufficient conditions for its

solvability has been made. Let us first quickly review the main contributions on the problem under consideration. The *linear* model matching problem was completely solved in the early seventies, first by Moore and Silverman (1972), using Silverman's structure algorithm (Silverman, 1969) and then by Morse (1973) using a geometric setting. Later it was realized (Morse, 1976; Emre and Hautus, 1980) that the model matching problem is equivalent to a certain disturbance decoupling problem with disturbance measurements. Finally, Malabre (1982) stated equivalent solvability conditions in terms of certain structures at infinity.

A first step towards the solution of the nonlinear model matching problem was taken in (Isidori, 1985a), where sufficient conditions were given in case the model is a linear system. A very important and innovative contribution for the general nonlinear MMP was given in the paper of Di Benedetto and Isidori (1986). In this paper a detailed problem formulation and sufficient conditions for the existence of a local solution were given. Moreover under some rather restrictive assumptions on the model and plant necessary and sufficient conditions for the existence of a local solution were given. Essentially the approach taken in Di Benedetto and Isidori (1986) was very much inspired by the recently developed nonlinear differential geometric theory, see for instance the survey by Isidori (1985b), and it forms a direct generalization of the geometric linear theory (Morse, 1976; Emre and Hautus, 1980; Malabre, 1982). Following the approach of Di Benedetto and Isidori (1986) other partial solutions have appeared (Di Benedetto, 1988a, 1988b; Huijberts, 1989a). To what extent the machinery of Di Benedetto and Isidori (1986) allows for a

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complete answer to the problem is still an unsolved problem.

In the present paper we take a different approach for studying the nonlinear MMP. Throughout we assume the model to be input-output decouplable by static state feedback (see Section 4 for a precise definition). Although this assumption is certainly restrictive, it can be argued that in practical circumstances it is often desirable. Given the forementioned assumption an obvious condition for the local solvability of the MMP is that the plant P is input-output decouplable by adding a dynamic precompensator. The dynamic input-output decoupling problem has been solved by Descusse and Moog (1985) and later on in a slightly different way by Nijmeijer and Respondek (1988). The solvability of the MMP now may be checked by using a minimal order decoupling precompensator for the plant P .

The present method has an important feature that it is completely analytic and avoids concepts such as controlled invariant distributions and so on, as were used in Di Benedetto and Isidori (1986). All computations are straightforward in terms of the data of P and M and may be executed by formula manipulation, using e.g. MACSYMA, MAPLE or REDUCE. There is however another essential aspect of our solution we want to emphasize. Namely, we show that under generic conditions—a mathematical phrasing of almost always—the nonlinear MMP is solvable around equilibria of model M and plant P if and only if the corresponding linear MMP is solvable for the linearized model LM and plant LP . In our opinion this has important practical implications, in that in engineering practice one often studies a specific control problem by addressing the problem on the linearization around a given working point. The result given here may be viewed as an *a posteriori* justification of this methodology. In this way the paper fits in the philosophy developed by Gras and Nijmeijer (1989), where the relation between the input-output decoupling problem for a nonlinear system and its linearization around a working point have been investigated. In fact, as a byproduct we extend the results of that paper by discussing the dynamic input-output decoupling problem under linearization.

Finally, after having submitted this paper, we received a very interesting preprint by Moog *et al.* (1989) in which, using differential algebraic tools, a solution of the nonlinear MMP is given that generalizes the results of Moore and Silverman (1972) to nonlinear systems, and which apparently matches with the sufficiency results given here.

The paper is organized as follows. In the following section we will formulate the linear model matching problem and give conditions for the solvability of this problem. In Section 3 a formulation of the nonlinear model matching problem will be given along the lines of Di Benedetto and Isidori (1986). Furthermore we will, after we have treated the results of Di Benedetto and Isidori (1986), state our main theorem. This main theorem holds true under generic conditions on the plant P . The conditions naturally appear in the solution of the input-output decoupling problem for the plant P . This solution will be given in Section 4. Moreover, we will give some results on the minimal order linear input-output decoupling problem in Section 4. In Section 5 the main theorem will be proved and a characterization of models that can be matched starting from a specific plant will be given. In Section 6 the theory will be illustrated by means of two examples. The first one, which is borrowed from Di Benedetto and Isidori (1986), see also Di Benedetto (1988a), illustrates the constructiveness of our method. The second one, which is taken from De Luca and Ulivi (1988), illustrates our method in case the plant is a voltage frequency-controlled induction motor. In Section 7, final conclusions will be drawn.

2. THE LINEAR MODEL MATCHING PROBLEM

Consider a linear plant P , described by equations of the form:

$$P \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \quad (1)$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, output $y \in \mathbb{R}^m$ and matrices A , B , C of appropriate dimensions. Also, let a linear model M be given, which is described by the equations:

$$M \begin{cases} \dot{x}^M = A^M x^M + B^M u^M \\ y^M = C^M x^M \end{cases} \quad (2)$$

with state $x^M \in \mathbb{R}^{n^M}$, input $u^M \in \mathbb{R}^{m^M}$, output $y^M \in \mathbb{R}^{m^M}$ and matrices A^M , B^M , C^M of appropriate dimensions.

The compensator Q used to control P is a linear system described by equations of the form:

$$Q \begin{cases} \dot{x}_c = Kx_c + Lx + Mu^M \\ u = Fx_c + Gx + Hu^M \end{cases} \quad (3)$$

with state $x_c \in \mathbb{R}^v$ and matrices K , L , M , F , G , H of appropriate dimensions. The composition of (1) and (3) is denoted by $P \circ Q$.

Then the model matching problem consists of finding (if possible) an integer v and a

compensator Q of the form (3) such that the transfer matrices of M and $P \circ Q$ coincide, so that one is able to design a precompensator Q for the plant P such that the input-output behaviour of $P \circ Q$ matches that of the model M .

A way to solve this problem is as follows. Define the matrices

$$\begin{aligned} A^E &= \begin{bmatrix} A & 0 \\ 0 & A^M \end{bmatrix} & B^E &= \begin{bmatrix} B & 0 \\ 0 & B^M \end{bmatrix} \\ C^E &= [C \quad -C^M]. \end{aligned} \tag{4}$$

Theorem 1. (Morse, 1973). The model matching problem is solvable if and only if

$$\text{Im} \begin{bmatrix} 0 \\ B^M \end{bmatrix} \subset \text{Im} \begin{bmatrix} B \\ 0 \end{bmatrix} + \mathcal{V}^* \tag{5}$$

where \mathcal{V}^* is the maximal (A^E, B^E) -invariant subspace in $\text{Ker } C^E$. \square

The above problem also admits a direct interpretation in terms of the structure at infinity, cf. Malabre (1982).

Theorem 2. The model matching problem is solvable if and only if the system (1) and the system defined by the matrices (A^E, B^E, C^E) of (4) have the same structure at infinity.

3. THE NONLINEAR MODEL MATCHING PROBLEM

The definition of the nonlinear model matching problem will follow the same lines as the definition of the linear model matching problem. We employ the definition of Di Benedetto and Isidori (1986).

Consider a nonlinear plant P , described by equations of the form:

$$P \begin{cases} \dot{x} = f(x) + \sum_{i=1}^m u_i g_i(x) \\ y_i = h_i(x) \quad (i \in \{1, \dots, m\}) \end{cases} \tag{6}$$

with state $x \in \mathcal{X}$, an open neighbourhood in \mathbb{R}^n , inputs $u_i \in \mathbb{R}$, outputs $y_i \in \mathbb{R}$, f, g_1, \dots, g_m are real analytic vector fields on \mathbb{R}^n and h_1, \dots, h_m are real analytic functions.

Furthermore, let a nonlinear model M be given, which is described by the equations:

$$M \begin{cases} \dot{x}^M = f^M(x^M) + \sum_{i=1}^m u_i^M g_i^M(x^M) \\ y_i^M = h_i^M(x^M) \quad (i \in \{1, \dots, m\}) \end{cases} \tag{7}$$

with state $x^M \in \mathcal{X}^M$, an open neighbourhood in \mathbb{R}^{n^M} , inputs $u_i^M \in \mathbb{R}$, outputs $y_i^M \in \mathbb{R}$, f^M, g_1^M, \dots, g_m^M are real analytic vector fields on \mathbb{R}^{n^M} and h_1^M, \dots, h_m^M are real analytic functions.

The compensator Q used to control P is a

nonlinear system described by equations of the form:

$$Q \begin{cases} \dot{x}_c = a(x_c, x) + b(x_c, x)u^M \\ u = c(x_c, x) + d(x_c, x)u^M \end{cases} \tag{8}$$

with state $x_c \in \mathcal{X}_c$, an open neighbourhood in \mathbb{R}^v , and real analytic a, b, c, d . The composition of (6) and (8) is again denoted by $P \circ Q$.

In Section 2 we saw that in the case of linear systems the objective of model matching is to design a compensator Q such as to impose the coincidence between the transfer matrix of M and that of $P \circ Q$. In the case of nonlinear systems, where the input-output behaviour may be described in terms of Volterra series expansions, the object of model matching is to impose the coincidence of corresponding Volterra kernels.

Recall that the output $y(t) = (y_1(t), \dots, y_m(t))$ of a nonlinear system of the form (6) has a Volterra series expansion of the form [see e.g. Isidori (1985b)]

$$\begin{aligned} y(t) &= w_0(t, x_0) + \sum_{i=1}^m \int_0^t w_i(t, \tau_1, x_0) u_i(\tau_1) d\tau_1 \\ &+ \sum_{i_1, i_2=1}^m \int_0^t \int_0^{\tau_1} w_{i_1 i_2}(t, \tau_1, \tau_2, x_0) u_{i_1}(\tau_1) \\ &\times u_{i_2}(\tau_2) d\tau_1 d\tau_2 + \dots \end{aligned} \tag{9}$$

where x_0 is the initial state at time $t = 0$.

Let $w_{j_1 \dots j_i}^M(t, \tau_1, \dots, \tau_i, x_0^M)$ denote the (j_1, \dots, j_i) -th kernel of model M and similarly $w_{j_1 \dots j_i}^{P \circ Q}(t, \tau_1, \dots, \tau_i, (x_0, x_{c0}))$ the (j_1, \dots, j_i) -th kernel of the compensated plant $P \circ Q$. Since $w_{j_1 \dots j_i}^M$ depends on the initial state x_0^M of M and $w_{j_1 \dots j_i}^{P \circ Q}$ on the initial state (x_0, x_{c0}) of $P \circ Q$, when imposing the coincidence between these kernels one must specify how x_0^M and (x_0, x_{c0}) are chosen. Depending on this choice, one may formulate different matching problems. The most usual definition of the model matching problem, due to Di Benedetto and Isidori (1986), is given below.

Definition 1. Nonlinear model matching problem (MMP). Given a plant $P = (f, g, h)$, a model $M = (f^M, g^M, h^M)$ and a point $(x_0, x_0^M) \in \mathcal{X} \times \mathcal{X}^M \subset \mathbb{R}^n \times \mathbb{R}^{n^M}$, find neighbourhoods U of x_0 and U^M of x_0^M , an integer v , an open subset V of $\mathcal{X}_c \subset \mathbb{R}^v$, a compensator $Q = (a, b, c, d)$ with a, b, c, d real analytic functions defined on $V \times U$, a map $F: U \times U^M \rightarrow V$, with the property that

$$\begin{aligned} w_{j_1 \dots j_i}^{P \circ Q}(t, \tau_1, \dots, \tau_i, (x, F(x, x^M))) \\ = w_{j_1 \dots j_i}^M(t, \tau_1, \dots, \tau_i, x^M) \end{aligned} \tag{10}$$

for all $i \geq 1$, for all $1 \leq j_i \leq m$ and for all $(x, x^M) \in U \times U^M$. \square

In Di Benedetto and Isidori (1986) an attempt has been made to generalize the solution of the linear problem, cf. Section 2, to obtain a local solution to the nonlinear MMP. This is done by defining an extended system E associated with model and plant:

$$E \begin{cases} \dot{x}^E = f^E(x^E) + \sum_{i=1}^m u_i g_i^E(x^E) + \sum_{i=1}^m u_i^M p_i^E(x^E) \\ y_i^E = h_i^E(x^E) \quad (i \in \{1, \dots, m\}) \end{cases} \quad (11)$$

where $x^E = (x, x^M)$ and

$$f^E(x^E) = \begin{bmatrix} f(x) \\ f^M(x^M) \end{bmatrix}, \quad g_i^E(x^E) = \begin{bmatrix} g_i(x) \\ 0 \end{bmatrix}$$

$$p_i^E(x^E) = \begin{bmatrix} 0 \\ g_i^M(x^M) \end{bmatrix}, \quad h_i^E(x^E) = h_i(x) - h_i^M(x^M)$$

Define: (12)

$$\mathcal{G}^E = \text{span} \{g_1^E, \dots, g_m^E\} \quad (13)$$

$$\mathcal{P}^E = \text{span} \{p_1^E, \dots, p_m^E\}.$$

Theorem 3. (Di Benedetto and Isidori, 1986). The following equivalent conditions are sufficient for local solvability of MMP.

1. $\mathcal{P}^E \subset \Delta^* + \mathcal{G}^E$ (14)

where Δ^* is the maximal (f^E, g^E) -invariant distribution contained in $\text{Ker } dh^E$.

2. The structure at infinity of the extended system (11) (w.r.t. the inputs u_1, \dots, u_m) and that of the plant (6) are the same. \square

Now define relative degrees $\rho_1(x), \dots, \rho_m(x)$ for the plant P

$$\rho_i(x) = \inf \{k \in \mathbb{N} \text{ s.t. } \mathcal{L}_{g_i} \mathcal{L}_f^{k-1} h_i(x) \neq 0 \text{ for some } j\}. \quad (15)$$

In the same way we define relative degrees $\rho_i^M(x^M)$ for the model M , $\rho_i^{P \circ Q}(x, x_c)$ for the compensated plant $P \circ Q$, and $\rho_i^E(x^E)$ for the extended system E .

The following will be a standing assumption throughout the paper.

(A1) All relative degrees defined above are finite and constant on an open and dense submanifold, i.e. $\rho_i(x) = \rho_i < \infty$, $\rho_i^M(x^M) = \rho_i^M < \infty$, $\rho_i^{P \circ Q}(x, x_c) = \rho_i^{P \circ Q} < \infty$, $\rho_i^E(x^E) = \rho_i^E < \infty$ on an open and dense submanifold.

Lemma 1. Let Q locally solve MMP for (M, P) . Then $\rho_i^{P \circ Q} = \rho_i^M$.

Proof. Let $P \circ Q$ be given by

$$\begin{cases} \dot{\bar{x}} = \bar{f}(\bar{x}) + \bar{g}(\bar{x})u^M \\ \bar{y} = \bar{h}(\bar{x}) = h(x) \end{cases} \quad (16)$$

where $\bar{x} \in \mathcal{X} \times \mathcal{X}_c$. The fact that Q solves MMP for (M, P) implies that u^M does not influence the outputs $h_i^E(x^E) = h_i(x) - h_i^M(x^M)$ of the extended system E , i.e. we have solved a disturbance decoupling problem. By analyticity of the data this implies that for all $k \in \mathbb{N}$ and for all $i, j \in \{1, \dots, m\}$: $\mathcal{L}_{\bar{g}_i} \mathcal{L}_{\bar{f}}^k \bar{h}_i(\bar{x}) = \mathcal{L}_{g_i^M} \mathcal{L}_f^k h_i^M(x^M)$ and hence in particular $\rho_i^{P \circ Q} = \rho_i^M$. \square

The decoupling matrix $A(x)$ of P is defined on the open and dense submanifold where (A1) is satisfied in the following way:

$$(A(x))_{ij} = \mathcal{L}_{g_j} \mathcal{L}_f^{\rho_i-1} h_i(x) \quad (i, j \in \{1, \dots, m\}). \quad (17)$$

Similarly we define decoupling matrices $A^M, A^{P \circ Q}, A^E$ for $M, P \circ Q, E$ respectively.

Theorem 4. If the decoupling matrix $A(x)$ of P has full rank for all x on an open and dense submanifold of \mathcal{X} , then MMP is locally solvable if and only if

$$\rho_i \leq \rho_i^M \quad \text{for all } i \in \{1, \dots, m\}. \quad (18)$$

Moreover, in this case (18) is equivalent to the conditions in Theorem 3.

Proof. See Di Benedetto and Isidori (1986) and Huijberts (1989a). \square

Next we formulate our main result on the MMP, of which the proof will be given in Section 5.

Theorem 5. Consider an analytic plant P and an analytic model M . Let x_0 and x_0^M be equilibrium points of, respectively, P and M . Assume the decoupling matrix of the model has full rank around x_0^M . Let LP, LM denote the linearizations of P around x_0 and M around x_0^M respectively. Then under generic conditions on P the MMP is locally solvable for (M, P) if and only if it is solvable for (LM, LP) . \square

Roughly phrased, the theorem says that the MMP is ‘‘almost always’’ solvable for (M, P) provided the problem is solvable for the linearization (LM, LP) . We refer to e.g. Tchoñ (1986) for a precise mathematical definition of a generic property, and we will identify checkable and verifiable conditions on P for which the above result holds true. These conditions naturally appear in the solution of the input–output decoupling problem for the plant P . This problem will be discussed in the next section.

Notice that Theorem 5 in principle only

addresses the MMP in a neighbourhood of the equilibria x_0 and x_0^M . However, it is rather straightforward to see that the analyticity of P and M imply that whenever a local solution of the MMP exists around (x_0, x_0^M) , then there exists a local solution around points in an open and dense submanifold of $\mathcal{X} \times \mathcal{X}^M$. The theory developed in the following sections can be directly extended to this situation.

4. THE INPUT-OUTPUT DECOUPLING PROBLEM

We consider a nonlinear system (6). Then the (dynamic) input-output decoupling problem is defined as follows

Definition 1. Input-output decoupling problem. Find (if possible) an integer ν , a compensator of the form (8) for (6) and an initial state $x_{c0} \in \mathbb{R}^\nu$ for the compensator such that the overall system is input-output decoupled. That is, each of the new inputs influences one and only one of the outputs.

It is well known [cf. Isidori (1985b)] that the input-output decoupling problem for analytic systems is locally solvable by static state feedback (i.e. we can take $\nu = 0$) if and only if the decoupling matrix of (6) has full rank for all x . The problem of dynamic input-output decoupling was studied by Singh (1980, 1981) via a generalization of Hirschorn's nonlinear version (Hirschorn, 1979) of the Silverman structure algorithm (Silverman, 1969). In Descusse and Moog (1985) an interesting extension to strongly left-invertible nonlinear systems of linear dynamic decoupling, as was used in Wang (1970), was given. A more transparent algorithm for the input-output decoupling problem was given in Nijmeijer and Respondek (1988), thereby generalizing and expanding the algorithm of Descusse and Moog (1985). Essentially this algorithm (as well as the others) works on an open and dense submanifold of points (x, x_c) in $\mathcal{X} \times \mathcal{X}_c$. Our basic assumption will be that given the equilibrium point x_0 for P there exists an equilibrium point (x_0, x_{c0}) for the precompensated system $P \circ Q$ at which the algorithm can be effectively applied. To make this more transparent we shortly discuss the first step of the algorithm of Nijmeijer and Respondek (1988). Given the plant P we compute its decoupling matrix $A(x)$ according to equation (17). Clearly the rank of $A(x)$ is constant on an open and dense submanifold \mathcal{X}^1 of \mathcal{X} . Assume $x_0 \in \mathcal{X}^1$. Next a regular static state feedback.

$$u = \alpha(x) + \beta(x)\bar{u} \tag{19}$$

is applied to P , and which leaves the equilibrium

point x_0 invariant. Note that applying such a feedback does not change the rank of the decoupling matrix $A(x)$. The first step of the algorithm proceeds with adding a bank of integrators

$$\begin{cases} \dot{z}_{i1} = z_{i2} \\ \vdots \\ \dot{z}_{ik_i} = v_i \\ \bar{u}_i = z_{i1} \end{cases} \tag{20}$$

for $i = 1, \dots, m$ and certain integers $k_i \in \mathbb{N}$. Note that for some indices i , $k_i = 0$, which shows that some of the inputs \bar{u} are not integrated. Obviously, adding (19), (20) to P yields a system on $\mathcal{X} \times \mathcal{Z}$, where $\mathcal{Z} = (z_{11}, \dots, z_{1k_1}, \dots, z_{m1}, \dots, z_{mk_m})$ and this system possesses an equilibrium point $(x_0, z_0) \in \mathcal{X} \times \mathcal{Z}$. We now compute the decoupling matrix of P together with (19), (20), to be denoted as $A(x, z)$. Again this decoupling matrix has constant rank on an open and dense submanifold $(\mathcal{X} \times \mathcal{Z})^1$ of $\mathcal{X} \times \mathcal{Z}$. We assume (x_0, z_0) belongs to $(\mathcal{X} \times \mathcal{Z})^1$. The algorithm of Nijmeijer and Respondek (1988) now repeats the above steps of applying state feedbacks (19) and adding a bank of integrators as in (20). Moreover the algorithm terminates after a finite number of steps. Our basic assumption can now be stated as follows:

(A2) Consider the system (6) around the equilibrium point $x_0 \in \mathcal{X}$. Then we assume that at each step of the above algorithm the decoupling matrix $A(x, z)$ has constant rank in a neighbourhood of the equilibrium point (x_0, z_0) .

Essentially (A2) says that the algorithm of Nijmeijer and Respondek (1988) can be applied on a neighbourhood of the equilibrium point x_0 . Given the analyticity of (6) this condition is generically met for vector fields f, g_1, \dots, g_m and functions h_1, \dots, h_m .

Remark 1. Strictly speaking, the equilibria of a vector field $f(x, u)$ are all (x_0, u_0) satisfying $f(x_0, u_0) = 0$, whereas in assumption (A2) we restrict ourselves to equilibria of the form $(x_0, 0)$, i.e. we let $u_0 = 0$. If we do not restrict ourselves to the case that $u_0 = 0$ the set of equilibrium points obviously gets larger. This freedom will be explored in Example 6.2 where it is shown that the conditions for solvability of MMP are met around an equilibrium (x_0, u_0) where $u_0 \neq 0$, while the conditions are not met around an equilibrium of the form $(x_0, 0)$. Note however that an equilibrium of the form (x_0, u_0) where $u_0 \neq 0$ can be transformed into an equilibrium of the form $(x_0, 0)$ by applying a preliminary feedback.

The following proposition shows the importance of (A2) in that under this assumption the

solvability of the input-output decoupling problem for (6) around x_0 is completely equivalent to the solvability of the input-output decoupling problem for its linearization around x_0 .

Proposition 1. Consider a nonlinear system (6). Let x_0 be an equilibrium point of the system that satisfies (A2). Then the input-output decoupling problem is solvable for (6) if and only if it is solvable for the linearization of (6) around x_0 .

Proof. (Necessity)

Assume that the input-output decoupling problem is solvable for (6) by means of a compensator of the form (8). Let x_{c0} be such that (x_0, x_{c0}) is an equilibrium point of $P \circ Q$. Since the compensator solves the input-output decoupling problem and (A2) holds, the decoupling matrix of $P \circ Q$ has full rank at (x_0, x_{c0}) . This also implies that the decoupling matrix of the linearization of $P \circ Q$ around (x_0, x_{c0}) has full rank [see e.g. Gras and Nijmeijer (1989)]. Observe that the linearization of $P \circ Q$ around (x_0, x_{c0}) is nothing else but the linearization of $P \circ Q$ around x_0 compensated by the linearization of (8) around (x_0, x_{c0}) . Hence the input-output decoupling problem is also solvable for the linearization of P around x_0 .

(Sufficiency)

Assume that the input-output decoupling problem is solvable for the linearization of (6) around x_0 . It is easy to see that, since (A2) holds, the algorithm from Nijmeijer and Respondek (1988) can be applied at the same time to (6) and the linearization of (6) around x_0 , meaning that at each round the feedback applied to the linearization is the linearization of the feedback applied to (6) and that we add the same bank of integrators to (6) and the linearization. \square

In the rest of this section, we will give some results on the input-output decoupling problem for linear systems. So, consider a linear system of the form (1). It is well known from the literature [see e.g. Wang (1970), Falb and Wolovich (1967)] that the input-output decoupling problem is solvable for (1) if and only if (1) is invertible [see Descusse and Moog (1985) for a generalization to nonlinear systems].

Apart from the relative degrees as defined in (15) we can also define another type of characteristic numbers for (1), called the essential orders. The notion of essential orders is based on the work by Cremer (1971). Here we follow Commault *et al.* (1986) [for a generalization to nonlinear systems, see Glumineau and Moog, (1989)].

Definition 2. (Cremer, (1971). Given a matrix W , the i th row w_i of W is called essential if w_i cannot be represented by a linear combination of other rows of W . \square

Now let us denote the $(\mu m, \mu m)$ -matrix:

$$\Gamma_\mu = \begin{bmatrix} CB & 0 & \cdots & \cdots & 0 \\ CAB & CB & 0 & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ CA^{\mu-1}B & CA^{\mu-2}B & \cdots & \cdots & CB \end{bmatrix}. \quad (21)$$

Definition 3. Consider an invertible linear system (1). Let c_i denote the i th row of C . The essential order of the i th output of (1) is the integer ϵ_i defined by:

$$\epsilon_i = \inf\{\mu \in \mathbb{N} \text{ s.t. } [c_i A^{\mu-1} B \cdots c_i B 0 \cdots 0] \text{ is essential in } \Gamma_n\}. \quad (22)$$

Theorem 6. Consider an invertible linear system (1). Then we have:

1. $\epsilon_i \geq \rho_i$
2. ϵ_i, ρ_i cannot be decreased by static or dynamic feedback
3. The input-output decoupling problem is solvable by static state feedback if and only if $\epsilon_i = \rho_i$
4. Among all possible decoupling compensators there is at least one for which the essential orders are not increased.

Proof. See Commault *et al.*, 1986; Dion and Commault, 1985; Descusse *et al.* 1986. \square

Remark 2. The assertions of the above theorem also hold for nonlinear systems (6). For a proof, see Glumineau and Moog (1989), and Xia (1989). \square

We have as an immediate consequence (cf. Commault *et al.*, 1986; Dion and Commault, 1985):

Corollary 1. Consider an invertible linear system (1). Then for this system there is a decoupling precompensator of minimal dimension such that the relative degrees of the decoupled system equal the essential orders of (1).

5. PROOF OF THE MAIN THEOREM

In this section we will first give a proof of the main theorem. After that we will make some comments on the characterization of models that can be matched starting from a specific plant.

Assume we are given a plant P with equilibrium point x_0 and a model M with equilibrium point x_0^M such that the decoupling matrix $A^M(x^M)$ of the model has full rank in a neighbourhood of x_0^M . For convenience we will first give a reformulation of our main theorem, where the generic conditions on P are made explicit.

Theorem 7. Consider an analytic plant P and an analytic model M . Let x_0 and x_0^M be equilibrium points of respectively P and M . Assume that P satisfies (A2) and that the decoupling matrix of the model has full rank around x_0^M . Let LP , LM denote the linearizations of P around x_0 and M around x_0^M respectively. Then the MMP is locally solvable for (M, P) if and only if it is solvable for (LM, LP) . \square

Before proving the main theorem we will prove some preliminaries.

Lemma 2. Let Q be a compensator that locally solves the MMP for (M, P) , where the decoupling matrix $A^M(x^M)$ has full rank around the equilibrium x_0^M . Then $A^{P \circ Q}(x, x_c)$ has full rank on an open and dense submanifold of $\mathcal{X} \times \mathcal{X}_c$.

Proof. Let $P \circ Q$ be given by:

$$\begin{cases} \dot{\bar{x}} = \bar{f}(\bar{x}) + \bar{g}(\bar{x})u^M \\ \bar{y} = \bar{h}(\bar{x}) = h(x) \end{cases} \quad (23)$$

where $\bar{x} \in \mathcal{X} \times \mathcal{X}_c$. Recall from Lemma 1 that the fact that Q solves MMP implies that $\rho_i^{P \circ Q} = \rho_i^M$. Since Q solves MMP, for any x_*^M there is an \bar{x}_* such that for any u^M and any $k \in \mathbb{N}$: $\mathcal{L}_{\bar{g}}^k \mathcal{L}_{\bar{f}}^k \bar{h}(\bar{x}) = \mathcal{L}_{g^M}^k \mathcal{L}_{f^M}^k h^M(x^M)$ along the trajectories of $P \circ Q$ and M starting at \bar{x}_* and x_*^M respectively. Hence in particular there is an \bar{x} for which $A^{P \circ Q}(\bar{x})$ has full rank. Then by analyticity of the data: $A^{P \circ Q}(\bar{x})$ has full rank on an open and dense submanifold of $\mathcal{X} \times \mathcal{X}_c$. \square

This leads to:

Proposition 2. Consider an analytic plant P and an analytic model M , where the decoupling matrix $A^M(x^M)$ has full rank around the equilibrium x_0^M . Then MMP is locally solvable for (M, P) if and only if there is a dynamic compensator Q such that $P \circ Q$ is decouplable by regular static state feedback and $\rho_i^{P \circ Q} \leq \rho_i^M$.

Proof. (Necessity)

If Q locally solves MMP, then we know by Lemma 2 that $A^{P \circ Q}$ has full rank on an open and

dense submanifold. Hence $P \circ Q$ is decouplable by static state feedback. Furthermore by Theorem 4: $\rho_i^{P \circ Q} \leq \rho_i^M$.

(Sufficiency)

Follows immediately from Theorem 4. \square

We now come to the proof of our main theorem:

Proof (of Theorem 7). (Necessity)

If MMP is solvable for (M, P) then by Proposition 2 there is a dynamic compensator Q such that $P \circ Q$ is decouplable by regular static state feedback and $\rho_i^{P \circ Q} \leq \rho_i^M$. Since (A2) holds, this implies that there is a compensator LQ for LP such that $LP \circ LQ$ is decouplable by regular static state feedback and $\rho_i^{LP \circ LQ} \leq \rho_i^M$. Hence by Proposition 2, MMP is solvable for (LM, LP) .

(Sufficiency)

Follows immediately by reversing the reasoning of the necessity part. \square

Proposition 2 also enables us to characterize models that can be matched starting from a specific plant.

Theorem 8. Consider an analytic plant P with equilibrium point x_0 satisfying (A2). Let the essential orders of its linearization around x_0 be denoted by ϵ_i . Then MMP is locally solvable for (M, P) where M is an analytic system and the decoupling matrix $A^M(x^M)$ has full rank around the equilibrium x_0^M , if and only if $\epsilon_i \leq \rho_i^M$.

Proof. (Necessity)

Assume that MMP is solvable for (M, P) where the decoupling matrix $A^M(x^M)$ has full rank around the equilibrium x_0^M . Then, by Theorem 7 MMP is also solvable for (LM, LP) . By Proposition 2 this implies that there is a dynamic compensator LQ such that $LP \circ LQ$ is decouplable by regular static state feedback and $\rho_i^{LP \circ LQ} \leq \rho_i^M$. By Corollary 1 we have $\rho_i^{LP \circ LQ} \geq \epsilon_i$ and thus: $\epsilon_i \leq \rho_i^M$.

(Sufficiency)

Consider a model M for which the decoupling matrix $A^M(x^M)$ has full rank around the equilibrium x_0^M and that satisfies $\epsilon_i \leq \rho_i^M$. By Corollary 1 there is a decoupling compensator LQ for LP such that the relative degrees of $LP \circ LQ$ are just ϵ_i . By Proposition 1 (or rather the proof of it) this implies that there is a decoupling compensator Q for P such that the relative degrees of $P \circ Q$ also equal ϵ_i . Thus there is a decoupling compensator Q for P such that $\rho_i^{P \circ Q} = \epsilon_i \leq \rho_i^M$ and hence by Proposition 2 we find that the MMP is solvable for (M, P) . \square

stator and rotor self-inductances and M is the mutual inductance. The speed ω can be considered as a slowly varying parameter, due to the large separation of time-scales between the mechanical and the electromagnetic dynamics. In the sequel we will assume it to be constant.

Defining $\bar{x} = (x_1, \dots, x_4)$ and $x = (\bar{x}, x_5)$, the state equations are written as

$$\dot{x} = \begin{bmatrix} A\bar{x} \\ 0 \end{bmatrix} + \begin{bmatrix} g_1(x_5) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad (36)$$

where

$$A = \begin{bmatrix} -(\alpha + \beta) & -\omega & \frac{\beta}{L_s} & \frac{\omega}{\sigma L_s} \\ \omega & -(\alpha + \beta) & -\frac{\omega}{\sigma L_s} & \frac{\beta}{L_s} \\ -\alpha\sigma L_s & 0 & 0 & 0 \\ 0 & -\alpha\sigma L_s & 0 & 0 \end{bmatrix}$$

$$g_1(x_5) = \begin{bmatrix} \cos x_5 \\ \frac{\sigma L_s}{\sin x_5} \\ \frac{\sin x_5}{\sigma L_s} \\ \cos x_5 \\ \sin x_5 \end{bmatrix} \quad (37)$$

and

$$\alpha = \frac{R_s}{\sigma L_s}, \quad \beta = \frac{R_r}{\sigma L_r}, \quad \sigma = 1 - \frac{M^2}{L_s L_r}.$$

Suitable outputs for the system are defined in terms of the stator flux and the torque. Hence, the following nonlinear output functions will be used

$$h_1(x) = \Phi_s^2 = x_3^2 + x_4^2 \quad (38)$$

$$h_2(x) = T_m = x_2 x_3 - x_1 x_4.$$

If we look for equilibria of the form $(x_0, 0)$ (cf. Remark 1), we find $x_0 = (\bar{x}, x_5) = (0, x_5)$, since A is invertible. In these equilibria the relative degrees are infinite, so (A2) does not hold in these equilibria. It is easily seen that with

$$x_0 = \left(-\frac{1}{\alpha\sigma L_s}, 0, -\frac{\beta^2\sigma + \omega^2}{\alpha(\beta^2\sigma^2 + \omega^2)}, \frac{\beta\omega(\sigma - 1)}{\alpha(\beta^2\sigma^2 + \omega^2)}, 0 \right),$$

$u_0 = (1, 0)$, (x_0, u_0) is an equilibrium. This equilibrium can be transformed into an equilibrium of the form (x_0, v_0) with $v_0 = 0$ by introducing the preliminary feedback

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ x_5 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}. \quad (39)$$

The relative degrees of the system after we have applied the preliminary feedback equal 1

and 1 on an open and dense submanifold containing x_0 , provided $M \neq 0$ and $M^2 \neq \frac{L_r^2 \omega^2 - R_r^2}{L_s \omega^2} L_s$, and the decoupling matrix on this submanifold is given by

$$A(x) = \begin{bmatrix} 2x_3 \cos x_5 + 2x_4 \sin x_5 & 0 \\ \left(\frac{x_3}{\sigma L_s} - x_1 \right) \sin x_5 - \left(\frac{x_4}{\sigma L_s} - x_2 \right) \cos x_5 & 0 \end{bmatrix}. \quad (40)$$

A straightforward calculation, using REDUCE shows that (A2) is satisfied around the equilibrium point x_0 . The linearization of the system around x_0 is given by

$$LP \begin{cases} \dot{x} = Fx + Gv \\ y = Hx \end{cases} \quad (41)$$

where

$$F = \begin{bmatrix} -(\alpha + \beta) & -\omega & \frac{\beta}{L_s} & \frac{\omega}{\sigma L_s} & 0 \\ \omega & -(\alpha + \beta) & -\frac{\omega}{\sigma L_s} & \frac{\beta}{L_s} & \frac{1}{\sigma L_s} \\ -\alpha\sigma L_s & 0 & 0 & 0 & 0 \\ 0 & -\alpha\sigma L_s & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$G = \begin{bmatrix} \frac{1}{\sigma L_s} & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$H = \begin{bmatrix} 0 & 0 \\ -\frac{\beta\omega(\sigma - 1)}{\alpha(\beta^2\sigma^2 + \omega^2)} & -\frac{\beta^2\sigma + \omega^2}{\alpha(\beta^2\sigma^2 + \omega^2)} \\ \frac{-2\beta^2\sigma + \omega^2}{\alpha(\beta^2\sigma^2 + \omega^2)} & 2\frac{\beta\omega(\sigma - 1)}{\alpha(\beta^2\sigma^2 + \omega^2)} & 0 \\ 0 & \frac{1}{\alpha\sigma L_s} & 0 \end{bmatrix}.$$

It can easily be checked that the essential orders of LP equal 2 and 2. Hence any two-input-two-output system that is decouplable by static state feedback and has relative degrees at least equal to 2 and 2 can be taken as a model. Using REDUCE, compensators have been calculated for several models. The results have been omitted here, because of their lengthiness. We refer to e.g. Huijberts (1989b) for more worked examples.

7. CONCLUSIONS

A local solution to the nonlinear model matching problem in case the model is decouplable by static state feedback has been given. The method to obtain this solution is completely analytic.

An implication of the method that is proposed in this paper is that if we start from an equilibrium point of $P \circ Q$ that satisfies (A2) and from an equilibrium point of the model, we in fact have perfect matching: also the zero-th order kernels of the Volterra expansions of $P \circ Q$ and M coincide.

It should be noted also that in this paper results from Isidori (1985a) are generalized for a linear model that is decouplable by static state feedback.

A problem that remains unsolved is the model matching problem for models that are not decouplable by static state feedback. In this case the method of this paper does give some necessary conditions: the number of input-output channels of P that is decouplable by dynamic state feedback must be greater than or equal to the number of input-output channels of M that is decouplable by static state feedback. Further research is required to obtain necessary and sufficient conditions in this case.

Another problem that remains unsolved is the problem of internal stability of the compensated plant after we have solved the nonlinear MMP. Up till now, this problem has only been addressed by Byrnes *et al.* (1988) in the case where the plant is a SISO-system and by Huijberts (1989a), in the case where the plant is decouplable by static state feedback. The problem consists in the fact that, even if we start from an internally stable plant and an internally stable model, we may very well introduce unobservable unstable modes in the closed loop. In order to solve this problem, further investigation of the structure of a model matching configuration is needed, especially concerning the "fixed" and "free" modes of such a configuration. For linear systems this investigation has already been performed in Morse (1973). For nonlinear systems this question is undoubtedly much more difficult. So far, only results about "fixed" modes in the solution of the input-output decoupling problem have been obtained in Isidori and Grizzle (1988). It is not clear if a similar analysis is applicable for the here considered nonlinear MMP. We leave this open for future research.

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