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Tandem fluid queues fed by homogeneous on–off sources

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Abstract

We consider a tandem fluid model with multiple consecutive buffers. The input of buffer $j+1$ is the output from buffer j, while the first buffer is fed by a, possibly infinite, number of independent homogeneous on–off sources. The sources have exponentially distributed silent periods and generally distributed active periods. Under the assumption that the input rate of one source is larger than the maximum output rate of the first buffer, we are able to characterize the output from each buffer. Due to this fact we find (i) an equation for the Laplace–Stieltjes transform of the marginal content distribution of any buffer $j \geqslant 2$, (ii) explicit expressions for corresponding moments, and (iii) an explicit expression for the correlation between two buffer contents, again from the second buffer on. These results make use of a key observation concerning the aggregate contents of several consecutive buffers. For the case in which the active periods of the sources are exponential, the Laplace–Stieltjes transform is inverted. If there is only one source, all results are also valid for the first buffer.

MSC: 60K25

Keywords: Tandem fluid queue; Buffer content; Output process; On–off source

1. Introduction

In the area of modern telecommunication systems, fluid queues are often used as burst scale models for multiplexers, see e.g. [16]. In such models, a fluid queue (or fluid buffer) receives input at a rate which is determined by some stochastic process $X(t)$. Often this process is integer-valued and the input rate is ic_0 at times when $X(t) = i$. In addition, the buffer leaks fluid at a fixed rate c_1 as long as it is not empty. The physical interpretation of such models is as follows. A multiplexer consists of a number of input lines, a buffer and an output line. The parameters c_0 and c_1 can be thought to be the transmission rates of each input line and of the output line, respectively. At time t, there are $X(t) \geqslant 0$ sources active, each of which is generating fluid (information), feeding this into the buffer at a constant rate c_0. Whenever the total input rate exceeds the capacity of the output line, fluid is temporarily stored in the buffer. As long as there is

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fluid in the buffer, the output line transmits at full capacity c_1.

The sources are often assumed to be on–off sources. When active, such a source generates fluid (at rate c_0), and when silent, no fluid is generated. The consecutive active and silent periods of each source constitute an alternating renewal process. Moreover, the sources are often assumed to be homogeneous and independent. In that case, the number of active sources $X(t)$ behaves like the number of customers in a certain queue. Assuming that the silent periods of the sources are exponential and the active periods have a general distribution, this queue is either a so-called M/G/N/N/N queue if the number of sources, N, is finite, or it is an M/G/∞ queue if $N = \infty$. Classical examples of these models can be found in [3,14], respectively. As an aside, we note that in some papers the fluid queue driven by an M/G/∞ queue is called an M/G/1 *gradual input* queue, see e.g. [6,13].

Although many papers consider these types of fluid queues (see e.g. the survey paper [4] and the references therein), there are hardly any *explicit* results concerning the buffer content distribution when the on times of the sources have a general distribution. For example, for the fluid queue driven by an M/G/∞ queue the mean buffer content seems to be known only when $c_0 = c_1$ (see [19,13]).

Besides the buffer content, it is important to characterize the output process, one reason being that this knowledge enables the analysis of a tandem system. Some results have been obtained in this area, one important result being that whenever $c_0 \geqslant c_1$, the output process looks like another on–off source with exponential silent periods and generally distributed active periods. Rubinovitch [17] and Cohen [6] found that if $c_0 = c_1$, then the active periods on the output line are distributed like busy periods in a certain M/G/1 queue. Recent studies [5,2] show that similar results are valid also in the case $c_0 > c_1$. However, when $c_0 < c_1$, the output process is much more complicated. In this case, the output has been characterized only for the model with exponential sources, see [1].

In this paper we use these results to consider a tandem fluid queue fed by a finite or an infinite number of independent homogeneous on–off sources with exponential silent periods and generally distributed active periods. Thus, the input rate to the first buffer is modulated by the appropriate queue length process $X(t)$, and the output from buffer j is the input to buffer $j+1$. In view of the previous paragraph it is no surprise that we assume $c_0 \geqslant c_1$, leaving the opposite case for future research. As an aside, we remark that this tandem fluid queue is essentially different from ordinary tandem queues, where each workload process has jumps. Moreover, the sizes of these jumps are all random, whereas in our model all randomness relates to the behaviour of the sources, the network itself being deterministic.

Kella and Whitt [9] considered a tandem fluid model where the flow between consecutive buffers is deterministic (as in our model) but the input flow to the first buffer is instantaneous. They assumed that the input process is a non-decreasing Lévy process. Their main results concern the case where the input process is a compound Poisson process. This can be seen as a limiting case of our model with a single source ($N = 1$): let the rate c_0 tend to ∞ and the active periods tend to 0 in such a way that the burst sizes (i.e., the product of c_0 and the length of an active period) converge to a proper random variable. In some more recent papers, see e.g. [10,11], Kella considers more general tandem fluid networks, also with instantaneous input and hence different from ours. In [12], some Markov-modulated fluid networks are considered. For a particular model in this setting, namely a two-node tandem fluid queue driven by one on–off source with exponential on times and off times, the joint content distribution was found explicitly in [15,18].

Our main results in this paper are as follows. We show that the output from each buffer in the tandem looks like an on–off source with exponential off times and on times distributed as busy periods in an M/G/1 queue. Due to this fact, we find (i) an implicit equation for the Laplace–Stieltjes transform of the marginal content distribution of any buffer $j \geqslant 2$, (ii) explicit expressions for corresponding moments, and (iii) an explicit expression for the correlation between two buffer contents, again from the second buffer on. For the case in which the active periods of the sources are exponential, the Laplace–Stieltjes transform is inverted, as in [18]. If there is only one source, all results are also valid for the first buffer.

The paper is organized as follows. First, in Section 2, we introduce the tandem fluid queue and make some key observations that hold under rather general conditions. The most important observation gives a

relation between the behaviour of the aggregate contents of several consecutive buffers and the behaviour of the content of a single buffer in a related tandem system. In Section 3, we present some (known) results concerning the model with a single source and a single buffer. These results serve as a basis for the main results which are presented in Section 4, where we consider the tandem fluid queues driven by the M/G/N/N/N and M/G/∞ queues.

2. Key observations

Consider a tandem fluid queue driven by a stochastic process $X(t)$. In later sections we will further specify this process, but since the observations we make in this section hold in a more general sense, we will only assume here that $X(t)$ takes values in $\{0,1,\ldots,N\}$ for some $N>0$ (possibly $N=\infty$). Assume that there are M fluid buffers connected in series. Let c_j denote the leak rate (i.e. the maximum output rate) of the jth buffer. The output from buffer j is the input to buffer $j+1$. The input rate to the first buffer is $X(t)c_0$.

We denote the content of buffer j at time t by $Z_j(t)$. We assume that the system is stable, and let Z_j be distributed according to the limiting distribution of $Z_j(t)$ as $t\to\infty$. Since we are interested in this limiting behaviour, we may assume that $Z_j(0)=0$ for all j, and we will henceforth do so.

We assume, without loss of generality, that

$$Nc_0 > c_1 > c_2 > \cdots > c_M. \tag{2.1}$$

When this condition does not hold, one or more buffers will always be empty in stationarity, so that these buffers can be removed from the tandem (ensuring validity of (2.1) in the modified system) before employing the analysis in this paper. We can now make the following observation.

Proposition 2.1. *Consider the tandem fluid queue model described above. Let $i<j\leqslant M$. Then, for all $t\geqslant 0$,*

$$Z_i(t)>0 \Rightarrow Z_j(t)>0.$$

Proof. Suppose $Z_i(t)>0$ for some $i<M$ and $t>0$. Define $t_0=\sup\{s<t\,|\,Z_i(s)=0\}$. Then buffer $i+1$ has experienced an inflow at rate c_i during the interval $(t_0,t]$. Since $c_i>c_{i+1}$, this has resulted in an increase of the fluid level in buffer $i+1$, so that $Z_{i+1}(t)>0$. The proposition now follows by induction. □

Finally, if $N>1$, we assume that

$$c_0 \geqslant c_1. \tag{2.2}$$

This is a rather restrictive assumption, but it guarantees that the output from the first buffer looks like another on–off source.

Under Assumptions (2.1) and (2.2), it now follows formally that the evolutions of the processes $Z_j(t)$ are given by

$$Z_1(t)=\int_0^t (c_0X(u)-c_1\mathbf{1}\{Z_1(u)>0\})\,\mathrm{d}u, \tag{2.3}$$

$$Z_j(t)=\int_0^t (c_{j-1}\mathbf{1}\{Z_{j-1}(u)>0\} - c_j\mathbf{1}\{Z_j(u)>0\})\,\mathrm{d}u, \quad j\geqslant 2, \tag{2.4}$$

where $\mathbf{1}\{A\}$ is the indicator function of the event $\{A\}$. We now present another observation, characterizing the distribution of a sum of the contents of consecutive buffers.

Corollary 2.2. *Consider the tandem fluid queue model described above. Let $j\geqslant i\geqslant 1$. Then*

$$Z_i+Z_{i+1}+\cdots+Z_j \sim \tilde{Z}_i,$$

where $\tilde{Z}_i$ refers to the stationary version of the content of buffer i in a modified tandem fluid queue model with i buffers, where the input rate into the first buffer is modulated by $X(t)$ in the same way as before, but the leak rates of the buffers are $c_1,c_2,\ldots,c_{i-1},c_j$.

Proof. We couple the original and the modified tandem system such that they are regulated by the same process $X(t)$. Assume first that $i\geqslant 2$. For the modified system, we find that $\tilde{Z}_i(t)$ satisfies

$$\tilde{Z}_i(t)=\int_0^t (c_{i-1}\mathbf{1}\{\tilde{Z}_{i-1}(u)>0\} - c_j\mathbf{1}\{\tilde{Z}_i(u)>0\})\,\mathrm{d}u. \tag{2.5}$$

In the original system, we find, by summing (2.4) for $k = i, \ldots, j$, that

$$Z_{ij}(t) \equiv Z_i(t) + Z_{i+1}(t) + \cdots + Z_j(t), \tag{2.6}$$

satisfies

$$Z_{ij}(t) = \int_0^t (c_{i-1}\mathbf{1}\{Z_{i-1}(u) > 0\} - c_j\mathbf{1}\{Z_j(u) > 0\})\,\mathrm{d}u. \tag{2.7}$$

By applying Proposition 2.1, we can see that $Z_{ij}(u) > 0$ (that is, $Z_k(u) > 0$ for some $k = i, \ldots, j,$) implies $Z_j(u) > 0$, while clearly also the converse is true. Therefore we can replace $\mathbf{1}\{Z_j(u) > 0\}$ by $\mathbf{1}\{Z_{ij}(u) > 0\}$ in (2.7). It follows that $\tilde{Z}_i(t) = Z_{ij}(t)$ for all $t \geqslant 0$. A similar reasoning holds when $i = 1$, replacing $c_{i-1}\mathbf{1}\{Z_{i-1}(u) > 0\}$ by $c_0X(u)$ in both (2.5) and (2.7). Hence the proposition follows. □

Notice in particular that, by taking $i = 1$, we find that $Z_1 + \cdots + Z_j$ is distributed as the content of a fluid buffer with leak rate c_j and driven by $X(t)$.

3. Preliminaries: single source, single buffer

In this section we present some known results concerning the characterization of the output process and the buffer content distribution in a system with only one source and one buffer. These results will be utilized later in this paper.

Consider a fluid queue driven by the process $X(t)$, where $X(t)$ is now taken to be the number of customers present in an M/G/1/1/1 queue. In other words, we make the assumption that there is a single on–off source, with exponentially distributed silent periods and generally distributed active periods. Let S_0 and A_0 denote a typical silent period and a typical active period of the source, respectively. We denote $\lambda = 1/E[S_0]$, $\beta(\theta) = E[\mathrm{e}^{-\theta A_0}]$, and $\beta_k = E[A_0^k]$. Following the notation of Section 2, we denote the output rate of the source by c_0 and the leak rate of the buffer by c_1. Since there is only one source, we may assume that

$$c_0 > c_1. \tag{3.1}$$

3.1. Output from the buffer

Under Assumption (3.1) the output from the buffer looks like another on–off source: when non-empty, the buffer is leaking out with rate c_1, and when empty, no fluid flows out. Thus, the empty periods (non-empty periods) of the buffer are the same as the silent periods (active periods) of the output rate process. This fluid system is stable if and only if $c_0\beta_1 < c_1(\beta_1 + 1/\lambda)$, or equivalently, if

$$\rho \equiv \frac{c_0}{c_1}\frac{\lambda\beta_1}{1 + \lambda\beta_1} < 1. \tag{3.2}$$

Let S_1 and A_1 denote a typical empty period and a typical non-empty period of the buffer, respectively. It is easy to see that empty periods are independently and exponentially distributed with mean

$$E[S_1] = 1/\lambda. \tag{3.3}$$

The non-empty periods are characterized by the following lemma. For the proof, see e.g. Proposition 1 in [2].

Lemma 3.1. *Non-empty periods A_1 are distributed as busy periods in an* M/G/1 *queue with arrival rate $\lambda(c_0 - c_1)/c_0$ and Laplace–Stieltjes transform of the service time distribution given by $\beta(\theta c_0/c_1)$. In addition, this* M/G/1 *queue is stable if and only if $\rho < 1$.*

3.2. Content of the buffer

We now turn to the stationary distribution of the content of the buffer. Let Z denote the (stationary) buffer content. First we note that, under the assumption that $\rho < 1$, the buffer content process is regenerative with cycles consisting of a non-empty period and the following empty period. It follows immediately that

$$P\{Z > 0\} = \frac{E[A_1]}{E[A_1] + E[S_1]} = \rho. \tag{3.4}$$

The following lemma is due to Corollary 3 in Kella and Whitt [8].

Lemma 3.2. *Let V denote the stationary version of the workload in an* M/G/1 *queue with arrival rate λ/c_1 and Laplace–Stieltjes transform of the service time distribution given by $\beta((c_0 - c_1)\theta)$. This queue is stable if and only if $\rho < 1$. In the stable case, for all $z \geqslant 0$,*

$$P\{Z > z\} = \gamma P\{V > z\},$$

where

$$\gamma \equiv \frac{P\{Z>0\}}{P\{V>0\}} = \frac{c_0}{c_0-c_1}\frac{1}{1+\lambda\beta_1}. \tag{3.5}$$

Corollary 3.3. *If* $\rho<1$, *then*

$$E[e^{-\theta Z}] = 1-\gamma+\gamma\frac{(c_1-(c_0-c_1)\lambda\beta_1)\theta}{c_1\theta-\lambda+\lambda\beta((c_0-c_1)\theta)}.$$

Proof. From Lemma 3.2 it follows that $E[e^{-\theta Z}]=1-\gamma+\gamma E[e^{-\theta V}]$. Then apply the Pollaczek–Khintchine formula for $E[e^{-\theta V}]$. □

Corollary 3.4. *If* $\rho<1$, *then*

$$E[Z] = \frac{\beta_2}{2\beta_1}\frac{c_0-c_1}{1+\lambda\beta_1}\frac{\rho}{1-\rho},$$

$$E[Z^2] = \frac{\beta_3}{3\beta_1}\frac{(c_0-c_1)^2}{1+\lambda\beta_1}\frac{\rho}{1-\rho} + 2\left(\frac{\beta_2}{2\beta_1}\right)^2\frac{(c_0-c_1)^3}{c_0(1+\lambda\beta_1)}\left(\frac{\rho}{1-\rho}\right)^2.$$

Proof. From Lemma 3.2 it follows that $E[Z^k]=\gamma E[V^k]$ for all $k\geqslant 1$. Then apply the Pollaczek–Khintchine formulas for the first moments of V. □

4. Main results: multiple sources, multiple buffers

In this section we present new results concerning a tandem fluid queue fed by multiple on–off sources. First, we characterize the output from each of the buffers (Theorem 4.1). Then, we derive an implicit formula for the Laplace–Stieltjes transform (Theorem 4.4) and explicit expressions for the first moments (Theorem 4.5) of the marginal content distribution of any buffer $j\geqslant 2$. The Laplace–Stieltjes transform is found explicitly and inverted for the case in which the active periods of the sources are exponentially distributed (Theorem 4.6). Furthermore, we obtain an explicit expression for the correlation coefficient between the contents of any two buffers $i,j\geqslant 2$ (Theorem 4.7). The results are formulated for a finite number of sources. In Section 4.5 we explain how they extend to the case $N=\infty$. If there is only one source, all the results also hold for the first buffer ($j=1$).

We take the process $X(t)$ that modulates the tandem fluid queue now to be the number of customers in an M/G/N/N/N queue ($N<\infty$). So, there is a series of fluid buffers, the first of which is fed by N-independent homogeneous on–off sources with exponential silent periods and generally distributed active periods. The notation used to describe these sources is the same as in Section 3. For the notation used to describe the tandem system, we refer to Section 2. In addition, we denote, for all k,

$$\tilde{\beta}_k \equiv E[(c_0A_0)^k] = \beta_k c_0^k. \tag{4.1}$$

Note that c_0A_0 is the amount of fluid generated in an active period A_0 of a source. As in Section 2, we make Assumptions (2.1) and (2.2) concerning the buffer rates. Furthermore, we define, for all $j\geqslant 1$,

$$\rho_j = \frac{\bar{c}}{c_j}, \qquad \kappa_j = \frac{\bar{c}}{c_j-\bar{c}}, \tag{4.2}$$

where $\bar{c}$ is the mean input rate to the first buffer,

$$\bar{c} = Nc_0\frac{\lambda\beta_1}{1+\lambda\beta_1}. \tag{4.3}$$

Notice that κ_j can also be given as

$$\kappa_j = \frac{\rho_j}{1-\rho_j}, \qquad j\geqslant 1. \tag{4.4}$$

In addition, we let

$$\kappa_0 = \frac{\bar{c}}{Nc_0-\bar{c}}. \tag{4.5}$$

Notice that, if $N=1$, then $\kappa_0=\lambda\beta_1$.

4.1. Output from the jth buffer

As mentioned in Section 2, under Assumptions (2.1) and (2.2), the output from any buffer $j\geqslant 1$ looks like an on–off source: the empty periods (non-empty periods) of the buffer are the same as the silent periods (active periods) of the corresponding output rate process. Since the first buffer can be empty only if all the N sources are silent, and buffer $j+1$ can be empty only if buffer j is empty (by Proposition 2.1), we deduce that, for any $j\geqslant 1$, the empty periods S_j of buffer j are independently and exponentially distributed with mean

$$E[S_j] = 1/(N\lambda). \tag{4.6}$$

The non-empty periods A_j are characterized in the following theorem.

Theorem 4.1. *Let $j \geqslant 1$. Non-empty periods A_j are distributed as busy periods in an* M/G/1 *queue with arrival rate $\lambda(Nc_0 - c_j)/c_0$ and Laplace–Stieltjes transform of the service time distribution given by $\beta(\theta c_0/c_j)$. In addition, this* M/G/1 *queue is stable if and only if $\rho_j < 1$.*

Proof. For $j=1$, the theorem is proved in [5,2]. However, due to Corollary 2.2 (take $i=1$), the result immediately generalizes to any $j \geqslant 1$. □

We get the following corollaries for the Laplace–Stieltjes transform $\alpha_j(\theta) = E[\mathrm{e}^{-\theta A_j}]$ and the first moments of A_j, which will be needed for Theorem 4.5.

Corollary 4.2. *Let $j \geqslant 1$. If $\rho_j < 1$, then $\alpha_j(\theta)$ is the unique solution, with the property $|\alpha_j(\theta)| \leqslant 1$, of the following implicit equation*:

$$x = \beta\left(\frac{c_0}{c_j}\left[\theta + \lambda\left(N - \frac{c_j}{c_0}\right)(1-x)\right]\right), \quad Re\,\theta \geqslant 0.$$

Corollary 4.3. *Let $j \geqslant 1$. If $\rho_j < 1$, then*

$$E[A_j] = \frac{\kappa_j}{N\lambda},$$

$$E[A_j^2] = \beta_2 \frac{c_j}{c_0}\left(\frac{\kappa_j}{N\lambda\beta_1}\right)^3,$$

$$E[A_j^3] = \beta_3 \frac{c_j}{c_0}\left(\frac{\kappa_j}{N\lambda\beta_1}\right)^4 + 3\lambda\beta_2^2 \frac{c_j}{c_0}\left(N - \frac{c_j}{c_0}\right)\left(\frac{\kappa_j}{N\lambda\beta_1}\right)^5.$$

4.2. Content of the jth buffer

Now we consider the (stationary) buffer content Z_j. We note that results concerning the first buffer ($j=1$) have only been derived for $N=1$, see Section 3 (unless the on times of the sources are exponentially distributed, see [3]). Therefore, we assume in this subsection that $j \geqslant 2$, or that $j=1$ and $N=1$.

By Theorem 4.1, the input to buffer j looks like an on–off source with exponential silent periods and generally distributed active periods. This implies that we may apply the results of Section 3 to get formulas for the Laplace–Stieltjes transform and the first moments of Z_j. All we have to do, is replace c_0, c_1, S_0, S_1, A_0 and A_1 (and hence λ, $\beta(\cdot)$ and β_k) by c_{j-1}, c_j, S_{j-1}, S_j, A_{j-1}, and A_j (and $N\lambda$, $\alpha_{j-1}(\cdot)$ and $E[A_{j-1}^k]$), respectively.

First, the stability condition given in Lemma 3.2 translates into

$$\frac{c_{j-1}}{c_j}\frac{N\lambda E[A_{j-1}]}{1 + N\lambda E[A_{j-1}]} < 1,$$

which, using Corollary 4.3 and the fact that $\rho_{j-1}c_{j-1} = \rho_j c_j$, is easilly seen to be equivalent to $\rho_j < 1$. Note that the interpretation of ρ_j is given by

$$P\{Z_j > 0\} = \frac{E[A_j]}{E[A_j] + E[S_j]} = \rho_j.$$

Theorem 4.4. *Let $j \geqslant 2$, or let $j=1$ and $N=1$. If $\rho_j < 1$, then*

$$E[\mathrm{e}^{-\theta Z_j}] = 1 - \gamma_j + \gamma_j \frac{(c_j - (c_{j-1} - c_j)\kappa_{j-1})\theta}{c_j\theta - N\lambda + N\lambda\alpha_{j-1}((c_{j-1} - c_j)\theta)},$$

where α_{j-1} is given in Corollary 4.2 for $j \geqslant 2$, $\alpha_0(\cdot) = \beta(\cdot)$, and

$$\gamma_j \equiv \frac{P\{Z_j > 0\}}{P\{V_j > 0\}} = \frac{c_{j-1}}{c_{j-1} - c_j}(1 - \rho_{j-1}). \tag{4.7}$$

Proof. In both cases, the result is a consequence of Corollary 3.3. For $j=1$ and $N=1$, it is immediate, while for $j \geqslant 2$ we apply the substitutions as mentioned. □

Theorem 4.5. *Let $j \geqslant 2$, or let $j=1$ and $N=1$. If $\rho_j < 1$, then*

$$E[Z_j] = \frac{\tilde{\beta}_2}{2\tilde{\beta}_1}\left(\frac{1}{1+\lambda\beta_1}\right)^2(\kappa_j - \kappa_{j-1}),$$

$$E[Z_j^2] = \frac{\tilde{\beta}_3}{3\tilde{\beta}_1}\left(\frac{1}{1+\lambda\beta_1}\right)^3\frac{(\kappa_j - \kappa_{j-1})^2}{\kappa_j} + 2\left(\frac{\tilde{\beta}_2}{2\tilde{\beta}_1}\right)^2\left(\frac{1}{1+\lambda\beta_1}\right)^4 \times \frac{(\kappa_j - \kappa_{j-1})^2(\kappa_{j-1} + \kappa_j - 2\lambda\beta_1)}{\kappa_j}.$$

Proof. For $j = 1$ and $N = 1$, the result can easily be verified using Corollary 3.4. Now assume $j \geqslant 2$. We invoke Corollary 3.4 and substitute the moments of A_{j-1} using Corollary 4.3. After strenuous rewriting and using (4.1) we find the claimed results. □

We notice that, apart from the moments, $\tilde{\beta}_k$, these expressions depend on the active period distribution only through the mean.

4.3. Content of the jth buffer when the active periods are exponential

When we assume that not only the silent periods but also the active periods of the sources are distributed according to an exponential distribution, with intensity μ say ($=1/\beta_1$), it is possible to invert the transform in Theorem 4.4.

Theorem 4.6. *Let $j \geqslant 2$, or let $j = 1$ and $N = 1$. Furthermore, let the active periods of the sources be exponentially distributed with intensity μ. If $\rho_j < 1$, then*

$$P\{Z_j \in (y, y + \mathrm{d}y)\} = (1 - \rho_j)\delta_0(y)\,\mathrm{d}y + (1 - \rho_j)\mathrm{e}^{-\eta_j y} \times \left(\frac{c_{j-1}}{c_j} \frac{N\lambda}{c_{j-1} - c_j} - \frac{1}{2} \frac{Nc_0\omega_j}{Nc_0 - c_j} \times \int_0^y \mathrm{e}^{-(\Theta_j - \eta_j)u} \frac{I_1(u\sqrt{\omega})}{u\sqrt{\omega}}\,\mathrm{d}u \right) \mathrm{d}y,$$

where δ_0 denotes the Dirac measure at 0 and I_1 the modified Bessel function of the first kind of order 1. Furthermore, ρ_j is defined as in (4.2) and the other parameters are given by

$$\Theta_j = \frac{N\lambda + (\mu - \lambda)c_{j-1}/c_0}{c_{j-1} - c_j}, \tag{4.8}$$

$$\eta_j = \frac{N\mu}{Nc_0 - c_j} - \frac{N\lambda}{c_j}, \tag{4.9}$$

$$\omega_j = \frac{4\lambda\mu c_{j-1}(Nc_0 - c_{j-1})}{c_0^2(c_{j-1} - c_j)^2}. \tag{4.10}$$

Proof. Since A_{j-1} is now distributed as the busy period of an M/M/1 system, its Laplace–Stieltjes transform $\alpha_{j-1}(\theta)$ can be found explicitly. After putting things together and rewriting, we obtain

$$E[\mathrm{e}^{-(\theta - \Theta_j)Z_j}] = 1 - \rho_j + (1 - \rho_j)\left(\frac{c_{j-1}}{c_j} \frac{N\lambda}{c_{j-1} - c_j} \frac{1}{\theta - (\Theta_j - \eta_j)} - \frac{1}{2} \frac{Nc_0}{Nc_0 - c_j} \frac{\theta - \sqrt{\theta^2 - \omega_j}}{\theta - (\Theta_j - \eta_j)} \right). \tag{4.11}$$

Inversion can be done by using (28) in [7, p. 235]. □

We mention that the density of Z_1 with $N = 1$ is exponential, since $\omega_1 = 0$ in this case.

4.4. Correlation between buffer contents

Returning to the case where the active periods of the sources are generally distributed, we now consider the correlation between the buffer contents Z_i and Z_j for some $j > i$. As before, we exclude the first buffer from this analysis in the general case, unless $N = 1$. Throughout this subsection, we assume that $\beta_k < \infty$ for $k = 1, 2, 3$.

Theorem 4.7. *Let $j > 1 \geqslant 2$, or let $j > i = 1$ and $N = 1$. If $\rho_j < 1$, then*

$$\mathrm{Corr}[Z_i, Z_j] = \frac{\sqrt{\kappa_i}}{\kappa_{j-1}\sqrt{\kappa_j}} \times \frac{b(\kappa_{i-1} + \kappa_i) + \kappa_{i-1}^2 + \kappa_{i-1}\kappa_i + \kappa_i^2}{\sqrt{(2b + 2\kappa_{i-1} + \kappa_i)(2b + 2\kappa_{j-1} + \kappa_j)}}, \tag{4.12}$$

where the constant b is defined as follows:

$$b = \frac{2\beta_3\beta_1}{3\beta_2^2}(1 + \lambda\beta_1) - 2\lambda\beta_1. \tag{4.13}$$

Proof. By definition,

$$\mathrm{Corr}[Z_i, Z_j] = \frac{\mathrm{Cov}[Z_i, Z_j]}{\sqrt{\mathrm{Var}[Z_i]\mathrm{Var}[Z_j]}},$$

where $\mathrm{Cov}[\cdot,\cdot]$ and $\mathrm{Var}[\cdot]$ refer to covariance and variance, respectively. The variances $\mathrm{Var}[Z_i]$ and $\mathrm{Var}[Z_j]$ can be derived from the formulas presented in Theorem 4.5. As before, we let $Z_{ij} = Z_i + \cdots + Z_j$, with the convention that $Z_{ij} = 0$ if $j < i$. Since

$$2Z_iZ_j = Z_{ij}^2 + Z_{i+1,j-1}^2 - Z_{i,j-1}^2 - Z_{i+1,j}^2,$$

we obtain

$$2\,\mathrm{Cov}[Z_i, Z_j] = \mathrm{Var}[Z_{ij}] + \mathrm{Var}[Z_{i+1,j-1}] - \mathrm{Var}[Z_{i,j-1}] - \mathrm{Var}[Z_{i+1,j}].$$

By Corollary 2.2, all these variances can also be derived from the formulas presented in Theorem 4.5. After some straightforward manipulations the claimed result can be obtained. □

Proposition 4.8. *Let $j > i \geqslant 2$, or let $j > i = 1$ and $N = 1$. If $\rho_j < 1$, then*

$$\mathrm{Corr}[Z_i, Z_j] > 0.$$

Proof. In this proof we make use of the fact that

$$\kappa_M > \kappa_{M-1} > \cdots > \kappa_1 > 0. \tag{4.14}$$

In fact, by (2.1) we can show that $\kappa_i > \lambda\beta_1$ for any $i \geqslant 1$. By further taking into account that $\beta_3\beta_1 \geqslant \beta_2^2$ for any active period distribution, we get the following lower bound for b:

$$b = \frac{2\beta_3\beta_1}{3\beta_2^2}(1 + \lambda\beta_1) - 2\lambda\beta_1 \geqslant \frac{2}{3} - \frac{4}{3}\kappa_{i-1}, \quad i = 1, \ldots, M.$$

It now follows that

$$\begin{aligned} b(\kappa_{i-1} + \kappa_i) + \kappa_{i-1}^2 + \kappa_{i-1}\kappa_i + \kappa_i^2 &\geqslant \tfrac{2}{3}(\kappa_{i-1} + \kappa_i) - \tfrac{4}{3}\kappa_{i-1}(\kappa_{i-1} + \kappa_i) \\ &\quad + \kappa_i^2 + \kappa_{i-1}(\kappa_{i-1} + \kappa_i) \\ &\geqslant \tfrac{2}{3}(\kappa_{i-1} + \kappa_i) - \tfrac{2}{3}\kappa_i^2 + \kappa_i^2 \geqslant 0, \end{aligned}$$

so that $\mathrm{Corr}[Z_i, Z_j]$ clearly is positive as well. □

Consider now two consecutive buffers i and $i+1$ ($i \geqslant 2$, or $i \geqslant 1$ and $N = 1$). When the source characteristics are fixed, the correlation coefficient $\mathrm{Corr}[Z_i, Z_{i+1}]$ depends just on the rates c_{i-1}, c_i and c_{i+1}. We now fix c_{i-1} and c_i for the time being (such that $c_{i-1} > c_i$), and vary c_{i+1} between $\bar{c}$ and c_i. It can be shown that

(i) $\mathrm{Corr}[Z_i, Z_{i+1}] \to 0$ whenever $c_{i+1} \to \bar{c}$, and
(ii) $\mathrm{Corr}[Z_i, Z_{i+1}]$ is increasing and continuous as a function of c_{i+1}.

Let then $f(c_i)$ denote the limit of $\mathrm{Corr}[Z_i, Z_{i+1}]$ as $c_{i+1} \to c_i$. It can be shown that $f(c_i)$ is continuous on the interval $(\bar{c}, c_{i-1})$, with $f(\bar{c}) = 1/\sqrt{3}$ and $f(c_{i-1}) = 1$, while in between $f(c_i) \geqslant 1/2$. It follows that, with c_{i-1} fixed, the correlation coefficient takes all values from 0 to 1, as c_i and c_{i+1} vary in the range $c_{i-1} > c_i > c_{i+1} > \bar{c}$. In addition, we find that, in any neighbourhood of the point $(c_i, c_{i+1}) = (\bar{c}, \bar{c})$, the correlation coefficient takes all values from 0 to $1/\sqrt{3}$. As an aside, we remark that in corresponding instantaneous input models, the correlation between the first two buffers is limited to the interval $(0, 1/\sqrt{3})$, cf. [9–11].

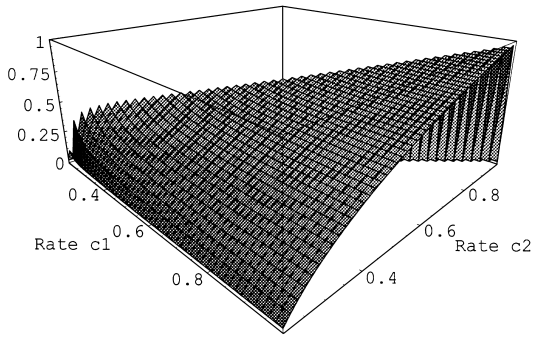

Fig. 1. $\mathrm{Corr}[Z_1, Z_2]$ as a function of the rates c_1 and c_2.

4.4.1. Numerical example

As a numerical example we consider a tandem fluid model with one source and two buffers. Silent and active periods of the source are exponentially distributed with means $1/\lambda = 1/4$ and $\beta_1 = 1$. The source rate and the mean input rate are $c_0 = 1$ and $\bar{c} = 1/5$, respectively. Thus, we have the following constraints for the leak rates of the two buffers: $1 > c_1 > c_2 > 1/5$. The correlation coefficient between the buffer contents, $\mathrm{Corr}[Z_1, Z_2]$, is plotted as a function of the rates c_1 and c_2 in Fig. 1. A warning might be in place here, since it is not clear from this figure that the correlation coefficient takes all values from 0 to $1/\sqrt{3}$ in any neighbourhood of the point $(c_1, c_2) = (\bar{c}, \bar{c})$. In Fig. 2, the function $f(c_1)$, that is the limit of $\mathrm{Corr}[Z_i, Z_{i+1}]$ as $c_2 \to c_1$, is plotted for this particular example.

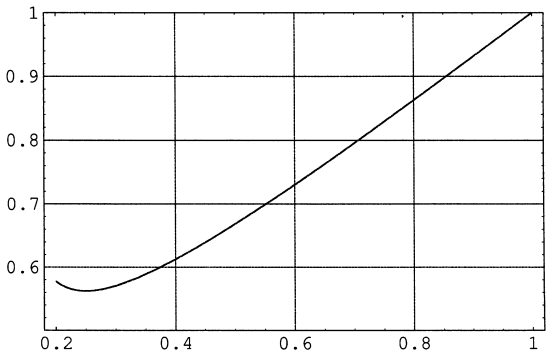

Fig. 2. The limit $f(c_1)$ of $\mathrm{Corr}[Z_1, Z_2]$ as a function of the rate c_1.

4.5. Infinite number of sources

Consider then the tandem fluid queue driven by the M/G/∞ queue ($N = \infty$). So, there is a series of fluid buffers, the first of which is fed by random bursts arriving according to a Poisson process with intensity ν. Note that bursts in this model play the same role as the active periods in the previous model with finitely many sources. Clearly, the present model can be seen as a limiting case of the previous one: let $N \to \infty$ and $\lambda \to 0$ in such a way that $N\lambda \to \nu$.

The central observation here is that, under Assumptions (2.1) and (2.2), the output from each buffer has a similar characterization as before: the empty periods S_j of buffer j are independently and exponentially distributed with mean $E[S_j] = 1/\nu$, and the non-empty periods A_j are distributed as busy periods in an M/G/1 queue with arrival rate ν and Laplace–Stieltjes transform of the service time distribution given by $\beta(\theta c_0/c_j)$. For $j = 1$, these results were proved in [5]. However, again due to Corollary 2.2 (take $i = 1$), the result immediately generalizes to any $j \geqslant 1$.

It follows that all the other results derived earlier in this section have their counterparts in the present model: just replace λ by ν/N and then let $N \to \infty$.

4.6. Tandem fluid model presented in [9]

As mentioned in Section 1, the tandem fluid model presented in [9] can be seen as a limiting case of our model with a single source ($N = 1$): the rate c_0 tends to ∞ and the active periods tend to 0. It follows that, for example, the correlation between the contents of any two buffers i and j (in the model of [9]) has the same formula (4.12) as in our model but with $b = 2\tilde{\beta}_3\tilde{\beta}_1/(3\tilde{\beta}_2^2)$. The κ's are defined as in (4.2) with $\bar{c} = \lambda\tilde{\beta}_1$ (and $\kappa_0 = 0$). As mentioned in [9], the correlation between the first two buffers is always in the interval $(0, 1/\sqrt{3})$. However, the correlation between two consecutive buffers from the second buffer on can have any value from 0 to 1.

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