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Bilinear State Space Systems for Nonlinear Dynamical Modelling

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Summary: We discuss the identification of multiple input, multiple output, discrete-time bilinear state space systems. We consider two identification problems. In the first case, the input to the system is a measurable white noise sequence. We show that it is possible to identify the system by solving a nonlinear optimization problem. The number of parameters in this optimization problem can be reduced by exploiting the principle of separable least squares. A subspace-based algorithm can be used to generate initial estimates for this nonlinear identification procedure. In the second case, the input to the system is not measurable. This makes it a much more difficult identification problem than the case with known inputs. At present, we can only solve this problem for a certain class of single input, single output bilinear state space systems, namely bilinear systems in phase variable form.

1 Introduction

Most real-life systems, including physical and biological systems show nonlinear dynamical behavior. Their behavior in time depends not only on the inputs to the system, but also on the state of the system. The state completely describes the time history of the system. It describes the influence of past inputs and outputs on the current output of the system. The output of the system at a certain time instant is completely described by its state and inputs at that time instant. A mathematical model of a nonlinear dynamical system that takes the state of the system into account is called a nonlinear state space model. It can be described as follows:

$$x_{k+1} = f(x_k, u_k)$$
$$y_k = h(x_k)$$

where $y_k \in \mathbb{R}^l$ is the output of the system, $u_k \in \mathbb{R}^m$ the input of the system, x_k the state of the system, $f : X \times \mathbb{R}^m \to X$ is a smooth one-to-one function having a smooth inverse and $h : X \to \mathbb{R}$ a smooth function. Note that this model is a discrete-time model with k denoting time. Although in real-life, systems almost always behave continuously in time, in practice the measurements of the system are sampled leading to discrete time signals. In this paper we will therefore only consider discrete-time models.

To obtain a state space model of a nonlinear dynamical systems, we have to estimate the mappings f and h using only measurements of the system's output and possibly measurements of the input. In other words, we have to identify the system.

One approach to estimate a model is to use a model structure that can be considered as general approximator to the mappings f and h. Implementations of this approach are, among others, neural networks, radial basis function networks, and local linear models. For a general overview see Sjöberg *et al.* [1]. Due the complex structure of these models, they can be difficult to identify and analyze. In estimating most of these models a non-convex nonlinear optimization problem has to be solved. However, with some engineering work, often very good results can be obtained with these models.

Another approach is to choose a certain simple structure for the model. Sometimes available knowledge about the system can be used to choose this structure. In other cases a certain simple structure is chosen because it provides a reasonable approximation and is easy to identify and analyze. An example of a simple structure is a linear state space model. In a number of real-life application these models have proven to yield satisfactory performance. A linear state space model has the following structure

$$x_{k+1} = Ax_k + Bu_k$$

$$y_k = Cx_k + Du_k$$

were $x_k \in \mathbb{R}^n$ is the state. In this paper we focus on bilinear state space models. These models are a simple nonlinear extension of the linear models and have the following structure

$$x_{k+1} = Ax_k + F(u_k \otimes x_k) + Bu_k \tag{1}$$

$$y_k = Cx_k + Du_k \tag{2}$$

where $\otimes$ denotes the Kronecker product. Note that the bilinear term can also be written as

$$F(u_k \otimes x_k) = \sum_{i=1}^{m} F_i [u_k]_i \, x_k$$

where $[\cdot]_i$ denotes the ith entry and the matrix F has been partitioned as $F =: [F_1\ F_2 \cdots F_m]$ with $F_i \in \mathbb{R}^{n\times n}$. The bilinear system got its name from the fact that if you fix the input the system is linear in the state and if you fix the state it is linear in the input. Bilinear models are capable of modelling certain nonlinear dynamics, while having a relatively simple structure that makes them attractive for identification and analysis. Due to the special structure of the bilinear state space system, a lot of similarities exist with linear state space systems [2], [3]. However, as we will show in this paper identification of these models is not a trivial matter.
This paper is organized as follows. Section 2 describes a identification method for bilinear state space systems with measurable inputs. In section 3 we discuss the identification of bilinear state space systems when the input is unmeasurable. Only for a special class of bilinear systems a solution is provided.

2 Identification of Bilinear Systems with Measurable Inputs

In this section we present a method to identify the bilinear system (1)–(2). It is assumed that the input to the system is a white noise sequence and that the bilinear system is observable with respect to the definition used in [4]. The difficulty in identifying a state space model is that in general the state sequence is unknown, only measurements of the input and output are available. We can however derive an expression for the output in terms of the initial state of the system and the inputs. In this way, we in fact eliminate the state sequence. The output at time instant k is given by

$$y_k = C\prod_{h=1}^{k-1} A_h x_1 + \left(\sum_{\tau=1}^{k-1} u_\tau^T \otimes C \prod_{h=\tau+1}^{k-1} A_h\right)\text{vec}(B) + u_k^T \otimes \text{vec}(D)$$

where

$$A_k := A + \sum_{i=1}^{m} [u_k]_i\, F_i$$

and

$$\prod_{h=j}^{k} A_h := \begin{cases} A_k A_{k-1} \cdots A_{j-1} A_j & \text{j} \leq \text{k} \\ 1 & \text{j} > \text{k} \end{cases}$$

We want to write y_k as a function of a set of nonlinear parameters, denoted by θ_n, and a set of linear ones, denoted by θ_l. Therefore, we have to parameterize the matrices A, B, C, D and F. The matrices A and C are parameterized in a special way. The pair (A, C) is transformed such that the observability Gramian equals identity, and the observability matrix is in lower triangular form with positive entries on the diagonal. From this special form of (A, C) a balanced parameterization is calculated. This has been explained for the SISO case in [5]. The parameterization has nl parameters which are stored in θ_n. The matrix F is parameterized by its entries, these n^2m parameters are also stored in θ_n. We parameterize the matrices B and D by their entries, we store these entries in θ_l.

$$\theta_l := \begin{bmatrix} \mathrm{vec}(B) \\ \mathrm{vec}(D) \end{bmatrix}$$

The parameterization might look a bit *ad hoc*. However, to the knowledge of the authors, a balanced parameterization for discrete-time bilinear state space systems has not been developed yet. It seems reasonable to use a balanced parameterization for linear systems to parameterize A and C, and by lack of a better choice, to fully parameterize F.
Let us introduce the following matrices

$$Y_{1,N} := [y_1^T, y_2^T, \ldots, y_N^T]^T$$

$$V_{1,N} := [v_1^T, v_2^T, \ldots, v_N^T]^T$$

$$\Gamma_N(\theta_n) := \left[C^T, (CA_1)^T, \ldots, \left(C \prod_{h=1}^{N-1} A_h \right)^T \right]^T$$

$$\Phi_N(\theta_n) := \begin{bmatrix} 0 & u_1^T \otimes I_l \\ u_1^T \otimes C & u_2^T \otimes I_l \\ \vdots & \vdots \\ \sum_{\tau=1}^{N-1} u_\tau^T \otimes C \prod_{h=\tau+1}^{N-1} A_h & u_N^T \otimes I_l \end{bmatrix}$$

Now eq. (2) can be written as

$$Y_{1,N} = \Gamma_N(\theta_n)\, x_1 + \Phi_N(\theta_n)\, \theta_l$$

for k ranging from 1 to N. To determine θ_n and θ_l, we formulate the following optimization problem

$$\min_{\theta_l, \theta_n} ||Y_{1,N} - \Phi_N(\theta_n)\, \theta_l||_2^2$$

This optimization problem has a very special structure: it is linear in θ_l and nonlinear in θ_n. To solve this problem, we can exploit the principle of separable least squares described by Golub and Pereyra [6] to reduce the number of parameters in the nonlinear optimization. First, we compute $\hat{\theta}_n$ by solving

$$\min_{\theta_n} ||Y_{1,N} - \Phi_N(\theta_n)\, \Phi_N^\dagger(\theta_n) Y_{1,N}||_2^2 \tag{3}$$

where $\Phi_N^\dagger(\theta_n)$ denotes the pseudo inverse of $\Phi_N(\theta_n)$. Since this optimization problem does not have an analytical solution, we have to solve it numerically. For this we use the Levenberg-Marquardt iterative procedure.

The gradients which are needed for the Levenberg-Marquardt method, are approximated numerically using finite differences. Although it is possible to derive an analytical expression for these gradients, their evaluation creates a huge computational load. The algorithm is much more efficient when the finite difference approximation is used. Second, we calculate the linear parameters as $\theta_l = \Phi_N^{\dagger}(\theta_n) Y_{1,N}$.

In the optimization problem (3) we have to constrain the parameters θ_n such that the bilinear system corresponding to θ_n and θ_l is stable at every iteration. Note that if the input is a white noise sequence, the system is stable if the eigenvalues of the matrix A and of the matrix $A \otimes A + \sum_{i=1}^{m} \mathbf{E}[[u_k]_i^2] \, F_i \otimes F_i$ have magnitudes smaller than one ($\mathbf{E}[\cdot]$ denotes statistical expected value) [7]. We deal with this constraint by introducing a simple barrier function. The object function in (3) is modified as follows: If θ_n is such that the system becomes unstable, then the object function is replaced by $||Y_{1,N}||_2^2$.

Because of the nonlinear optimization involved, we have no guarantee of finding the parameters corresponding to the global minimum of the error. However, the algorithm has a good chance of converging to the global optimum, if we have initial estimates of A, C and F which are already close to the original system matrices. One way to obtain initial guesses for these matrices is to first estimate a linear model using for example subspace identification techniques [8], [9], and then use the A and C matrices of this model as initial guesses. For the F matrix, one can try a number of random initializations. This procedure will only work when the dominant part of the dynamics behaves linearly. A much better way to obtain initial estimates of A, C and F has been described in [10]. The method described there, is a subspace-based algorithm that is computationally efficient and yields an estimate of the system order.

In [11] we show that the identification algorithm presented in this section can also be used when there is measurement noise present and when there is process noise present.

We have to mention that Favoreel *et al.* [12] (see also [13]) have presented a identification method for bilinear systems with measurable white noise inputs, that completely avoids the need for numerically solving a nonlinear optimization problem. However, we argue in [14] that this method can only be used for systems of low order with only a few inputs and outputs, because of the enormous amount of memory required; the memory requirements grow exponentially with the order of the system. We also presented in [14] a Monte Carlo simulation that shows that the method presented in this section yields models that are more accurate than the ones obtained by the method of Favoreel *et al.*

3 Identification of Bilinear Systems with Unmeasurable Inputs

In this section we discuss the identification of the bilinear system (1)–(2) when the input is an unmeasurable white noise sequence. It is assumed that the bilinear system is observable with respect to the definition used in [4]. When the input is not measurable, the only information that is available to identify a bilinear state space model are the measurements of the output. Compared to the case where the input is available, we have two unknown quantities: the state sequence and the input sequence. This makes it a much more difficult identification problem than the case with known inputs. After reading the previous section it should be clear to the reader that the method described there cannot be used to identify a bilinear system when the input is not available. In fact we are dealing with the problem of finding a nonlinear time series model.

Let us analyze the identification problem when the input is not available. For simplicity we assume that we are dealing with a single input and single output, and that the matrix D equals 1. Now we can eliminate the input in the state equation by substituting $u_k = y_k - Cx_k$. We obtain:

$$x_{k+1} = (A - BC)\, x_k + F(y_k - Cx_k)\, x_k + By_k \tag{4}$$

What we observe is that the system becomes quadratic in the state. To find the system matrices, we have to minimize the cost function

$$\sum_{k=1}^{N} u_k^2 = \sum_{k=1}^{N} (y_k - Cx_k)^2$$

with respect to the matrices A, B, C, and F under the constraint that the system (4) is stable. So we end up with a nonconvex constrained optimization problem. There are two major problems with this optimization problem.

- It is difficult to obtain a closed expression for the output that does not depend on the input u_k. To come up with such an expression, we have to iterate equation (4) for x_k.
- It is not clear what conditions we have to impose on the matrices A, B, C, and F such that the system (4) is stable. We cannot use the condition used in section 2, because this requires the input u_k to be white. It is important to realize that it cannot be guaranteed that the input u_k is white during the iterations. At present, only some preliminary results regarding stability of discrete-time quadratic systems are available. In [15] conditions on the parameters are derived for a single input, single output quadratic system without a cross-product between the input and output. Unfortunately, these results not useful in our framework because in equation (4) there is a product between y_k and x_k and of course x_k is multivariable.

At present, we have not been able to overcome these problems.

For a certain class of single input, single output bilinear state space systems, it is possible to derive a simple input-output model and identify the system matrices. These systems are called bilinear systems in phase variable form and have the following structured system matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_n \end{bmatrix} \qquad F = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ f_1 & f_2 & \cdots & f_n \end{bmatrix}$$

$$B = [0 \cdots 0 \;\; b]^T \qquad C = [1 \;\; 0 \cdots 0]$$

Due to this special structure it is easy to determine the state sequence from the output measurements. This simplifies the identification problem a lot. We have that $[x_k]_i = y_{k+i-1} - u_{k+i-1}$ where $[\cdot]_i$ denotes the ith element. It is easy to derive the following input-output description

$$y_{k+n} = \sum_{i=1}^{n} a_i(y_{k+i-1} - u_{k+i-1}) + \sum_{i=1}^{n} f_i(y_{k+i-1} - u_{k+i-1})\, u_k + bu_k + u_{k+n}$$

Bilinear input-output systems that have some similarity with the bilinear state space system in phase variable form are considered in for example [7].

To identify the system, we minimize the cost function

$$\sum_{k=1}^{N-n} u_{k+n}^2 = \sum_{k=1}^{N-n} \left(y_{k+n} - \sum_{i=1}^{n} a_i(y_{k+i-1} - u_{k+i-1}) \right.$$
$$\left. - \sum_{i=1}^{n} f_i(y_{k+i-1} - u_{k+i-1})\, u_k - bu_k \right)^2$$

with respect to the parameters a_i, f_i, and b where we take as initial condition $u_i = 0$, i = 1, 2, …, n. We use the Levenberg-Marquardt iterative procedure to solve this problem. The gradients which are needed are approximated numerically using finite differences. The method is a bit *ad hoc*, because we do not constrain the system to be stable during the iterations.

We have no guarantee of finding the parameters corresponding to the global minimum of u_k. However, the algorithm has a good chance of converging to the global optimum, if we have good initial estimates of a_i, f_i, and b. We do not have an elegant way of finding these initial estimates. One way to obtain initial guesses of a_i is to first estimate a linear model using for example subspace identification techniques [8], [9], and then convert the A matrix to the form described above.

4 Conclusions

We have discussed the identification of multiple input, multiple output, discrete-time bilinear state space systems. When the input to the system is a measurable white noise sequence, it is possible to identify the system by solving a nonlinear optimization problem. The dimension of the parameters in this optimization problem can be reduced by exploiting the principle of separable least squares. When the input to the system is not measurable, the identification problem becomes more difficult. At present, no general solution has been found. However for a certain class of single input, single output bilinear state space systems, namely bilinear systems in phase variable form, it is possible to derive a simple identification algorithm.

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