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Simplices by Point-Sliding and the Yamnitsky-Levin Algorithm

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Abstract: Yamnitsky and Levin proposed a variant of Khachiyan's ellopsoid method for testing feasibility of systems of linear inequalities that also runs in polynomial time but uses simplices instead of ellipsoids. Starting with the n-simplex S and the half-space $\{x|a^T x \leq \beta\}$, the algorithm finds a simplex S_{YL} of small volume that encloses $S \cap \{x|a^T x \leq \beta\}$. We interpret S_{YL} as a simplex obtainable by point-sliding and show that the smallest such simplex can be determined by minimizing a simple strictly convex function. We furthermore discuss some numerical results. The results suggest that the number of iterations used by our method may be considerably less than that of the standard ellipsoid method.

Key Words: Ellipsoid method, linear programming, simplex, volume.

1 Introduction

The ellipsoid method finds a feasible solution for a system of linear inequalities – provided one exists. Khachiyan [1979] showed that the running time of the algorithm is bounded from above by a polynomial in the size of the input data. From a theoretical point of view this method is "efficient". However, the ellipsoid method has not shown good results in practice (cf. Bland et al. [1981]). Yamnitsky and Levin [1982] proposed a variant using simplices instead of ellipsoids. In the present paper, we investigate this variant more in detail. We interpret the update simplex of Yamnitsky and Levin in each iteration as a simplex obtained by *point-sliding*. Our main result (Theorem 4.4) shows that the best update simplex obtainable by point-sliding can be easily determined once the minimum of a simple strictly convex function is found. For short, we refer to this algorithmic approach as the *simplices method*. The paper works out the details of the theory and discusses some numerical results. The results suggest that in practice the number of iterations used by the simplices method may be considerably less than that of the ellipsoid method.

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In Section 2, we review the ellipsoid method. In analogy with the ellipsoid method we introduce in Section 3 the concept of *half-simplices*. In each step of the simplices method, we construct a simplex containing the half-simplex. This is done by point-sliding as will be explained in Section 4. Under some conditions, a half-simplex forms a simplex as shown in Section 5. If the half-simplex is not already a simplex, we can still obtain a "reasonably small" simplex containing the half-simplex by rotating some hyperplane around a $(n-2)$-dimensional polytope. We summarize our computational results in Section 6.

2 The Ellipsoid Method

The ellipsoid method determines the feasibility of the system

$$Ax < b \tag{1}$$

of linear inequalities with integral coefficients, where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. The algorithm is initialized with the ellipsoid

$$E_0 := \{x \in \mathbb{R}^n \mid \|x\| \leq n2^L\} \ ,$$

which is in this case a sphere and contains a solution of (1) if one exists at all, as follows from the next (well-known) lemma. (L denotes the *input length* of the system, *i.e.*,

$$L = nm + \lceil \log_2 |N| \rceil + 1 \ ,$$

where N is the product of all non-zero coefficients occurring in A and b.)

Lemma 2.1: If (1) *has a solution, then the system*

$$Ax < b \ ,$$
$$-2^L \leq x_i \leq 2^L \ , \qquad 1 \leq i \leq n \ ,$$

has a solution, and the volume of the solution set of that system is at least $2^{-(n+1)L}$.

Proof: See Gács and Lovász [1981], p. 63. □

Starting with E_0, the ellipsoid method constructs a sequence of successively smaller ellipsoids. At some iteration k we have an ellipsoid E_k with center c_k,

$$E_k = \{x \in \mathbb{R}^n | (x - c_k)^T B_k^{-1} (x - c_k) \leq 1\} ,$$

which contains a solution of (1) if a solution exists. Check if c_k is a solution of (1). If so, stop. If not, pick an inequality in (1) which is violated by c_k:

$$a^T c_k \geq \beta$$

(a^T is a row in A and β an element of b). The hyperplane $\{x \in \mathbb{R}^n | a^T x = a^T c_k\}$ cuts E_k into two half-ellipsoids. By the half-ellipsoid $\frac{1}{2}E_k$ we mean the set of all points in E_k that satisfies $a^T x \leq a^T k$. Updating ellipsoid E_k by

$$c_{k+1} := c_k - \frac{1}{n+1} \frac{B_k a}{\sqrt{a^T B_k a}} \tag{2}$$

$$B_{k+1} := \frac{n^2}{n^2 - 1}\left[B_k - \frac{2}{n+1} \frac{(B_k a)(B_k a)^T}{a^T B_k a}\right] \tag{3}$$

yields an ellipsoid E_{k+1} that is the smallest ellipsoid containing $\frac{1}{2}E_k$. We denote by vol E_k the volume of E_k. The next lemma gives the reduction in volume at each iteration.

Lemma 2.2:

$$\frac{\text{vol } E_{k+1}}{\text{vol } E_k} = \frac{n}{n+1}\left(\frac{n^2}{n^2 - 1}\right)^{(n-1)/2} \tag{4}$$

Proof: See Gács and Lovász [1981], p. 67 (see also Zorychta [1982] for an analysis of volume ratios obtained by "deep cuts"). □

If the ellipsoid method stops in iteration k, c_k is a solution of (1). Using Lemma 2.1 and Lemma 2.2 one can prove that (1) is not solvable if the method does not stop after at most $6n^2L$ iterations (see, e.g., Gács and Lovász [1981], p. 62).

Yamnitsky and Levin [1982] introduced a variant of the ellipsoid method with simplices taking over the role of ellipsoids. In the following, we will describe an improvement on this method.

3 Simplices and Half-Simplices

At each iteration in the simplices method, we have a n-simplex S that contains a solution of (1), if one exists. We check feasibility of the center of S, which is defined as the point

$$v_c = \frac{1}{n+1} \sum_{k=0}^{n} v_k$$

where $v_0, \ldots, v_n$ are the vertices of S. If the center v_c is feasible, we are done. Otherwise, a violated constraint $a^T v_c \geq \beta$ is chosen. The hyperplane

$$h := \{x | a^T x = \beta\}$$

cuts S into two half-simplices, and by $\frac{1}{2}S$ we denote the one satisfying the constraint:

$$\frac{1}{2}S := S \cap \{x | a^T x \leq \beta\} \ .$$

We generate a new simplex S' by point-sliding (see below) containing $\frac{1}{2}S$, and continue. This process is repeated until either the center of our current simplex is feasible or the volume of the simplex is so small that the system of inequalities must be infeasible.

For all points $x \in \mathbb{R}^n$ we define the error function

$$e(x) = \beta - a^T x \ .$$

We assume that v_0 is the vertex of S that maximizes the error function. An algorithm for computing S' that fixes v_0 and slides each other vertex v_k' along the line $v_0 v_k$ is called a *point-sliding method*. In the next section, we will present an algorithm that constructs a simplex of the smallest possible volume, obtainable by point-sliding and enclosing $\frac{1}{2}S$.

4 Point-Sliding

For notational convenience, let us transform S into the standard simplex. So let $S = conv\{v_0, v_1, \dots, v_n\}$ be an arbitrary n-simplex and assume that the center v_c of S violates the constraint $a^T x < \beta$. We assume that v_0 is the vertex that maximizes $e(x) = \beta - a^T x$.

Using the affine transformation $\tau \circ T_{v_0}: \mathbb{R}^n \to \mathbb{R}^n$ with

$$\begin{aligned} \tau(x) &= [v_1 - v_0 \dots v_n - v_0]^{-1} x \\ T_{v_0}(x) &= x - v_0 \ , \end{aligned} \tag{5}$$

we transform the simplex S into the simplex

$$\bar{S} = conv\{0, e_1, \dots, e_n\} = \{x | x \geq 0, e^T x \leq 1\} \ , \tag{6}$$

where $e^T = (1, \dots, 1)$.

Since $\beta - a^T v_0 > 0$, the affine transformation $\tau \circ T_{v_0}$ transforms the half-space $\{x | a^T x < \beta\}$ into $\{x | \bar{a}^T x < 1\}$, where

$$\bar{a}^T = \frac{a^T[v_1 - v_0 \dots v_n - v_0]}{\beta - a^T v_0} \ . \tag{7}$$

Note $\bar{a} > 0$ and $\bar{v}_0 = 0$ is the vertex of $\bar{S}$ that maximizes $\bar{e}(x) = 1 - \bar{a}^T x$, *i.e.*, $\bar{e}(0) > \bar{e}(e_i)$, $1 \leq i \leq n$.

Note that the transformation takes $\frac{1}{2}S$ into

$$\frac{1}{2}\bar{S} = \{x | x \geq 0, e^T x \leq 1, \bar{a}^T x \leq 1\} \ . \tag{8}$$

We denote by $\bar{f}_0$ the facet of $\bar{S}$ which is *opposite* to $\bar{v}_0 = 0$, *i.e.*,

$$\bar{f}_0 := \{x \in \bar{S} | e^T x = 1\} \ .$$

Definition 4.1: We say that $S' = conv(v'_0, \dots, v'_n)$ is obtained from $\bar{S}$ by point-sliding, *fixing $\bar{v}_0 = 0$, if for some $\alpha_i > 0$*

$$v_0' = 0$$

$$v_i' = \alpha_i e_i \qquad (i \geq 1) \ .$$

Equivalently, $S' = \{x | x \geq 0, d'^T x \leq 1\}$ *for some* $d' \geq 0$.

Define, for $0 \leq t \leq 1$:

$$S(t) := \{x | x \geq 0, d(t)^T x \leq 1\} \tag{9}$$

where

$$d(t) := (1 - t)\bar{a} + te \ .$$

Thus $\frac{1}{2}\bar{S} \subseteq S(t)\ \forall t \in [0, 1]$. We say that $S(t)$ is obtained by *rotating* $\bar{f}_0$ *around* P, where

$$P := \bar{f}_0 \cap \{x | \bar{a}^T x = 1\} \ .$$

If we denote by $v_0(t), \ldots, v_n(t)$ the vertices of $S(t)$, then

$$v_0(t) = 0 \tag{10}$$

$$v_i(t) = \alpha_i(t) e_i \qquad (i \geq 1) \ , \tag{11}$$

where

$$\alpha_i(t) = \|v_i(t)\| = \frac{1}{t + (1 - t)\bar{a}_i} \ .$$

(The latter equality can be obtained by multiplying equation (11) by $d(t)$.) So we get

$$r(t) := \frac{\text{vol } S(t)}{\text{vol } \bar{S}} = \prod_{i=1}^{n} \alpha_i(t) \ . \tag{12}$$

Lemma 4.2: The function $r(t)$ *is strictly convex.*

Proof: Define $f(t) := \ln r(t)$. So, $r(t) = e^{f(t)}$ and $r''(t) = f''(t)e^{f(t)} + (f'(t))^2 e^{f(t)}$. If $f''(t) > 0$ then $r''(t) > 0$. A straightforward calculation shows that

$$f''(t) = \sum_{i=1}^{n} \frac{(1-\bar{a}_i)^2}{(t+(1-t)\bar{a}_i)^2} > 0 \ ,$$

since $\bar{a} > 0$ and $\bar{a} \neq e$, and thus $r(t)$ is strictly convex. □

The next theorem states that minimizing $r(t)$ yields the unique simplex with minimum volume obtainable by point-sliding. The proof of the theorem is based on the following lemma.

Lemma 4.3: (Affine form of Farkas' Lemma) *Let the system $Ax \leq b$ of linear inequalities have at least one solution, and suppose that the linear inequality $c^T x \leq \delta$ holds for each x satisfying $Ax \leq b$. Then for some $\delta' \leq \delta$ the linear inequality $c^T x \leq \delta'$ is a nonnegative linear combination of the inequalities in the system $Ax \leq b$.*

Proof: See Schrijver [1986], p. 93. □

We can now state our main result.

Theorem 4.4: Let t^* be the unique minimum of $r(t)$. Then the simplex $S(t^*)$ has minimum volume among all simplices that contain $\frac{1}{2}\bar{S}$ and are obtainable by point-sliding, fixing $\bar{v}_0 = 0$.

Proof: By definition, $S(t^*)$ has minimum volume among all simplices that contain $\frac{1}{2}\bar{S}$ and are obtained by rotating $\bar{f}_0$ around P. Let now $\tilde{S}$ be an arbitrary simplex of minimum volume among all simplices containing $\frac{1}{2}\bar{S}$ that are being obtainable by point-sliding, fixing $\bar{v}_0$. The Theorem follows if we can verify the following

Claim: $\tilde{S}$ is obtained by rotating $\bar{f}_0$ around P.

Since $\tilde{S}$ is obtained by point-sliding, we have

$$\tilde{S} = \{x | x \geq 0, \tilde{a}^T x \leq 1\} \ .$$

By Farkas' Lemma, the inequality $\tilde{d}^T x \leq 1$ can be written as a nonnegative linear combination of the inequalities defining $\frac{1}{2}\bar{S}$, *i.e.*, there exist nonnegative numbers μ, ν, $\lambda_1, \ldots, \lambda_n$ such that

$$\tilde{d} = \mu e + \nu \bar{a} - \sum_{i=1}^{n} \lambda_i e_i \quad \textit{and} \quad \mu + \nu \leq 1 \ .$$

Since $\tilde{S}$ has minimum volume among all "point-sliding" simplices, we may assume that, in fact, $\mu + \nu = 1$ (otherwise $\frac{1}{2}\bar{S} \subseteq \{x | x \geq 0, \tilde{d}^T \leq \mu + \nu\} \subseteq \tilde{S}$ would contradict the assumed minimality).

But then $\hat{d} = \mu e + \nu \bar{a}$ defines a simplex $\hat{S} := \{x | x \geq 0, \hat{d}^T x \leq 1\}$, obtained by rotating $\bar{f}_0$ around P (with $t = \mu$).

By assumption, vol $\tilde{S} \leq$ vol $\hat{S}$. Also $\frac{1}{2}\bar{S} \subseteq \hat{S} \subseteq \tilde{S}$ holds and thus $\hat{S} = \tilde{S}$. So $\tilde{S}$ is indeed obtained by rotating $\bar{f}_0$ around P. □

5 The Algorithm

Our algorithm computes in each step a new simplex $S(t^*)$, where $t^* \in [0, 1]$ minimizes $r(t)$. The main steps of the algorithm are as follows.

A_0: Take S as a simplex containing the body defined by the inequalities

$$-2^L \leq x_i \leq 2^L \ , \qquad 1 \leq i \leq n \ .$$

A_1: Calculate the center v_c of S;
IF v_c satisfies system (1) THEN STOP (solution found);
ELSE choose a violated inequality $a^T v_c \geq \beta$ in system (1);
Determine v_0 as the vertex v maximizing $e(v) = \beta - \bar{a}^T v$;
Take $\bar{S}$ as in (6) and $\frac{1}{2}\bar{S}$ as in (8) with $\bar{a}$ as in (7);
Determine $t^* \in [0, 1]$ so that $r(t)$ is minimized by $t = t^*$;
Compute $S(t^*)$ from equation (9).

A_2: Set $S := (\tau \circ T_{v_0})^{-1} S(t^*)$;
IF vol $S < 2^{-(n+1)L}$ THEN STOP (system (1) has no solution);
ELSE calculate the new vertices of S and return to step A_1.

In general, $P \subseteq \bar{f}_0$ is not necessarily $(n-2)$-dimensional. In fact, it may happen that P is empty, in which case the term "rotating $\bar{f}_0$ around P" may be somewhat misleading. However, the case dim $P < n - 2$ is extremely "easy", as can be seen from the following lemma.

Lemma 5.1: Let relint $f_0 := \{x \in \mathbb{R}^n | \exists \varepsilon > 0\colon B(x, \varepsilon) \cap \text{aff.hull}\, \bar{f}_0 \subseteq \bar{f}_0\}$ *be the relative interior of* $\bar{f}_0$. *Consider the statements*

(i) $\dim P < n - 2$,
(ii) $P \cap \text{relint}\, \bar{f}_0 = \varnothing$,
(iii) $\bar{f}_0 \subseteq h^+ = \{x | \bar{a}^T x \geq 1\}$,
(iv) $\frac{1}{2}\bar{S}$ *is a simplex.*

Then (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv).

Proof: $\frac{1}{2}\bar{S}$ is a simplex if and only if one of its defining inequalities is redundant. Obviously, the only possibly redundant inequality is $e^T x \leq 1$. This is redundant if and only if (iii) holds. Thus (iii)$\Leftrightarrow$(iv). The equivalence of (ii) and (iii) is straightforward. Finally, if (ii) does not hold, then $h \cap \text{relint}\, \bar{f}_0 \neq \varnothing$ and therefore $\dim P = n - 2$. □

Note that if property (iv) in the preceding lemma holds, then the algorithm computes $S(t) = \frac{1}{2}\bar{S}$ as the simplex of minimum volume (corresponding to $t = 0$). (Moreover, in practice one need not minimize $r(t)$ but just determines $S(0)$ if one knows that the error satisfies $e(v_k) \leq 0$ for all $1 \leq k \leq n$. In that case namely, $\frac{1}{2}S$ already is a simplex.)

Because the function $r(t)$ is strictly convex (by Lemma 4.2), we minimize it using the method of interval reduction in step A_1 of the algorithm.

Yamnitsky and Levin [1982] compute the new simplex S_{YL} for each iteration by point-sliding with the parameters

$$\alpha_i = \frac{n^2}{n^2 - 1 + \bar{a}_i}\,, \qquad 1 \leq i \leq n\ .$$

(This simplex is also obtainable by rotating $\bar{f}_0$ around P.) Yamnitsky and Levin (see also the expositions of Akgül [1984] and Chvátal [1983]) proved that in this case

$$\frac{\text{vol}\, S_{YL}}{\text{vol}\, \bar{S}} < e^{-1/2(n+1)^2}\ . \tag{13}$$

Note that this upper bound is worse than the corresponding term for the ellipsoid method given in Lemma 2.2.

In our notation, $S_{YL} = S(t_{YN})$, where $t_{YN} = 1 - 1/n^2$. With the optimal choice t^* we will, therefore, obtain an update simplex $S(t^*)$ with a volume reduction at least as good. From this observation, a polynomial bound on the number of iterations of the algorithm can be derived as follows.

Assuming full dimensionality, the feasibility region of the system $Ax < b$ of inequalities is contained in the cube $Q = [-2^L, 2^L]^n$. Hence there exists a simplex

$$S_0 = x_0 + \operatorname{conv}\{0, \alpha e_1, \ldots, \alpha e_n\} \supseteq Q$$

where $x_0 = (-2^L, \ldots, -2^L)$ and $\alpha = n2^{n(L+1)}$. Straightforward computation yields

$$\operatorname{vol} S_0 = \frac{1}{n}\alpha^n = \frac{n^n}{n!}2^{n(L+1)} .$$

Thus, by Lemma 2.1, the ratio ρ between the volumes of S_0 and the feasibility region of $Ax < b$ is bounded from above by

$$\rho \le \frac{n^n}{n!}2^{n(L+1)}2^{(n+1)L} \le e^n 2^{2(n+1)L} \le e^{2(n+1)L} .$$

Since the volume reduction in each iteration is at least $e^{-1/2n^2}$, we obtain the upper bound

$$K \le 4n^2(n+1)L$$

on the number of iterations of our algorithm. (We remark that a rigorous analysis exhibits polynomial running time also when rounding errors are taken into account (cf. Bartels [1995] for the central cut version in the Yamnitsky and Levin model)).

We would like to remark, however, that in practice one can hope to find much smaller initial simplices, *e.g.*, by simply choosing $n + 1$ inequalities from the system $Ax < b$ (cf. Akgül [1984]). However, we did not take advantage of this possibility in our computational experiments. There, the initial ellipsoid and the initial simplex are taken to have roughly the same volume because we are interested in comparing the number of iterations of the two methods.

6 Computational Results

We have implemented our algorithm and compared it with an implementation of the deep cut ellipsoid method. The results are reported in the extended

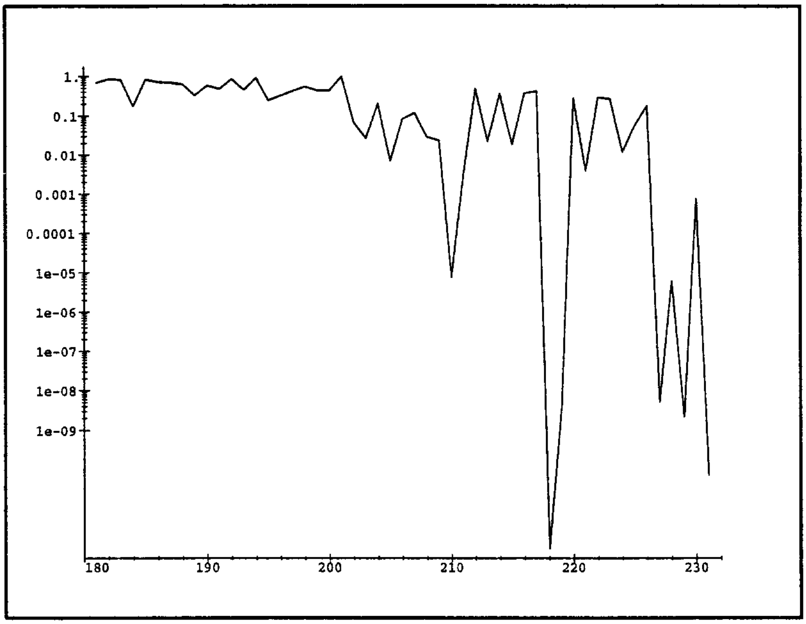

Fig. 6.1. Volume reduction per iteration (example with $n = 20$ and $m = 60$)

abstract Faigle et al. [1996]. It appears that the ellipsoid method generally converges considerably slower than the simplices method. It may be interesting to point out, however, that the volume reduction achieved in the iterative steps of the simplices method is not homogeneous as illustrated in the following figure (m is the number of inequalities and n the number of variables in the typical random example, the last 51 of the 230 iterations are recorded).

From the figure we notice that the simplices method accelerates just before convergence. It quite often occurred that the volume reduction was quite substantial (a reduction factor of less than 0.01), and then most of the time the half-simplex was a simplex.

With $n = 70$ and $m = 160$, the running time that was used by the deep cut ellipsoid method was around 4350 seconds and by the simplices method around 25 seconds in our implementation.

A drawback of the ellipsoid method is that it takes no advantage of sparsity, when dealing with a sparse coefficient-matrix. We do not expect that the simplices method is an improvement with respect to sparsity. Another practical problem with the ellipsoid method arises from the fact that it does not lend itself easily to sensitivity analysis or to the addition or deletion of constraints or variables. Being a volume reduction method as well, the simplices method suffers from the same problem. On the other hand, this method seems to be more numerically stable than the ellipsoid method.

Ecker et al. [1985] and Frenk et al. [1994], for example, showed that the ellipsoid method can be a practical tool for solving convex and quasiconvex continuous location problems (and possibly for more general problems), whereby seperation hyperplanes are used to obtain an optimal solution. In that case, it may be of practical advantage to use a variant of the ellipsoid method with simplices instead of ellipsoids.

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