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# A generalization of Ore's Theorem involving neighborhood unions

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*Abstract*

Let $G$ be a graph of order $n$. Settling conjectures of Chen and Jackson, we prove the following generalization of Ore's Theorem: If $G$ is 2-connected and $|N(u)\cup N(v)|\geqslant\frac{1}{2}n$ for every pair of nonadjacent vertices $u, v$, then either $G$ is hamiltonian, or $G$ is the Petersen graph, or $G$ belongs to one of three families of exceptional graphs of connectivity 2.

## 1. Introduction

We use Bondy and Murty [3] for terminology and notation not defined here and consider simple graphs only.

Let $G$ be a graph of order $n$. If $G$ has a Hamilton cycle (a cycle containing every vertex of $G$), then $G$ is called *hamiltonian.* A cycle $C$ of $G$ is called a *dominating cycle*, or briefly *D-cycle*, if $V(G)-V(C)$ is an independent set of vertices in $G$. The number of vertices in a maximum independent set of $G$ is denoted by $\alpha(G)$ and the set of vertices adjacent to a vertex $v$ by $N_G(v)$; $d_G(v):=|N_G(v)|$ is the *degree* of the vertex $v$ and $\delta(G)$ denotes $\min\{d_G(v)\mid v\in V(G)\}$. If $G$ is noncomplete, then $\mathrm{NC}(G)$ denotes $\min\{|N_G(u)\cup N_G(v)|\mid uv\notin E(G)\}$; if $G$ is complete, we set $\mathrm{NC}(G)=n-1$. If no ambiguity can arise, we sometimes write $\alpha$ instead of $\alpha(G), N(v)$ instead of $N_G(v)$, etc.

The earliest degree condition for a graph to be hamiltonian is due to Dirac.

**Theorem 1** (Dirac [5]). *If $G$ is a graph of order $n$ with $\delta(G)\geqslant\frac{1}{2}n>1$, then $G$ is hamiltonian.*

Ore generalized Theorem 1 as follows.

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**Theorem 2** (Ore [10]). *If $G$ is a graph of order $n$ with $d(u)+d(v)\geqslant n\geqslant 3$ for every pair $u, v$ of nonadjacent vertices, then $G$ is hamiltonian.*

We will refer to Theorem 2 as Ore's Theorem.

In recent literature on hamiltonian graph theory, many results appear in which certain vertex sets are required to have large neighborhood unions instead of large degree-sums. Three such results are the following.

**Theorem 3** (Faudree, Gould, Jacobson and Schelp [8]). *If $G$ is a 2-connected graph of order $n$ and $\mathrm{NC}(G)\geqslant\frac{1}{3}(2n-1)$, then $G$ is hamiltonian.*

**Theorem 4** (Faudree, Gould, Jacobson and Lesniak [7]). *If $G$ is a 2-connected graph of order $n$ and $\mathrm{NC}(G)\geqslant n-\delta(G)$, then $G$ is hamiltonian.*

**Theorem 5** (Faudree, Gould, Jacobson and Lesniak [6]). *If $G$ is a 2-connected graph of order $n$ and $|N(u)\cup N(v)|\geqslant\frac{1}{2}n$ for all distinct $u, v\in V(G)$, then, for $n$ sufficiently large, $G$ is hamiltonian.*

None of these results generalizes Ore's Theorem. In fact, Theorems 3 and 1 are incomparable in the sense that neither theorem implies the other. Theorems 4 and 5 are more general than Theorem 1 (for $n$ sufficiently large in the case of Theorem 5), but are incomparable to Theorem 2.

In Bauer, Fan and Veldman [1] it is shown that a recent result in Bauer, Morgana, Schmeichel and Veldman [2] is a common generalization of Theorems 2, 3 and 4 (and Theorem 1). Another common generalization of Theorems 2, 3 and 4 is also established in Bauer, Fan and Veldman [1]. Both of these results involve degree-sums of triples of independent vertices.

In Broersma and Veldman [4] the following generalization of Theorems 2, 3, 4 and 5 is obtained (for graphs that contain a $D$-cycle).

**Theorem 6** (Broersma and Veldman [4]). *If $G$ is a 2-connected graph of order $n$ and $G$ contains a $D$-cycle, then $G$ contains a $D$-cycle of length at least $\min\{n, 2\mathrm{NC}(G)\}$, unless $G$ is the Petersen graph.*

Recently, in Jackson [9] it was shown that the bound $\frac{1}{3}(2n-1)$ in Theorem 3 can be lowered to $\frac{1}{2}(n+3)$, unless $G$ belongs to one of three families of exceptional graphs (the families $\mathscr{G}_n, \mathscr{H}_n, \mathscr{F}_n$ defined below, but with $\frac{1}{2}n$ replaced by $\frac{1}{2}(n+3)$ in the definition of $\mathscr{F}_n$). Moreover, it was proved that a 3-connected graph of order $n$ with $\mathrm{NC}\geqslant\frac{1}{2}(n+1)$ is hamiltonian.

We use Theorem 6 to prove the following generalization of Theorems 2, 3, 4 and 5, which was conjectured in Jackson [9].

**Theorem 7.** *If $G$ is a 2-connected graph of order $n$ and $\mathrm{NC}(G)\geqslant\frac{1}{2}n$, then either $G$ is hamiltonian, or $G$ is the Petersen graph, or $G\in\mathscr{G}_n\cup\mathscr{H}_n\cup\mathscr{F}_n$.*

Here, the classes $\mathscr{G}_n, \mathscr{H}_n$ and $\mathscr{J}_n$ are defined as follows. For positive integers $n$ and $i$, let $\mathscr{K}_n^{(i)}$ denote the class of graphs of order $n$ consisting of three disjoint complete graphs, where each of the components has order at least $i$, i.e., $\mathscr{K}_n^{(i)} := \{K_p + K_q + K_r \mid p+q+r=n, p, q, r \geq i\}$. Let $\mathscr{G}_n^*$ denote the family of all graphs of order $n$ which can be obtained as the join of $K_2$ and a graph in $\mathscr{K}_{n-2}^{(1)}$. Let $\mathscr{H}_n^*$ denote the family of all graphs of order $n$ which can be obtained from the join of $K_1$ and a graph $H$ in $\mathscr{K}_{n-1}^{(2)}$ by adding the edges of a triangle between three vertices from different components of $H$. Let $\mathscr{J}_n^*$ denote the family of all graphs of order $n$ which can be obtained from a graph $H$ in $\mathscr{K}_n^{(3)}$ by adding the edges of two triangles between two disjoint triples of vertices, each containing one vertex of each component of $H$.

If $H$ is a spanning subgraph of $G$, we write $H \leq G$. Now

$$\mathscr{F}_n := \{G \mid G \text{ is 2-connected, } |V(G)| = n \text{ and } \mathrm{NC}(G) \geq \tfrac{1}{2}n\},$$

$$\mathscr{G}_n := \{G \in \mathscr{F}_n \mid G \leq G_1, \text{ for some } G_1 \in \mathscr{G}_n^*\},$$

$$\mathscr{H}_n := \{G \in \mathscr{F}_n \mid G \leq G_1, \text{ for some } G_1 \in \mathscr{H}_n^*\},$$

$$\mathscr{J}_n := \{G \in \mathscr{F}_n \mid G \leq G_1, \text{ for some } G_1 \in \mathscr{J}_n^*\}.$$

It is easy to check that all graphs in $\mathscr{G}_n \cup \mathscr{H}_n \cup \mathscr{J}_n$ are nonhamiltonian. Hence, Theorem 7 implies that, apart from the Petersen graph, the graphs of $\mathscr{G}_n \cup \mathscr{H}_n \cup \mathscr{J}_n$ are the only nonhamiltonian graphs in $\mathscr{F}_n$.

All graphs in $\mathscr{G}_n \cup \mathscr{H}_n \cup \mathscr{J}_n$ have connectivity 2. So Theorem 7 has the following consequence, which was conjectured by Chen (see Jackson [9]).

**Corollary 8.** *If $G$ is a 3-connected graph of order $n$ and $NC(G) \geq \frac{1}{2}n$, then either $G$ is hamiltonian, or $G$ is the Petersen graph.*

The complete bipartite graphs $K_{\frac{1}{2}(n-1), \frac{1}{2}(n+1)}, n \geq 5$, form an infinite family of 2-connected nonhamiltonian graphs for which $\mathrm{NC} = \frac{1}{2}(n-1)$.

## 2. Proof of Theorem 7

By Theorem 6 it is sufficient to prove the following lemma.

**Lemma 9.** *If $G$ is a 2-connected graph of order $n$, $\mathrm{NC}(G) \geq \frac{1}{2}n$ and $G \notin \mathscr{G}_n \cup \mathscr{H}_n \cup \mathscr{J}_n$, then $G$ contains a D-cycle.*

In the proof of Lemma 9 we use some additional terminology and notation. If $C$ is a cycle of $G$, we denote by $\vec{C}$ the cycle $C$ with a given orientation. If $u, v \in V(C)$, then $u\vec{C}v$ denotes the consecutive vertices on $C$ from $u$ to $v$ in the direction specified by $\vec{C}$. The same vertices, in reverse order, are given by $v\overleftarrow{C}u$. We will consider $u\vec{C}v$ and $v\overleftarrow{C}u$ both as paths and as vertex sets. We use $u^+$ to denote the successor of $u$ on $\vec{C}$ and $u^-$

to denote its predecessor; $u^{++}:=(u^+)^+$ and $u^{--}:=(u^-)^-$. If $A\subseteq V(C)$, then $A^+:=\{v^+ \mid v\in A\}$ and $A^-:=\{v^- \mid v\in A\}$.

Let $t$ be a positive integer. A cycle $C$ of $G$ is called a *$D_t$-cycle* if every component of $G-V(C)$ has order smaller than $t$. This means that a $D_1$-cycle is a Hamilton cycle and a $D_2$-cycle is a $D$-cycle.

Let $X, X_1, X_2$ be subgraphs of $G$. By $N(X)$ we denote the set of vertices in $V(G)-V(X)$ that are adjacent to at least one vertex of $X$. We call $X_1$ and $X_2$ *remote* if $V(X_1)\cap V(X_2)=\emptyset$ and $N(X_1)\cap V(X_2)=\emptyset$. By $\omega_t(X)$ we denote the number of components of $X$ with at least $t$ vertices.

If $\vec{C}$ is an oriented cycle of $G$ and $v\in V(C)$, then we call a subgraph $X$ of $G$ a *$(\vec{C}, v, t)$-subgraph* if each of the following requirements holds:

(i) $X$ is connected and has order $t$;

(ii) $\emptyset\neq V(X)\cap V(C)=v\vec{C}w$ for some vertex $w\in V(C)$;

(iii) if $X'$ satisfies (i) and (ii), then $V(X)\cap V(C)\subseteq V(X')\cap V(C)$.

**Proof of Lemma 9.** Assume that $G$ is a 2-connected graph, $|V(G)|=n$, $\mathrm{NC}(G)\geqslant\frac{1}{2}n$ and that $G$ contains no $D$-cycle. We distinguish two main cases and a number of subcases, in each of which we either reach contradictions with the assumptions, or the conclusion $G\in\mathscr{G}_n\cup\mathscr{H}_n\cup\mathscr{J}_n$.

Set $\lambda+1:=\min\{i \mid G \text{ has a } D_i\text{-cycle}\}$, so that $\lambda\geqslant 2$. Let $C_\lambda$ be a $D_{\lambda+1}$-cycle of $G$ for which $\omega_\lambda(G-V(C_\lambda))$ is minimum. Fix an orientation on $C_\lambda$. Since $G$ has no $D_\lambda$-cycle, $G-V(C_\lambda)$ has a component $X_0$ of order $\lambda$. Let $a_1,\dots,a_k$ be the neighbors of $X_0$, occurring on $\vec{C}_\lambda$ in the order of their indices. Since $G$ is 2-connected, we have $k\geqslant 2$.

We distinguish two cases.

*Case* 1: $k\geqslant 3$.

Let in every segment $T_i:=a_i^+\vec{C}_\lambda a_{i+1}^-$ (indices mod $k$) the vertex $t_i$ be the first vertex such that $a_i^-t_i\notin E(G)$. This vertex exists because the choice of $C_\lambda$ implies $a_i^-a_{i+1}^-\notin E(G)$.

Now we can find a $(\vec{C}_\lambda, t_1, \lambda)$-subgraph $X_1$, a $(\vec{C}_\lambda, t_2, \lambda)$-subgraph $X_2$ and $(\vec{C}_\lambda, a_i^+, \lambda)$-subgraphs $X_i$ for $i=3,\dots,k$, such that $X_0,\dots,X_k$ are mutually remote. The proofs of these assertions are copies of those in the proof of Theorem 2 in Veldman [11], only every time in Veldman [11] a path $a_1^-a_1u_{01}$ or $a_2^-a_2u_{02}$ is used, we now must use the path $a_1^-t_1^-\overleftarrow{C}_\lambda a_1u_{01}$ or $a_2^-t_2^-\overleftarrow{C}_\lambda a_2u_{02}$ (here $u_{0i}$ is a vertex in $X_0$ such that $a_iu_{0i}\in E(G)$, for $i=1,\dots,k$).

We make a number of observations. These observations follow by definition, or they are proved by contradiction. In the latter cases we give a cycle contradicting the choice of $C_\lambda$ if we assume the contrary to the observation. Here $a_1Pa_2$ is a path joining $a_1$ and $a_2$ with all internal vertices in $X_0$.

(a) $t_1a_1^-\notin E(G)$ (by definition of $t_1$);

(b) $t_2a_2^-\notin E(G)$ (by definition of $t_2$);

(c) if $v\in a_1^+\vec{C}_\lambda t_1^-$, then $t_2v\notin E(G)$

$(a_1^- v^- \overleftarrow{C}_\lambda a_1 P a_2 \overrightarrow{C}_\lambda t_2^- a_2^- \overleftarrow{C}_\lambda v t_2 \overrightarrow{C}_\lambda a_1^-)$;

(d) if $v \in a_2^+ \overrightarrow{C}_\lambda t_2^-$, then $t_1 v \notin E(G)$

$(a_1^- t_1^- \overleftarrow{C}_\lambda a_1 P a_2 \overrightarrow{C}_\lambda v^- a_2^- \overleftarrow{C}_\lambda t_1 v \overrightarrow{C}_\lambda a_1^-)$;

(e) if $v \in t_1^+ \overrightarrow{C}_\lambda a_2^-$ and $t_1 v \in E(G)$, then $t_2 v^- \notin E(G)$

$(a_1^- t_1^- \overleftarrow{C}_\lambda a_1 P a_2 \overrightarrow{C}_\lambda t_2^- a_2^- \overleftarrow{C}_\lambda v t_1 \overrightarrow{C}_\lambda v^- t_2 \overrightarrow{C}_\lambda a_1^-)$;

(f) if $v \in t_2^+ \overrightarrow{C}_\lambda a_1^-$ and $t_2 v \in E(G)$, then $t_1 v^- \notin E(G)$

$(a_1^- t_1^- \overleftarrow{C}_\lambda a_1 P a_2 \overrightarrow{C}_\lambda t_2^- a_2^- \overleftarrow{C}_\lambda t_1 v^- \overleftarrow{C}_\lambda t_2 v \overrightarrow{C}_\lambda a_1^-)$;

(g) if $v \in V(G) - (V(C_\lambda) \cup V(X_0))$ and $t_1 v \in E(G)$, then $t_2 v \notin E(G)$

$(a_1^- t_1^- \overleftarrow{C}_\lambda a_1 P a_2 \overrightarrow{C}_\lambda t_2^- a_2^- \overleftarrow{C}_\lambda t_1 v t_2 \overrightarrow{C}_\lambda a_1^-)$.

Define

$$T_0 := V(G) - V(C_\lambda),$$

$$T_{11} := a_1^+ \overrightarrow{C}_\lambda t_1^-, \quad T_{12} := t_1 \overrightarrow{C}_\lambda a_2^-, \quad T_{21} := a_2^+ \overrightarrow{C}_\lambda t_2^-, \quad T_{22} := t_2 \overrightarrow{C}_\lambda a_3^-$$

and for $i = 1, \ldots, k$

$$X_{i1} := V(X_i) \cap V(C_\lambda), \quad \lambda_i := |X_{i1}|, \quad X_{i2} := V(X_i) - X_{i1}.$$

Let $x \in V(X_0)$, then

$$N(x) \subseteq (V(X_0) - \{x\}) \cup \{a_1, \ldots, a_k\}.$$

Because of (g) and the definition of the $X_i$'s we have

$$(N(t_1) \cap T_0) \cap (N(t_2) \cap T_0) = \emptyset,$$

$$(N(t_1) \cap T_0) \cup (N(t_2) \cap T_0) \subseteq T_0 - \left( V(X_0) \cup \bigcup_{i=3}^{k} X_{i2} \right),$$

hence $|N(t_1) \cap T_0| + |N(t_2) \cap T_0| \leqslant |T_0| - \lambda - \sum_{i=3}^k (\lambda - \lambda_i)$.

Because of (c) we have

$$N(t_2) \cap T_{11} = \emptyset,$$

so $|N(t_1) \cap T_{11}| + |N(t_2) \cap T_{11}| \leqslant |T_{11}|$.

Because of (b) and (e) we can conclude

$$(N(t_1) \cap T_{12}) \cap (N(t_2) \cap T_{12})^+ = \emptyset,$$

$$(N(t_1) \cap T_{12}) \cup (N(t_2) \cap T_{12})^+ \subseteq T_{12} - \{t_1\},$$

hence $|N(t_1) \cap T_{12}| + |N(t_2) \cap T_{12}| \leqslant |T_{12}| - 1$.

From (d) we can conclude

$$N(t_1)\cap T_{21}=\emptyset,$$

so $|N(t_1)\cap T_{21}|+|N(t_2)\cap T_{21}|\leqslant|T_{21}|$.

Because of (f) it follows that

$$(N(t_1)\cap T_{22})\cap(N(t_2)\cap T_{22})^- =\emptyset,$$

$$(N(t_1)\cap T_{22})\cup(N(t_2)\cap T_{22})^- \subseteq T_{22},$$

hence $|N(t_1)\cap T_{22}|+|N(t_2)\cap T_{22}|\leqslant|T_{22}|$.

From (f) and the definition of the $X_i$'s we can conclude for $i=3,\ldots,k-1$

$$(N(t_1)\cap T_i)^+\cap(N(t_2)\cap T_i)=\emptyset,$$

$$(N(t_1)\cap T_i)^+\cup(N(t_2)\cap T_i)\subseteq(T_i\cup\{a_{i+1}\})-X_{i1},$$

hence $|N(t_1)\cap T_i|+|N(t_2)\cap T_i|\leqslant|T_i|+1-\lambda_i$ for $i=3,\ldots,k-1$.

Finally, we have because of (a) and (f)

$$(N(t_1)\cap T_k)^+\cap(N(t_2)\cap T_k)=\emptyset,$$

$$(N(t_1)\cap T_k)^+\cup(N(t_2)\cap T_k)\subseteq T_k-X_{k1},$$

which gives $|N(t_1)\cap T_k|+|N(t_2)\cap T_k|\leqslant|T_k|-\lambda_k$.

Combining all this we get

$$\begin{aligned}
n&\leqslant 2\mathrm{NC}(G)\\
&\leqslant|N(t_1)\cup N(x)|+|N(t_2)\cup N(x)|\\
&\leqslant|N(t_1)-N(X_0)|+|N(t_2)-N(X_0)|\\
&\quad+2|V(X_0)-\{x\}|+2|\{a_1\ldots,a_k\}|\\
&\leqslant|T_0|-\lambda-\sum_{i=3}^{k}(\lambda-\lambda_i)+|T_{11}|+|T_{12}|-1+|T_{21}|+|T_{22}|\\
&\quad+\sum_{i=3}^{k-1}(|T_i|+1-\lambda_i)+|T_k|-\lambda_k+2(\lambda-1)+2k\\
&=\sum_{i=0}^{k}|T_i|-\lambda-(k-2)\lambda-1+(k-3)+2\lambda-2+2k\\
&=n-k+3\lambda-k\lambda-6+3k\\
&=n-k\lambda+3\lambda+2k-6\\
&=n-(k-3)(\lambda-2).
\end{aligned}$$

Because $k\geqslant 3$ and $\lambda\geqslant 2$, we must have equality throughout. In particular this means

$$(N(t_1)\cap T_{22})\cup(N(t_2)\cap T_{22})^- = T_{22},$$

hence $a_3^- \in N(t_1)$;

$$(N(t_1)\cap T_i)^+ \cup (N(t_2)\cap T_i) = (T_i \cup \{a_{i+1}\}) - V(X_i),$$

hence $a_{i+1}^- \in N(t_1)$ for $i=3,\dots,k-1$ and $N(x)=\{a_1,\dots,a_k\}\cup(V(X_0)-\{x\})$.

Because of symmetry we must have for $i=1,\dots,k$

$$t_i a_j^- \in E(G) \quad \text{for } j=1,\dots,i-1,i+2,\dots,k \text{ (indices mod } k),$$

and we must have for all $x_0 \in V(X_0)$

$$N(x_0)=\{a_1,\dots,a_k\}\cup(V(X_0)-\{x_0\}).$$

Now choose two vertices $x_1$ and $x_2$ in $X_0$. Assume that $a_1^+ a_1^- \notin E(G)$. Then $t_1 = a_1^+$ and the cycle

$$a_1^- t_2 \vec{C}_\lambda a_3^- a_1^+ \vec{C}_\lambda a_2^- t_2^- \overleftarrow{C}_\lambda a_2 x_1 a_1 x_2 a_3 \vec{C}_\lambda a_1^-$$

contradicts the choice of $C_\lambda$, hence $a_1^+ a_1^- \in E(G)$. By symmetry this means $a_i^- a_i^+ \in E(G)$ for $i=1,\dots,k$.

If $a_2^+ a_2^{--} \in E(G)$, then the cycle

$$a_2^{--} a_2^+ \vec{C}_\lambda a_3^- t_3^- \overleftarrow{C}_\lambda a_3 x_1 a_2 a_2^- t_3 \vec{C}_\lambda a_2^{--}$$

contradicts the choice of $C_\lambda$, so $a_2^+ a_2^{--} \notin E(G)$. This means we can consider $a_2^{--}$ as some kind of '$t$-vertex' for $C_\lambda$ with the reverse orientation. In particular $a_2^{--} a_3^+ \in E(G)$.

If $a_1^- a_2^{--} \in E(G)$, then the cycle

$$a_1^- a_2^{--} \overleftarrow{C}_\lambda a_1 x_1 a_2 a_2^- a_2^+ \vec{C}_\lambda a_1^-$$

contradicts the choice of $C_\lambda$, hence $a_1^- a_2^{--} \notin E(G)$. This means $t_1 \neq a_2^-$, which in particular means that $t_1$ is equal to, or precedes $a_2^{--}$ on the segment $a_1^+ \vec{C}_\lambda a_2^-$. We know $t_1 a_3^- \in E(G)$, so we can form the cycle

$$a_1^- t_1^- \overleftarrow{C}_\lambda a_1 x_1 a_3 x_2 a_2 a_2^- a_2^+ \vec{C}_\lambda a_3^- t_1 \vec{C}_\lambda a_2^{--} a_3^+ \vec{C}_\lambda a_1^-,$$

the final contradiction in this case.

*Case 2*: $k=2$.

As in Case 1 we can find a $(\vec{C}_\lambda, a_1^+, \lambda)$-subgraph $X'_1$ and a $(\vec{C}_\lambda, a_2^+, \lambda)$-subgraph $X'_2$ such that $X_0, X'_1$ and $X'_2$ are mutually remote. This shows $2 \leqslant |V(X_0)| \leqslant \frac{1}{3}(n-2)$. If $X_0$ is noncomplete, then two nonadjacent vertices $x_1$ and $x_2$ of $X_0$ satisfy $|N(x_1)\cup N(x_2)| \leqslant \frac{1}{3}(n-2) < \frac{1}{2}n$, a contradiction. Hence $X_0$ is complete.

Set $\mu+1 := \min\{i \mid G \text{ has a } D_i\text{-cycle containing } V(X_0)\}$, so that $\mu \geqslant \lambda$. Let $C_\mu$ be a $D_{\mu+1}$-cycle of $G$ containing $V(X_0)$ for which $\omega_\mu(G-V(C_\mu))$ is minimum, and let $Y_0$ be a component of $G-V(C_\mu)$ of order $\mu$. Fix an orientation on $C_\mu$. Let $b_1,\dots,b_l$ be the neighbors of $Y_0$, occurring on $\vec{C}_\mu$ in the order of their indices and such that $V(X_0) \subseteq b_1^+ \vec{C}_\mu b_2^-$ and $a_1$ precedes $a_2$ on $b_1 \vec{C}_\mu b_2$.

We distinguish two subcases.

*Case* 2.1: $l \geqslant 3$.

Choose $t_l$ in $b_l^+ \vec{C}_\mu b_1^-$ as in Case 1. Set $T_i := b_i^+ \vec{C}_\mu b_{i+1}^-(i=1,\ldots,l$, indices mod $l)$. As in Case 1, we can find $(\vec{C}_\mu, b_i^+, \mu)$-subgraphs $Y_i$ for $2 \leqslant i \leqslant l-1$ and a $(\vec{C}_\mu, t_l, \mu)$-subgraph $Y_l$ such that $Y_0, Y_2, \ldots, Y_l$ are mutually remote. Define $T_0 := V(G) - V(C_\mu)$ and $\mu_i := |V(Y_i) \cap V(C_\mu)|(i=2,\ldots,l)$.

Let $y \in V(Y_0)$ and $x \in V(X_0)$. Using observations analogous to those in Case 1 we obtain the following inequalities. We have

$$N(y) \subseteq (V(Y_0) - \{y\}) \cup \{b_1, \ldots, b_l\},$$

hence $|N(y)| \leqslant \mu - 1 + l$.

Because $t_l b_1^+ \notin E(G)$ we get

$$(N(x) \cap T_1) \cap (N(t_l) \cap T_1) \subseteq \begin{cases} \{a_2\}, & \text{if } a_1 \in \{b_1, b_1^+\}, \\ \{a_1, a_2\}, & \text{if } a_1 \notin \{b_1, b_1^+\}, \end{cases}$$

$$(N(x) \cap T_1) \cup (N(t_l) \cap T_1) \subseteq \begin{cases} T_1 - \{x\}, & \text{if } a_1 \in \{b_1, b_1^+\}, \\ T_1 - \{b_1^+, x\}, & \text{if } a_1 \notin \{b_1, b_1^+\}, \end{cases}$$

hence $|N(x) \cap T_1| + |N(t_l) \cap T_1| \leqslant |T_1|$.

Next we observe that $|N(x) \cap T_i| = 0$ for $i = 0, 2, 3, \ldots, l$.

Moreover, for $i = 2, \ldots, l-2$ we get

$$N(t_l) \cap T_i \subseteq T_i - (T_i \cap V(Y_i)),$$

so $|N(t_l) \cap T_i| \leqslant |T_i| - \mu_i$ for $i = 2, \ldots, l-2$.

Since $b_{l-1}^- t_l \notin E(G)$,

$$N(t_l) \cap T_{l-1} \subseteq T_{l-1} - ((T_{l-1} \cap V(Y_{l-1})) \cup \{b_{l-1}^-\}),$$

hence

$$|N(t_l) \cap T_{l-1}| \leqslant \begin{cases} |T_{l-1}| - \mu_{l-1} - 1, & \text{if } \mu_{l-1} \neq |T_{l-1}|, \\ |T_{l-1}| - \mu_{l-1}, & \text{if } \mu_{l-1} = |T_{l-1}|. \end{cases}$$

Because $N(t_l) \cap T_l \subseteq T_l - \{t_l\}$, we get $|N(t_l) \cap T_l| \leqslant |T_l| - 1$.

Finally, we have

$$N(t_l) \cap T_0 \subseteq T_0 - V(Y_0) - \bigcup_{i=2}^{l-1} (V(Y_i) - (V(Y_i) \cap V(C_\mu))),$$

so $|N(t_l) \cap T_0| \leqslant |T_0| - \mu - \sum_{i=2}^{l-1} (\mu - \mu_i)$.

Hence it follows that

$$\begin{aligned} n &\leqslant 2\mathrm{NC}(G) \\ &\leqslant |N(y) \cup N(x)| + |N(y) \cup N(t_l)| \\ &\leqslant |N(x) - N(Y_0)| + |N(t_l) - N(Y_0)| + 2(\mu - 1 + l) \end{aligned}$$

$$\leqslant |T_0|-\mu-\sum_{i=2}^{l-1}(\mu-\mu_i)+|T_1|+\sum_{i=2}^{l-2}(|T_i|-\mu_i)+|T_{l-1}|-\mu_{l-1}$$

$$+|T_l|-1+2(\mu-1+l)$$

$$=\sum_{i=0}^{l}|T_i|-\mu-(l-2)\mu-1+2\mu-2+2l$$

$$=n-l+3\mu-l\mu-3+2l$$

$$=n-l\mu+3\mu+l-3$$

$$=n-(l-3)(\mu-1).$$

This implies $l=3$, so that equality holds throughout. In particular, $\mu_2=|T_2|$ or, equivalently, $b_2^+\vec{C}_\mu b_3^- \subseteq V(Y_2)$. By symmetry arguments, a $(\overleftarrow{C}_\mu, b_1^-, \mu)$-subgraph $Y_1'$ satisfies $b_3^+\vec{C}_\mu b_1^- \subseteq V(Y_1')$. Let $K:=\{a_1,a_2,b_1,b_2,b_3\}$. Then

$$|N(x)\cup N(y)|=\tfrac{1}{2}n=\lambda+\mu-2+|K|$$

and

$$n\geqslant\lambda+3\mu+|K|\geqslant 2(\lambda+\mu)+|K|=n+4-|K|,$$

so that $|K|\geqslant 4$. Without loss of generality assume $a_2\neq b_2$. Then $t_3b_2^-\in E(G)$, and the cycle

$$b_1Qb_2\vec{C}_\mu t_3 b_2^-\overleftarrow{C}_\mu b_1,$$

where $b_1Qb_2$ is a path from $b_1$ to $b_2$ with all internal vertices in $Y_0$, contradicts the choice of $C_\mu$.

*Case* 2.2: $l=2$.

Obviously, $2\leqslant\lambda\leqslant\mu=|V(Y_0)|\leqslant\frac{1}{2}(n-4)$. If $Y_0$ is noncomplete, then two nonadjacent vertices $y_1$ and $y_2$ of $Y_0$ sastisfy

$$|N(y_1)\cup N(y_2)|\leqslant\tfrac{1}{2}(n-4)<\tfrac{1}{2}n,$$

a contradiction. Hence $Y_0$ is complete.

Let $K:=\{a_1,a_2,b_1,b_2\}$ and let $R$ be the subgraph of $G$ induced by $V(G)-(V(X_0)\cup V(Y_0)\cup K)$. With $x\in V(X_0)$ and $y\in V(Y_0)$, we have

$$\tfrac{1}{2}n\leqslant|N(x)\cup N(y)|\leqslant|V(X_0)|-1+|V(Y_0)|-1+|K|,$$

so that

$$|V(X_0)|+|V(Y_0)|+|K|\geqslant\tfrac{1}{2}n+2$$

and

$$|V(R)|\leqslant n-(\tfrac{1}{2}n+2)=\tfrac{1}{2}n-2.$$

If $\alpha(R)\geqslant 3$, then $\frac{1}{2}n\leqslant \mathrm{NC}(G)\leqslant\frac{1}{2}n-5+|K|\leqslant\frac{1}{2}n-1$, a contradiction. Hence $\alpha(R)\leqslant 2$.

Set $v+1:=\min\{i \mid G$ has a $D_i$-cycle containing $V(X_0)\cup V(Y_0)\}$, so that $v\geqslant\mu$. Let $C_v$ be a $D_{v+1}$-cycle of $G$ containing $V(X_0)\cup V(Y_0)$ for which $\omega_v(G-V(C_v))$ is minimum, and let $Z_0$ be a component of $G-V(C_v)$ of order $v$. Choose an orientation on $C_v$. Let $z\in V(Z_0)$. Then $(N(Z_0))^+\cup\{z\}$ is an independent set containing at most two vertices outside $R$. Because $\alpha(R)\leqslant 2$, this means $|N(Z_0)|\leqslant 3$. Moreover, if $|N(Z_0)|=3$, then $X_0$ and $Y_0$ are in different components of $C_v-N(Z_0)$.

*Case* 2.2.1: $|N(Z_0)|=3$.

The vertices $c_1,c_2,c_3\in N(Z_0)$ can be chosen in such a way that $V(X_0)\subseteq c_1^+\vec{C}_v c_2^-$ and $V(Y_0)\subseteq c_2^+\vec{C}_v c_3^-$. Without loss of generality assume $a_1$ precedes $a_2$ on $c_1\vec{C}_v c_2$, and $b_1$ precedes $b_2$ on $c_2\vec{C}_v c_3$. Now $\alpha(R)\leqslant 2$ implies that $Z_0$ is complete.

Define $L:=\{a_1,a_2,b_1,b_2,c_1,c_2,c_3\}$. Then $n-\lambda-\mu-v-|L|\geqslant v$, because $R-(L\cup V(Z_0))$ contains a $(\vec{C}_v,c_3^+,v)$-subgraph; hence $\lambda+\mu+2v\leqslant n-|L|$. Let $x\in V(X_0)$, $y\in V(Y_0)$ and $z\in V(Z_0)$. Then we have

$$\begin{aligned}
n&\leqslant 2\,\mathrm{NC}(G)\\
&\leqslant |N(x)\cup N(z)|+|N(y)\cup N(z)|\\
&\leqslant \lambda-1+v-1+|\{a_1,a_2,c_1,c_2,c_3\}|\\
&\quad +\mu-1+v-1+|\{b_1,b_2,c_1,c_2,c_3\}|\\
&=\lambda+\mu+2v-4+|\{a_1,a_2,c_1,c_2,c_3\}|+|\{b_1,b_2,c_1,c_2,c_3\}|\\
&\leqslant n-|L|-4+|\{a_1,a_2,c_1,c_2,c_3\}|+|\{b_1,b_2,c_1,c_2,c_3\}|\\
&=n-4-|\{a_1,c_1\}|-|\{a_2,b_1,c_2\}|-|\{b_2,c_3\}|\\
&\quad +|\{a_1,c_1\}|+|\{a_2,c_2\}|+1+|\{b_1,c_2\}|+|\{b_2,c_3\}|+1\\
&=n-2-|\{a_2,b_1,c_2\}|+|\{a_2,c_2\}|+|\{b_1,c_2\}|\\
&=n-1,
\end{aligned}$$

a contradiction.

*Case* 2.2.2: $|N(Z_0)|=2$.

Let $N(Z_0)=\{c_1,c_2\}$ and define $L:=\{a_1,a_2,b_1,b_2,c_1,c_2\}$. First suppose $V(X_0)$ and $V(Y_0)$ are in the same component of $C_v-N(Z_0)$. Without loss of generality, assume $V(X_0)\subseteq c_1^+\vec{C}_v c_2^-$, $V(Y_0)\subseteq c_1^+\vec{C}_v c_2^-$, $a_1$ precedes $a_2$ on $c_1\vec{C}_v c_2$, $a_2$ is equal to or precedes $b_1$ on $c_1\vec{C}_v c_2$, and $b_1$ precedes $b_2$ on $c_1\vec{C}_v c_2$.

Because $R-(L\cup V(Z_0))$ contains a $(\vec{C}_v,c_2^+,v)$-subgraph, we have $n-\lambda-\mu-v-|L|\geqslant v$, hence $\lambda+\mu+2v\leqslant n-|L|$. Let $x\in V(X_0)$, $y\in V(Y_0)$ and $z\in V(Z_0)$. Then we have

$$\begin{aligned}
n&\leqslant 2\,\mathrm{NC}(G)\\
&\leqslant |N(x)\cup N(z)|+|N(y)\cup N(z)|
\end{aligned}$$

$$\leqslant \lambda-1+\nu-1+|\{a_1,a_2,c_1,c_2\}|$$
$$+\mu-1+\nu-1+|\{b_1,b_2,c_1,c_2\}|$$
$$=\lambda+\mu+2\nu-4+|\{a_1,c_1\}|+2+|\{b_2,c_2\}|+2$$
$$\leqslant n-|L|+|\{a_1,c_1\}|+|\{b_2,c_2\}|$$
$$=n-|\{a_1,c_1\}|-|\{a_2,b_1\}|-|\{b_2,c_2\}|+|\{a_1,c_1\}|+|\{b_2,c_2\}|$$
$$=n-|\{a_2,b_1\}|$$
$$\leqslant n-1.$$

This gives a contradiction, so $V(X_0)$ and $V(Y_0)$ aren't in the same component of $C_\nu-N(Z_0)$.

Without loss of generality, assume $V(X_0)\subseteq c_1^+\vec{C}_\nu c_2^-$, $V(Y_0)\subseteq c_2^+\vec{C}_\nu c_1^-$, $a_1$ precedes $a_2$ on $c_1\vec{C}_\nu c_2$, and $b_2$ precedes $b_1$ on $c_2\vec{C}_\nu c_1$. Recall that

$$V(R):=V(G)-(V(X_0)\cup V(Y_0)\cup\{a_1,a_2,b_1,b_2\}).$$

If $V(R)=V(Z_0)\cup\{c_1,c_2\}$, then a contradiction with the choice of $C_\nu$ is avoided only if $G\in\mathcal{G}_n\cup\mathcal{H}_n\cup\mathcal{J}_n$.

Hence assume $V(R)\neq V(Z_0)\cup\{c_1,c_2\}$ and define $Z_1:=R-(V(Z_0)\cup\{c_1,c_2\})$. Since $\alpha(R)\leqslant 2, Z_1$ is complete. Obviously, the choice of $C_\nu$ implies

$$V(Z_1)\cap a_2\vec{C}_\nu c_2=\emptyset \quad\text{or}\quad V(Z_1)\cap b_1\vec{C}_\nu c_1=\emptyset,$$

and

$$V(Z_1)\cap c_2\vec{C}_\nu b_2=\emptyset \quad\text{or}\quad V(Z_1)\cap c_1\vec{C}_\nu a_1=\emptyset.$$

We distinguish two subcases.

*Case* 2.2.2.1: $V(Z_1)\cap V(C_\nu)\neq\emptyset$.

Without loss of generality, assume $\{c_2^+,c_2^-\}\cap V(Z_1)\neq\emptyset$. First suppose $c_2^-\in V(Z_1)$. By the choice of $C_\nu$ we must have $b_1=c_1^-$ or $b_1=c_1$. Moreover, if $b_1=c_1^-$, then $b_1\notin N(c_2^-)$.

We have $|V(Z_1)|=n-\lambda-\mu-\nu-|L|$ and $|L|=|\{a_1,b_1,c_1\}|+|\{b_2,c_2\}|+1$. Let $x\in V(X_0), y\in V(Y_0)$ and $z\in V(Z_0)$. Then

$$n\leqslant 2\mathrm{NC}(G)$$
$$\leqslant|N(x)\cup N(c_2^-)|+|N(y)\cup N(z)|$$
$$\leqslant|V(X_0)|-1+|V(Z_1)|-1+|\{a_1,a_2,b_2,c_1,c_2\}|$$
$$+|V(Y_0)|-1+|V(Z_0)|-1+|\{b_1,b_2,c_1,c_2\}|$$
$$=\lambda-1+n-\lambda-\mu-\nu-|L|-1+|\{a_1,c_1\}|+1+|\{b_2,c_2\}|$$
$$+\mu-1+\nu-1+|\{b_1,c_1\}|+|\{b_2,c_2\}|$$

$$=n-3-|\{a_1,b_1,c_1\}|-|\{b_2,c_2\}|-1+|\{a_1,c_1\}|+|\{b_1,c_1\}|$$
$$+2|\{b_2,c_2\}|$$
$$=n-4-|\{a_1,b_1,c_1\}|+|\{a_1,c_1\}|+|\{b_1,c_1\}|+|\{b_2,c_2\}|$$
$$=n-3+|\{b_2,c_2\}|$$
$$\leqslant n-1,$$

a contradiction.

If $c_2^- \notin V(Z_1)$, then $c_2^+ \in V(Z_1)$, and we reach a similar contradiction by considering

$$|N(y)\cup N(c_2^+)|+|N(x)\cup N(z)|.$$

*Case* 2.2.2.2: $V(Z_1)\cap V(C_v)=\emptyset$.

Assume $C_v$ is chosen longest subject to the restrictions imposed on $C_v$. The $Z_1$ has no consecutive neighbors on $C_v$. In particular, for $i=1,2, N(Z_1)$ contains at most one of the vertices $b_i$ and $c_i$. Let $x\in V(X_0), y\in V(Y_0), z\in V(Z_0)$ and $z_1\in V(Z_1)$. Then

$$n\leqslant 2\mathrm{NC}(G)$$
$$\leqslant |N(x)\cup N(z_1)|+|N(y)\cup N(z)|$$
$$\leqslant |V(X_0)|-1+|V(Z_1)|-1+|\{a_1,a_2,c_1,c_2\}|$$
$$+|V(Y_0)|-1+|V(Z_0)|-1+|\{b_1,b_2,c_1,c_2\}|$$
$$=n-4-|L|+|\{a_1,a_2,c_1,c_2\}|+|\{b_1,b_2,c_1,c_2\}|$$
$$=n-4+|\{c_1,c_2\}|$$
$$=n-2,$$

a contradiction. □

## 3. An analogue for traceable graphs

We close with an analogue of Theorem 7 for *traceable* graphs, i.e., graphs containing a Hamilton path.

Let

$$\mathscr{G}'_n:=\{G-v\mid G\in\mathscr{G}^*_{n+1} \text{ and } d_G(v)=n\},$$
$$\mathscr{H}'_n:=\{G-v\mid G\in\mathscr{H}^*_{n+1} \text{ and } d_G(v)=n\}.$$

Define

$$\mathscr{F}^\dagger_n:=\{G\mid G \text{ is connected, } |V(G)|=n \text{ and } \mathrm{NC}(G)\geqslant\tfrac{1}{2}(n-1)\},$$
$$\mathscr{G}^\dagger_n:=\{G\in\mathscr{F}^\dagger_n\mid G\leqslant G_1, \text{ for some } G_1\in\mathscr{G}'_n\},$$
$$\mathscr{H}^\dagger_n:=\{G\in\mathscr{F}^\dagger_n\mid G\leqslant G_1, \text{ for some } G_1\in\mathscr{H}'_n\}.$$

**Theorem 10.** *If $G$ is a connected graph of order $n$ and* $\mathrm{NC}(G) \geq \frac{1}{2}(n-1)$, *then either $G$ is traceable or* $G \in \mathscr{G}_n^\dagger \cup \mathscr{H}_n^\dagger$.

**Proof.** Let $G$ be connected and $\mathrm{NC}(G) \geq \frac{1}{2}(n-1)$. Consider the graph $G^+$ obtained from $G$ by adding a new vertex $v$ and joining $v$ to all vertices of $G$. It is clear that $G^+$ is 2-connected and

$$\mathrm{NC}(G^+) \geq \tfrac{1}{2}(n-1)+1 = \tfrac{1}{2}(n+1) = \tfrac{1}{2}|V(G^+)|.$$

From Theorem 7 and the fact that $G^+$ contains a vertex of degree $n$, we conclude that $G^+$ is hamiltonian, or $G^+ \in \mathscr{G}_{n+1} \cup \mathscr{H}_{n+1}$. It follows that $G$ is traceable, or $G \in \mathscr{G}_n^\dagger \cup \mathscr{H}_n^\dagger$. □

Theorem 10 generalizes analogues of Theorems 1 to 4 for traceable graphs and the following result.

**Corollary 11** (Faudree, Gould, Jacobson and Schelp [8]). *If $G$ is a 2-connected graph of order $n$ and* $\mathrm{NC}(G) \geq \frac{1}{2}(n-1)$, *then $G$ is traceable.*

**Proof.** All graphs in $\mathscr{G}_n^\dagger \cup \mathscr{H}_n^\dagger$ have connectivity 1. □

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