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Degree sums and subpancyclicity in line graphs ☆

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Abstract

A graph is called subpancyclic if it contains a cycle of length k for each k between 3 and the circumference of the graph. In this paper, we show that if the degree sum of the vertices along each 2-path of a graph G exceeds $(n+6)/2$, or if the degree sum of the vertices along each 3-path of G exceeds $(2n+16)/3$, then its line graph $L(G)$ is subpancyclic. Simple examples show that these bounds are best possible. Our results shed some light on the content of a famous Metaconjecture of Bondy. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

The graphs considered in this paper are finite simple graphs. We follow the notation of Bondy and Murty [3], unless otherwise stated.

Let G be a finite simple graph, and H be a subgraph of G. Then $V(H)$ and $E(H)$ denote the set of vertices and edges of H, and $\bar{E}(H)$ denotes the set of edges of G that are incident with vertices of H. We denote $|V(H)|$, $|E(H)|$ and $|\bar{E}(H)|$ by $v(H)$, $\varepsilon(H)$, $\bar{\varepsilon}(H)$, respectively. If S is a subgraph of H or a subset of $V(H)$, then the *degree* of S in H, denoted by $d_H(S)$, is defined to be the degree sum of vertices in S, i.e., $d_H(S)=\sum_{u\in V(S)} d_H(u)$, or just $d(S)$ if $G=H$. The *distance* between two vertices x and y, denoted by $d(x,y)$, is the length of a shortest path between x and y. The *distance between two subgraphs* G_1, G_2 of H, denoted by $d_H(G_1,G_2)$, is defined to be $\min\{d_H(x,y): x\in V(G_1) \text{ and } y\in V(G_2)\}$. The *diameter* of H is the maximum distance between two vertices of H.

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We use $H - E(S)$ to denote the edge-induced subgraph $H[E(H) \setminus E(S)]$ of H. We denote the nontrivial component of H by $H-S$ if $H-E(S)$ has at most one nontrivial component of H. By C_k we denote the cycle of length k. Let $\lambda(G)=\{k: G \text{ has a } C_k\}$. We use $\mathrm{cr}(G)$ to denote the *circumference* of G, i.e., the length of a longest cycle of G. G is said *subpancyclic* if $\lambda(G)=[3,\mathrm{cr}(G)]=\{3,4,\ldots,\mathrm{cr}(G)\}$. G is called *pancyclic* if it is subpancyclic and hamiltonian. C is called a *circuit* of a graph G if C is an Eulerian subgraph of G, i.e., a connected subgraph in which every vertex has even degree. Note that by this definition (the trivial subgraph induced by) a single vertex is also a circuit.

Harary and Nash-Williams characterized those graphs whose line graphs are hamiltonian.

Theorem 1 (Harary and Nash-Williams [7]). *The line graph $L(G)$ of a graph G is hamiltonian if and only if G contains a circuit C such that $\bar{\varepsilon}(C)=\varepsilon(G)\geqslant 3$.*

A more general result is the following.

Theorem 2 (Broersma [4]). *The line graph $L(G)$ of a graph G contains a cycle of length $k\geqslant 3$ if and only if G contains a circuit C such that $\varepsilon(C)\leqslant k\leqslant \bar{\varepsilon}(C)$.*

Define

$$\rho_i(G)=\min\{d(P)\ :\ P \text{ is a path of length } i \text{ in } G\}.$$

Obviously, $\delta(G)=\rho_0(G)$. As introduced in [1], let $f_i(n)$ be the smallest integer such that for any graph G of order n with $\rho_i(G)>f_i(n)$, the line graph $L(G)$ of G is pancyclic whenever $L(G)$ is hamiltonian. Van Blanken et al. [1] proved that $f_0(n)$ has an order of magnitude: $n^{1/3}$. It was shown that $f_1(n)=[(\sqrt{8n+1}+1)/2]$ (if $n\geqslant 600$) [12] and that $f_3(n)\leqslant n-1$ (if $n\geqslant 40$) [13]. In this paper, we obtain that if $n\geqslant 76$, then $f_2(n)=[(n+6)/2]$ and $f_3(n)=[(2n+16)/3]$. These results show that $f_i(n)$ has the interesting order of magnitude: $n^{1/(3-i)}$ for $0\leqslant i\leqslant 2$.

Moreover, we give the following more general results.

Theorem 3. *Let G be a graph of order n ($n\geqslant 76$). If G satisfies one of the following conditions*:

(i) $\rho_2(G)>(n+6)/2$;
(ii) $\rho_3(G)>(2n+16)/3$;

then $L(G)$ is subpancyclic and the results are all best possible.

Conditions (i) and (ii) of Theorem 3 cannot be improved in the following sense.

Let $s=(m-2)/2$ ($m\equiv 0 \pmod 2$) and $t=(n-1)/3$ ($n\equiv 1 \pmod 3$). Define two graphs G_1 of order m and G_2 of order n, as follows respectively: vertex sets $V(G_1)=(\bigcup_{i=1}^{s}\{u_i,v_i\})\cup\{x,y\}$ and $V(G_2)=(\bigcup_{i=1}^{t}\{x_i,y_i,z_i\})\cup\{w\}$, edge sets $E(G_1)=$

$\bigcup_{i=1}^{s}\{xu_i, u_iv_i, v_iy\}$ and $E(G_2)=\bigcup_{i=1}^{t}\{wx_i, x_iy_i, y_iz_i, z_iw\}$. It is obvious that G_1 and G_2 are two graphs such that $\rho_2(G_1)=s+4=(m+6)/2$ and $\rho_3(G_2)=2t+6=(2n+16)/3$, respectively. But, by Theorem 2, $3s-1 \in [3, \varepsilon(G_1)] \setminus \lambda(L(G_1))$ and $4t-1 \in [3, \varepsilon(G_2)] \setminus \lambda(L(G_2))$ which imply that $L(G_1)$ and $L(G_2)$ are not subpancyclic.

It follows from Theorem 1 that $L(G_1)$ and $L(G_2)$ are all hamiltonian. This also implies the following corollary.

Corollary 4. *If* $n \geqslant 76$, *then* $f_2(n)=[(n+6)/2]$ *and* $f_3(n)=[(2n+16)/3]$.

Corollary 4 improves the results of [11,13] and shows that those graphs in [8–10] and [14] are pancyclic. We only give one example as follows.

Theorem 5 (Liu et al. [9]). *Let G be a simple graph with* $\varepsilon(G)\geqslant 3$ *and let G be not a* 3-*path. If* $\sum_{i=1}^{4} d(u_i) \geqslant 2n-2$ *for any four vertices such that* $u_1u_2, u_3u_4 \in E(G)$, *then* $L(G)$ *is hamiltonian and the result is best possible.*

Combining Theorems 3 and 5 we obtain the following.

Corollary 6. *If G is a simple graph of order* $n \geqslant 76$ such that $\sum_{i=1}^{4} d(u_i) \geqslant 2n-2$ *for any four vertices with* $u_1u_2, u_3u_4 \in E(G)$, *then* $L(G)$ *is pancyclic.*

Corollary 6 supports the famous Metaconjecture of Bondy (see e.g. [2]) that almost every nontrivial condition which implies that a graph is hamiltonian also implies that the graph is pancyclic. The results shown in the above as well as in [5,6,8,10] show that degree sums conditions along paths required on a graph which ensure that its line graph is subpancyclic are considerably weaker than those required to ensure that its line graph is hamiltonian. This sheds some light on the Metaconjecture.

In general, results involving degree sums are directly derived from results involving the minimum degree of the graph. The result in [12] shows an exception to this rule. Our results show that the results involving degree sums of the vertices along a 2-path or a 3-path do not imply immediately the corresponding results involving the minimum edge degree.

2. The proof of Theorem 3

We will complete the proof by contradiction.

Assuming G is a graph of order n which satisfies the conditions of Theorem 3 but its line graph $L(G)$ is not subpancyclic, we let

$$k=\max\{i\colon i \in [3, \mathrm{cr}(L(G))] \setminus \lambda(L(G))\}.$$

Then it follows from Theorem 2 that

Claim 1. *G does not contain a circuit* C_0 *with* $\varepsilon(C_0) \leqslant k \leqslant \bar{\varepsilon}(C_0)$.

It is obvious that $L(G)$ contains a cycle C_{k+1} of length $k+1$. It follows from Theorem 2 that G contains a circuit C with $\varepsilon(C) \leqslant k+1 \leqslant \bar{\varepsilon}(C)$. By Claim 1, $\varepsilon(C) = k+1$. Since C is a circuit, there exist edge-disjoint cycles $D_1, D_2, \ldots, D_r$ such that $C = \bigcup_{i=1}^{r} D_i$ and r is maximized.

Hence,

$$\text{if } r \geqslant 2, \text{ then } |V(D_i) \cap V(D_j)| \leqslant 2 \quad \text{for } \{i,j\} \subseteq \{1,2,\ldots,r\}. \tag{1}$$

Let $UP_i(C) = \{P\colon P \text{ is a path of length } i \text{ in } C\}$.

Proof of (i) in Theorem 3. Since $\rho_2(G) > (n+6)/2 \geqslant 41$,

$$\varepsilon(C) = k+1 \geqslant \Delta(G) + 2 \geqslant \rho_2(G)/3 + 2 > (n+18)/6 \geqslant 14. \tag{2}$$

We will consider the following two cases:

Case 1: $r = 1$, i.e., *C is a cycle of length $k+1$*. First, we show a needed claim.

Claim 2. *G does not contain a cycle C' with $\varepsilon(C)/2 < \varepsilon(C') \leqslant k$.*

Proof of Claim 2. Otherwise, in $\sum_{P \in UP_2(C)} d(P)$, every edge in $\bar{E}(C')$ is counted at most 6 times. Hence, by (2) and (i),

$$\begin{aligned}
\bar{\varepsilon}(C') &\geqslant \sum_{P \in UP_2(C)} (d(P) - 6)/6 + \varepsilon(C') \\
&\geqslant (\rho_2 - 6)\varepsilon(C')/6 + \varepsilon(C') \\
&= \rho_2 \varepsilon(C')/6 \\
&> \rho_2 \varepsilon(C)/12 \geqslant k+1.
\end{aligned}$$

On the other hand, $\varepsilon(C') \leqslant k$. Theorem 2 implies that $L(G)$ contains a C_k, a contradiction. □

So, C has no chord. By $\rho_2 \geqslant 42$, C cannot be a hamiltonian cycle of G. Let u be a vertex in $V(G) \backslash V(C)$. By Claim 2, u is adjacent to at most three vertices of C. Hence, by (2),

$$\bar{\varepsilon}(C) \leqslant 3|V(G) \backslash V(C)| + \varepsilon(C) = 3(n - \varepsilon(C)) + \varepsilon(C) < (8n - 18)/3. \tag{3}$$

On the other hand, since C has no chord,

$$\begin{aligned}
\bar{\varepsilon}(C) &\geqslant \sum_{P \in UP_2(C)} (d(P) - 6)/3 + \varepsilon(C) \\
&\geqslant (\rho_2 - 6)\varepsilon(C)/3 + \varepsilon(C) \\
&= (\rho_2 - 3)\varepsilon(C)/3 > n(n+18)/36,
\end{aligned}$$

which contradicts (3) since $n \geqslant 76$, and proves Case 1.

Case 2: $r \geqslant 2$. Let H be the graph with $V(H) = \{D_1, D_2, \ldots, D_r\}$ and $D_iD_j \in E(H)$ if and only if $V(D_i) \cap V(D_j) \neq \emptyset$. Since C is a circuit, H is connected. Without loss of generality, we assume that D_1 and D_r are two vertices of H such that

$$d_H(D_1, D_r) = \text{dia}(H). \tag{4}$$

Obviously each of $\{D_1, D_r\}$ is not cut-vertex of H, hence $C^1 = \bigcup_{i=2}^{r} D_i$ and $C^r = \bigcup_{i=1}^{r-1} D_i$ are two circuits of G.

Let

$$E_1(D_i) = E(D_i) \cap \bar{E}(C^i) \quad \text{and} \quad E_2(D_i) = E(D_i) \backslash E_1(D_i),$$

$$V_1(D_i) = V(D_i) \cap V(C^i) \quad \text{and} \quad V_2(D_i) = \{u, v \colon uv \in E_2(D_i)\},$$

where $i \in \{1, r\}$.

For any path P of C, let $d_2(P) = d(P) - d_C(P)$. Since $\bar{\varepsilon}(C^i) \geqslant \varepsilon(C) - |E_2(D_i)| = k + 1 - |E_2(D_i)|$,

$$|V_2(D_i)| - 1 \geqslant |E_2(D_i)| \geqslant 2 \quad \text{for } i \in \{1, r\}. \tag{5}$$

Otherwise C^i is a circuit with $\varepsilon(C^i) \leqslant k \leqslant \bar{\varepsilon}(C^i)$ which contradicts Claim 1.

Since $\bar{\varepsilon}(C^t) \geqslant \varepsilon(C) - |E_2(D_t)| + |\bar{E}(D_s) \backslash E(C)|$,

$$|\bar{E}(D_s) \backslash E(C)| \leqslant |E_2(D_t)| - 2, \tag{6}$$

where $\{s, t\} = \{1, r\}$. Otherwise C^t is a circuit with $\varepsilon(C^t) \leqslant k \leqslant \bar{\varepsilon}(C^t)$, which again contradicts Claim 1.

We now present two more claims:

Claim 3. *Taking any path $P = uvw$ of C with $uv \in E_2(D_s)$, we obtain*

$$d_C(w) > n/2 - |E_2(D_t)| + 1 \tag{7}$$

and

$$|E_2(D_t)| = |V_2(D_t)|/2 \quad \textit{and} \quad d_C(w) > n/2 - |V_2(D_t)|/2 + 1, \tag{8}$$

where $\{s, t\} = \{1, r\}$.

Proof of Claim 3. Let $P = uvw$ be a path of C with $uv \in E_2(D_s)$. Then

$$|\bar{E}(D_s) \backslash E(C)| \geqslant d(u) + d(v) - 4 + d_2(w).$$

Hence by (6),

$$d(u) + d(v) + d_2(w) \leqslant |E_2(D_t)| - 2 + 4 = |E_2(D_t)| + 2.$$

Since $d(u) + d(v) + d(w) > (n + 6)/2$,

$$d_C(w) > (n + 6)/2 - (d(u) + d(v) + d_2(w)) \geqslant (n + 6)/2 - (|E_2(D_t)| + 2),$$

i.e., (7) is true.

In order to obtain (8), we only need to prove the following claim.

$$\text{each component of } C[E_2(D_1)\cup E_2(D_r)] \text{ is a path of length one.} \tag{9}$$

Otherwise, there would exist an $s\in\{1,r\}$ and a path $P_0=u_0v_0w_0x$ of D_s such that $\{u_0,v_0,w_0\}\subseteq V_2(D_s)$ and $x\in V_1(D_s)$. By (5) and (7),

$$d_C(w_0) > n/2-|V_2(D_t)|+2 \quad \text{where } \{s,t\}=\{1,r\}. \tag{10}$$

By (10) and $d_C(w_0)=2$, $|V_2(D_t)|>n/2\geqslant 38$. Hence by (1) and $|V_2(D_t)|>38$, there exists a path $P_0'=u_0'v_0'w_0'$ in D_t such that $u_0'v_0'\in E_2(D_t)$ and $w_0'\notin V_1(D_s)$. Obviously,

$$|N_C(w_0)\cap N_C(w_0')|\leqslant 1. \tag{11}$$

From (5) and (7), we obtain

$$d_C(w_0') > n/2-|V_2(D_s)|+2. \tag{12}$$

Hence, by (10) and (12),

$$d_C(w_0)+d_C(w_0') > n-|V_2(D_1)|-|V_2(D_r)|+4. \tag{13}$$

On the other hand,

$$|(V_2(D_1)\cup V_2(D_r))\setminus(N_C(w_0)\cup N_C(w_0'))|\geqslant|V_2(D_1)|+|V_2(D_r)|-5. \tag{14}$$

Using (11) and (14), we obtain

$$\begin{aligned} d_C(w_0)+d_C(w_0') &= |N_C(w_0)\cup N_C(w_0')|+|N_C(w_0)\cap N_C(w_0')| \\ &\leqslant n-2-(|V_2(D_1)|+|V_2(D_r)|-5)+1 \\ &\leqslant n-(|V_2(D_1)|+|V_2(D_r)|)+4. \end{aligned} \tag{15}$$

Eqs. (13) and (15) are contradictory. This implies that (8) and (9) are true, which completes the proof of Claim 3. □

Claim 4. *There exist two vertices* $w\in V_1(D_1)\cap N_C(V_2(D_1))$ *and* $w'\in V_1(D_r)\cap N_C(V_2(D_r))$ *such that* $d_C(w,w')\geqslant 2$.

Proof of Claim 4. Let w_1uvw_2 and $x_1u'v'x_2$ be two paths in D_1 and D_r respectively, such that $uv\in E_2(D_1)$ and $u'v'\in E_2(D_r)$. It follows from (9) that $\{w_1,w_2,x_1,x_2\}\subseteq V_1(D_1)\cup V_1(D_r)$. It is easy to see that there exist two vertices $w\in\{w_1,w_2\}$ and $w'\in\{x_1,x_2\}$ with $ww'\notin E(C)$. Otherwise

$$C'=\begin{cases} C-E(x_1u'v'x_2x_1) & \text{if } w_1\in\{x_1,x_2\},\\ C-E(x_1u'v'x_2w_1x_1) & \text{otherwise,}\end{cases}$$

is a circuit with $\varepsilon(C')\leqslant k\leqslant\bar{\varepsilon}(C')$, a contradiction which implies that $d_C(w,w')\geqslant 2$. □

Due to Claim 4, we only need to consider the following two subcases:

Case 2.1: There exist two vertices $w\in V_1(D_1)\cap N_C(V_2(D_1))$ and $w'\in V_1(D_r)\cap N_C(V_2(D_r))$ such that $d_C(w,w')\geqslant 3$.

Then $N_C(w)\cap N_C(w')=\emptyset$. Obviously,

$$|(V_2(D_1)\cap V_2(D_r))\backslash(N_C(w)\cup N_C(w'))|\geqslant|V_2(D_1)|+|V_2(D_r)|-8.$$

Therefore,

$$\begin{aligned} d_C(w)+d_C(w') &= |N_C(w)\cup N_C(w')|+|N_C(w)\cap N_C(w')| \\ &\leqslant n-2-(|V_2(D_1)|+|V_2(D_r)|-8) \\ &= n+6-|V_2(D_1)|-|V_2(D_r)|. \end{aligned} \tag{16}$$

By (8),

$$d_C(w)+d_C(w')>n-(|V_2(D_1)|+|V_2(D_r)|)/2+2. \tag{17}$$

Using (16) and (17), we obtain

$$|V_2(D_1)|+|V_2(D_r)|<8,$$

which is a contradiction, since $|V_2(D_i)|=2|E_2(D_i)|\geqslant 4$ for $i\in\{1,r\}$ by (5) and (8).

Case 2.2: $d_C(w,w')\leqslant 2$ for any pair of vertices $\{w,w'\}$ with $w\in V_1(D_1)\cap N_C(V_2(D_1))$ and $w'\in V_1(D_r)\cap N_C(V_2(D_r))$.

We will prove that $V(D_1)\cap V(D_r)=\emptyset$. Otherwise, we can take a vertex $w_0\in V(D_1)\cap V(D_r)$, let $w_0w_1w_2\cdots w_huvw$ and $w_0u_1u_2\cdots u_fu'v'w'$ be two paths in D_1 and D_r respectively, such that $\{uv,u'v'\}\subseteq E_2(D_1)\cup E_2(D_r)$ and $\{w_0,w_1,w_2,\ldots,w_h\}\cap V_2(D_1)=\{w_0,u_1,u_2,\ldots,u_f\}\cap V_2(D_r)=\emptyset$. In a way similar to that of the proof of Claim 4, we obtain $w_hw'\notin E(C)$ and $u_fw\notin E(C)$. By (9), $\{w_h,w,u_f,w'\}\subseteq(V_1(D_1)\cup V_1(D_r))\cap(N_C(V_2(D_1)\cup N_C(V_2(D_r)))$. Let $P_C(x,y)$ denote a shortest path between x and y in C. Hence $|V(P_C(x,y))|\leqslant 2$ for $x\in\{w_h,w\}$ and $y\in\{u_f,w'\}$. Since $\mathrm{dia}(H)=d_H(D_1,D_r)=1$,

$$C'=C-(E(P_C(w_h,w'))\cup E(P_C(w,w'))\cup E(w_huvw))$$

is a circuit with $\varepsilon(C')\leqslant k\leqslant\bar{\varepsilon}(C')$, which contradicts Claim 1.

So $V(D_1)\cap V(D_r)=\emptyset$. By Claim 4, we can take two vertices w,w' with $d_C(w,w')=2$ such that $w\in V_1(D_1)\cap N_C(V_2(D_1))$ and $w'\in V_1(D_r)\cap N_C(V_2(D_r))$. Obviously,

$$|(V_2(D_1)\cup V_2(D_r))\backslash(N_C(w)\cup N_C(w'))|\geqslant|V_2(D_1)|+|V_2(D_r)|-4.$$

Therefore, if $\beta=|N_C(w)\cap N_C(w')|<4$, then

$$\begin{aligned} d_C(w)+d_C(w') &= |N_C(w)\cup N_C(w')|+|N_C(w)\cap N_C(w')| \\ &\leqslant n-2-(|V_2(D_1)|+|V_2(D_r)|-4)+\beta \\ &\leqslant n-|V_2(D_1)|-|V_2(D_r)|+6. \end{aligned} \tag{18}$$

Using (17) and (18), we obtain

$$|V_2(D_1)|+|V_2(D_r)|<8,$$

which is a contradiction, since $|V_2(D_i)|=2|E_2(D_i)|\geqslant 4$ for $i\in\{1,r\}$ by (5) and (8).

If $\beta = |N_C(w) \cap N_C(w')| \geqslant 4$, then $\{wx, xw', wy, yw'\}$ is a nontrivial cutset of C for any pair of vertices $\{x, y\} \subseteq N_C(w) \cap N_C(w')$. Otherwise $C' = C - E(xw'ywx)$ is a circuit such that $\varepsilon(C') \leqslant k \leqslant \bar{\varepsilon}(C')$, a contradiction. Hence, we can take a nontrivial component of $C - E(xw'ywx)$, denoted by $Q(x, y)$, which does not contain w and w'. Hence,

$$|V(Q(x, y))| \geqslant 4. \tag{19}$$

Otherwise, $Q'(x, y) = C - Q(x, y)$ is a circuit such that $\varepsilon(Q'(x, y)) \leqslant k \leqslant \bar{\varepsilon}(Q'(x, y))$, a contradiction.

Obviously,

$$|V(Q(x, y)) \backslash (N_C(w) \cup N_C(w'))| \geqslant 2$$

and

$$V(Q(a, b)) \cap V(Q(c, d)) = \emptyset$$

for any four vertices $\{a, b, c, d\} \subseteq N_C(w) \cap N_C(w')$.

This implies, from (19), that

$$\begin{aligned} |V(G) \backslash (N_C(w) \cup N_C(w'))| &\geqslant (|V_2(D_1)| + |V_2(D_r)| - 4) + 2 + [\beta/2] \times 2 \\ &\geqslant |V_2(D_1)| + |V_2(D_r)| + \beta - 3. \end{aligned}$$

Therefore,

$$\begin{aligned} d_C(w) + d_C(w') &= |N_C(w) \cup N_C(w')| + |N_C(w) \cap N_C(w')| \\ &\leqslant n - (|V_2(D_1)| + |V_2(D_r)| + \beta - 3) + \beta \\ &\leqslant n - (|V_2(D_1)| + |V_2(D_r)|) + 3. \end{aligned} \tag{20}$$

Using (17) and (20), we obtain

$$|V_2(D_1)| + |V_2(D_r)| < 2$$

which contradicts (5).

This completes the proof of (i) in Theorem 3. □

Proof of (ii) in Theorem 3. Since $\rho_3(G) > (2n + 16)/3 \geqslant 56$,

$$\varepsilon(C) = k + 1 \geqslant \Delta(G) + 2 \geqslant \rho_3(G)/4 + 2 > (n + 20)/6 \geqslant 16. \tag{21}$$

We will consider the following two cases:

Case 1: $r = 1$, i.e., *C is a cycle of length $k + 1$*. First, we show that Claim 2 is also true here.

Otherwise, in $\sum_{P \in UP_3(C)} d(P)$, every edge in $\bar{E}(C')$ is counted at most 8 times. Hence by (21) and (ii),

$$\begin{aligned} \bar{\varepsilon}(C') &\geqslant \sum_{P \in UP_3(C)} (d(P) - 8)/8 + \varepsilon(C') \\ &\geqslant (\rho_3 - 8)\varepsilon(C')/8 + \varepsilon(C') \\ &= \rho_3 \varepsilon(C')/8 \geqslant \rho_3 \varepsilon(C)/16 \geqslant k + 1. \end{aligned}$$

On the other hand, $\varepsilon(C')\leqslant k$. Theorem 2 implies that $L(G)$ contains a C_k, a contradiction. This shows that Claim 2 is true.

So C has no chord. By $\rho_3\geqslant 57$, C cannot be a hamiltonian cycle of G. Let u be a vertex in $V(G)\backslash V(C)$. By Claim 2, u is adjacent to at most three vertices of C. Hence, by (21),

$$\bar{\varepsilon}(C)\leqslant 3|V(G)\backslash V(C)|+\varepsilon(C)=3(n-\varepsilon(C))+\varepsilon(C)<(8n-20)/3. \tag{22}$$

On the other hand, since C has no chord, using (21) we obtain

$$\begin{aligned}\bar{\varepsilon}(C) &\geqslant \sum_{P\in UP_3(C)}(d(P)-8)/4+\varepsilon(C)\\ &\geqslant (\rho_3-8)\varepsilon(C)/4+\varepsilon(C)=(\rho_3-4)\varepsilon(C)/4\\ &> (n+20)(n+2)/36,\end{aligned}$$

which contradicts (22) since $n\geqslant 76$.

Case 2: $r\geqslant 2$.

As in the proof of (i) in Theorem 3, H is defined to be the graph with $V(H)=\{D_1,D_2,\ldots,D_r\}$ and $D_iD_j\in E(H)$ if and only if $V(D_i)\cap V(D_j)\neq\emptyset$, and D_1,D_r are two vertices of H with (4). Hence (5) and (6) are also true here.

By (5), $|V(D_i)|\geqslant 4$ for $i\in\{1,r\}$. Hence we can prove the following claim.

Claim 5. *Let P be a path of length* 3 *in D_s. We obtain*

$$d_C(P)>(2n+22)/3-|E_2(D_t)| \tag{23}$$

and

$$d_C(P)>(2n+25)/3-|V_2(D_t)|, \tag{24}$$

where $\{s,t\}=\{1,r\}$.

Proof of Claim 5. Let P be a path of length 3 in D_s. Then

$$|\bar{E}(D_s)\backslash E(C)|\geqslant d(P)-d_C(P).$$

Hence by (6) and (ii), $d_C(P)>(2n+16)/3-(|E_2(D_t)|-2)=(2n+22)/3-|E_2(D_t)|$, i.e., (23) is true. Eq. (24) is easily obtained from (5) and (23). □

We consider the following two subcases to obtain contradictions.

Subcase 2.1: There exist two paths $P=uvxy$ and $P'=u'v'x'y'$ of length 3 in D_1 and D_r respectively, such that $\{uv,u'v'\}\subseteq E_2(D_1)\cup E_2(D_r)$ and $V(P)\cap V(P')=\emptyset$.

Let

$$S=\{x,y,x',y'\},$$

$$N_i(C)=\{w\in V(C)\colon |N_C(w)\cap S|=i\}\quad \text{for } i\in\{1,2,3,4\},$$

$$N_{2,1}=\{w\in N_2(C)\colon |N_C(w)\cap\{x,y\}|=|N_C(w)\cap\{x',y'\}|=1\},$$

$$N_{2,2} = \{w \in N_2(C)\colon |N_C(w) \cap \{x, y\}| = 2 \text{ and } |N_C(w) \cap \{x', y'\}| = 0\},$$

$$N_{2,3} = \{w \in N_2(C)\colon |N_C(w) \cap \{x', y'\}| = 2 \text{ and } |N_C(w) \cap \{x, y\}| = 0\},$$

$$M_1 = N_C(x) \cap N_C(x') \cap N_2,$$

$$M_2 = N_C(x) \cap N_C(y') \cap N_2,$$

$$M_3 = N_C(y) \cap N_C(x') \cap N_2,$$

$$M_4 = N_C(y) \cap N_C(y') \cap N_2,$$

$$n_j = |N_{2,j}| \quad \text{for } j \in \{1,2,3\} \quad \text{and} \quad m_i = |M_i| \quad \text{for } i \in \{1,2,3,4\}.$$

It is easy to see that

$$N_2(C) = N_{2,1} \cup N_{2,2} \cup N_{2,3} \tag{25}$$

and

$$|N_2| = \sum_{i=1}^{3} n_i \quad \text{and} \quad n_1 = \sum_{i=1}^{4} m_i. \tag{26}$$

We now prove the following three claims.

Claim 6. $|N_3 \cup N_4| \leqslant 1$.

Proof of Claim 6. Otherwise, let $\{w, w'\} \subseteq N_3 \cup N_4$. Obviously,

$$\{w, w'\} \subseteq (N_C(x) \cap N_C(y)) \cup (N_C(x') \cap N_C(y')).$$

Without loss of generality, we assume that $wx, wy \in E(C)$. Hence,

$$C' = C - \{wx, wy, xy\}$$

is a circuit with $\varepsilon(C) - 3 = \varepsilon(C') \leqslant k \leqslant \bar{\varepsilon}(C')$, a contradiction. □

Claim 7. *For $i \in \{1,2,3,4\}$, if $m_i \geqslant 3$, then there exist at least $m_i - 1$ cut-vertices of C in M_i.*

Proof of Claim 7. For any pair of vertices $\{w, w'\} \subseteq M_i$, $C[S \cup \{w, w'\}]$ has a 4-cycle, denoted by $C(w, w')$, which contains the vertices w and w', but not the edge ww'. It is easy to see that $C' = C - E(C(w, w'))$ has at least two nontrivial components in C. Otherwise $\varepsilon(C) - 4 = \varepsilon(C') \leqslant k \leqslant \bar{\varepsilon}(C') = \bar{\varepsilon}(C)$, which contradicts Claim 1.

Since $m_i \geqslant 3$, for any pair of vertices $\{w, w'\} \subseteq M_i, w, w'$ is not in the same component of $C - E(C(w, w'))$ which does not contain any element of S. Hence there exist at least $m_i - 1$ cut-vertices of C in M_i. □

Let W_i denote the cut-vertex set of C in M_i. Taking any element x_i of W_i, we can take a nontrivial component $C - \{x_i\}$, denoted by $Q_{i,x}$, which does not contain any element of S for $i \in \{1,2,3,4\}$. It is easy to see that

$$|V(Q_{i,x})| \geqslant 3 \quad \text{and} \quad |\{Q_{i,x}\colon x \in W_i\}| = |W_i| \quad \text{for } i \in \{1,2,3,4\}. \tag{27}$$

Otherwise $C'_{i,x} = C - Q_{i,x}$ is a circuit with $\varepsilon(C) - 3 = \varepsilon(C'_{i,x}) \leqslant k \leqslant \bar{\varepsilon}(C'_{i,x})$, a contradiction.

Claim 8. *For $j \in \{2,3\}$, if $n_j \geqslant 2$, then each vertex of $N_{2,j}$ is a cut-vertex of C.*

Proof of Claim 8. Otherwise, there would exist a $j \in \{2,3\}$, say, $j = 2$, such that $N_{2,2}$ has a vertex w_0 which is not a cut-vertex of C. Hence,

$$C' = C - \{w_0x,\ w_0y,\ xy\}$$

has exactly one nontrivial component C'', and such that C'' is a circuit with $\varepsilon(C'') \leqslant k \leqslant \bar{\varepsilon}(C'')$, a contradiction. □

Let W' denote the cut-vertices set of C in $N_{2,2} \cup N_{2,3}$ such that for any element $y \in W', C - y$ has a nontrivial component which does not contain any element of S. It is easy to see that

$$|W'| \geqslant n_2 + n_3 - 2. \tag{28}$$

Hence, taking any element $y \in W'$, we can take a nontrivial component of $C - y$, denoted by Q'_y, which does not contain any element of S. It is easy to see that

$$|V(Q'_y)| \geqslant 3 \quad \text{for } y \in W' \quad \text{and} \quad |\{Q'_y\colon y \in W'\}| = |W'|. \tag{29}$$

Otherwise $C'_y = C - Q'_y$ is a circuit such that $\varepsilon(C'_y) \leqslant k \leqslant \bar{\varepsilon}(C'_y)$, a contradiction.
Obviously,

$$A \cap B = \emptyset \text{ for any pair of } \{A,B\} \subseteq \{V_2(D_1),\ V_2(D_r)\}$$
$$\cup \{Q'_x : x \in W'\} \cup \left(\bigcup_{i=1}^{4} \{Q_{i,x} : x \in W_i\} \right). \tag{30}$$

Let $I_1 = \{i\colon 1 \leqslant i \leqslant 4 \text{ and } m_i \geqslant 3\}$ and $I_2 = \{i\colon 1 \leqslant i \leqslant 4 \text{ and } m_i \leqslant 2\}$.
Using Claims 6–8 and (25)–(30), we obtain

$$d_C(S) = \left| \bigcup_{i=1}^{4} N_i \right| + |N_2| + 2|N_3| + 3|N_4|$$

$$\leqslant \left| \bigcup_{t=1}^{4} N_i \right| + \sum_{t=1}^{3} n_i + 3$$

$$\leqslant \begin{cases} n-\left(|V_2(D_1)|+|V_2(D_r)|-12+\sum_{i\in I_1}3(m_i-1)\right. \\ \qquad \left.+3(n_2+n_3-2)\right)+\sum_{i=1}^{3}n_i+3 \quad \text{if } n_2+n_3\geqslant 2, \\ n-\left(|V_2(D_1)|+|V_2(D_r)|-12\right. \\ \qquad \left.+\sum_{i\in I_1}3(m_i-1)\right)+\sum_{i=1}^{3}n_i+3 \quad \text{otherwise} \end{cases}$$

$$\leqslant \begin{cases} n-|V_2(D_1)|-|V_2(D_r)|-3\sum_{i\in I_1}m_i \\ \qquad -2(n_2+n_3)\ +\sum_{i=1}^{4}m_i+3|I_1|+21 \quad \text{if } n_2+n_3\geqslant 2, \\ n-|V_2(D_1)|-|V_2(D_r)|-3\sum_{i\in I_1}m_i+\sum_{i=1}^{4}m_i \\ \qquad +(n_2+n_3)+15 \quad \text{otherwise} \end{cases}$$

$$= \begin{cases} n-|V_2(D_1)|-|V_2(D_r)|-2\sum_{i\in I_1}m_i \\ \qquad -2(n_2+n_3)+\sum_{i\in I_2}m_i+3|I_1|+21 \quad \text{if } n_2+n_3\geqslant 2, \\ n-|V_2(D_1)|-|V_2(D_r)|-2\sum_{i\in I_1}m_i \\ \qquad +\sum_{i\in I_2}m_i+(n_2+n_3)+15 \quad \text{otherwise.} \end{cases}$$

Hence, we obtain

$$d_C(S)\leqslant n-|V_2(D_1)|-|V_2(D_r)|+24. \tag{31}$$

On the other hand, by (24),

$$d_C(S)>(4n+50)/3-|V_2(D_1)|-|V_2(D_r)|. \tag{32}$$

By $n\geqslant 76$, (31) and (32) are contradictory.

Subcase 2.2. If $P=uvxy$ and $P'=u'v'x'y'$ are two paths of length 3 in D_1 and D_r, respectively, such that $\{uv,u'v'\}\subset E_2(D_1)\cup E_2(D_r)$, then $V(P)\cap V(P')\neq\emptyset$.

Hence it follows from (5) that $|V(D_1)|=|V(D_r)|=4$ and $|V(D_1)\cap V(D_r)|=1$. By (4), $d_H(D_i,D_j)=\text{dia}(H)=1$ for $\{i,j\}=\{1,2,\ldots,r\}$. This implies that H is a complete graph. If there exists a j with $|V(D_j)|\geqslant 5$, then D_1 and D_j play the same roles as D_1

and D_r. We derive a contradiction in a way similar to that in the proof of Subcase 2.1. So $|V(D_j)| \leqslant 4$ for $j \in \{2,3,\ldots,r-1\}$. It is easy to see that

$$|V(D_i)| = 4 \quad \text{and} \quad |V(D_i) \cap V(D_j)| = 1 \quad \text{for } \{i,j\} \subseteq \{1,2,\ldots,r\}. \tag{33}$$

Otherwise $C_i' = C - D_i$ is a circuit such that $\varepsilon(C_i') \leqslant \varepsilon(C) - 3 \leqslant k \leqslant \bar{\varepsilon}(C_i') = \bar{\varepsilon}(C) - 1$ for $i \in \{1,2,\ldots,r\}$, which contradicts Claim 1.

Let $P = uvxy$ be a path of D_1 with $\{uv, vx\} = E_2(D_1)$. By (5) and (33), $|V_2(D_r)| = 3$. Hence by (24),

$$d_C(y) > (2n+16)/3 - 6 = (2n-2)/3. \tag{34}$$

On the other hand, by (33), $d_C(y) \leqslant n - 1 - d_C(y)/2$, i.e., $d_C(y) \leqslant (2n-2)/3$ which contradicts (34).

This completes the proofs of (ii) and of Theorem 3. □

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