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# TECHNICAL NOTE

# Note on Prime Representations of Convex Polyhedral Sets[1]

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**Abstract.** Consider a convex polyhedral set represented by a system of linear inequalities. A prime representation of the polyhedron is one that contains no redundant constraints. We present a sharp upper bound on the difference between the cardinalities of any two primes.

**Key Words.** Convex polyhedral sets, linear inequalities, minimal representation, prime representation, redundancy.

## 1. Introduction

Consider a convex polyhedral set $P$ with initial representation denoted by the augmented matrix $[A|b]$, that is,

$$P = \{x \in R^n \mid Ax \leq b;\ A \in R^{m \times n}\}.$$

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We say that the representation $[A_1|b_1]$ of $P$ is a reduction of $[A|b]$ if $[A_1|b_1]$ is obtained from $[A|b]$ by removing at least one redundant constraint (Refs. 1 and 2). The reduction $[A_1|b_1]$ is called a prime (Ref. 3) if it contains no redundant constraints, and is called a minimal prime if it is a prime with minimum cardinality, that is, number of inequalities. We present a sharp upper bound on the difference between the cardinalities of any two primes.

We first note that, if the original representation contains no implicit equalities and no duplicate constraints, then there is a unique prime and it is the minimal representation as defined by Telgen (Ref. 4). Also, if there are no implicit equalities, but there are duplicate constraints, then there is more than one prime, but they are all minimal representations. Finally, if there are implicit equalities, then the primes are not necessarily minimal representations. In fact, in order to obtain a minimal representation, Telgen (Ref. 4) has shown that the implicit equalities must be replaced with explicit equalities.

Since the prime derived by an algorithm depends upon the order in which the constraints are classified, it is possible for primes with different cardinalities to be obtained for the same polyhedral set. The results of this paper can determine whether or not the observed difference is possible, or simply due to implementation error. If the observed difference is correct, the results can be used to provide an upper bound on the dimension of the polyhedral set.

## 2. Results

Consider the following example. Let

$$P=\{0\}\subseteq R^2,$$

with the original representation

$$[A|b]=\left[\begin{array}{rr|r} 1 & 0 & 0\\ -1 & 0 & 0\\ 0 & 1 & 0\\ 0 & -1 & 0\\ 1 & -1 & 0\\ 1 & 1 & 0 \end{array}\right].$$

The following three prime representations of $P$ are reductions of $[A|b]$:

$$[A_1|b_1]=\left[\begin{array}{cc|c} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{array}\right], \qquad [A_2|b_2]=\left[\begin{array}{cc|c} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{array}\right],$$

$$[A_3|b_3]=\left[\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & 0 \end{array}\right].$$

A minimal representation of $P$ is the set of constraints

$$\{x_1=0, x_2=0\}.$$

Theorem 2.1 gives an upper bound on the difference between the cardinalities of any two primes. We first require the following lemma.

**Lemma 2.1.** Let $[A|0]$ be a prime representation of $P=\{0\}\subseteq R^n$, where $A=[a_1,\ldots,a_m]^T\in R^{m\times n}$. Then, $n+1\leq m\leq 2n$.

**Proof.** If $\text{rank}(A)<n$, then there exists an $x\neq 0$ such that $Ax=0$, which contradicts $P=\{0\}$. Thus, $A$ has full rank and $m\geq n$. If $m=n$, then $A$ is nonsingular and there exists an $x\neq 0$ such that $Ax=b<0$, which again contradicts $P=\{0\}$. Therefore, $m\geq n+1$.

Suppose that $m>2n$. Farkas' lemma (Ref. 5) implies that $P=\{0\}$ is equivalent to

$$R^n=K(A):=\{A^Tx\,|\,x\geq 0\}.$$

We need only show that there exists a matrix $A^*$, whose rows are a proper subset of the rows of $A$, such that $K(A^*)=R^n$. The proof is by induction on $n$. The result is true if $n=1$.

Since $K(A)=R^n$, there exists an $x\geq 0$ such that $-a_m=A^Tx$. Let

$$A_m^T=[a_1,\ldots,a_{m-1}],$$

and define $x_m\in R^{m-1}$ by

$$(x_m)_i=(x)_i/(1+(x)_m), \qquad \text{for } i=1,\ldots,m-1.$$

Then,

$$-a_m=A_m^Tx_m, \qquad x_m\geq 0.$$

Let $r$ be the minimum number of rows of $A_m$ such that $-a_m^T$ is a positive linear combination of those rows. Without loss of generality suppose that

$$-a_m\in K^+(B)=\{B^Tx\,|\,x>0\},$$

where

$$B^T = [a_1, \ldots, a_r].$$

Let $s = \text{rank}(B)$, and note that $s = \text{rank}(C)$, where $C^T = [B^T, a_m]$. Since $[A|0]$ is prime, there is a permutation of the columns of $A^T$ which results in $s \geq 2$. Without loss of generality, we assume that this is the case.

Let

$$-a_m = B^T x, \qquad x > 0,$$

and let

$$y = x/e^T x,$$

where $e$ is a vector of ones. Then,

$$-a_m/e^T x = B^T y, \qquad 0 < y < e,$$

and

$$e^T y = 1.$$

Since we have a barycentric representation of $-a_m/e^T x \in K^+(B)$ using $r$ vectors, Caratheodory's theorem (Ref. 5) and the fact that $r$ is minimal implies that $r \leq s + 1$.

Let

$$D^T = [d_1, \ldots, d_{m-r-1}],$$

where $d_i$ is the orthogonal projection of $a_{r+i}$ onto the null space $N(C)$ of $C$. Let $v \in N(C)$, and write

$$v = A^T x, \qquad x \geq 0.$$

This is equivalent to

$$v = C^T x_1 + E^T x_2, \qquad x_1 \geq 0, \qquad x_2 \geq 0,$$

where

$$E^T = [a_{r+1}, \ldots, a_{m-1}].$$

Therefore,

$$v = C^T x_1 + E^T x_2, \qquad x_1 \geq 0, \qquad x_2 \geq 0,$$

But

$$C^T x_1 + (E^T - D^T) x_2 = v - D^T x_2,$$

and

$$v - D^T x_2 \in N(C) \cap R(C^T) = \{0\},$$

where $R(C^T)$ is the range space of $C^T$. Thus,

$$v = D^T x_2 \quad \text{and} \quad N(C) \subseteq K(D).$$

Since, by construction,

$$K(D) \subseteq N(C),$$

we have

$$K(D) = N(C).$$

Since $\dim(N(C)) = n - s$, the induction hypothesis states that at most $2(n-s)$ vectors are needed to define $K(D)$. Since $s \geq 2$, it follows that $m - r - 1 > 2(n-s)$. Therefore, we can eliminate at least one of the $m - r - 1$ rows of $D$ to get a matrix $D^*$ with

$$K(D^*) = N(C).$$

Finally, this implies that we can delete the corresponding rows in $A$ to get $A^*$ with

$$K(A^*) = R(C^T) + K(D^*) = R^n. \qquad \square$$

**Theorem 2.1.** Let $k$ be the dimension of the convex polyhedral set $P$. Let $[A_1 | b_1]$ and $[A_2 | b_2]$, with cardinalities $m_1$ and $m_2$, respectively, be two prime representations of $P$ that are reductions of $[A | b]$. If $k = n$, then $|m_1 - m_2| = 0$. Otherwise, $|m_1 - m_2| \leq n - k - 1$.

**Proof.** If $k = n$, then there are no implicit equalities in $[A | b]$. As noted above, all primes are therefore minimal representations, i.e., $|m_1 - m_2| = 0$.

Now assume that $k < n$. Let $W$ be the $k$-dimensional subspace generated by $P$. Since $P$ has full dimension in $W$, the cardinality of a prime representation of $P$ in $W$ is unique. Thus, the only variation in the cardinality of primes is due to the number of constraints used in reducing the dimension of $P$ from $n$ to $k$. This is equivalent to the possible variation in the number of constraints $t$ that reduce the dimension of a polyhedron from $(n-k)$ to zero. Lemma 2.1 implies that $(n-k)+1 \leq t \leq 2(n-k)$. Thus, for $k < n$, the maximum variation is $2(n-k) - ((n-k)+1) = n - k - 1$. $\square$

To prove that the bounds given by the theorem are sharp, note that the prime representation of $P = \{0\} \subseteq R^n$, given by

$$\left\{ x \,|\, x_i \leq 0,\ i = 1, \ldots, n;\ \sum_{i=1}^{n} x_i \geq 0 \right\},$$

has cardinality $n+1$, while the prime representation

$$\{x \mid 0 \leq x_i \leq 0, i = 1, \ldots, n\}$$

has cardinality $2n$.

The theorem can be used to provide an upper bound on the dimension of a convex polyhedral set. In the example, we had $m_1 = 4$, $m_2 = 3$, and $n = 2$. Thus, $k \leq 2 - |3-2| - 1 = 0$, which implies that the polyhedron has dimension $k = 0$.

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