Discrete Mathematics 233 (2001) 55–63

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

Strengthening the closure concept in claw-free graphs

Hajo Broersma[a,*], Zdeněk Ryjáček[b]

[a]*Faculty of Mathematical Sciences, University of Twente, P.O. Box 217, 7500 AE Enschede, Netherlands*

[b]*Department of Mathematics, University of West Bohemia, Univerzitní 22, 306 14 Plzeň, Czech Republic*

Abstract

We give a strengthening of the closure concept for claw-free graphs introduced by the second author in 1997. The new closure of a claw-free graph G defined here is uniquely determined and preserves the value of the circumference of G. We present an infinite family of graphs with n vertices and $\frac{3}{2}n-1$ edges for which the new closure is the complete graph K_n. © 2001 Elsevier Science B.V. All rights reserved.

MSC: 05C45; 05C35

Keywords: Closure; Cycle closure; Claw-free graph; Circumference; Hamiltonian graph

1. Introduction

We consider finite simple undirected graphs $G=(V(G),E(G))$. For concepts and notation not defined here we refer the reader to [1]. We denote by $c(G)$ the *circumference* of G, i.e. the length of a longest cycle in G, by $N_G(x)$ the *neighborhood* of a vertex x in G (i.e., $N_G(x)=\{y \in V(G)|\ xy \in E(G)\}$), and we denote $N_G[x]=N_G(x)\cup\{x\}$. For a nonempty set $A \subseteq V(G)$, the *induced subgraph on A* is denoted by $\langle A\rangle_G$, the notation $G-A$ stands for $\langle V(G)\backslash A\rangle_G$ (if $A \neq V(G)$) and we put $N_G(A)=\{x \in V(G)|\ N(x)\cap A \neq \emptyset\}$ and $N_G[A]=N_G(A)\cup A$. For a subgraph X of G we denote $N_G(X)=N_G(V(X))$ and $N_G[X]=N_G[V(X)]$.

If F is a graph, then we say that a graph G is F-free if G does not contain a copy of F as an induced subgraph. The graph $K_{1,3}$ will be called the *claw* and in the special case $F=K_{1,3}$ we say that G is *claw-free* (instead of F-free). The *line*

* Corresponding author.

E-mail addresses: broersma@math.utwente.nl (H. Broersma), ryjacek@kma.zcu.cz (Z. Ryjáček).

PII: S0012-365X(00)00226-0

graph of a graph H is denoted by $L(H)$. If $G = L(H)$, then we also say that H is the *line graph preimage* of G and denote $H = L^{-1}(G)$. It is well known that for any connected line graph $G \not\simeq K_3$ its line graph preimage is uniquely determined.

Let T be a closed trail in G. We say that T is a *dominating closed trail* (DCT), if $V(G) \setminus V(T)$ is an independent set in G (or, equivalently, if every edge of G has at least one vertex on T). Harary and Nash-Williams [6] proved the following result, relating the existence of a DCT in a graph to the hamiltonicity of its line graph.

Theorem A (Harary and Nash-Williams [6]). *Let H be a graph with $|E(H)| \geqslant 3$ without isolated vertices. Then $L(H)$ is hamiltonian if and only if H contains a DCT.*

A special case is that $H = K_{1,r}$ for some $r \geqslant 3$; then $L(H) = K_r$ and the DCT in H consists of a single vertex.

For a vertex $x \in V(G)$, set $B_x = \{uv \mid u, v \in N(x), uv \notin E(G)\}$ and $G'_x = (V(G), E(G) \cup B_x)$. The graph G'_x is called the *local completion of G at x*. It was proved in [8] that if G is claw-free, then so is G'_x, and if $x \in V(G)$ is a *locally connected vertex* (i.e., $\langle N(x) \rangle_G$ is a connected graph), then $c(G) = c(G'_x)$. A locally connected vertex x with $B_x \neq \emptyset$ is called *eligible* (in G) and the set of all eligible vertices of G is denoted by $V_{\mathrm{EL}}(G)$.

We say that a graph F is a *closure of G*, denoted $F = \mathrm{cl}(G)$ (see [8]), if $V_{\mathrm{EL}}(F) = \emptyset$ and there is a sequence of graphs $G_1, \ldots, G_t$ and vertices $x_1, \ldots, x_{t-1}$ such that $G_1 = G$, $G_t = F$, $x_i \in V_{\mathrm{EL}}(G_i)$ and $G_{i+1} = (G_i)'_{x_i}$, $i = 1, \ldots, t-1$ (equivalently, $\mathrm{cl}(G)$ is obtained from G by a series of local completions at eligible vertices, as long as this is possible). The following basic result was proved in [8].

Theorem B (Ryjáček [8]). *Let G be a claw-free graph. Then*

(i) $\mathrm{cl}(G)$ *is well-defined* (*i.e., uniquely determined*),
(ii) *there is a triangle-free graph H such that* $\mathrm{cl}(G) = L(H)$,
(iii) $c(G) = c(\mathrm{cl}(G))$.

Consequently, a claw-free graph G is hamiltonian if and only if so is its closure $\mathrm{cl}(G)$. A claw-free graph G for which $G = \mathrm{cl}(G)$ will be called *closed*. Clearly, G is closed if and only if $V_{\mathrm{EL}}(G) = \emptyset$, i.e. if every vertex $x \in V(G)$ is either *simplicial* ($\langle N(x) \rangle_G$ is a clique), or is *locally disconnected* ($\langle N(x) \rangle_G$ is disconnected, implying that, since G is claw-free, $\langle N(x) \rangle_G$ consists of two vertex disjoint cliques). It is easy to observe that G is a closed claw-free graph if and only if G is claw-free and $(K_4 - e)$-free. This implies that if G is closed claw-free, then so is every induced subgraph of G. It is also straightforward to check that for any edge e of a closed claw-free graph the largest clique containing e is uniquely determined. The order of the largest clique in a closed claw-free graph G containing a given edge e will be denoted by $\omega_G(e)$.

The closure concept for claw-free graphs has been studied intensively since it has been introduced in [8]. It is known to preserve a number of graph properties and values of graph parameters, and has found many applications. Interested readers can find more information e.g. in the survey paper [3].

In the following section, we introduce a strengthening of this closure concept, and we show that this new closure is again uniquely determined and that it preserves the value of the circumference of G.

2. The cycle closure

Let G be a closed claw-free graph and let C be an induced cycle in G of length k. We say that the cycle C is *eligible in G* if $4 \leqslant k \leqslant 6$ and $\omega_G(e) = 2$ for at least $k - 3$ nonconsecutive edges $e \in E(C)$ (or, equivalently, if the k-cycle $L^{-1}(C)$ in $H = L^{-1}(G)$ contains at least $k - 3$ nonconsecutive vertices of degree 2).

For an eligible cycle C in G set $B_C = \{uv \mid u, v \in N_G[C], uv \notin E(G)\}$. The graph G'_C with vertex set $V(G'_C) = V(G)$ and edge set $E(G'_C) = E(G) \cup B_C$ is called the *C-completion* of G at C.

The following proposition shows that the C-completion of a closed claw-free graph at an eligible cycle C is again claw-free and has the same circumference. Note that a C-completion of a closed claw-free graph is not necessarily closed (for example, the graph G with $V(G) = \{a, b, c, d, e, f, g\}$ and $E(G) = \{ab, bc, cd, de, ef, fa, ga, gb, gd, ge\}$ is closed and claw-free, the 4-cycle $C = agefa$ is eligible in G, but G'_C is not closed since $b, d \in V_{\mathrm{EL}}(G'_C)$).

Proposition 1. *Let G be a closed claw-free graph, let C be an eligible cycle in G and let G'_C be the C-completion of G. Then*

(i) *G'_C is claw-free,*
(ii) $c(G'_C) = c(G)$.

Proof. (i) Let $H = \langle\{z, y_1, y_2, y_3\}\rangle_{G'_C}$ be a claw. Then $1 \leqslant |E(H) \cap B_C|$ since G is claw-free, and $|E(H) \cap B_C| \leqslant 1$ since $\langle N[C]\rangle_{G'_C}$ is a clique. Let $zy_1 \in B_C$. Then $z \in N[C]$, implying $zu \in E(G)$ for some $u \in V(C)$. Then obviously $uy_2, uy_3 \notin E(G)$ (otherwise H is not a claw in G'_C), but then $\langle\{z, u, y_2, y_2\}\rangle_G$ is a claw in G, a contradiction.

(ii) Obviously $c(G'_C) \geqslant c(G)$ since every cycle in G is a cycle in G'_C. To prove the converse, it is sufficient to show that for every longest cycle C'_1 in G'_C there is a cycle C_1 in G with $V(C_1) = V(C'_1)$. This is clear if $E(C'_1) \cap B_C = \emptyset$; hence, suppose $E(C'_1) \cap B_C \neq \emptyset$. Since C'_1 is longest and $\langle N[C]\rangle_{G'_C}$ is a clique, $N[C] \subset V(C'_1)$, implying that $\langle V(C'_1)\rangle_{G'_C}$ is the C-completion of $\langle V(C'_1)\rangle_G$. Since every induced subgraph of a closed claw-free graph is again claw-free and closed, it is sufficient to show that if G'_C is hamiltonian then so is G.

Let $H = L^{-1}(G)$ and suppose that C is a k-cycle ($4 \leqslant k \leqslant 6$). Since C is eligible in G, the k-cycle $L^{-1}(C)$ in H contains $k-3$ nonconsecutive vertices x_i, $i = 1,\dots,k-3$, of degree 2. Let x_i^-, x_i^+ be the predecessor and successor of x_i on $L^{-1}(C)$, respectively.

It is straightforward to check that G'_C can be equivalently obtained by the following construction:

(i) denote by H' the graph obtained from H by replacing the path $x_i^- x_i x_i^+$ by the edge $x_i^- x_i^+$, $i = 1,\dots,k-3$;
(ii) denote by a_i the vertices of $L(H')$ corresponding to the edges $x_i^- x_i^+$, $i=1,\dots,k-3$;
(iii) construct a graph $\bar{G}$ from $L(H')$ by a series of consecutive local completions at the vertices $a_1,\dots,a_{k-3}$;
(iv) add $k-3$ vertices $z_1,\dots,z_{k-3}$ to $\bar{G}$ and turn the set $\{z_1,\dots,z_{k-3}\} \cup N_{\bar{G}}[\{a_1,\dots,a_{k-3}\}]$ into a clique.

Note that step (i) turns C into a triangle, and hence the vertices $a_1,\dots,a_{k-3}$ are locally connected in $L(H')$.

By the main result of [8], by the above considerations and by Theorem A, it is sufficient to show that if H' contains a DCT, then so does H. Let T be a DCT in H'.

Suppose first that $k = 4$ and, for simplicity, set $x = x_1$. If $x^-x^+ \in E(T)$, then, replacing in T the edge x^-x^+ by the path x^-xx^+, we have a DCT in H. Hence suppose $x^-x^+ \notin E(T)$. Since T is dominating, $|\{x^-,x^+\} \cap V(T)| \geqslant 1$. If both x^-, x^+ are on T, then T is dominating in H. Hence we can suppose $x^- \in V(T)$ and $x^+ \notin V(T)$. If $x^-x^{++} \in E(T)$, then we replace in T the edge x^-x^{++} by the path $x^-xx^+x^{++}$, and if $x^-x^{++} \notin E(T)$, then we add to T the 4-cycle $x^-xx^+x^{++}x^-$. In both cases, we have a DCT in H.

Let now $k = 5$ and suppose the notation is chosen such that $x_1^+ = x_2^-$. If $x_1^-x_1^+ \in E(T)$ and $x_2^-x_2^+ \in E(T)$, then, replacing in T the edges $x_1^-x_1^+$ and $x_2^-x_2^+$ by the paths $x_1^-x_1x_1^+$ and $x_2^-x_2x_2^+$, we have a DCT in H. If $x_1^-x_1^+ \notin E(T)$ and $x_2^-x_2^+ \notin E(T)$, then for $x_1^-x_2^+ \in E(T)$ we replace in T the edge $x_1^-x_2^+$ by the path $x_1^-x_1x_1^+x_2x_2^+$, and for $x_1^-x_2^+ \notin E(T)$ we add to T the cycle $x_1^-x_1x_1^+x_2x_2^+x_1^-$. In both cases, we have a DCT in H (note that at least two of the vertices x_1^-,x_1^+,x_2^+ are on T since T is dominating). Up to symmetry, it remains to consider the case when $x_1^-x_1^+ \in E(T)$ and $x_2^-x_2^+ \notin E(T)$. Then for $x_1^-x_2^+ \in E(T)$ the trail T is a DCT in H, and for $x_1^-x_2^+ \notin E(T)$ we get a DCT in H by replacing in T the edge $x_1^-x_1^+$ by the path $x_1^-x_2^+x_2x_2^-(=x_1^+)$. Thus, in all cases we have a DCT in H.

Finally, let $k = 6$ and choose the notation such that $x_1^+ = x_2^-$ and $x_2^+ = x_3^-$. If at least two of the edges $x_1^+x_2^+$, $x_2^+x_3^+$, $x_3^+x_1^+$ are on T (say, $x_1^+x_2^+$, $x_2^+x_3^+$ are on T), then, replacing in T the edges $x_1^+x_2^+$ and $x_2^+x_3^+$ by the paths $x_1^+x_2x_2^+$ and $x_2^+x_3x_3^+$, we get a DCT in H. If none of the edges $x_1^+x_2^+$, $x_2^+x_3^+$, $x_3^+x_1^+$ is on T, then we get a DCT in H by adding to T the cycle $x_1x_1^+x_2x_2^+x_3x_3^+x_1$ (note that again at least two of the vertices

x_1^+, x_2^+, x_3^+ are on T since T is dominating). Hence, it remains to consider the case that exactly one of these edges, say, $x_1^+x_2^+$, is on T, but in this case we obtain a DCT in H by replacing in T the edge $x_1^+x_2^+$ by the path $x_1^+x_1x_3^+x_3x_2^+$. □

Now, we can define the main concept of this paper which strengthens the closure concept introduced in [8].

Definition 2. Let G be a claw-free graph. We say that a graph F is a cycle closure of G, denoted $F = \mathrm{cl}_C(G)$, if there is a sequence of graphs $G_1, \ldots, G_t$ such that

(i) $G_1 = \mathrm{cl}(G)$,
(ii) $G_{i+1} = \mathrm{cl}((G_i)'_C)$ for some eligible cycle C in G_i, $i = 1, \ldots, t-1$,
(iii) $G_t = F$ contains no eligible cycle.

Thus, $\mathrm{cl}_C(G)$ is obtained from $\mathrm{cl}(G)$ by recursively performing C-completion operations at eligible cycles and each time closing the resulting graphs with the closure defined in [8], as long as this is possible (i.e., as long as there is some eligible cycle). It is easy to see that $\mathrm{cl}_C(G)$ can be computed in polynomial time.

It follows immediately from the definition that $E(\mathrm{cl}(G)) \subseteq E(\mathrm{cl}_C(G))$ for any claw-free graph G. We show that $\mathrm{cl}_C(G)$ is well-defined (i.e., uniquely determined) and that the cycle closure operation preserves the value of the circumference of G.

Theorem 3. *Let G be a claw-free graph. Then*

(i) $\mathrm{cl}_C(G)$ *is well-defined*,
(ii) $c(G) = c(\mathrm{cl}_C(G))$.

From Theorem 3 we immediately have the following consequence.

Corollary 4. *Let G be a claw-free graph. Then*

(i) *G is hamiltonian if and only if $\mathrm{cl}_C(G)$ is hamiltonian*;
(ii) *if $\mathrm{cl}_C(G)$ is complete, then G is hamiltonian.*

Before proving Theorem 3, we first prove the following lemma.

Lemma 5. *Let G be a closed claw-free graph, let C, C_1 be two eligible cycles in G and let $G' = \mathrm{cl}(G'_C)$, where G'_C is the C-completion of G at C. Then either $\langle V(C_1)\rangle_{G'}$ is a clique, or there is a cycle C_2 such that $V(C_2) \subseteq V(C_1)$, C_2 is eligible in G' and, in the graph $G'' = (G')'_{C_2}$, $\langle V(C_1)\rangle_{G''}$ is a clique.*

This implies, in particular, that all vertices of C_1 are locally connected in G' or G'', respectively.

Proof. The last statement follows obviously from the eligibility of C_1 in G and the completeness of $\langle V(C_1)\rangle_{G'}$ or $\langle V(C_1)\rangle_{G''}$, respectively. To prove the first statement, denote by $k=|V(C_1)|$ and let $e_i=a_ia_i^+$ $(i=1,\ldots,k-3)$ be the nonconsecutive edges of C_1 with $\omega_G(e_i)=2$. Suppose the notation is chosen such that $a_1^+=a_2^-$ if $k\geqslant 5$ and, moreover, $a_2^+=a_3^-$ if $k=6$. We can suppose that $\langle V(C_1)\rangle_{G'}$ is not a clique (otherwise we are done) and that C_1 is not eligible in G' (otherwise we are done with $C_2=C_1$).

Suppose that $\omega_{G'}(e_i)=2$ for all i, $1\leqslant i\leqslant k-3$. Since C_1 is not eligible, C_1 is not an induced cycle in G'. For $k=4$ this immediately implies that $\langle V(C_1)\rangle_{G'}$ is a clique (since G is closed), a contradiction. For $k=5$, the only chord in C_1 is $a_1a_2^+$ (all other chords would imply $\omega_{G'}(e_i)\geqslant 3$ for some i), but then we are done with $C_2=a_1a_1^+a_2a_2^+a_1$. For $k=6$, any chord in C_1 implies $\omega_{G'}(e_i)\geqslant 3$ for some i (using the fact that G' is claw-free). Hence, we can suppose that $\omega_{G'}(e_i)\geqslant 3$ for some i, $1\leqslant i\leqslant k-3$. By symmetry, suppose that $\omega_{G'}(e_1)\geqslant 3$. We claim the following.

Claim 1. *Let $e=aa^+$ be an edge of C_1 such that $\omega_G(e)=\omega_{G'_C}(e)=2$ but $\omega_{G'}(e)\geqslant 3$. Then either $aa^{++}\in E(G')$, or $a^-a^+\in E(G')$.*

Proof of Claim 1. Suppose that $\omega_{G'}(e)\geqslant 3$. By the definition of G', there is a sequence of graphs $F_1,\ldots,F_\ell$ and vertices $x_1,\ldots,x_{\ell-1}$ such that $F_1=G'_C$, $F_\ell=G'$, $x_1\in V_{\mathrm{EL}}(F_i)$ and $F_{i+1}=(F_i)'_{x_i}$, $i=1,\ldots,\ell-1$. Let j $(1\leqslant j\leqslant \ell-1)$ be the smallest integer for which $\omega_{F_j}(e)\geqslant 3$. Then there is a vertex $c\in V(G)$ such that $ca,ca^+\in E(F_j)$, but at least one of ca,ca^+ is not in $E(F_{j-1})$.

Let first $ca\notin E(F_{j-1})$. Then $cx_{j-1},ax_{j-1}\in E(F_{j-1})$. Clearly $x_{j-1}a^+\notin E(F_{j-1})$ (otherwise $\omega_{F_{j-1}}(e)\geqslant 3$) and $a^-a^+\notin E(F_{j-1})$ (otherwise there is nothing to prove). Since $\langle\{a,a^-,a^+,x_{j-1}\}\rangle_{F_{j-1}}$ is not a claw, we have $x_{j-1}a^-\in E(F_{j-1})$. From $x_{j-1}a^+\notin E(F_{j-1})$ we also have $ca^+\in E(F_{j-1})$, since otherwise cannot be $ca^+\in E(F_j)$. But then $a^+cx_{j-1}a^-$ is an (a^+,a^-)-path in $N_{F_j}(a)$, implying $a\in V_{\mathrm{EL}}(F_j)$, from which, since $G'=\mathrm{cl}(F_j)$, we have $a^-a^+\in E(G')$.

If $ca^+\notin E(F_{j-1})$, then symmetrically $aa^{++}\in E(G')$. Hence the claim follows. □

Claim 2. *Let $e=aa^+$ be an edge of C_1 such that $\omega_G(e)=2$ and $\omega_{G'_C}(e)\geqslant 3$. Then $\langle\{a^-,a,a^+,a^{++}\}\rangle_{G'}$ is a clique.*

Proof of Claim 2. Let $c\in V(G)$ be such that $ca,ca^+\in E(G'_C)$. By symmetry, suppose $ca^+\notin E(G)$. Then $c,a^+\in N_G[C]$. Let d be a neighbor of a^+ on C, and denote by K^+ (K^-) the largest clique in G, containing the edge a^+a^{++} (a^-a), respectively. Since $\langle\{a^+,a^{++},a,d\}\rangle_G$ cannot be a claw and $da,a^{++}a\notin E(G)$ (since $\omega_G(e)=2$), we have $da^{++}\in E(G)$, implying, since G is closed, $d\in V(K^+)$. Since $cd,ca^+\in E(G'_C)$ and G' is closed, we have $aa^{++}\in E(G')$. For $k=4$ this immediately implies that $\langle V(C_1)\rangle_{G'}$ is a clique, hence $|V(C_1)|\geqslant 5$.

Now we consider the edge ca. If $ca\notin E(G)$, then, by a symmetric argument, we have $a^-a^+\in E(G')$ and we are done since G' is closed. Hence $ca\in E(G)$. Since $\langle\{a,c,a^+,a^-\}\rangle_G$ cannot be a claw and $ca^+\notin E(G)$, either $a^-a^+\in E(G)$ (and we are done), or $ca^- \in E(G)$, implying $c \in V(K^-)$. But then, since $ca^+ \in E(G'_C)$ and G' is closed, again $a^-a^+ \in E(G')$ and hence also $a^-a^{++} \in E(G')$. This proves Claim 2. □

Now for $k=4$ from $\omega_{G'}(e_1) \geqslant 3$ and from Claims 1 and 2 we immediately have that $\langle V(C_1)\rangle_{G'}$ is a clique.

Let $k=5$. If $\omega_{G'_C}(e_1) \geqslant 3$, then $\langle V(C_1)\rangle_{G'}$ is a clique by Claim 2 and since G' is closed. Thus, let $\omega_{G'_C}(e_1)=2$. By Claim 1, $a_1^-a_1^+ \in E(G')$ or $a_1a_2 \in E(G')$. If both these edges are present or if $\omega_{G'}(e_2) \geqslant 3$, then clearly $\langle V(C_1)\rangle_{G'}$ is a clique. Otherwise, we set $C_2 = a_1^-a_1^+a_2a_2^+a_1^-$ (if $a_1^-a_1^+ \in E(G')$) or $C_2 = a_1a_2a_2^+a_1^-a_1$ (if $a_1a_2 \in E(G')$).

Finally, suppose that $k=6$. We show that $\omega_{G'_C}(e_1)=2$. If $\omega_{G'_C}(e_1) \geqslant 3$ and $\omega_{G'}(e_2) \geqslant 3$ or $\omega_{G'}(e_3) \geqslant 3$, then, by Claims 1 and 2 and since G' is closed, $\langle V(C_1)\rangle_{G'}$ is a clique. If $\omega_{G'_C}(e_1) \geqslant 3$ and $\omega_{G'}(e_2)=\omega_{G'}(e_3)=2$, then we are done with $C_2=a_2a_2^+a_3a_3^+a_2$. Hence $\omega_{G'_C}(e_1)=2$. By a symmetric argument we can prove that also $\omega_{G'_C}(e_2)=\omega_{G'_C}(e_3)=2$. By the assumption $\omega_{G'}(e_1) \geqslant 3$ and by Claim 1, at least one of the chords $a_1^-a_1^+$, a_1a_2 is present. Now, if both $a_2^-a_2^+ \in E(G')$ and $a_2a_3 \in E(G')$, then, since G' is closed, also $a_2^-a_3 \in E(G')$, which together with any of the chords $a_1^-a_1^+$, a_1a_2 implies that $\langle V(C_1)\rangle_{G'}$ is a clique. Hence at most one of $a_2^-a_2^+$, a_2a_3 is present. Symmetrically, at most one of $a_3^-a_3^+$, a_3a_1 is present. Hence we have at least one of the chords $a_1^-a_1^+$, a_1a_2, at most one of $a_2^-a_2^+$, a_2a_3, and at most one of $a_3^-a_3^+$, a_3a_1. Then it is straightforward to check that in each of the possible cases either $\langle V(C_1)\rangle_{G'}$ is a clique or we can find a required cycle C_2.

Proof of Theorem 3. (i) Let F_1, F_2 be two cycle closures of G, suppose $E(F_1) \setminus E(F_2) \neq \emptyset$ and let $G_1,\ldots,G_t$ be the sequence of graphs that yields F_1. Let $e=xy \in E(G_j) \setminus E(F_2)$ be chosen such that j is as small as possible. Since $e \in E(G_j)$, either $x,y \in N[C]$ for some eligible cycle C in G_{j-1}, or there is a sequence of vertices $x_1,\ldots,x_k$ and graphs $H_1,\ldots,H_k$ such that $H_1=(G_{j-1})'_C$, x_i is eligible in H_i, $H_{i+1}=(H_i)'_{x_i}$, $i=1,\ldots,k$, and $x,y \in N_{H_k}(x_k)$. By Lemma 5 (in the first case) and since obviously a locally connected vertex remains locally connected after adding edges to the graph (in the second case), we have $xy \in E(F_2)$, a contradiction.

(ii) Part (ii) follows immediately from Proposition 1 and from the main result of [8]. □

Example 1. The graph in Fig. 1(a) shows that Proposition 1 fails if we require only one edge e with $\omega_G(e)=2$ in a C_5 or if we admit the two edges to be consecutive. The graph in Fig. 1(b) gives a similar example for a C_6 (elliptical parts represent cliques of order at least three).

Example 2. Linderman [7] proved that the minimum number of edges of a claw-free graph G of order n with a complete closure $\mathrm{cl}(G)$ equals $2n-3$. The graph in Fig. 2 is an example of a claw-free graph G of order $n \equiv 0 \pmod 6$ with a complete cycle closure $\mathrm{cl}_C(G)$ and with only $\frac{3}{2}n-1$ edges.

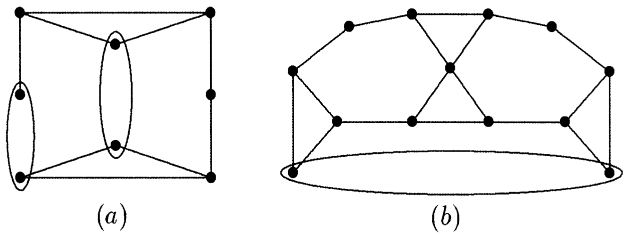

Fig. 1.

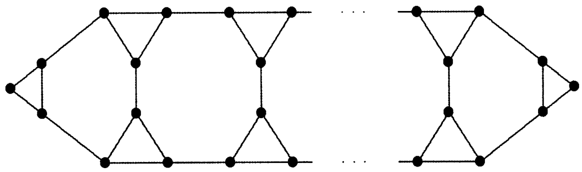

Fig. 2.

Remarks. (i) The graph in Fig. 2 is a closed claw-free graph that contains neither a C_4 nor a $K_4 - e$ as an induced subgraph. This implies that the closure concepts based on neighborhood conditions for the vertices of an induced $K_4 - e$ introduced in [2,4] cannot be applied to add new edges to this graph (while its cycle closure is a complete graph). On the other hand, the closures from [2,4] do not assume claw-freeness of the original graph, and yield additional edges in graphs for which the closure of [8] and the cycle closure are not defined.

(ii) Catlin [5] has introduced a powerful reduction technique that reduces the order of the line graph preimage, preserving the existence of a spanning closed trail, and, with some restrictions, of a DCT in this preimage. Considering the graph $H = K_{2,t}$ for $t \geqslant 3$, it is not difficult to check that H is equal to its reduction (i.e. Catlin's reduction technique is not applicable), $L(H)$ is a closed claw-free graph (hence the closure technique introduced in [8] is also not applicable), but the cycle closure of $L(H)$ is a complete graph. This example shows that the cycle closure technique is not a special case of Catlin's reduction technique. Moreover, it is not known whether the reduction of a graph in the sense of Catlin's technique can be obtained in polynomial time. The same holds for the refinement of Catlin's technique due to Veldman [9].

References

[1] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, Macmillan, London and Elsevier, New York, 1976.

[2] H.J. Broersma, A note on K_4-closures in hamiltonian graph theory, Discrete Math. 121 (1993) 19–23.

[3] H.J. Broersma, Z. Ryjáček, I. Schiermeyer, Closure concepts — a survey, Graphs and Combin. 16(1) (2000) 17–48.

[4] H.J. Broersma, H. Trommel, Closure concepts for claw-free graphs, Discrete Math. 185 (1998) 231–238.
[5] P.A. Catlin, A reduction method to find spanning Eulerian subgraphs, J. Graph Theory 12 (1988) 29–44.
[6] F. Harary, C.St.J.A. Nash-Williams, On eulerian and hamiltonian graphs and line graphs, Canad. Math. Bull. 8 (1965) 701–709.
[7] W. Linderman, Edge extremal graphs with hamiltonian properties, Ph.D. Thesis, University of Memphis, USA, 1998.
[8] Z. Ryjáček, On a closure concept in claw-free graphs, J. Combin. Theory Ser. B 70 (1997) 217–224.
[9] H.J. Veldman, On dominating and spanning circuits in graphs, Discrete Math. 124 (1994) 229–239.