Long Dominating Cycles and Paths in Graphs with Large Neighborhood Unions

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ABSTRACT

Let G be a graph of order n and define $NC(G) = \min\{|N(u) \cup N(v)| \mid uv \notin E(G)\}$. A cycle C of G is called a *dominating cycle* or *D-cycle* if $V(G) - V(C)$ is an independent set. A *D-path* is defined analogously. The following result is proved: if G is 2-connected and contains a D-cycle, then G contains a D-cycle of length at least $\min\{n, 2NC(G)\}$ unless G is the Petersen graph. By combining this result with a known sufficient condition for the existence of a D-cycle, a common generalization of Ore's Theorem and several recent "neighborhood union results" is obtained. An analogous result on long D-paths is also established.

1. TERMINOLOGY AND NOTATIONS

We use [3] for terminology and notations not defined here, and consider simple graphs only. Throughout, let G be a graph of order n.

If G has a Hamilton cycle (a cycle containing every vertex of G), then G is called *hamiltonian.* G is *traceable* if G has a Hamilton path (a path containing every vertex of G). A cycle C of G is called a *dominating cycle*, or briefly D-*cycle*, if $V(G) - V(C)$ is an independent set of vertices in G. A *dominating path* or D-*path* is analogously defined. Two edges e_1 and e_2 of G are called *remote* if they are nonadjacent, and there is no edge of G joining an end of e_1 and one of e_2. The *degree* of an edge uv of G is the number of vertices in $V(G)$-$\{u,v\}$ adjacent to at least one of the vertices u and v.

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The length of a longest cycle in G is denoted by $c(G)$, the order of a longest path by $p(G)$, the number of vertices in a maximum independent set by $\alpha(G)$, and the set of vertices adjacent to a vertex v by $N(v)$. We denote by $\sigma_k(G)$ the minimum value of the degree-sum of any k pairwise nonadjacent vertices; if $k > \alpha(G)$, we set $\sigma_k(G) = k(n - 1)$. Instead of $\sigma_1(G)$ we use the more common notation $\delta(G)$. We denote by $\sigma_k'(G)$ the minimum value of the degree-sum of any k pairwise remote edges; if G does not contain k pairwise remote edges, then $\sigma_k'(G) = k(n - 2)$. If G is noncomplete, then $NC(G)$ denotes $\min\{|N(u) \cup N(v)| \,|uv \notin E(G), u \neq v\}$; if G is complete, we set $NC(G) = n - 1$. If $|E(G)| > 0$, then $NC'(G)$ denotes $\min\{|N(u) \cup N(v)| \,|uv \in E(G)\}$; otherwise, $NC'(G) = 0$. By $NC''(G)$ we denote $\min\{|N(u) \cup N(v)| \,|u, v \in V(G), u \neq v\}$. If no ambiguity can arise, we sometimes write α instead of $\alpha(G)$, σ_k instead of $\sigma_k(G)$, etc.

We now define two special classes of graphs. For $n \geq 5$, the graph G_n is defined as the join of K_2 and the graph of order $n - 2$ consisting of three disjoint complete graphs, the orders of which pairwise differ by at most one. For $n \geq 4$, the graph H_n is obtained from G_{n+1} by deleting a vertex of degree n.

2. MAIN RESULT AND CONSEQUENCES

A slightly stronger version of the following result was recently established.

Theorem 1 [1]. If G is 2-connected and $\sigma_3(G) \geq n + 2$, then $c(G) \geq \min\{n, 2NC(G)\}$.

It was shown in [1] that Theorem 1 is a common generalization of results in [5], [6], and [7].

A key ingredient in the proof of Theorem 1 is the following result of Bondy:

Theorem 2 [2]. If G is 2-connected and $\sigma_3(G) \geq n + 2$, then every longest cycle of G is a D-cycle.

By the role of Theorem 2 in the proof of Theorem 1 we were led to investigate whether the conclusion of Theorem 1 still holds if G is only required to be 2-connected and to have a D-cycle. Our main result is as follows.

Theorem 3. If G is 2-connected and contains a D-cycle, then G contains a D-cycle of length at least $\min\{n, 2NC(G)\}$ unless G is the Petersen graph.

Note that $c(G) = 2NC(G) - 1$ if G is the Petersen graph.

The proof of Theorem 3 is postponed to Section 3.

The conclusion of Theorem 3 cannot be strengthened, as shown by complete bipartite graphs: for $2 \leq r \leq s$ we have $c(K_{r,s}) = 2r = 2NC(K_{r,s})$.

Furthermore, the requirement that G contain a D-cycle, cannot be omitted: for $n \geq 8$ the graph G_n contains no D-cycle, while $c(G_n) = NC(G_n) + 2$ if $n \equiv 2 \pmod 3$ and $c(G_n) = NC(G_n) + 3$ otherwise.

By combining Theorem 3 with Theorem 2 we obtain Theorem 1.

The following condition for the existence of a D-cycle occurs in [8].

Theorem 4 [8]. If G is 2-connected and $\sigma_3'(G) \geq n - 1$, then G contains a D-cycle.

It was observed in [8] that the hypothesis of Theorem 4 is weaker than the hypothesis of Theorem 2. (Note that, on the other hand, the conclusion of Theorem 4 is weaker than the conclusion of Theorem 2.) Thus by combining Theorem 3 with Theorem 4 we obtain a result that is more general than Theorem 1.

Theorem 5. If G is 2-connected and $\sigma_3'(G) \geq n - 1$, then $c(G) \geq \min\{n, 2NC(G)\}$ unless G is the Petersen graph.

Corollary 6. If G is 2-connected and $NC'(G) \geq \frac{1}{3}(n + 5)$, then $c(G) \geq \min\{n, 2NC(G)\}$ unless G is the Petersen graph.

Proof. If G is 2-connected and $NC'(G) \geq \frac{1}{3}(n + 5)$, then

$$\sigma_3'(G) \geq 3\sigma_1'(G) = 3(NC'(G) - 2) \geq n - 1. \quad \blacksquare$$

Corollary 6 complements and partially improves the following result, since clearly $NC''(G) \leq \min\{NC(G), NC'(G)\}$:

Theorem 7 [4]. If G is 2-connected and $NC''(G) \leq \frac{1}{2}n$, then $c(G) \geq 2NC''(G) - 2$. For $NC''(G) \leq \frac{1}{3}(n + 4)$, the result is sharp in the sense that longer cycles are not implied by the conditions.

An immediate consequence of Corollary 6 is the following:

Corollary 8. If G is 2-connected, $NC'(G) \geq \frac{1}{3}(n + 5)$, and $NC(G) \geq \frac{1}{2}n$, then G is hamiltonian unless G is the Petersen graph.

Corollary 8 improves the following result.

Theorem 9 [4]. If G is 2-connected and $NC''(G) \geq \frac{1}{2}n$, then, for n sufficiently large, G is hamiltonian.

Theorem 3 has the following analogue:

Theorem 10. If G is connected and contains a D-path, then G contains a D-path of order at least $\min\{n, 2NC(G) + 1\}$.

Proof. Apply Theorem 3 to the join of G and K_1. ∎

Again the complete bipartite graphs show that the conclusion of Theorem 10 cannot be strengthened. Furthermore, the requirement that G contain a D-path cannot be omitted, as shown by the graph H_n for $n \geq 7$.

Theorem 10 admits corollaries similar to those of Theorem 3.

3. PROOF OF THE MAIN RESULT

Throughout this section we assume that

—G is 2-connected and nonhamiltonian,
—C is a longest D-cycle of G for which $\max\{d(v)|v \in V(G) - V(C)\}$ is as large as possible,
—$|V(C)| \leq 2NC - 1$.

We first introduce some additional notations. By $\vec{C}$ we denote the cycle C with a given orientation. Let $u, v \in V(C)$. By $u\vec{C}v$ we denote the consecutive vertices on C from u to v in the direction specified by $\vec{C}$. The same vertices, in reverse order, are given by $v\overleftarrow{C}u$. We will consider $u\vec{C}v$ and $v\overleftarrow{C}u$ both as paths and as vertex sets. We use u^+ to denote the successor of u on $\vec{C}$ and u^- to denote its predecessor. We write u^{++} instead of $(u^+)^+$ and u^{--} instead of $(u^-)^-$. If $S \subseteq V(C)$, then $S^+ = \{x^+|x \in S\}$ and $S^- = \{x^-|x \in S\}$. We write $uv \in P_C(G)$ if u and v are connected by a path of length at least 2 that is internally disjoint from C.

Before proving Theorem 3 we establish a number of lemmas, the first four of which have become so standard in hamiltonian graph theory that we omit their proofs.

Lemma 11. If $v \in V(C)$, then $vv^+ \notin P_C(G)$.

Lemma 12. If $x \in V(G) - V(C)$ and $v_1, v_2 \in N(x)$, then $v_1^+v_2^+, v_1^-v_2^- \notin E(G) \cup P_C(G)$.

Lemma 13. Let $x \in V(G) - V(C)$, v_1, $v_2 \in N(x)$ and $v \in v_2^+\vec{C}v_1^-$. If $v_1^+v \in E(G) \cup P_C(G)$, then $v_2^+v^+ \notin E(G) \cup P_C(G)$. If $v_2^-v \in E(G) \cup P_C(G)$, then $v_1^-v^- \notin E(G) \cup P_C(G)$.

Lemma 14. Let $x \in V(G) - V(C)$, $v_1, v_2 \in N(x)$ and $v \in v_2^+\vec{C}v_1^-$. If $v_1^+v \in E(G) \cup P_C(G)$, then $v_2^-v^-, v_2^-v^+ \notin E(G) \cup P_C(G)$.

Lemma 15. Let $x \in V(G) - V(C)$ and $v_1, v_2 \in N(x)$. If $v_1^+ v_2^{++} \in P_C(G)$, then $N(v_2^+) \cap (V(G) - (V(C) \cup \{x\})) \neq \emptyset$. If $v_1^- v_2^{--} \in P_C(G)$, then $N(v_2^-) \cap (V(G) - (V(C) \cup \{x\})) \neq \emptyset$.

Proof. By symmetry, we need only prove the first part of the lemma. Suppose $v_1^+ v_2^{++} \in P_C(G)$ and $N(v_2^+) \cap (V(G) - (V(C) \cup \{x\})) = \emptyset$. Let $v_1^+ x_1 \cdots x_r v_2^{++}$ be a (v_1^+, v_2^{++})-path that is internally disjoint from C ($r \geq 1$). By Lemma 11, $x \notin \{x_1, \cdots, x_r\}$. Now the cycle $v_1 x v_2 \overleftarrow{C} v_1^+ x_1 \cdots x_r v_2^{++} \overrightarrow{C} v_1$ is a D-cycle longer than C, a contradiction. ∎

Lemma 16. Let $x_1, x_2 \in V(G) - V(C), A_1 = N(x_1), A_2 = N(x_2)$ and $A = A_1 \cup A_2$. Then $|A| = NC$ and either $|A \cap A^+| = 1$ and $A \cup A^+ = V(C)$ or $|A \cap A^+| = 2$ and $|A \cup A^+| \geq |V(C)| - 1$.

Proof. By Lemma 12, $|A_1 \cap A_2^+| \leq 1$ and $|A_2 \cap A_1^+| \leq 1$. Using Lemma 11 we conclude that $|A \cap A^+| \leq 2$. Hence

$$2NC - 1 \geq |V(C)| \geq |A \cup A^+| = |A| + |A^+| - |A \cap A^+| \geq 2|A| - 2,$$

implying that $|A| \leq NC$ and hence $|A| = NC$. The rest of the lemma also follows. ∎

Lemma 17. Let $x \in V(G) - V(C)$ and $y \in V(C)$. Then $xy^+ \notin E(G)$ or $xy^- \notin E(G)$.

Proof. Suppose $xy^+, xy^- \in E(G)$. If $N(y) \cap (V(G) - (V(C) \cup \{x\})) = \emptyset$, then $N(x) \cup N(y) \subseteq V(C)$ and, by Lemmas 11, 12, and 14, $v^+ \notin N(x) \cup N(y)$ whenever $v \in N(x) \cup N(y)$, implying that $|N(x) \cup N(y)| \leq \frac{1}{2}|V(C)| < NC$, a contradiction. Hence y has a neighbor x_1 in $V(G) - (V(C) \cup \{x\})$. Set $A = N(x) \cup N(x_1)$. By Lemmas 11 and 12, $y^{++}, y^{--} \notin A$. From Lemma 16 we conclude that $y^{+++} \in A$ or $y^{---} \in A$. Assume without loss of generality that $y^{+++} \in A$ and set $w = y^{+++}$. Then w^- has a neighbor x_2 in $V(G) - (V(C) \cup \{x\})$: assuming the contrary, we obtain a contradiction as in the beginning of the proof if $w \in N(x)$, while we contradict Lemma 15 if $w \in N(x_1)$. By Lemma 12, $x_2 \neq x_1$.

We claim that $N(x_2) \cap (V(C) - (A \cup A^+)) = \emptyset$. This is clear if $V(C) = A \cup A^+$. Otherwise, by Lemma 16, $V(C) - (A \cup A^+)$ contains a unique vertex z and $z^+ \in A$. If $w, z^+ \in N(x)$, then $x_2 z \notin E(G)$ by Lemma 12. If $w \in N(x)$ and $z^+ \in N(x_1)$, then $x_2 z \notin E(G)$ by Lemma 14. If $w \in N(x_1)$ and $z^+ \in N(x)$, then $x_2 z \notin E(G)$ by Lemma 13. Finally, if $w, z^+ \in N(x_1)$, then $x_2 z \notin E(G)$ by Lemma 12. Hence, indeed, $N(x_2) \cap (V(C) - (A \cup A^+)) = \emptyset$.

By Lemmas 12 and 13, $N(x_2) \cap (A^+ - \{w^-\}) = \emptyset$. Now if $w \notin N(x_1)$, it follows that $N(x_1) \cup N(x_2) \subseteq (A \cup \{w^-\}) - \{y^+, w\}$, whence $|N(x_1) \cup$

$N(x_2)| < |A| = NC$, a contradiction. If $w \in N(x_1)$, then $N(x_1) \cup N(x_2) \subseteq (A \cup \{w^-\}) - \{y^+, y^-\}$, and we again obtain a contradiction. ∎

Lemma 18. If $x_1, x_2 \in V(G) - V(C)$, then $N(x_1) \cap N(x_2) = \emptyset$.

Proof. Suppose $N(x_1) \cap N(x_2)$ contains a vertex y. From Lemma 16 we conclude that y^{++} or y^{--} is in $N(x_1) \cup N(x_2)$. In either case we contradict Lemma 17. ∎

Proof of Theorem 3. We distinguish two main cases and a number of subcases, in each of which we either reach contradictions with the assumptions at the beginning of this section or the conclusion that G is the Petersen graph.

Case 1. $|V(G) - V(C)| \geq 2$.

If $|V(G) - V(C)| \geq 4$, then by Lemma 18 $V(G) - V(C)$ contains two vertices z_1, z_2 with $|N(z_1) \cup N(z_2)| \leq \frac{1}{2}|V(C)| < NC$, a contradiction. Hence $|V(G) - V(C)| \leq 3$.

Let x_1 and x_2 be two vertices in $V(G) - V(C)$. Set $A_i = N(x_i)$ $(i = 1, 2)$ and $A = A_1 \cup A_2$. By Lemma 16 there are two possibilities.

Case 1.1. $|A \cap A^+| = 1$ and $A \cup A^+ = V(C)$.

Let y be the unique vertex in $A \cap A^+$. Assume without loss of generality that $x_1y^-, x_2y \in E(G)$. Set $w = y^{++}, z = w^{++}$. By Lemma 17, $w \in A_1$ and $z \in A_2$. By Lemma 15, w^+ has a neighbor x_3 in $V(G) - (V(C) \cup \{x_1, x_2\})$. Since G is 2-connected, x_3 has a neighbor v on $V(C)$ with $v \neq w^+$. By Lemmas 17 and 18, $v \in z^{++}\vec{C}y^{--}$ and $v \notin A$. Hence $v \in A^+$. If $v^- \in A_1$, then we contradict Lemma 12 (with $x = x_1$). If $v^- \in A_2$, then we contradict Lemma 13 (with $x = x_1$).

Case 1.2. $|A \cap A^+| = 2$ and $|A \cup A^+| \geq |V(C)| - 1$.

Set $A \cap A^+ = \{y_1, y_2\}$. Assume without loss of generality that $y_1^- \in N(x_1)$, $y_1 \in N(x_2)$. Using Lemma 12 we conclude that $y_2 \in N(x_1)$, $y_2^- \in N(x_2)$. Since $|V(C) - (A \cup A^+)| \leq 1$, we may assume without loss of generality that $V(C) - (A \cup A^+) \subseteq y_1\vec{C}y_2^-$. Set $w = y_2^{++}$, $z = w^{++}$. Since $y_2\vec{C}y_1^- \subseteq A \cup A^+$, $w, z \in A$. By Lemma 17, $w \in A_2$ and $z \in A_1$. By Lemma 15, y_2^+ has a neighbor x_3 in $V(G) - (V(C) \cup \{x_1, x_2\})$ while w^+ has a neighbor x_4 in $V(G) - (V(C) \cup \{x_1, x_2\})$. Since $|V(G) - V(C)| \leq 3$, $x_3 = x_4$. But then we contradict Lemma 17 (with $x = x_3$).

Case 2. $|V(G) - V(C)| = 1$.

Since we have assumed that $|V(C)| \leq 2NC - 1$, we have $NC \geq \frac{1}{2}n$. Let x be the unique vertex in $V(G) - V(C)$ and $v_1, \ldots, v_k$ the neighbors of x, occurring on $\vec{C}$ in the order of their indices. For $i = 1,\dots,k$, set $u_i = v_i^+$, $w_i = v_{i+1}^-$ and $T_i = u_i\vec{C}w_i$ (indices mod k). We call the sets $T_1,\dots,T_k$ *segments* of C; T_i is a t-*segment* if $|T_i| = t$ ($i = 1,\dots,k$). We set $T = \bigcup_{i=1}^k T_i$. For a vertex v of G and an integer $i \in \{1,\dots,k\}$, we denote $N(v) \cap T_i$ by $N_i(v)$; by $N_T(v)$ we denote $\bigcup_{i=1}^k N_i(v)$.

By Lemma 17, C contains no 1-segments. Two possibilities remain.

Case 2.1. C contains a 2-segment.

Assume without loss of generality that T_1 is a 2-segment. Define the function f: $V(G) \to \{0,1,2\}$ by $f(v) = |N(v) \cap \{u_1, w_1\}|$. From Lemmas 12 and 14 we deduce that

$$f(u_i) + f(w_i) \le 1 \quad \text{for } i = 2,\dots,k. \tag{1}$$

Lemma 14 also implies that

$$\text{if} \quad v \in u_2\vec{C}w_k \quad \text{and} \quad f(v) = 2, \quad \text{then } f(v^-) = f(v^+) = 0. \tag{2}$$

From (1) and (2) we conclude that

$$|N_i(u_1)| + |N_i(w_1)| = \sum_{v \in T_i} f(v) \le |T_i| - 1 \quad \text{for } i = 2,\dots,k, \tag{3}$$

whence

$$|N_T(u_1)| + |N_T(w_1)| \le 2 + \sum_{i=2}^{k} (|T_i| - 1) = |V(C)| - 2k + 1.$$

Assuming without loss of generality that $|N_T(u_1)| \le |N_T(w_1)|$ we thus have

$$|N_T(u_1)| \le \tfrac{1}{2}(|V(C)| + 1) - k = \tfrac{1}{2}n - k,$$

implying that

$$\tfrac{1}{2}n \le NC \le |N(x) \cup N(u_1)| = k + |N_T(u_1)| \le \tfrac{1}{2}n. \tag{4}$$

Since all inequalities in (4) are in fact equalities, (3) also holds with equality:

$$\sum_{v \in T_i} f(v) = |T_i| - 1 \quad \text{for } i = 2,\dots,k. \tag{5}$$

We now show that

$$f(u_i^+) \ne 2 \quad \text{and} \quad f(w_i^-) \ne 2 \quad \text{for } i = 2,\dots,k. \tag{6}$$

Suppose, e.g., $f(u_i^+) = 2$ for some $i \in \{2,\ldots,k\}$. If $v \in u_i^+\vec{C}w_k$ and $u_iv, u_iv^+ \in E(G)$, then the cycle $v_1xv_i\overleftarrow{C}u_1u_i^+\vec{C}vu_iv^+\vec{C}v_1$ contradicts the choice of C. If $v \in u_1\vec{C}w_{i-1}$ and $u_iv, u_iv^+ \in E(G)$, then the cycle $v_1xv_i\overleftarrow{C}v^+u_iv\overleftarrow{C}u_1u_i^+\vec{C}v_1$ contradicts the choice of C. Together with Lemmas 11–14 these observations show that $v^+ \notin N(x) \cup N(u_i)$ whenever $v \in N(x) \cup N(u_i)$. But then $|N(x) \cup N(u_i)| \le \frac{1}{2}|V(C)| < \frac{1}{2}n \le NC$. This contradiction establishes (6).

In view of (6), (5) holds only if

$$\text{for } i = 2,\ldots,k, \text{ either } f(u_i) = 0 \text{ and } f(v) = 1 \text{ for every } v \in T_i - \{u_i\} \text{ or } f(w_i) = 0 \text{ and } f(v) = 1 \text{ for every } v \in T_i - \{w_i\}. \quad (7)$$

If, for some $i \in \{2,\ldots,k\}$, $N_i(u_1) \ne \emptyset$ and $N_i(w_1) \ne \emptyset$, then by (7) T_i contains a vertex v such that either $u_1v, w_1v^+ \in E(G)$ or $u_1v^+, w_1v \in E(G)$, contradicting Lemma 14. Hence

$$\text{for } i = 2,\ldots,k, \quad \text{either } N_i(u_1) = \emptyset \quad \text{or } N_i(w_1) = \emptyset. \quad (8)$$

Combining (7) and (8) and using Lemma 12 we conclude that

$$\text{for } i = 2,\ldots,k, \text{ either } f(u_i) = 0 \text{ and } u_1v \in E(G) \text{ for every } v \in T_i - \{u_i\} \text{ or } f(w_i) = 0 \text{ and } w_1v \in E(G) \text{ for every } v \in T_i - \{w_i\}. \quad (9)$$

We now show that all segments of C are 2-segments. Suppose there exist integers r and t with $2 \le r \le k$ and $t \ge 3$ such that T_r is a t-segment. In view of (9) we may assume, without loss of generality, that $f(u_r) = 0$ and $u_1v \in E(G)$ for every $v \in T_r - \{u_r\}$. By (9) and Lemma 14, $N_i(w_1) = \emptyset$ for $i = 2,\ldots,r$. If $r < k$, then $N_i(w_1) = \emptyset$ for $i = r+1,\ldots,k$ also, otherwise $w_1u_s \in$ E(G) for some $s \in \{r+1,\ldots,k\}$ and the cycle $v_1xv_s\overleftarrow{C}w_ru_1w_r^-\overleftarrow{C}$ $w_1u_s\vec{C}v_1$ contradicts the choice of C. It follows that

$$|N(x) \cup N(w_1)| = k + |N_T(w_1)| = k + 1 \le \tfrac{1}{3}(n-2) + 1 < \tfrac{1}{2}n.$$

This contradiction shows that C indeed contains 2-segments only, implying that $n = 3k + 1$.

Set $m = \max\{i | u_1w_i \in E(G)\}$. Then, by (9) and Lemma 14, $u_1w_i \in E(G)$ for $1 \le i \le m$, while $w_1u_j \in E(G)$ for $j = 1$ and $m + 1 \le j \le k$. If $m < \frac{1}{2}(k+1)$, then

$$|N(x) \cup N(u_1)| = k + |N_T(u_1)| = k + m < \tfrac{1}{2}(3k+1) = \tfrac{1}{2}n,$$

a contradiction. If $m > \frac{1}{2}(k+1)$, then

$$|N(x) \cup N(w_1)| = k + |N_T(w_1)| = k + k - m + 1 < \tfrac{1}{2}(3k+1) = \tfrac{1}{2}n,$$

again a contradiction. Hence $m = \frac{1}{2}(k+1)$ and k is odd.

We have shown that $u_1w_i \in E(G)$ for $1 \le i \le m$. By the same token, $u_kw_i \in E(G)$ for $1 \le i \le m-1$. Now by Lemma 13, $u_1v_i \notin E(G)$ for $i = 2,\dots,m$. By Lemma 14, $u_1v_i \notin E(G)$ for $i = m+1,\dots,k$. Hence $N(u_1) \subseteq \{w_1,\dots,w_m,v_1\}$ and, similarly, $N(u_2) \subseteq \{w_2,\dots,w_{m+1},v_2\}$. Thus

$$\tfrac{1}{2}n \le |N(u_1) \cup N(u_2)| \le m + 3 = \tfrac{1}{2}(k+1) + 3 = \tfrac{1}{2}n - k + 3.$$

It follows that $k = 3$ and $N(u_1) = \{w_1, w_2, v_1\}$. An argument of symmetry gives us $N(u_i)$ and $N(w_i)$ for each $i \in \{1,2,3\}$. We conclude that the Petersen graph is a spanning subgraph of G. Since the Petersen graph is a maximal nonhamiltonian graph, G itself must be the Petersen graph.

Case 2.2. C contains no 2-segments.

Then $d(x) \le \frac{1}{4}|V(C)| < \frac{1}{4}n$. If $v \in T$, then $|N(x) \cup N(v)| \ge \frac{1}{2}n$, implying that $d(v) > \frac{1}{4}n$. From the maximality of $\max\{d(v)|v \in V(G) - V(C)\}$, we conclude that

$$\text{if } v \in T, \text{ then } G \text{ contains no cycle with vertex set } V(G) - \{v\}. \quad (10)$$

For $i = 1,2$, let y_i be the first vertex of T_i such that $y_iv_i^- \notin E(G)$. Set $Y_{11} = N_1(y_1)$, $Y_{12} = (N_1(y_2))^+$, $Y_{21} = (N_2(y_1))^+$, $Y_{22} = N_2(y_2)$, and for $i = 3,\dots,k$, $Y_{i1} = N_i(y_1)$, $Y_{i2} = (N_i(y_2))^-$. By a variation of Lemma 12 we have

$$\begin{aligned} &y_1v \notin E(G) \quad \text{for} \quad v \in u_2\vec{C}y_2,\ y_2v \notin E(G) \quad \text{for} \quad v \in u_1\vec{C}y_1, \quad \text{and} \\ &y_iu_j \notin E(G) \quad \text{for} \quad i = 1,2 \quad \text{and} \quad j = 3,\dots,k. \end{aligned} \quad (11)$$

(If, e.g., $y_1y_2 \in E(G)$, then the cycle $v_1xv_2\vec{C}y_2^-w_1\overleftarrow{C}y_1y_2\vec{C}w_ky_1^-\overleftarrow{C}v_1$ contradicts the choice of C). From (11) and a similar variation of Lemma 13 we deduce that $Y_{i1} \cap Y_{i2} = \varnothing$ ($i = 1,\dots,k$). By (11) and the way y_2 was chosen, $Y_{11} \cup Y_{12} \subseteq T_1 - \{y_1\}$. By (11) and (10) (with $v = u_i$), $Y_{i1} \cup Y_{i2} \subseteq T_i - \{u_i\}$ ($i = 3,\dots,k-1$). If $k \ge 3$, then $Y_{21} \cup Y_{22} \subseteq (T_2 - \{y_2\}) \cup \{v_3\}$ (by (11)) and $Y_{k1} \cup Y_{k2} \subseteq T_k - \{u_k, w_k\}$ (by (11), (10), and the way y_1 was chosen). If $k = 2$, then the choice of y_1 implies $Y_{21} \cup Y_{22} \subseteq T_2 - \{y_2\}$. In both cases we conclude that

$$\begin{aligned} |N_T(y_1)| + |N_T(y_2)| &= \sum_{i=1}^{k} (|N_i(y_1)| + |N_i(y_2)|) = \sum_{i=1}^{k} (|Y_{i1}| + |Y_{i2}|) \\ &= \sum_{i=1}^{k} |Y_{i1} \cup Y_{i2}| \le |T| - k = n - 2k - 1. \end{aligned} \quad (12)$$

On the other hand, we have

$$\tfrac{1}{2}n \le |N(x) \cup N(y_i)| = |N(x)| + |N_T(y_i)| = k + |N_T(y_i)| \qquad (i = 1,2),$$

whence $|N_T(y_1)| + |N_T(y_2)| \ge n - 2k$, contradicting (12). ∎

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