Prolongation structure of the KdV equation in the bilinear form of Hirota

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Abstract. The prolongation structure of the KdV equation in the bilinear form of Hirota is determined, the resulting Lie algebra is realised and the Bäcklund transformation obtained from the prolongation structure is derived. The results are compared with those found by Wahlquist and Estabrook and by Hirota.

1. Introduction

In their well known article [1] Wahlquist and Estabrook developed a prolongation technique for nonlinear evolution equations and applied it to the Korteweg–de Vries (KdV) equation. This resulted in a Lie algebra structure, which has been obtained independently by Estabrook [2] and van Eck [3], the latter merely using the algebraic foundations of this Lie algebra.

In [4] Hirota shows how nonlinear evolution equations may be transformed into bilinear differential equations. For this purpose it is convenient to write the KdV equation in potential form

$$4u_t = u_{xxx} + 6u_x^2. \tag{1}$$

From equation (1) we obtain the bilinear equation

$$(D_1^4 - 4D_1D_3)\tau \cdot \tau = 0 \tag{2}$$

where

$$(D_1^4 - 4D_1D_3)\tau \cdot \tau = \left(\frac{\partial^4}{\partial y^4} - 4 \frac{\partial}{\partial y} \frac{\partial}{\partial t'} \right) \tau(x+y, t+t')\tau(x-y, t-t') \Big|_{y=0, t'=0}$$

by putting

$$u = \frac{\partial}{\partial x} \log \tau(x, t). \tag{3}$$

In this paper we will investigate the prolongation structure of the KdV equation in Hirota's bilinear form (2). Equation (2) is easily seen to admit an additional symmetry with group action $(x, t, \tau) \mapsto (x, t, \lambda\tau)$ which has no counterpart for the ordinary KdV.

This investigation is motivated by the conjecture that this additional symmetry might be reflected in the prolongation structure.

Moreover, equation (2) is directly connected to the τ-function approach of nonlinear evolution equations [5]. In a certain sense these τ functions are more primitive than the corresponding functions u, namely as an element of a group orbit τ stands closer to the underlying symmetry group than u itself. This might also be reflected in the prolongation structure.

In section 2 we carry out the actual prolongation, after which in section 3 we obtain the resulting Lie algebra, in the sense that we give a faithful representation of it. Finally, in section 4 we derive the Bäcklund transformation obtained from the prolongation structure, which appears to be equal to the one found by Hirota.

2. Prolongation

Writing out equation (2) yields

$$\tau\tau_{xxxx} - 4\tau_x\tau_{xxx} + 3\tau_{xx}^2 - 4\tau\tau_{xt} + 4\tau_x\tau_t = 0. \tag{4}$$

In order to express equation (4) by means of an ideal of differential forms we introduce

$$p \equiv \tau_x \qquad q \equiv p_x = \tau_{xx} \qquad r \equiv q_x = \tau_{xxx} \tag{5}$$

whereupon equation (4) may be written as the first-order equation

$$\tau r_x - 4pr + 3q^2 - 4\tau p_t + 4\tau_t p = 0. \tag{6}$$

The set of first-order equations (5) and (6) may now be expressed by the following set of differential 2-forms

$$\alpha_1 = d\tau \wedge dt - p\,dx \wedge dt$$

$$\alpha_2 = dp \wedge dt - q\,dx \wedge dt$$

$$\alpha_3 = dq \wedge dt - r\,dx \wedge dt$$

$$\alpha_4 = \tau\,dr \wedge dt + 4\tau\,dp \wedge dx - 4p\,d\tau \wedge dx + (3q^2 - 4pr)\,dx \wedge dt$$

where a solution manifold of equations (5) and (6) satisfies the condition

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$$

and vice versa. In order to ensure integrability of this system Cartan's theory of differential forms [6] imposes the condition $dI \subset I$ on $I = I(\alpha_1, \ldots, \alpha_4)$, the ideal generated by the α_i in the exterior algebra with basis $\{dx, dt, d\tau, dp, dq, dr\}$; this condition is easily seen to be satisfied.

The prolongation method of Wahlquist and Estabrook may be described as follows: add a Lie algebra valued prolongation form

$$\omega = dy + F(\tau, p, q, r)\,dx + G(\tau, p, q, r)\,dt \tag{7}$$

to the original system of differential forms and demand ω satisfies the condition

$$d\omega + [G, F]\, dx \wedge dt \in I(\alpha_1, \ldots, \alpha_4). \tag{8}$$

Condition (8) gives rise to the following system of overdetermined differential equations

$$\tau F_\tau + p F_p = 0 \tag{9}$$

$$F_p - 4G_r = 0 \tag{10}$$

$$F_q = 0 \tag{11}$$

$$F_r = 0 \tag{12}$$

$$\tau p G_\tau + \tau q G_p + \tau r G_q + (4pr - 3q^2)G_r + \tau[G, F] = 0. \tag{13}$$

Using the method of characteristics, one derives from equations (9), (11) and (12) that

$$F = F(p/\tau).$$

The remaining equations were solved using a computer package for computations in prolongation theory described in [7]. If we define

$$P = p/\tau \qquad Q = q/\tau \qquad R = r/\tau \tag{14}$$

we find that

$$F = x_1 + 4Px_2 + 4P^2x_3 \tag{15}$$

$$G = (R + PQ - 2P^3)x_2 + (2PR - Q^2 - P^4)x_3 + x_4 - 2Px_5 - 2P^2x_6 - (Q + P^2)x_7 \tag{16}$$

where we have introduced

$$x_7 = [x_2, x_1] \qquad x_6 = [x_2, x_7] \qquad x_5 = \tfrac{1}{2}[x_1, x_7]. \tag{17}$$

In addition we find the commutation relations

$$[x_1, x_3] = 2x_2 \qquad [x_1, x_4] = 0 \qquad [x_1, x_5] = 2[x_2, x_4]$$

$$[x_2, x_3] = x_3 \qquad [x_2, x_6] = x_7 \qquad [x_3, x_4] = 3[x_2, x_5] + x_5. \tag{18}$$

In the next section we will realise the Lie algebra determined by relations (17) and (18).

Remark. From definition (14) we see that

$$P_x = Q - P^2 \qquad Q_x = R - PQ \tag{19}$$

whereupon equation (6) may be written as follows

$$R_x = 4P_t + 3PR - 3Q^2. \tag{20}$$

It is easily checked that this set of equations leads to the same prolongation structure as derived for equations (5) and (6). We see from this that both the KdV in Hirota's bilinear form and the prolongation structure belonging to it, are really determined by $\tau_x/\tau = (\log \tau)_x = u$ and not by τ itself. In fact, as we will see in section 5, the results found here are the same as for the KdV equation in potential form.

Equations (19) and (20), however, will be useful in section 4, where we derive the Bäcklund transformation belonging to the prolongation structure.

3. Realisation of the prolongation algebra

In this section we will determine the structure of the prolongation algebra of the KdV in Hirota's bilinear form and give a realisation of it, using the theory of Kac–Moody algebras. Henceforth we will denote this prolongation algebra, i.e. the Lie algebra generated by the letters $x_1, \ldots, x_7$ and subjected to relations (17) and (18), by HIR. An important tool for our purpose is a gradation. The following proposition is immediate.

Proposition 1. HIR admits a $\mathbb{Z}$ gradation with $\deg(x_1) = 1$, $\deg(x_2) = 0$, $\deg(x_3) = -1$, $\deg(x_4) = 3$, $\deg(x_5) = 2$, $\deg(x_6) = 1$ and $\deg(x_7) = 1$ such that relations (17) and (18) are homogeneous.

Using the computer package already mentioned and introducing new letters $x_8, \ldots, x_{15}$ we have computed (part of) the commutator table of HIR, which can be found in table 1 and in principle may be checked by hand using the Jacobi identity.

Table 1. Commutator table of HIR.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
x_1	0	$-x_7$	$2x_2$	0	$2x_8$	0	$2x_5$	x_9	$2x_{11}$	0
x_2		0	x_3	x_8	0	x_7	x_6	x_{10}	0	x_8
x_3			0	x_5	$-x_7 - x_6$	$-2x_2$	$2x_2$	$-x_5$	$-2(x_8 + x_{10})$	x_5
x_4				0	x_{11}	0	x_9	x_{12}	x_{13}	0
x_5					0	$-2x_8$	$-2x_{10}$	x_{14}	0	$-x_{11}$
x_6						0	$2x_5$	x_9	$2x_{11}$	0
x_7							0	0	$-2x_{14}$	$-x_9$
x_8								0	x_{15}	$-x_{12}$
x_9									0	$-x_{13}$
x_{10}										0

Using this table and the gradation of proposition 1 we find that HIR contains a subalgebra isomorphic to $A_1 = \mathrm{sl}(2)$, with standard basis $\{e, f, h\}$, where

$$e = \tfrac{1}{2}(x_7 - x_6) \qquad f = x_3 \qquad h = -2x_2 \tag{21}$$

and standard relations

$$[e, f] = h \qquad [h, e] = 2e \qquad [h, f] = -2f.$$

Moreover we see that HIR contains a two-dimensional Abelian subalgebra

$$H_{\mathrm{Hir}} = \langle x_1 - x_6, x_4 - x_{10} \rangle \tag{22}$$

commuting with all generators of HIR, hence consisting of central elements.

We recall from van Eck [3] that the prolongation algebra of the ordinary KdV is isomorphic to $H_{\mathrm{KdV}} \times (\mathbb{C}[\lambda] \otimes A_1)$, the direct product of a five-dimensional Heisenberg algebra H_{KdV} and the tensor product $\mathbb{C}[\lambda] \otimes A_1$. HIR will appear to have a similar structure, namely $H_{\mathrm{Hir}} \times (\mathbb{C}[\lambda] \otimes A_1)$, with H_{Hir} and A_1 as defined in (21) and (22).

To prove this we will need some results from the theory of Kac–Moody algebras, in particular the defining relations and the realisation of the infinite-dimensional Lie

algebra $A_1^{(1)'}$, i.e. the derived algebra of the infinite-dimensional Lie algebra $A_1^{(1)}$ [8]. We have the following.

Proposition 2 (Kac). (a) $A_1^{(1)'}$ is the Lie algebra with generators $e_0, f_0, h_0, e_1, f_1, h_1$ and defining relations $(i, j = 0, 1)$:

$$[h_i, h_j] = 0 \qquad\qquad [e_i, f_j] = \delta_{ij} h_i$$

$$[h_i, e_j] = a_{ij} e_j \qquad\qquad [h_i, f_j] = -a_{ij} f_j$$

$$(\mathrm{ad}\, e_i)^{1-a_{ij}} e_j = 0 \qquad (\mathrm{ad}\, f_i)^{1-a_{ij}} f_j = 0 \qquad (i \ne j)$$

where $a_{ii} = 2$ and $a_{ij} = -2$ $(i \ne j)$.

(b) $A_1^{(1)'}$ is isomorphic to the algebra $(\mathbb{C}[\lambda^{-1}, \lambda] \otimes A_1) \oplus \mathbb{C}c$ where the isomorphism is given by

$$e_0 \mapsto \lambda \otimes f \qquad f_0 \mapsto \lambda^{-1} \otimes e \qquad h_0 \mapsto -1 \otimes h + c$$

$$e_1 \mapsto 1 \otimes e \qquad f_1 \mapsto 1 \otimes f \qquad h_1 \mapsto 1 \otimes h$$

and the commutator on $(\mathbb{C}[\lambda^{-1}, \lambda] \otimes A_1) \oplus \mathbb{C}c$ by

$$[(\lambda^{k_1} \otimes x) \oplus \lambda_1 c, (\lambda^{k_2} \otimes y) \oplus \lambda_2 c] = (\lambda^{k_1 + k_2} \otimes [x, y]) \oplus k_1 \delta_{k_1, -k_2} (x|y) c$$

with $(.|.)$ an invariant bilinear symmetric form on A_1 with

$$(x|y) = \begin{cases} 1 & \text{if } (x, y) = (e, f), (f, e) \\ 2 & \text{if } (x, y) = (h, h) \\ 0 & \text{otherwise.} \end{cases}$$

Equating to zero f_0 and h_0 yields the following.

Proposition 3. The Lie algebra with generators e_0, e_1, f_1, h_1 and defining relations

$$[e_1, f_1] = h_1 \qquad\qquad [h_1, e_0] = -2e_0$$

$$[h_1, e_1] = 2e_1 \qquad\qquad [h_1, f_1] = -2f_1 \qquad\qquad (23)$$

$$(\mathrm{ad}\, e_1)^3 e_0 = (\mathrm{ad}\, e_0)^3 e_1 = 0$$

is isomorphic to the algebra $\mathbb{C}[\lambda] \otimes A_1$ where the isomorphism is given by

$$e_0 \mapsto \lambda \otimes f \qquad e_1 \mapsto 1 \otimes e$$

$$f_1 \mapsto 1 \otimes f \qquad h_1 \mapsto 1 \otimes h. \qquad\qquad (24)$$

We are now in a position to give a realisation of HIR.

Theorem 1. HIR is isomorphic to the algebra $H_{\text{Hir}} \times (\mathbb{C}[\lambda] \otimes A_1)$.

Proof. Put $H_{\text{Hir}} = \langle c_1, c_2 \rangle$ and let $L(e_0, e_1, f_1, h_1, c_1, c_2)$ be the free Lie algebra on generators $e_0, e_1, f_1, h_1, c_1, c_2$ and consider the Lie algebra morphism $\phi : L(e_0, e_1, f_1, h_1, c_1, c_2) \rightarrow$ HIR given by (cf formulae (21) and (22))

$$
\begin{aligned}
e_0 &\mapsto -\tfrac{1}{2}(x_7 + x_6) & e_1 &\mapsto \tfrac{1}{2}(x_7 - x_6) & f_1 &\mapsto x_3 \\
h_1 &\mapsto -2x_2 & c_1 &\mapsto x_1 - x_6 & c_2 &\mapsto x_4 - x_{10}.
\end{aligned}
\tag{25}
$$

Using table 1 it is easy to prove that ϕ leaves invariant relations (23), hence there is a Lie algebra morphism $\phi' : H_{\text{Hir}} \times (\mathbb{C}[\lambda] \otimes A_1) \rightarrow$ HIR. On the other hand if we denote $\lambda^k \otimes x$ by $\lambda^k x$, the Lie algebra morphism $\psi : L(x_1, \ldots, x_7) \rightarrow H_{Hir} \times (\mathbb{C}[\lambda] \otimes A_1)$ given by

$$
\begin{aligned}
x_1 &\mapsto -(e + \lambda f) - c_1 & x_2 &\mapsto -\tfrac{1}{2}h & x_3 &\mapsto f \\
x_4 &\mapsto -(\lambda e + \lambda^2 f) - c_2 & x_5 &\mapsto \lambda h & x_6 &\mapsto -(e + \lambda f) \\
x_7 &\mapsto e - \lambda f
\end{aligned}
\tag{26}
$$

preserves the relations (17) and (18), hence there exists a Lie algebra morphism $\psi' : \text{HIR} \rightarrow H_{Hir} \times (\mathbb{C}[\lambda] \otimes A_1)$. ϕ' and ψ' are easily seen to be each other's inverse.   □

4. Bäcklund transformation

Bäcklund transformations can be obtained from prolongation structures by realising the prolongation algebra in a appropiate way and putting

$$
y_x = -F \qquad y_t = -G.
$$

In this way new solutions for the differential equation for which the prolongation structure was computed may be constructed [1].

Using the nonlinear representation of sl(2)

$$
e = -y^2 \qquad f = 1 \qquad h = 2y
$$

we find for F and G

$$
F = -\lambda + (y - 2P)^2
$$
$$
G = -\lambda^2 - \lambda(-y^2 + 4yP - 3P^2 - Q) - y^2(P^2 - Q) - y(R + PQ - 2P^3) - P^4 - Q^2 + 2PR
$$

which leads to the equations

$$
y_x = \lambda - (y - 2P)^2
\tag{27}
$$
$$
y_t = \lambda^2 + \lambda(-y^2 + 4yP - 3P^2 - Q) + y^2(P^2 - Q) + y(R + PQ - 2P^3) + P^4 + Q^2 - 2PR
\tag{28}
$$

The compatibility condition $y_{xt} = y_{tx}$ is satisfied if P, Q and R are solutions of the KdV in Hirota's bilinear form ((19) and (20)). Solving P from equation (27) and substituting the result into equation (28) we find that y has to satisfy

$$
y_{xx}^2 = \tfrac{4}{3}(y_x - \lambda)(y_{xxx} + 3y_x^2 - 4y_t).
\tag{29}
$$

On the other hand we can take the realisation of sl(2)

$$e = v^2 \qquad f = -1 \qquad h = 2v$$

yielding

$$v_x = -\lambda + (2P + v)^2 \tag{30}$$

$$v_t = -\lambda^2 + \lambda(+v^2 + 4vP + 3P^2 + Q) - v^2(P^2 - Q) + v(R + PQ - 2P^3) - P^4 - Q^2 + 2PR. \tag{31}$$

In the same way as before we find that v has to satisfy

$$v_{xx}^2 = \tfrac{4}{3}(v_x + \lambda)(v_{xxx} - 3v_x^2 - 4v_t). \tag{32}$$

Now a simple observation shows that $v = -y$ transforms equation (32) into equation (29). Differentiating with respect to x gives $v_x = -y_x$ from which we conclude that

$$\lambda - (y - 2P)^2 = \lambda - (2\tilde{P} + v)^2 = \lambda - (y - 2\tilde{P})^2$$

where $\tilde{P}$ is another solution of equations (19) and (20). From this we find that $\tilde{P} = P$ or $\tilde{P} = y - P$. One easily checks by substitution that $\tilde{P} = y - P$ indeed satisfies equations (19) and (20) iff P does, hence we have found the auto-Bäcklund transformation

$$\tilde{P} = y - P. \tag{33}$$

Differentiating equation (33) with respect to x and using equation (27) to eliminate y_x we obtain the same Bäcklund transformation in a more familiar form

$$(\tilde{P} + P)_x + (\tilde{P} - P)^2 = \lambda \tag{34}$$

which is the same as the one found by Wahlquist and Estabrook [1] and by Hirota [4].

5. Conclusions

If we take a closer look at equation (20), which is equivalent to the equation (6) for τ, and use equation (19) to eliminate Q and R, we find that P has to satisfy

$$4P_t = P_{xxx} + 6P_x^2$$

which is the KdV equation in potential form (1). Now the potential form of the KdV equation has already been discussed from the prolongation point of view by Kaup [9]. In fact, the present equations (15) and (16) are the same as equation (5) in Kaup's paper, but in different notation. Also the realisation that (30) and (31) are the same as Kaup's equation (12). However, in Kaup's paper the explicit form of the prolongation algebra has not been determined.

Hence we have found in a natural context that from the prolongation point of view Hirota's formalism is in fact nothing more than a way to obtain results, which could have been obtained without applying Hirota's formalism. As a consequence of this,

the additional symmetry of the KdV equation in Hirota's bilinear form does not lead to extra information obtained from the prolongation structure of this equation.

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