Queueing Syst (2013) 75:1–17
DOI 10.1007/s11134-012-9336-4

Cost sharing of cooperating queues in a Jackson network

Judith Timmer · **Werner Scheinhardt**

Received: 26 April 2012 / Revised: 7 November 2012 / Published online: 28 December 2012

Abstract We consider networks of queues in which the independent operators of individual queues may cooperate to reduce the amount of waiting. More specifically, we focus on Jackson networks in which the total capacity of the servers can be redistributed over all queues in any desired way. If we associate a cost to waiting that is linear in the queue lengths, it is known from the literature how the operators should share the available service capacity to minimize the long run total cost. This paper deals with the question whether or not (the operators of) the individual queues will indeed cooperate in this way, and if so, how they could share the cost in the new situation such that each operator never pays more than his own cost without cooperation. For the particular case of a tandem network with two or three nodes it is known from previous work that cooperation is indeed beneficial, but for larger tandem networks and for general Jackson networks this question was still open. The main result of this paper gives for any Jackson network an explicit cost allocation that is beneficial for all operators. The approach we use also works for other cost functions, such as the server utilization.

Keywords Jackson network · Cooperation · Cost allocation · Game theory · Capacity allocation

In the Netherlands, the three universities of technology have formed the 3TU.Federation. This article is the result of joint research in the 3TU.Centre of Competence NIRICT (Netherlands Institute for Research on ICT).

J. Timmer · W. Scheinhardt (✉)
Department of Applied Mathematics, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands
e-mail: w.r.w.scheinhardt@utwente.nl

J. Timmer
e-mail: j.b.timmer@utwente.nl

Mathematics Subject Classification (2000) 91A12 · 60K25

1 Introduction

Consider a queueing network consisting of different queues, and assume that each of these is operated by a different, independent operator. By working together (in some way), the operators can optimize the performance of the network (in some sense), leading to a social optimum with minimum total cost for the operators. On the other hand, individual operators will try to minimize their own cost, and will only cooperate if this is to their own benefit. This explains our idea of analyzing such networks using cooperative game theory.

In particular, we can view the independent operators as decision makers (or *players*) in a so-called *cooperative cost game*; see for example, [10]. In such a game, the players make binding agreements (as opposed to non-cooperative games) to jointly optimize the total cost they need to pay, and then try to share this cost by finding a fair cost allocation. Typically, a cost allocation is fair if it lowers the cost for each possible coalition (i.e., for any group of players). If this is not the case, then full cooperation is not beneficial, but there may still be partial cooperation between some (but not all) players.

When we try to model a queueing network as described above, there are a number of choices to be made. First of all, (i) one can think of a variety of ways in which the operators may work together, including sharing service capacity, sharing buffer capacity, or changing the routing structure; moreover, any of these can be done dynamically or statically (i.e., dependent on the current state of the network or not). Furthermore, (ii) different network topologies may be considered. Similarly, (iii) traffic characteristics, i.e., the behavior of the arrival process(es) and service demands, can be modeled in many ways. Finally, (iv) the performance of the total network and its individual queues, and the associated cost, can be measured in many ways.

In this paper, we study an initial model to investigate whether this line of research is useful to pursue. In this model, we make the following assumptions: (i) the different servers are able to share their service capacity, and do so in a static way; (ii, iii) the network is an n-node Jackson network see for example, [13]: customers arrive to (some of the) nodes according to independent Poisson processes, then (after service) move to another queue, according to some routing probabilities, etcetera, until they leave the system; service times are all exponentially distributed; (iv) we take the cost at queue i to be proportional to the long run expected queue length, or equivalently (by Little's law), proportional to the expected sojourn time of customers in queue i; furthermore, the total cost of a group of queues is just the sum of the costs of the individual queues (thus, there is no cost associated to the cooperation itself).

In earlier work [12], we treated a special case of our Jackson model, in which the network is a traditional tandem queue. We found that even this very elementary model exhibits some interesting and non-trivial behavior. In the current paper, we treat the more general model, but also find better results. In particular, we present a cost allocation which is shown to be beneficial for all operators.

In the queueing literature, there are many references in which the queues in a network cooperate (sharing capacity, pooling resources, etc.) to reach some form of optimality, e.g., [2,9]. However, in most cases the whole network is (implicitly) supposed to be run by a single operator. The combination of the queueing model with game theory, in which independent operators are only willing to cooperate if a good cost allocation can be found while they remain independent, is to the best of our knowledge hardly studied so far. Some related references are the following. González and Herrero [7] study several medical departments that may share an operating theater. The cost of each department is linear in the capacity needed to satisfy a maximum on the expected waiting time of its patients. It is shown that cooperation reduces the total cost, and that a cost allocation can be determined based on the Shapley value. In [4], García-Sanz et al. extend this model and study cooperation among Markovian queues that share a common server with preemptive priority discipline. The authors show that a cost allocation proportional to the arrival rates is fair. More recently, Anily and Haviv [3] study cooperation in service capacity management. A number of servers pool their capacities and customers. The cost of cooperation is the mean number of customers in the pooled system. It is shown that fair cost allocations always exist. In particular, servers with large capacities may receive payment for cooperation. In these three papers, the servers cooperate by means of pooling. One way in which our paper contributes is that we consider cooperation in a network of queues, while preserving the autonomy of the individual queues; we do not allow for pooling.

In [5], cooperative game theory is used to study resource allocation in dynamic ad-hoc networks, assuming that the cost function is superadditive. We also like to mention [6] where a cooperative game is considered in which countries can form coalitions to optimize the routing of (international) teletraffic streams. For a trial data set, they identify the most important members of the possible coalitions and the way in which benefits could be shared. Finally, we mention Altman et al. [1] who give an extensive survey on networking games. The models and papers discussed in this reference mostly deal with non-cooperative game theory, the only exception being a short section focused on bargaining games. This strengthens our belief that the problem formulation and approach in the current paper have not been studied before.

We end this section with a short overview of the remainder of the paper: in Sect. 2, we introduce the model in more detail with the optimal capacity allocation, and recall the basics of cooperative game theory. Based on these, we introduce the so-called *Jackson games* in Sect. 2.3, for which we then present the main result, i.e., an explicit cost allocation that is beneficial for all operators, see Theorem 3. In Sect. 3, we derive some additional results and present some examples, distinguishing between Jackson games with two, three, and more than three operators involved. In Sect. 4, we focus on a special case, viz. the tandem games as earlier presented in [12], and on an extension in which we choose the server utilization, rather than the expected queue length, as performance measure. We conclude in Sect. 5, also sketching some main lines for future research.

2 Jackson games: model, and main result

In this section, we first introduce our model in detail, and derive the optimal capacity allocation. Then we recall the basics of cooperative game theory, focusing on the solution concept of the *core*, which is the set of all cost allocations that are acceptable to all possible coalitions of queues. The last subsection shows how our Jackson model fits into the framework of cooperative game theory.

2.1 Model

We consider an n-node Jackson network, denoting the set of all queues by $N = \{1, 2, \ldots, n\}$. External customers arrive at queue i according to a Poisson process with rate λ_i^0. After finishing its service at queue i, any customer joins queue j with probability p_{ij} independent of all else, and leaves the network with probability $1 - \sum_{j\in N} p_{ij}$. We are only concerned with the local arrival rates λ_i to queue i, which follow from the traffic equations $\lambda_j = \lambda_j^0 + \sum_{i\in N} p_{ij}\lambda_i$ for all $j \in N$. The exponential service capacity at queue i is given by μ_i; to be more precise, we assume that upon arrival at queue i, all customers draw a random, exponentially distributed workload with mean 1, and the server at queue i is working at a rate of μ_i units of workload per unit of time. For stability, we assume $\mu_i > \lambda_i$. The cost incurred at queue i is represented by the long run expected queue length $\lambda_i/(\mu_i - \lambda_i)$. Furthermore, the total cost of any subset $S \subseteq N$ of queues is the sum of the costs of the individual queues in S.

Importantly, we assume that the queues in any subset S may cooperate to improve their performance and save on costs. Cooperation here means that the queues in S may redistribute their service capacities among each other. Denoting the service capacity of queue i after redistribution by m_i, this leads to the following optimization problem for the set S:

$$\begin{aligned} \min_{m_i, i\in S} \quad & \sum_{i\in S} \frac{\lambda_i}{m_i - \lambda_i} \\ \text{s.t.} \quad & \sum_{i\in S} m_i = \sum_{i\in S} \mu_i, \\ & m_i > \lambda_i, \qquad i \in S. \end{aligned} \tag{1}$$

To solve this, we omit the second constraint (which will turn out to be fulfilled automatically), and rewrite the problem as

$$\min_{\alpha, m_i, i\in S} \sum_{i\in S} \frac{\lambda_i}{m_i - \lambda_i} - \alpha \left(\sum_{i\in S} m_i - \sum_{i\in S} \mu_i \right), \tag{2}$$

where α is the Lagrange multiplier w.r.t. the first constraint.

Before turning to the solution, we introduce some additional notation. For each queue, we define the *relative excess capacity value*, or simply r value, as the value of the excess capacity, relative to the square root of the arrival rate, i.e.,

$$r_i = \frac{\mu_i - \lambda_i}{\sqrt{\lambda_i}}, \qquad i \in N. \tag{3}$$

We also generalize this concept to the r value of a set S as follows. Let

$$\bar{r}_S = \frac{\sum_{i\in S}(\mu_i - \lambda_i)}{\sum_{i\in S} \sqrt{\lambda_i}} = \sum_{i\in S} \frac{\sqrt{\lambda_i}}{\sum_{k\in S} \sqrt{\lambda_k}} r_i, \qquad S \subseteq N. \tag{4}$$

Both expressions will turn out to be helpful in the sequel; the first defines $\bar{r}_S$, generalizing (3), and the second gives $\bar{r}_S$ as a weighted average of the r values of the queues in S. Also, for example, for two disjoint sets S and T we have

$$\bar{r}_{S\cup T} = \frac{\sum_{i\in S}\sqrt{\lambda_i}}{\sum_{k\in S\cup T}\sqrt{\lambda_k}}\,\bar{r}_S + \frac{\sum_{i\in T}\sqrt{\lambda_i}}{\sum_{k\in S\cup T}\sqrt{\lambda_k}}\,\bar{r}_T. \tag{5}$$

Turning back to solving (2), we can take derivatives which leads to the solution [8, p. 63]

$$m_{i,S} = \lambda_i + \frac{\sqrt{\lambda_i}}{\sum_{k\in S}\sqrt{\lambda_k}}\sum_{k\in S}(\mu_k - \lambda_k) = \lambda_i + \sqrt{\lambda_i}\,\bar{r}_S \qquad i \in S, \tag{6}$$

which is denoted by $m_{i,S}$ to stress the dependence on the set S. In this solution, the total excess capacity $\sum_{k\in S}(\mu_k - \lambda_k)$ is distributed proportional to the square root of the arrival rate. Also,

$$\alpha_S = -\frac{\left(\sum_{k\in S}\sqrt{\lambda_k}\right)^2}{\left(\sum_{k\in S}(\mu_k - \lambda_k\right)^2} = -\frac{1}{\bar{r}_S^2} \tag{7}$$

is the Lagrange multiplier for subset S. The corresponding minimal cost for the set S of (cooperating) queues is

$$c(S) = \frac{\left(\sum_{k\in S}\sqrt{\lambda_k}\right)^2}{\sum_{k\in S}(\mu_k - \lambda_k)} = \frac{\sum_{k\in S}\sqrt{\lambda_k}}{\bar{r}_S}. \tag{8}$$

Notice that queue i contributes the amount $\sqrt{\lambda_i}\sum_{k\in S}\sqrt{\lambda_k}/\sum_{k\in S}(\mu_k - \lambda_k) = \sqrt{\lambda_i}/\bar{r}_S$ to the cost for S.

2.2 Preliminaries on cooperative cost games

A cooperative cost game is represented by a pair (N, c). The set $N = \{1, \ldots, n\}$ is the set of players. A coalition S is a (nonempty) group of players, that is, a nonempty subset of N. The cost function c assigns to each coalition S a certain cost $c(S)$.

In our analysis, we will need the concept of marginal vectors, and monotonicity. Let $\sigma = (\sigma(1), \ldots, \sigma(n))$ be a permutation of the player set, where $\sigma(k)$ is the player in position k. Denote by $P_\sigma(i) = \{j \in N | \sigma^{-1}(j) < \sigma^{-1}(i)\}$ the set of players in positions before player i. Now imagine that the players enter a room one by one in the ordering indicated by σ, and that each player has to pay the marginal contribution to the total cost when he enters the room. Then player i pays

$$m_i^\sigma(c) = c(P_\sigma(i) \cup \{i\}) - c(P_\sigma(i)). \tag{9}$$

The vector $m^\sigma(c) = (m_1^\sigma(c), \ldots, m_n^\sigma(c))$ is called the marginal vector[1] corresponding to the permutation σ. Further, a cost game is called *monotone increasing* (respectively decreasing) if $S \subseteq T$ implies $c(S) \leq c(T)$ (respectively $c(S) \geq c(T)$).

A game is *additive* if the coalitional costs are additive, $c(S) = \sum_{k \in S} c(\{k\})$. If for any two disjoint coalitions S and T of players it is beneficial to cooperate, we say that the game is *subadditive*. In this case, cooperation never leads to higher cost when compared to working separately:

$$c(S \cup T) \leq c(S) + c(T). \tag{10}$$

Remark 1 Notice that in a subadditive game the choice $T = N \setminus S$ in (10) implies $c(N) \leq c(S) + c(N \setminus S)$. Thus, if we split the coalition N of all players in two parts, namely the coalitions S and $N \setminus S$, then the total cost does not decrease. This is an incentive for all the players in coalition N to cooperate. □

The main question that remains is how the total joint cost $c(N)$ should be allocated among the players. A first step towards selecting a good and fair cost allocation is to consider allocations in the *core* $C(N, c)$ of the game (N, c), which is defined as

$$C(N, c) = \left\{ y \in \mathbb{R}^N \,\middle|\, \sum_{i \in N} y_i = c(N); \sum_{i \in S} y_i \leq c(S) \text{ for all } S \subset N \right\}.$$

If the cost is allocated among the players according to an allocation in the core, then any coalition S pays at most its own costs $c(S)$. Hence, no coalition has an incentive to break up the cooperation with coalition N.

We are now ready to view the Jackson network problem as a cooperative cost game.

2.3 Jackson games

Based on the optimal capacity allocation of a group of queues in our n-node Jackson network, see Sect. 2.1, we define a corresponding cooperative cost game. From now on we refer to this game as a Jackson game.

Definition A Jackson game is a cost game (N, c) with the set of queues $N = \{1, \ldots, n\}$ as player set. The cost $c(S)$ of coalition $S \subseteq N$ is given by (8).

Proposition 1 *The following properties hold.*

(i) *Jackson games are not monotone decreasing.*
(ii) *For all $n \geq 2$, n-node Jackson games may or may not be monotone increasing.*
(iii) *Jackson games are subadditive.*

[1] Where this is convenient, we will denote the permutation in the superscript without parentheses and commas; e.g., in Sect. 3.2 we write $m^{123}(c)$ instead of $m^{(1,2,3)}(c)$, etc.

Proof For (i), assume without loss of generality that $r_1 \le r_2$, and let $T = \{1, 2\}$ and $S = \{2\}$. Then we have by (4) that

$$\bar{r}_T = \frac{\sqrt{\lambda_1}}{\sqrt{\lambda_1} + \sqrt{\lambda_2}} r_1 + \frac{\sqrt{\lambda_2}}{\sqrt{\lambda_1} + \sqrt{\lambda_2}} r_2 \le r_2,$$

and hence, using (8),

$$c(S) = \frac{\sqrt{\lambda_2}}{r_2} < \frac{\sqrt{\lambda_1} + \sqrt{\lambda_2}}{r_2} \le \frac{\sqrt{\lambda_1} + \sqrt{\lambda_2}}{\bar{r}_T} = c(T).$$

To prove (ii), we construct two concrete examples. A (non-trivial) example of an n-node monotone increasing Jackson game can be found by choosing the capacities sufficiently close together, e.g., take $\lambda_i = 1$ and all μ_i inside the interval $[2-\varepsilon, 2+\varepsilon]$ for some $\varepsilon > 0$. Then the cost of any k-node coalition lies inside $[k/(1+\varepsilon), k/(1-\varepsilon)]$. By taking $\varepsilon < 1/(2k+1)$, we can ensure that these intervals do not overlap for different $k \le n$. On the other hand, any n-node Jackson game with $\lambda_1 = \lambda_2 = \lambda$ and $r_2 > 3r_1$ (i.e., $\mu_2 > 3\mu_1 - 2\lambda$) is not monotone increasing since for $T = \{1, 2\}$ we have by (8) and (4) that $c(T) = 2\sqrt{\lambda}/\bar{r}_T = 4\sqrt{\lambda}/(r_1 + r_2) < \sqrt{\lambda_1}/r_1 = c(\{1\})$.

To prove (iii), we observe that optimal capacity allocations for coalitions S and T induce a feasible capacity allocation for coalition $S \cup T$ in optimization problem (1). □

The fact that Jackson games are usually not monotone is not helpful for the analysis of the core. Before we move on to this in Sect. 3, we present some weaker monotonicity results. In the following proposition, we show that the total cost of any coalition may increase or decrease by adding queues with respectively sufficiently low or high capacity to the coalition.

Proposition 2 *Consider two coalitions S and T satisfying $S \subseteq T \subseteq N$. Then $c(S) \le c(T)$ is equivalent with*

$$\bar{r}_{T\setminus S} \le \left(1 + \frac{\sum_{k\in T}\sqrt{\lambda_k}}{\sum_{k\in S}\sqrt{\lambda_k}}\right)\bar{r}_S. \tag{11}$$

In the particular case where $T \setminus S$ only contains a single node, say $T = S \cup \{i\}$ with $i \notin S$, we have the following.

- (i) *A simple sufficient condition for $c(S) \le c(T)$ (increasing cost when adding queue i to S) is given by $r_i \le 2\bar{r}_S$.*
- (ii) *A simple sufficient condition for $c(S) > c(T)$ (decreasing cost when adding queue i to S) is that both $r_i > 3\bar{r}_S$ and $\sqrt{\lambda_i} \le \sum_{k\in S}\sqrt{\lambda_k}$ hold.*

Proof Using (8), the inequality $c(S) \le c(T)$ can be rewritten as

$$\sum_{k\in S}\sqrt{\lambda_k}\,\bar{r}_T \le \sum_{k\in T}\sqrt{\lambda_k}\,\bar{r}_S.$$

Substituting

$$\bar{r}_T = \frac{\sum_{k\in S}\sqrt{\lambda_k}}{\sum_{k\in T}\sqrt{\lambda_k}}\,\bar{r}_S + \frac{\sum_{k\in T\setminus S}\sqrt{\lambda_k}}{\sum_{k\in T}\sqrt{\lambda_k}}\,\bar{r}_{T\setminus S},$$

which is immediate from (5), we obtain

$$\sum_{k\in S}\sqrt{\lambda_k}\left(\frac{\sum_{k\in S}\sqrt{\lambda_k}}{\sum_{k\in T}\sqrt{\lambda_k}}\,\bar{r}_S + \frac{\sum_{k\in T\setminus S}\sqrt{\lambda_k}}{\sum_{k\in T}\sqrt{\lambda_k}}\,\bar{r}_{T\setminus S}\right) \le \sum_{k\in T}\sqrt{\lambda_k}\,\bar{r}_S.$$

Solving for $\bar{r}_{T\setminus S}$ leads to (11). Statements (i) and (ii) follow immediately. □

2.4 Main result

As mentioned before, we study the core of Jackson games. In particular, we want to know if fair cost allocations always exist. That is, if the core is always a nonempty set. We now present our main result, which specifies an allocation for general Jackson games, and prove that indeed it always belongs to the core.

Theorem 3 *Consider an n-node Jackson network. The corresponding Jackson game has a nonempty core. In particular, the cost allocation* $x := (x_1, \dots, x_n)$ *with*

$$x_i = \left(2\frac{\sqrt{\lambda_i}}{\sum_{j\in N}\sqrt{\lambda_j}} - \frac{\mu_i - \lambda_i}{\sum_{j\in N}(\mu_j - \lambda_j)}\right)c(N)$$

belongs to the core. For any coalition S, *the cost under this allocation is in fact strictly less than* $c(S)$, *unless* $\bar{r}_S = \bar{r}_N$.

Proof It follows from the definition of x that this allocation is efficient, i.e., $\sum_{i\in N} x_i = c(N)$. To verify the core condition for any coalition S, i.e., $\sum_{i\in S} x_i \le c(S)$, we use the definition of x to state the equivalent condition

$$2 \le \frac{\sum_{j\in N}\sqrt{\lambda_j}}{\sum_{j\in S}\sqrt{\lambda_j}}\left(\frac{c(S)}{c(N)} + \frac{\sum_{j\in S}(\mu_j - \lambda_j)}{\sum_{j\in N}(\mu_j - \lambda_j)}\right).$$

Since this can be rewritten, using (4) and (8), in the form

$$2 \le \frac{\bar{r}_N}{\bar{r}_S} + \frac{\bar{r}_S}{\bar{r}_N},$$

and since $a + a^{-1} \ge 2$ for any $a > 0$ (with equality holding only for $a = 1$), the statements of the theorem follow. □

Hence, a stable basis for cooperation always exists. Notice that the cost x_i allocated to a node i may be negative; the node is then payed for cooperation.

This theorem also indicates that if $\bar{r}_S < \bar{r}_N$ for all $S \subset N$, then the allocation lies in the interior of the core; small perturbations (under the condition that the sum of the allocation remains $c(N)$), do not get the allocation out of the core.

Interestingly, the cost allocation in Theorem 3 can also be given an interpretation by rewriting it as

$$x_i = \frac{\lambda_i}{m_{i,N} - \lambda_i} - \alpha_N(m_{i,N} - \mu_i), \tag{12}$$

where α_N is the Lagrange multiplier as in (7). Thus, in addition to their "own cost" *after* the capacity reallocation, all players pay (or receive) an additional amount which is linear in the capacity increase (or decrease) they experience, weighted with the (negative) shadow price of one unit of capacity. In fact, the form (12) can also be found from [11], where it is also shown that so-called "market games" have a nonempty core, which implies that the game is totally balanced. Indeed, our game is such a totally balanced market game.

3 Additional results

In this section, we first derive additional results on the core of two- and three-node networks. Then, we point out how larger networks differ from three-node networks, and present additional results for these networks.

3.1 Two-node Jackson games

According to Theorem 3, Jackson games corresponding to two-node networks have a nonempty core. This follows also from the subadditivity property (10) of Jackson games, see Remark 1.

We give an example to illustrate that the core can easily be found explicitly in this case.

Example 1 Consider a two-node network with arrival rates $\lambda_1 = 1$, $\lambda_2 = 4$, and service rates $\mu_1 = \mu_2 = 5$. The costs of the coalitions are $c(\{1\}) = 1/4$, $c(\{2\}) = 4$, and $c(N) = 9/5$. Since cooperation is worthwhile, cost savings are achieved: $c(N) < c(\{1\}) + c(\{2\})$. The core of the game is nonempty and equals

$$C(N, c) = \{(x, 9/5 - x) \,|\, -11/5 \le x \le 1/4\}.$$

□

We note that the cost allocation in Theorem 3 may also lead to negative cost in the case of widely varying r values. For instance, in the setting of Example 1, the resulting cost allocation is $x = (-18/75, 153/75)$, which has a negative cost component for queue 1. In such allocations, the first queue gets paid for its cooperation. Note that the first queue has the largest excess capacity; its contribution is so valuable that this queue may receive payment to cooperate. This phenomenon is not always present, e.g., when $\mu_1 = 2$, we get $c(\{1\}) = 1$, $c(\{2\}) = 4$, $c(N) = 9/2$ and $C(N, c) = \{(x, 9/2 - x) \,|\, 1/2 \leq x \leq 1\}$.

3.2 Three-node Jackson games

All the results in this subsection are based on orderings of the r values of the queues. First, we identify two cost allocations that belong to the core, but that are generally different from the one in Theorem 3.

Theorem 4 *If we assume*[2] *that*

$$r_1 \geq r_2 \geq r_3, \tag{13}$$

and $\sigma(1) = 2$, *i.e., queue 2 is in first position, then the marginal vector* $m^\sigma(c)$ *is a cost allocation that belongs to the core of the game.*

Proof There are two marginal vectors in which node 2 is in first position, namely $m^{213}(c)$ and $m^{231}(c)$. Consider the first marginal vector. The marginal contributions of the nodes (see (9)) are

$$\begin{aligned} m_1^{213}(c) &= c(\{1, 2\}) - c(\{2\}), \\ m_2^{213}(c) &= c(\{2\}), \\ m_3^{213}(c) &= c(N) - c(\{1, 2\}). \end{aligned}$$

We proceed by checking the core-conditions:

- $m_1^{213}(c) \leq c(\{1\})$: true by subadditivity;
- $m_2^{213}(c) \leq c(\{2\})$: true (with equality);
- $m_3^{213}(c) \leq c(\{3\})$: true by subadditivity;
- $m_1^{213}(c) + m_2^{213}(c) \leq c(\{1, 2\})$: true (with equality);
- $m_1^{213}(c) + m_3^{213}(c) \leq c(\{1, 3\})$: true by subadditivity;
- $m_2^{213}(c) + m_3^{213}(c) \leq c(\{2, 3\})$: see below;
- $m_1^{213}(c) + m_2^{213}(c) + m_3^{213}(c) = c(N)$: true.

It remains to prove the condition related to coalition {2, 3}, which can be written as

$$c(\{2\}) + c(N) - c(\{1, 2\}) \leq c(\{2, 3\}), \tag{14}$$

or as

$$\left(\frac{\lambda_2}{x_2}\right) + \left(\frac{\lambda_1}{\bar{x}_1} + \frac{\lambda_2}{\bar{x}_2} + \frac{\lambda_3}{\bar{x}_3}\right) \leq \left(\frac{\lambda_1}{\hat{x}_1} + \frac{\lambda_2}{\hat{x}_2}\right) + \left(\frac{\lambda_2}{\check{x}_2} + \frac{\lambda_3}{\check{x}_3}\right), \tag{15}$$

[2] We may assume this without loss of generality by relabeling the nodes. If two or three queues have equal r value, either of these can be chosen as “the” queue with middle r value.

where we define (recall also (6)),

$$\begin{aligned} x_i &= \mu_i - \lambda_i, && i = 1, 2, 3, \\ \bar{x}_i &= m_{i,\{1,2,3\}} - \lambda_i, && i = 1, 2, 3, \\ \hat{x}_i &= m_{i,\{1,2\}} - \lambda_i, && i = 1, 2, \\ \check{x}_i &= m_{i,\{2,3\}} - \lambda_i, && i = 2, 3. \end{aligned}$$

Each of these quantities can be interpreted as the excess capacity for queue i for the optimal capacity allocation in a certain coalition, namely:

x_i for coalition $\{i\}$ (i.e. without cooperation),
$\bar{x}_i$ for coalition $\{1, 2, 3\}$,
$\hat{x}_i$ for coalition $\{1, 2\}$,
$\check{x}_i$ for coalition $\{2, 3\}$.

For $\hat{x}_2$ we can write, based on (6)

$$\begin{aligned} \hat{x}_2 &= \frac{\sqrt{\lambda_2}}{\sqrt{\lambda_1} + \sqrt{\lambda_2}}(\mu_1 - \lambda_1 + \mu_2 - \lambda_2) \\ &= \frac{\sqrt{\lambda_2}}{\sqrt{\lambda_1} + \sqrt{\lambda_2}}(x_1 + x_2) \\ &\geq x_2, \end{aligned}$$

where we used $x_1\sqrt{\lambda_2} \geq x_2\sqrt{\lambda_1}$, which is due to the ordering in (13). Similarly, we have $\check{x}_2 \leq x_2$. As for $\bar{x}_1, \bar{x}_2, \bar{x}_3$, they satisfy $\bar{x}_1 + \bar{x}_2 + \bar{x}_3 = x_1 + x_2 + x_3$ and $\bar{x}_i + \lambda_i$, $i = 1, 2, 3$, are the optimal solution to (1) for $S = N$. Now we consider a suboptimal solution to this, given by

$$\tilde{x}_1 = \hat{x}_1, \quad \tilde{x}_2 = \hat{x}_2 + \check{x}_2 - x_2, \quad \tilde{x}_3 = \check{x}_3.$$

Notice that indeed $\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 = x_1 + x_2 + x_3$ as should, because $\hat{x}_1 + \hat{x}_2 = x_1 + x_2$ and $\check{x}_2 + \check{x}_3 = x_2 + x_3$. Furthermore we have $\tilde{x}_2 = \hat{x}_2 - (x_2 - \check{x}_2) \leq \hat{x}_2$ and similarly $\tilde{x}_2 \geq \check{x}_2$. We can now prove (15) as follows,

$$\begin{aligned} \left(\frac{\lambda_2}{x_2}\right) + \left(\frac{\lambda_1}{\bar{x}_1} + \frac{\lambda_2}{\bar{x}_2} + \frac{\lambda_3}{\bar{x}_3}\right) &\leq \frac{\lambda_2}{x_2} + \frac{\lambda_1}{\tilde{x}_1} + \frac{\lambda_2}{\tilde{x}_2} + \frac{\lambda_3}{\tilde{x}_3} \\ &= \frac{\lambda_2}{x_2} + \frac{\lambda_1}{\hat{x}_1} + \frac{\lambda_2}{\tilde{x}_2} + \frac{\lambda_3}{\check{x}_3} \\ &\leq \frac{\lambda_2}{\hat{x}_2} + \frac{\lambda_1}{\hat{x}_1} + \frac{\lambda_2}{\check{x}_2} + \frac{\lambda_3}{\check{x}_3}, \end{aligned}$$

where the first inequality is due to the optimality of $\bar{x}_i + \lambda_i$, $i = 1, 2, 3$, in (1) for $S = N$, and the second inequality is due to the convexity of the function λ_2/x in x and the fact that both x_2 and $\tilde{x}_2$ lie in the interval $[\check{x}_2, \hat{x}_2]$.

For the second marginal vector $m^{231}(c)$, the marginal contributions of the nodes are

$$\begin{aligned} m_1^{213}(c) &= c(N) - c(\{2,3\}), \\ m_2^{213}(c) &= c(\{2\}), \\ m_3^{213}(c) &= c(\{2,3\}) - c(\{2\}). \end{aligned}$$

As above, the core-conditions can be checked to hold, either with equality, or by subadditivity. The only exception is the condition related to coalition $\{1,2\}$: $c(N) - c(\{2,3\}) + c(\{2\}) \le c(\{1,2\})$, which is equivalent to (14) and therefore also holds. This proves the result. □

Notice that any convex combination of the two above-mentioned marginal vectors also belongs to the core because this is a convex set. All of these cost allocations are such that the queue with middle r value does not gain from cooperation. That is, the cost that is allocated to this queue is the same as its stand-alone cost $c(\{i\})$. Below we present a condition under which *all* cost allocations in the core have this property.

Proposition 5 *Consider a three-node Jackson network. If* $r_2 = \bar{r}_{\{1,3\}}$ *then*

(i) $c(\{2\}) + c(\{1,3\}) = c(N)$,
(ii) $c(\{2\})/c(N) = \sqrt{\lambda_2}/(\sqrt{\lambda_1} + \sqrt{\lambda_2} + \sqrt{\lambda_3})$,
(iii) *node 2 has no strict gain from cooperation, and*
(iv) *the core* $C(N,c)$ *is the convex hull of the two marginal vectors* $m^{213}(c)$ *and* $m^{231}(c)$.

Proof (i) Assuming $r_2 = \bar{r}_{\{1,3\}}$, we write

$$c(\{2\}) + c(\{1,3\}) = \frac{\sqrt{\lambda_2}}{r_2} + \frac{\sqrt{\lambda_1} + \sqrt{\lambda_3}}{\bar{r}_{\{1,3\}}} = \frac{\sqrt{\lambda_1} + \sqrt{\lambda_2} + \sqrt{\lambda_3}}{\bar{r}_{\{1,2,3\}}} = c(N),$$

where the first equality is due to (8), and the second equality follows after noting that (5) implies

$$\bar{r}_{\{1,2,3\}} = \frac{\sqrt{\lambda_2}}{\sum_{k\in N}\sqrt{\lambda_k}} r_2 + \frac{\sqrt{\lambda_1} + \sqrt{\lambda_3}}{\sum_{k\in N}\sqrt{\lambda_k}} \bar{r}_{\{1,3\}} = r_2 = \bar{r}_{\{1,3\}}.$$

Statement (ii) follows from applying (8) to find

$$\frac{c(\{2\})}{c(N)} = \frac{\sqrt{\lambda_2}/r_2}{(\sqrt{\lambda_1} + \sqrt{\lambda_2} + \sqrt{\lambda_3})/\bar{r}_{\{1,2,3\}}},$$

and then again noting that $r_2 = \bar{r}_{\{1,2,3\}}$.

For statement (iii), first note that any allocation x in the core should satisfy $x_2 \le c(\{2\})$ and $x_1 + x_3 \le c(\{1,3\})$. Together with the efficiency condition $x_1 + x_2 + x_3 = c(N)$ and statement (i) this leads to $x_2 = c(\{2\})$ for any allocation in the core. Hence, node 2 does not gain from cooperation.

Table 1 Costs of all coalitions for the game in Example 2

S	{1}	{2}	{3}	{4}	{1,2}	{1,3}	{1,4}	{2,3}	{2,4}	{3,4}
$c(S)$	1/7	1/5	1/3	1	1/3	2/5	1/2	1/2	2/3	1
S	{1,2,3}	{1,2,4}	{1,3,4}	{2,3,4}	{1,2,3,4}					
$c(S)$	3/5	9/13	9/11	1	1					

Finally, we turn to statement (iv). The equation $r_2 = \bar{r}_{\{1,3\}}$ implies that either $r_1 \geq r_2 \geq r_3$ or $r_3 \geq r_2 \geq r_1$. According to Theorem 4, any convex combination of the two marginal vectors $m^{213}(c)$ and $m^{231}(c)$ belongs to the core $C(N, c)$. By Weber [14], the core is a subset of the convex hull of all marginal vectors. Hence, in this case the core is the convex hull of $m^{213}(c)$ and $m^{231}(c)$. □

Remark 2 One may be inclined to argue that when a player does not gain from cooperation, as in the context of Proposition 5, he may refrain from cooperation to prevent extra benefit for the other players. However, it is easy to see (from the first statement) that his decision to cooperate or not does not affect the cost paid, or amount gained, by the other players.

3.3 Jackson games with more than three nodes

For three-node Jackson networks, we identified two marginal vectors that belong to the core, see Theorem 4. This approach cannot be extended to four-node networks, as the following example illustrates.

Example 2 Consider the four-node tandem Jackson network with arrival rates $\lambda_i = 1$, $i = 1, \ldots, 4$, and service rates $\mu_1 = 8$, $\mu_2 = 6$, $\mu_3 = 4$, and $\mu_4 = 2$. The costs of the various coalitions in the corresponding Jackson game are given in Table 1. The 24 marginal vectors are listed in Table 2. One can verify that none of these marginal vectors belongs to the core of the game. □

Next, we show some other simple structural results. If two queues i and j have equal r values, then their joint cost $c(\{i, j\})$ is the sum of their individual costs. This follows from the more general statement below by taking $S = \{i\}$ and $T = \{j\}$:

Proposition 6 *Consider an n-node Jackson network with $\bar{r}_S = \bar{r}_T$ for some disjoint coalitions $S, T \subset N$, $S \cap T = \emptyset$. Then $c(S \cup T) = c(S) + c(T)$ in the corresponding Jackson game.*

Proof First note that $\bar{r}_{S\cup T} = \bar{r}_S = \bar{r}_T$. Then by (8) we have

$$c(S \cup T) = \frac{\sum_{i\in S\cup T} \sqrt{\lambda_i}}{\bar{r}_{S\cup T}} = \frac{\sum_{i\in S} \sqrt{\lambda_i}}{\bar{r}_S} + \frac{\sum_{i\in T} \sqrt{\lambda_i}}{\bar{r}_T} = c(S) + c(T).$$

□

Table 2 The 24 marginal vectors of the game in Example 2

m^{1234}	m^{1243}	m^{1324}	m^{1342}	m^{1423}	m^{1432}	m^{2134}	m^{2143}	m^{2314}	m^{2341}	m^{2413}	m^{2431}
0.143	0.143	0.143	0.143	0.143	0.143	0.133	0.133	0.100	0	0.026	0
0.190	0.190	0.400	0.182	0.192	0.182	0.200	0.200	0.200	0.200	0.200	0.200
0.267	0.308	0.257	0.257	0.308	0.318	0.267	0.308	0.300	0.300	0.308	0.333
0.400	0.359	0.200	0.418	0.357	0.357	0.400	0.359	0.400	0.500	0.467	0.467
m^{3124}	m^{3142}	m^{3214}	m^{3241}	m^{3412}	m^{3421}	m^{4123}	m^{4132}	m^{4213}	m^{4231}	m^{4312}	m^{4321}
0.067	0.067	0.100	0	−0.182	0	−0.500	−0.500	0.026	0	−0.182	0
0.200	0.182	0.167	0.167	0.182	0	0.192	0.182	−0.333	−0.333	0.182	0
0.333	0.333	0.333	0.333	0.333	0.333	0.308	0.318	0.308	0.333	0	0
0.400	0.418	0.400	0.500	0.667	0.667	1	1	1	1	1	1

Top to bottom entries correspond to players 1–4

Next, if (and only if) all queues have equal r values, then there is a unique core allocation in which each queue pays its own cost as if in isolation, proportional to the square root of the arrival rate.

Proposition 7 *An n-node Jackson game is additive if and only if all nodes have equal r value. In this case, the core consists of a single allocation x with* $x_i = \lambda_i/(\mu_i - \lambda_i) = \sqrt{\lambda_i}/r$ *for all* i, *where r is the common r value of the nodes.*

Proof If $r_i = r$ for all $i \in N$, then also $\bar{r}_S = r$ for any coalition S. Therefore, the cost of coalition S equals

$$c(S) = \frac{\sum_{i \in S} \sqrt{\lambda_i}}{\bar{r}_S} = \sum_{i \in S} \frac{\sqrt{\lambda_i}}{r_i} = \sum_{i \in S} \frac{\lambda_i}{\mu_i - \lambda_i} = \sum_{i \in S} c(\{i\}).$$

Hence the game is additive, so there is a unique core allocation x with $x_i = \lambda_i/(\mu_i - \lambda_i)$ for all i.

The converse is a consequence of the last part of Theorem 3: when not all r values are equal, some nodes i will have $r_i \neq \bar{r}_N$ and hence $x_i < c(\{i\})$, while any remaining nodes (if any) will have $x_i = c(\{i\})$. Hence $c(N) = \sum_{i \in N} x_i < \sum_{i \in N} c(\{i\})$, so the game cannot be additive. □

4 Special case and extension

We now present a special case (in Sect. 4.1) and an extension (in Sect. 4.2) of our model, which are in particular interesting from the queueing perspective, rather than the game-theoretic perspective.

4.1 Tandem games

When all arrival rates are equal, $\lambda_i \equiv \lambda$, say, as is the case in a tandem queue network (see also [12]), some of the results take on a simpler form. The role of the r values as criterion value for many of the results is now simply played by the excess capacities $\mu_i - \lambda$, or indeed by the capacities μ_i. For convenience sake, we highlight the main results for this particular setting.

Let $|S|$ be the number of queues in coalition S, and $\bar{\mu}_S$ is the mean capacity of the queues in S. Then first of all, the optimal capacity allocation for coalition S is simply to share the total capacity equally between all queues in S, so (6) becomes $m_{i,S} = \bar{\mu}_S = \sum_{i \in S} \mu_i/|S|$, with total cost $c(S) = |S|\,\lambda/(\bar{\mu}_S - \lambda)$. Next, the monotonicity condition (11) in Proposition 2 simplifies to

$$\bar{\mu}_{T\setminus S} \le \bar{\mu}_S + \frac{|T|}{|S|}(\bar{\mu}_S - \lambda).$$

The main result in Theorem 3 shows a particular cost allocation that now simplifies to

$$x_i = \left(2 - \frac{\mu_i - \lambda}{\bar{\mu}_N - \lambda}\right)\frac{c(N)}{|N|}. \tag{16}$$

This belongs to the core of the game, and for any coalition S the cost under this allocation is strictly less than $c(S)$, unless $\bar{\mu}_S = \bar{\mu}_N$.

Theorem 4, and Propositions 5, 6, and 7 continue to hold under the same respective assumptions, in which "r values" can always be simply replaced by the corresponding "μ values". The second statement of Proposition 5 simply becomes $c(\{2\})/c(N) = 1/3$.

4.2 Utilization as cost

We now extend our results to a different cost structures, namely to the case in which the cost of queue i is given by the server utilization, instead of the expected number in queue. This may be useful when the operator is more interested in the direct cost of operation, rather than the delay performance for the customers. Thus, we take $c(i) = \lambda_i/\mu_i$ and replace the minimization problem for coalition S in (1) by

$$\begin{aligned} &\min_{m_i, i\in S} \sum_{i\in S} \frac{\lambda_i}{m_i} \\ &\text{s.t.} \sum_{i\in S} m_i = \sum_{i\in S}\mu_i, \quad m_i \ge \lambda_i. \end{aligned} \tag{17}$$

The second set of constraints $m_i \ge \lambda_i$ is included to ensure that the utilizations never exceed 1. In fact for stability of the queue sizes, we need these inequalities to be strict, but then a solution may not exist.

Since solving the solution to this problem is cumbersome, we restrict ourselves to the assumption that all arrival rates are equal ($\lambda_i = \lambda$, say), as in the previous subsection. The optimal capacity allocation is then to share the total capacity equally between all queues in S, i.e., $m_{i,S} = \bar{\mu}_S = \sum_{i \in S} \mu_i / |S|$. Notice that indeed $\mu_S > \lambda$ if for all $i \in N$ we have $\mu_i > \lambda$, as we assume. The total cost for coalition S is obviously given by $c(S) = |S| \lambda / \bar{\mu}_S$.

In this setting, it is possible to find a similar cost allocation that lies in the core as in Theorem 3, or rather as in (16). It is given by

$$x_i = \left(2 - \frac{\mu_i}{\bar{\mu}_N}\right) \frac{c(N)}{|N|}.$$

This belongs to the core of the game, and for any coalition S the cost under this allocation is strictly less than $c(S)$, unless $\bar{\mu}_S = \bar{\mu}_N$.

5 Conclusions and future work

5.1 Conclusions

We considered a Jackson network of queues in which each queue has an independent operator, and investigated whether or not these operators are willing to cooperate in order to reduce the total waiting cost, which is linear in the expected queue lengths (or equivalently, in the expected waiting times). Such a cooperation will involve sharing the individual service capacities, and then dividing the resulting total cost in some way. Our main conclusions are as follows:

- The core of the corresponding cooperative game is never empty. That is, there always exists a cost allocation such that the (operators of the) individual queues have an incentive to cooperate.
- One specific cost allocation has been found explicitly, see Theorem 3. This allocation is strictly beneficial for each coalition, unless the so-called relative excess capacity value of a coalition equals that of the grand coalition.
- The so-called *relative excess capacity value*, or simply r *value*, as just mentioned, turns out to be an important quantity throughout the analysis. It is defined as the (total) excess capacity of a (coalition of) queue(s), divided by (the sum of) the square root(s) of the arrival rate(s), see (3) and (4).
- For the case with two queues, the core can be characterized explicitly as the convex combination of the two marginal vector allocations; for the case with three queues, the core always contains two specific marginal vectors; for cases with more queues, the core may contain no marginal vectors at all.
- In the case of large asymmetries between queues in terms of arrival and/or service rates (more precisely, in the case of large differences in r value), core allocations can be such that some queue(s) have negative cost. In other words, such queues may receive payment from the other queues for the cooperation.
- Similar results are found when the cost is not based on expected waiting times or queue lengths, but on server utilizations.

5.2 Future work

As mentioned in Sect. 1, many assumptions can be made within our framework of "queueing network games". In particular, we intend to study another way of "sharing capacity", in which the routing pattern is changed such that "underloaded" queues can provide service to jobs that otherwise would have been routed to other (highly loaded) queues. Also, we will consider dynamic ways of cooperation, and stochastic cost structures.

References

1. Altman, E., Boulogne, T., El-Azouzi, R., Jiménez, T., Wynter, L.: A survey on networking games in telecommunications. Comput. Oper. Res. **33**(2), 286–311 (2006)
2. Andradóttir, S., Ayhan, H.: Throughput maximization for tandem lines with two stations and flexible servers. Oper. Res. **53**, 516–531 (2005)
3. Anily, S., Haviv, M.: Cooperation in service systems. Oper. Res. **58**(3), 660–673 (2010)
4. García-Sanz, M.D., Fernández, F.R., Fiestras-Janeiro, M.G., García-Jurado, I., Puerto, J.: Cooperation in Markovian queueing models. Eur. J. Oper. Res. **188**(2), 485–495 (2008)
5. Gibbens, R., Key, P.: Coalition games and resource allocation in ad-hoc networks. In: Liò, P., Yoneki, E., Crowcroft, J., Verma, D. (eds.) Bio-Inspired Computing and Communication, Volume 5151 of Lecture Notes in Computer Science, pp. 387–398. Springer, Berlin (2008)
6. Gibbens, R.J., Kelly, F.P., Cope, G.A., Whitehead, M.J.: Coalitions in the international network. In: Jensen A., Iversen V.B. (eds.) 13th International Teletraffic Congress (ITC), 1991, pp. 93–91 (1991)
7. González, P., Herrero, C.: Optimal sharing of surgical costs in the presence of queues. Math. Methods Oper. Res. **59**(3), 435–446 (2004)
8. Kleinrock, L.: Communication Nets: Stochastic Message Flow and Delay. McGraw-Hill, New York (1964)
9. Mandelbaum, A., Reiman, M.J.: On pooling in queueing networks. Manag. Sci. **44**, 971–981 (1998)
10. Peters, H.: Game Theory. Springer, Berlin (2008)
11. Shapley, L.S., Shubik, M.: On market games. J. Econ. Theory **1**(1), 9–25 (1969)
12. Timmer, J., Scheinhardt, W.: How to share the cost of cooperating queues in a tandem network? In: 22nd International Teletraffic Congress (ITC), 2010, pp. 1–7 (2010)
13. Walrand, J.: An Introduction to Queueing Networks. Prentice-Hall, New Jersey (1988)
14. Weber, R.J.: Probabilistic values for games. In: Roth, A.E. (ed.) The Shapley Value, Essays in Honor of Lloyd S. Shapley, pp. 101–119. Cambridge University Press, Cambridge (1988)