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On a family of values for TU-games generalizing the Shapley value[☆]

Tadeusz Radzik[a,*], Theo Driessen[b]

[a] Institute of Mathematics and Computer Science, Wrocław University of Technology, 50-370 Wrocław, Wybrzeże Wyspiańskiego 27, Poland

[b] Faculty of Electrical Engineering, Mathematics and Computer Science, Department of Applied Mathematics, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands

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ABSTRACT

In this paper we study a family of efficient, symmetric and linear values for TU-games, described by some formula generalizing the Shapley value. These values appear to have surprising properties described in terms of the axioms: Fair treatment, monotonicity and two types of acceptability. The results obtained are discussed in the context of the Shapley value, the solidarity value, the least square prenucleolus and the consensus value.

1. Introduction

It is well known that the Shapley value (Shapley, 1953) for TU-games can be characterized by the axioms of efficiency, symmetry, linearity and the null player axiom. In the considerations of this paper we replace the last axiom by three other ones: *fair treatment*, *monotonicity* and *social acceptability*. Roughly speaking, the fair treatment axiom requires that a player i should be awarded (by a value) not worse than a player j in a game if for every coalition S not containing players i and j, the worth of coalition $S \cup i$ is not smaller than the worth of coalition $S \cup j$. The monotonicity axiom requires for a value to award each player by at least zero in any monotonic game. Next, the social acceptability condition imposes a lower and an upper bound on what a null player obtains in unanimity games; that is, in a unanimity game every null player obtains at least zero, but at most what a non-null player receives in such a game.

The main purpose of the paper is to study how the family of efficient, symmetric and linear values for TU-games changes when we additionally assume some of those three axioms. It turns out that by adding the fair treatment and the monotonicity axioms to conditions defining that family, we obtain two subfamilies that can be described with the help of very surprising formulae. Next these two subfamilies are studied in terms of the social acceptability. The results of the paper are illustrated by and applied to the Shapley value, the solidarity value (Nowak and Radzik, 1994), the least square prenucleolus (Ruiz et al., 1996) and the consensus value (Ju et al., 2007).

The organization of the paper is as follows: In Section 2 we recall all the notions as definitions needed together with some auxiliary results from the literature. The results obtained in the paper and their discussion are contained in Sections 3 and 4. Section 5 is devoted to the proofs.

2. Preliminaries

Let $N = \{1, 2, \ldots, n\}$ with $n \geq 2$ be a fixed finite set of n players, called the *grand coalition*. Subsets of N are called *coalitions* while N is called the *grand coalition*.

The cardinality of a set X will be denoted by $|X|$. For brevity, throughout the paper, the cardinality of sets (coalitions) N, S and T will also be denoted by appropriate small letters n, s and t, respectively. All the set inclusions "$\subset$" are meant to be weak. Also, for notational convenience, we will write singleton $\{i\}$ as i.

A *(transferable utility) game* (on N) is any function $v : 2^N \to \mathcal{R}$ with $v(\emptyset) = 0$, where $\mathcal{R}$ denotes the set of real numbers. Then for any coalition S in N, $v(S)$ describes the *worth* of the coalition S when all the players in S collaborate. A game v is *monotonic* if $v(S) \leq v(T)$ for any $S \subset T \subset N$. The set of all games v is denoted by Γ.

For a coalition $T \subset N$, the *unanimity game* u_T is defined by $u_T(S) = 1$ for $S \supset T$ and $u_T(S) = 0$ otherwise, for $S \subset N$.

A *value* $\Phi(v) = (\Phi_1(v), \ldots, \Phi_n(v))$ on Γ is thought of as a vector-valued mapping $\Phi : \Gamma \to \mathcal{R}^n$, which uniquely determines, for each game $v \in \Gamma$, a distribution of the total wealth available to all the players $1, 2, \ldots, n$, through their participation in the game v. We quickly recall several basic properties a value Φ may have.

A value Φ is called *efficient* if $\sum_{i \in N} \Phi_i(v) = v(N)$ for all games v. If $\Phi(\alpha v + \beta w) = \alpha\Phi(v) + \beta\Phi(N, w)$ for all games v and w and for all reals α and β, a value Φ is called *linear*. If the last equality holds for $\alpha = \beta = 1$, a value is *additive*. A player i is called a *null player* (*dummy player*) in game v if $v(S \cup i) = v(S)$ ($v(S \cup i) = v(S) + v(i)$) for every coalition $S \subset N \setminus i$. If $\Phi_i(v) = 0$ in case of any null player i in game v, we say that a value

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[*] Corresponding author.
E-mail addresses: tadeusz.radzik@pwr.wroc.pl (T. Radzik), t.s.h.driessen@ewi.utwente.nl (T. Driessen).

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Φ satisfies the null player axiom. If a value Φ satisfies the equality $\Phi_{\pi i}(N, \pi v) = \Phi_i(v)$ for all $i \in N$ and every permutation π of the player set N, then we say that Φ satisfies the *anonymity* axiom, sometimes also called the *symmetry* axiom (here πv is defined as game πv by $\pi v(\pi(S)) = v(S)$ for $S \subset N$).

In this paper we will mainly discuss values which verify efficiency, symmetry and linearity. Hence, for brevity, every value satisfying those three properties will be shortly called an *ESL-value*.

Now we recall four *ESL*-values, essential for the illustration of the results obtained in the paper and their application: the *Shapley value*, the *solidarity value*, the *least square prenucleolus* and the *consensus value*. The first of these values is standard in cooperative game theory. The solidarity value was introduced in Nowak and Radzik (1994). In the recent paper Calvo (2008) proposed two variations of the non-cooperative model for games in coalitional form, introduced by Hart and Mas-Colell (1996), and found two new, very interesting NTU-values: the random marginal and random removal values. It turned out that for TU-games, the random marginal value coincides with the Shapley value and that, which was completely surprising, the random removal value coincides with the solidarity value. The third of these values, the least square prenucleolus, is an interesting proposal of a value presented in Ruiz et al. (1996). The fourth value, the consensus value, is a new solution recently proposed by Ju et al. (2007). Below, we describe these four in more detail.

The *Shapley value* Φ^{Sh}. It is the classical value on Γ (Shapley, 1953) determined by $\Phi^{Sh}(v) = (\Phi_1^{Sh}(v), \ldots, \Phi_n^{Sh}(v))$, where for $i \in N$,

$$\Phi_i^{Sh}(v) = \sum_{S \subset N \setminus i} \frac{s!(n-s-1)!}{n!} [v(S \cup i) - v(S)]. \tag{1}$$

It is known that the Shapley value Φ^{Sh} on Γ is the unique *ESL*-value which satisfies the null player axiom.

The *solidarity value* Φ^{So}. This is a value on Γ discussed in the paper (Nowak and Radzik, 1994). It is an *ESL*-value uniquely determined by some modification of the null player axiom, called *A-null player* axiom. We quickly recall this axiom. To express it we need to define, for any non-empty coalition $T \subset N$ and a game v, the quantity

$$A^v(S) = \frac{1}{s} \sum_{k \in S} [v(S) - v(S \setminus k)], \tag{2}$$

where s means the cardinality of S. Clearly, $A^v(S)$ can be seen as the *average marginal contribution* of a member of a coalition S. The axiom is as follows:

A-null player axiom: If $i \in N$ is an *A-null player* in a game v, that is, $A^v(S) = 0$ for every coalition S containing player i, then $\Phi_i(v) = 0$.

It is shown in Nowak and Radzik (1994) that for $v \in \Gamma$, the solidarity value is of the form $\Phi^{So}(v) = (\Phi_1^{So}(v), \ldots, \Phi_n^{So}(v))$, where for $i \in N$,

$$\Phi_i^{So}(v) = \sum_{S \ni i} \frac{(n-s)!(s-1)!}{n!} A^v(S). \tag{3}$$

It is left to the reader to verify with the help of (3) and (2) that the solidarity value can be written in the following equivalent form: For $i \in N$

$$\Phi_i^{So}(v) = \frac{v(N)}{n} - \frac{v(N \setminus i)}{n^2} + \sum_{S \subsetneq N \setminus i} \frac{s!(n-s-1)!}{n!} \left[\frac{v(S \cup i)}{s+2} - \frac{v(S)}{s+1} \right]. \tag{4}$$

The *least square prenucleolus* Φ^L. This value (Ruiz et al., 1996) is defined as the vector $\Phi^L(v) = (\Phi_1^L(v), \ldots, \Phi_n^L(v)) = (x_1^0, \ldots, x_n^0)$ minimizing the function

$$f(x_1, \ldots, x_n) = \sum_{\emptyset \neq S \subset N} \left\{ v(S) - \sum_{j \in S} x_j \right\}^2$$

subject to $\sum_{j \in N} x_j = v(N)$ over $(x_1, \ldots, x_n) \in \mathcal{R}^n$. The value Φ^L is an *ESL*-value.

The *consensus value* Φ^{Co}. This is an *ESL*-value $\Phi^{Co}(v) = (\Phi_1^{Co}(v), \ldots, \Phi_n^{Co}(v))$ (Ju et al., 2007), uniquely determined by the following *neutral dummy property*: For any dummy player i in a game v, $\Phi_i^{Co}(v) = \frac{v(i)}{2} + \frac{1}{2}\left\{v(i) + \frac{v(N) - \sum_{j \in N} v(j)}{n}\right\}$. It is proved that the consensus value Φ^{Co} is of the form

$$\Phi^{Co}(v) = \frac{1}{2}\Phi^{Sh}(v) + \frac{1}{2}E(v), \tag{5}$$

where Φ^{Sh} is the Shapley value and $E(v) = \{E_1(v), \ldots, E_n(v)\}$ denotes the *equal surplus solution* of v, i.e.,

$$E_i(v) = v(i) + \frac{v(N) - \sum_{j \in N} v(j)}{n} \quad \text{for } i \in N. \tag{6}$$

To end, we quote two very useful results from the literature, characterizing the class of *ESL*-values. The first one (Lemma 9 in Ruiz et al. (1998) is about some unique representation of values. We write it in the following form:

Proposition 1. *A value Φ is an ESL-value if and only if there exists a unique collection of constants $\{\rho_s | s = 1, 2, \ldots, n-1\}$ such that, for every game v the value payoff vector $(\Phi_i(v))_{i \in N}$ is of the following form:*

$$\Phi_i(v) = \frac{1}{n} v(N) + \sum_{\substack{S \subsetneq N \\ S \ni i}} \frac{\rho_s}{s} v(S) - \sum_{\substack{S \subsetneq N \\ S \not\ni i}} \frac{\rho_s}{n-s} v(S), \quad i \in N. \tag{7}$$

It is shown in Ruiz et al. (1998) that the least square prenucleolus Φ^L is of the form (7) with constants

$$\rho_s = \frac{s(n-s)}{n \cdot 2^{n-2}}, \quad s = 1, \ldots, n-1, \tag{8}$$

whence, by Proposition 1, it follows that Φ^L is an *ESL*-value.

The second result is an equivalent version of Proposition 1 (see Theorem 3 in Driessen and Radzik (2003)), which we write in the following form:

Proposition 2. *A value Φ is an ESL-value if and only if there exists a unique collection of constants $\{b_s | s = 1, 2, \ldots, n\}$ with $b_n = 1$ such that Φ is of the form*

$$\Phi_i(v) = \sum_{S \subset N \setminus i} \frac{s!(n-s-1)!}{n!} [b_{s+1} v(S \cup i) - b_s v(S)] \quad \text{for } i \in N, \tag{9}$$

(here we take $b_0 \equiv 0$, because of $v(\emptyset) = 0$).

One can easily check that if we rewrite formula (7) with the help of the new parameters $b_1, \ldots, b_{n-1}$, putting there

$$\rho_s = b_s / \binom{n}{s} \quad \text{for } s = 1, \ldots, n-1, \tag{10}$$

then (7) coincides with formula (9).

One can easily see that formula (9) generalizes formula (1) for the Shapley value, in the sense that the margin contribution $[v(S \cup i) - v(S)]$ has been replaced by a "modified margin contribution" $[b_{s+1} v(S \cup i) - b_s v(S)]$. Just the representation of *ESL*-values in the form (9) is basic for our considerations in the next sections.

3. First main result

In this section we present two of our main results (Theorems 1 and 2) about the family of *ESL*-values, defined on the set Γ of all cooperative games on an arbitrarily fixed grand coalition N. They discuss very surprising properties of values of the form (9). Because of their complexity, their proofs are given in the last section. The obtained results will be applied to the four values, the Shapley value, the solidarity value, the least square prenucleolus and the consensus value, introduced in the previous section.

The starting point for our considerations is the family of *ESL*-values $\Phi = (\Phi_i)_{i\in N}$ on Γ with their representation in the form (9). We begin with introducing the following two desirable properties of a value.

Fair treatment: Let $i, j \in N$ and $v \in \Gamma$. If $v(S \cup i) \geq v(S \cup j)$ for all $S \subset N \setminus i \setminus j$ then $\Phi_i(v) \geq \Phi_j(v)$.

Monotonicity: Let v be a *monotonic game*, that is satisfying $v(S) \leq v(T)$ whenever $S \subset T$. Then for each player $i \in N$, $\Phi_i(v) \geq 0$.

It is worth mentioning that the first property *fair treatment* appears in the literature under different names, such as *desirability* (see, e.g. Peleg and Sudhölter, 2003), or *local monotonocity* (see, e.g. Levinský and Silársky, 2004).

Theorem 1. *An ESL-value Φ verifies fair treatment, if and only if the constants b_s in its representation (9) satisfy:*

$$b_n = 1 \quad \text{and} \quad b_k \geq 0 \quad \text{for } k = 1, 2, \ldots, n-1. \tag{11}$$

Theorem 2. *An ESL-value Φ verifies fair treatment and monotonicity, if and only if the constants b_s in its representation (9) satisfy:*

$$b_n = 1 \quad \text{and} \quad 0 \leq b_k \leq 1 \quad \text{for } k = 1, 2, \ldots, n-1. \tag{12}$$

The above two theorems allow us to conclude the following corollary.

Corollary 1. (a) *The Shapley value and the solidarity value verify fair treatment and monotonicity.*
(b) *The least square prenucleolus and the consensus value verify fair treatment.*
(c) *The least square prenucleolus and the consensus value do not verify monotonicity for $|N| \geq 4$ and $|N| \geq 3$, respectively.*

Proof. Since the Shapley value, the solidarity value, the least square prenucleolus and the consensus value are *ESL*-values, therefore each of them has its representation in the form (9) with some constants b_s. Comparing (1) with (9), for the Shapley value we have

$$b_s = 1 \quad \text{for } s = 1, 2, \ldots, n, \tag{13}$$

while by comparing (4) and (9), we conclude that the constants b_s for the solidarity value Φ^{So} are of the form

$$b_n = 1 \quad \text{and} \quad b_s = \frac{1}{s+1} \quad \text{for } s = 1, 2, \ldots, n-1. \tag{14}$$

On the other hand, the least square prenucleolus Φ^L is of the form (9) with the constants

$$b_n = 1 \quad \text{and} \quad b_s = \frac{s}{2^{n-2}} \binom{n-1}{s}$$
$$\text{for } s = 1, 2, \ldots, n-1, \tag{15}$$

which easily follows by (8) and (10).

Further, one can easily check that the equal surplus solution $E(v)$ (described by (6)) has its representation in the form (9) with the constants

$$b_s = \begin{cases} n-1 & \text{if } s = 1, \\ 1 & \text{if } s = n, \\ 0 & \text{if } 1 < s < n. \end{cases} \tag{16}$$

Hence, using (5), (13) and (16), we immediately see that the consensus value Φ^{Co}, is also of the form (9), with the constants

$$b_s = \begin{cases} \dfrac{n}{2} & \text{if } s = 1, \\ 1 & \text{if } s = n, \\ \dfrac{1}{2} & \text{if } 1 < s < n. \end{cases} \tag{17}$$

By (13) and (14), (12) follows. Hence statement (a) is an immediate consequence of Theorem 2.

Further, the fact that the least square prenucleolus and the consensus value verify fair treatment, easily follows from (15), (17), (11) and Theorem 1. Therefore, statement (b) also is true.

To prove statement (c) for the least square prenucleolus, it suffices to show (because of Theorem 2) that for every $n \geq 4$, $b_s = \frac{s}{2^{n-2}}\binom{n-1}{s} > 1$ for some $1 \leq s \leq n-1$.

Namely, first consider the case $n = 2k$ with $k \geq 2$ and take $s = k$. Then $b_k = \frac{k}{2^{2k-2}}\binom{2k-1}{k}$ and we easily check that $b_2 = 3$ and $b_k < b_{k+1}$ for $k \geq 2$. Therefore $b_k > 1$ in case $n = 2k \geq 4$. For the second case $n = 2k+1$ with $k \geq 2$, we have $b_k = \frac{k}{2^{2k-1}}\binom{2k}{k}$, and similarly as before, we show that $b_k > 1$.

The statement (c) in the case of the consensus value is an immediate consequence of Theorems 1, 2, and (17). □

Remark 1. Theorems 1 and 2 are based on our earlier result of Proposition 2 with formula (10) from Driessen and Radzik (2003), and they are the first part of our main results. It should be mentioned here that similar results, closely related to these two theorems, were independently obtained (with the help of other tools) in the (so far unpublished) paper of Malawski (2008). Namely, his results can be written in the following way:

(1) An *ESL*-value Φ verifies *fair treatment* if and only if it is *coalitionally monotonic*, that is satisfying: for every coalition T and every two games v and w coinciding on all the coalitions $S \neq T$, $\Phi_i(v) \geq \Phi_i(v)$ for each $i \in T$ if $v(T) > w(T)$ (a consequence of Lemma 6 there);
(2) An *ESL*-value Φ is *coalitionally monotonic* if and only if it is of the form (7) with constants $\rho_s \geq 0$ for $s = 1, \ldots, n-1$ (a consequence of Lemma 3 there);
(3) An *ESL*-value Φ with the *fair treatment* and *monotonicity* properties is of the form (7) with constants ρ_s satisfying $0 \leq \rho_s \leq 1/\binom{n}{s}$ for $s = 1, \ldots, n-1$ (a consequence of Lemmas 6, 3 and 5 there).

Now, one can conclude that the above three statements together with Proposition 2 and formula (10) imply Theorem 1 and the part ($\Rightarrow$) of Theorem 2. However, the rest of our results (the part ($\Leftarrow$) of Theorem 2, and Theorems 3 and 4 given in the next section) cannot be derived from Malawski (2008), because they necessarily need the representation of *ESL*-values given by our Proposition 2, not present there. In Section 5 we directly prove Theorems 1 and 2, without any reference to considerations of that paper.

4. *ESL*-values and acceptability properties

In this section we continue our considerations about *ESL*-values, in the context of two other properties, *social acceptability* and *general acceptability* (defined below). The proofs of Theorems 3 and 4 describing necessary and sufficient conditions for an *ESL*-value to have such properties, are given the next section.

It is well known that the family of unanimity games constitutes a basis of the space of all TU-games. Besides, every unanimity game has a very simple and clear structure with two distinguished groups of the players (the null players and the rest). Thus the discussion about the properties of values reduced to the family of unanimity games gives an important extra information about the behavior of values. Obviously, for any unanimity game u_T, every non-null player $i \in T$ can be seen as a "productive" one, while the null players $j \in N \setminus T$ can be seen as "non-productive". Hence, it is very reasonable to require that every productive player should be awarded not worse (by a value Φ in unanimity games u_T) than any non-productive one. Additionally, from the social point of view, any non-productive player in u_T should be awarded with a payoff at least zero. This observation leads to the next property of a value $\Phi = (\Phi_i)_{i\in N}$ of TU-games, proposed by Joosten et al. (1994).

Social acceptability: Let u_T be any unanimity game with $\emptyset \neq T \subsetneq N$. Then

$$\Phi_i(u_T) \geq \Phi_j(u_T) \geq 0 \quad \text{for all } i \in T \text{ and } j \in N \setminus T. \tag{18}$$

Remark 2. Most of the values of cooperative games discussed in the literature are socially acceptable, that is, they satisfy (18) for all unanimity games u_T. In an obvious way, the Shapley value, the τ-value (Tijs, 1981) and the nucleolus (Schmeidler, 1969) are socially acceptable since they assign 0 to each null player $j \notin T$ and $1/|T|$ to each player $i \in T$ in any unanimity game u_T. The next theorem gives necessary and sufficient conditions for *ESL*-values to be socially acceptable.

Theorem 3. *An ESL-value Φ is socially acceptable if and only if the constants b_s in its representation (9) satisfy:*

$$0 \leq \left\{\frac{nt}{n-t} \Big/ \binom{n}{t}\right\} \cdot \sum_{k=t}^{n-1} \binom{k}{t} \frac{b_k}{k} \leq 1$$
$$\text{for } t = 1, 2, \ldots, n-1. \tag{19}$$

Obviously, not every *ESL*-value is socially acceptable. However, such values possess this property, when they additionally satisfy the monotonicity and fair treatment properties. Just this is expressed by the following corollary of Theorem 3.

Corollary 2. *Every ESL-value Φ with constants $b_1, \ldots, b_n$ satisfying (12) in its representation (9), is socially acceptable.*

Proof. Obviously, it suffices to show (19). Because of (12), the first inequality in (19) is trivial.

One can easily verify with the help of induction that

$$\sum_{s=t}^{n-1} \binom{s}{t} \cdot \frac{1}{s} = \frac{1}{t} \cdot \binom{n-1}{t} \quad \text{for } 1 \leq t < n. \tag{20}$$

Now, using (12) and (20), for $t = 1, 2, \ldots, n-1$, we can conclude as follows:

$$\left\{\frac{nt}{n-t} \Big/ \binom{n}{t}\right\} \cdot \sum_{s=t}^{n-1} \binom{s}{t} \frac{b_s}{s} \leq \left\{\frac{nt}{n-t} \Big/ \binom{n}{t}\right\} \cdot \sum_{s=t}^{n-1} \binom{s}{t} \frac{1}{s}$$
$$= \left\{\frac{nt}{n-t} \Big/ \binom{n}{t}\right\} \cdot \frac{1}{t} \cdot \binom{n-1}{t} = 1.$$

Therefore (19) holds, which completes the proof. □

Corollary 3. *The solidarity value, the least square prenucleolus and the consensus value are socially acceptable.*

Proof. The social acceptability of the solidarity value is an immediate consequence of (14), (12) and Corollary 2.

Further, note that the constants b_s for the least square prenucleolus and for the consensus value do not satisfy (12) (see (15) and (17)), and thereby we cannot use the result of Corollary 2.

Let $H(n,t) := \left\{\frac{nt}{n-t} / \binom{n}{t}\right\} \cdot \sum_{s=t}^{n-1} \binom{s}{t} \frac{b_s}{s}$. For the least square prenucleolus, the constants b_s are of the form (15), and thereby they satisfy (11). Therefore, for $1 \leq t \leq n-1$ we can conclude for this value as follows:

$$0 \leq H(n,t) = \frac{nt}{(n-t)2^{n-2}} \cdot \sum_{s=t}^{n-1} \left\{\binom{s}{t}\binom{n-1}{s} \Big/ \binom{n}{t}\right\}$$
$$= \frac{nt}{(n-t)2^{n-2}} \cdot \sum_{s=t}^{n-1} \left\{\frac{n-s}{n}\binom{n-t}{s-t}\right\}$$
$$= \frac{t}{2^{n-2}} \cdot \sum_{s=t}^{n-1} \left\{\frac{n-s}{n-t}\binom{n-t}{s-t}\right\}$$
$$= \frac{t}{2^{n-2}} \cdot \sum_{s=t}^{n-1} \binom{n-1-t}{n-1-s} = \frac{t}{2^{t-1}} \leq 1.$$

Thus (19) holds, and Theorem 3 proves the social acceptability of the least square prenucleolus.

Now, consider $H(n,t)$ for the consensus value, that is, with the constants b_s of the form (17). Obviously, $H(n,t) \geq 0$ for $t = 1, 2, \ldots, n-1$. On the other hand, using (18), one can verify that $H(n,1) = 1$ and $H(n,t) = \frac{1}{2}$ for all $1 \leq t < n$. Consequently (19) follows, whence, by Theorem 3, the consensus value also is socially acceptable. Thus the proof is completed. □

Remark 3. One can see as very desirable the following strengthening of the property of social acceptability. Namely, it is reasonable to require also from an efficient value of any unanimity game that a productive player should be awarded not worse when the number of all the productive players in the game is smaller. It leads to the next property for a value $\Phi = (\Phi_i)_{i\in N}$ of TU-games.

General acceptability: Let u_S and u_T be any two unanimity games with $\emptyset \neq S \subset T \subset N$. Then

$$\Phi_i(u_S) \geq \Phi_i(u_T) \quad \text{for all } i \in S. \tag{21}$$

The next theorem gives necessary and sufficient conditions for an *ESL*-value to be generally acceptable.

Theorem 4. *An ESL-value Φ is generally acceptable if and only if the constants b_s in its representation (9) satisfy:*

$$\sum_{k=s}^{n-1} \frac{n-k}{k} \cdot \binom{k}{s} b_k \geq 0 \quad \text{for } s = 1, 2, \ldots, n-1. \tag{22}$$

Theorem 4 immediately leads to the next corollary, which says (by Theorem 1) that an *ESL*-value with the fair treatment property also is generally acceptable.

Corollary 4. *Every ESL-value Φ with constants $b_1, \ldots, b_n$ satisfying (11) in its representation (9), is generally acceptable.*

The Shapley value is generally acceptable, because (21) is obviously satisfied for it. It turns out that the remaining three considered values also have the same property.

Corollary 5. *The solidarity value, the least square prenucleolus and the consensus value are generally acceptable.*

Proof. All the three values are of the form (9) with all the constants $b_s > 0$ and $b_n = 1$ as it follows from (14), (15) and (17). So, for each of them, inequalities (22) hold, ending the proof. □

Remark 4. The properties of fair treatment and monotonicity of *ESL*-values are closely related with the social acceptability and general acceptability. Namely, if an *ESL*-value verifies fair treatment, then it also is generally acceptable (a consequence of Corollary 4 and Theorem 1). However, if an *ESL*-value verifies fair treatment and monotonicity, then it is both socially acceptable and generally acceptable (a consequence of Corollaries 2 and 4 and Theorem 2). The last two properties are very desirable, and the four considered values, the Shapley value, the solidarity value, the least square prenucleolus and the consensus value, possess them as well as most values of cooperative games discussed in the literature. However, there exist *ESL*-values satisfying the fair treatment property, which are not socially acceptable. Namely, consider the *α-equal surplus solution* of the form $E^\alpha(v) = \{E_1^\alpha(v), \dots, E_n^\alpha(v)\}$, where

$$E_i^\alpha(v) = \alpha v(i) + \frac{v(N) - \sum_{j\in N} \alpha v(j)}{n} \quad \text{for } i \in N.$$

Obviously, this value generalizes the equal surplus solution $E(v)$ of the form (6), and is an *ESL*-value with a clear interpretation. It is not difficult to verify that for $0 \le \alpha \le 1$ value $E^\alpha(v)$ is both socially acceptable and generally acceptable, but for $\alpha > 1$ it is only generally acceptable, not socially acceptable. One can also notice that for $0 \le \alpha \le 1$, value $E^\alpha(v)$ is a convex combination of the equal surplus solution $E(v)$ and the *equal division solution* $\Phi^{Eq}(v) = \left\{\frac{v(N)}{n}, \dots, \frac{v(N)}{n}\right\}$, in the form $E^\alpha(v) = \alpha E(v) + (1-\alpha)\Phi^{Eq}(v)$. It is also worth mentioning that the class of values $E^\alpha(v)$, $0 \le \alpha \le 1$, coincides with the subclass $\phi^{\alpha,1}(v)$, $0 \le \alpha \le 1$, of some wider class of values $\phi^{\alpha,\beta}(v)$, $0 \le \alpha, \beta \le 1$, which was recently studied by van den Brink and Funaki (2009).

We end this remark with two interesting facts (in the context of value $E^\alpha(v)$). Namely, one can easily verify that in the case of 2-person games ($|N| = 2$), the Shapley value coincides with the consensus value and they are equal to 1-equal surplus solution, while the solidarity value coincides with $\frac{1}{2}$-equal surplus solution.

5. Proofs of Theorems 1–4

Proof of Theorem 1. ($\Leftarrow$) Fix a value Φ and assume that it is of the form (9) with constants b_s satisfying (11). We show that Φ possesses the fair treatment property.

We know that formula (9) is equivalent to formula (7) with constants ρ_s of the form (10). Therefore, we can assume (because of (11)) that the value Φ is of the form (7) with some constants

$$\rho_s \ge 0 \quad \text{for } s = 1, 2, \dots, n-1. \tag{23}$$

Let us fix $i, j \in N$ and assume that

$$v(T \cup i) \ge v(T \cup j) \quad \text{for all } T \subset N \setminus i \setminus j. \tag{24}$$

Then using (7), we can conclude as follows:

$$\Phi_i(v) - \Phi_j(v) = \left\{ \sum_{\substack{S\ni i\\ S\ni j}} \frac{\rho_s}{s} v(S) + \sum_{\substack{S\ni i\\ S\not\ni j}} \frac{\rho_s}{s} v(S) - \sum_{\substack{S\not\ni i\\ S\not\ni j}} \frac{\rho_s}{n-s} v(S) - \sum_{\substack{S\not\ni i\\ S\ni j}} \frac{\rho_s}{n-s} v(S) \right\}$$
$$- \left\{ \sum_{\substack{S\ni j\\ S\ni i}} \frac{\rho_s}{s} v(S) + \sum_{\substack{S\ni j\\ S\not\ni i}} \frac{\rho_s}{s} v(S) - \sum_{\substack{S\not\ni j\\ S\not\ni i}} \frac{\rho_s}{n-s} v(S) - \sum_{\substack{S\not\ni j\\ S\ni i}} \frac{\rho_s}{n-s} v(S) \right\}$$
$$= \sum_{\substack{S\ni i\\ S\not\ni j}} \frac{n\rho_s}{s(n-s)} v(S) - \sum_{\substack{S\ni j\\ S\not\ni i}} \frac{n\rho_s}{s(n-s)} v(S)$$
$$= n \left\{ \sum_{T\subset N\setminus i\setminus j} \frac{\rho_{t+1}}{(t+1)(n-t-1)} v(T\cup i) - \sum_{T\subset N\setminus i\setminus j} \frac{\rho_{t+1}}{(t+1)(n-t-1)} v(T\cup j) \right\},$$

whence

$$\Phi_i(v) - \Phi_j(v) = n \sum_{T\subset N\setminus i\setminus j} \frac{\rho_{t+1}}{(t+1)(n-t-1)} [v(T\cup i) - v(T\cup j)]. \tag{25}$$

But the last equality, in view of (23) and (24), implies that $\Phi_i(v) \ge \Phi_j(v)$. Thus we have proved that Φ has the fair treatment property, ending the first part of the proof of Theorem 1.

($\Rightarrow$) Now assume conversely that Φ is an *ESL*-value and has the fair treatment property. By Proposition 2, it follows that Φ is of the form (9). We will show that inequalities (11) must hold.

Let us fix $i, j \in N$ and $S \subset N \setminus i \setminus j$, and define the game $\tilde{v}$ on N by the following:

$$\tilde{v}(T) = \begin{cases} 1 & \text{if } T = S \cup i, \\ 0 & \text{if } T \ne S \cup i. \end{cases}$$

Then obviously the game $v = \tilde{v}$ satisfies (24) and (25). Therefore, by the fair treatment property, $\Phi_i(\tilde{v}) - \Phi_j(\tilde{v}) \ge 0$. On the other hand, (25) leads to the equality: $\Phi_i(\tilde{v}) - \Phi_j(\tilde{v}) = n\frac{\rho_{s+1}}{(s+1)(n-s-1)}$. Hence $\rho_{s+1} \ge 0$, and consequently $b_{s+1} \ge 0$, because of (10). This, in view of the arbitrarity of the set S, completes the proof of Theorem 1. □

Proof of Theorem 2. ($\Rightarrow$) Assume that Φ is an *ESL*-value and verifies fair treatment and monotonocity. Obviously, by Theorem 1, Φ is of the form (9) satisfying (11). Therefore, we need to show only that

$$b_k \le 1 \quad \text{for } k = 1, 2, \dots, n-1. \tag{26}$$

Let us arbitrarily fix $i \in N$ and k, $1 \le k \le n-1$, and define the game $\bar{v}$ on N by the following:

$$\bar{v}(T) = \begin{cases} 1 & \text{if } |T| \ge k+1 \text{ and } i \in T, \\ 1 & \text{if } |T| \ge k \text{ and } i \notin T, \\ 0 & \text{otherwise.} \end{cases}$$

Then, by (9), we can conclude as follows:

$$\Phi_i(\bar{v}) = \sum_{\substack{S\subset N\setminus i\\ |S|\ge k}} \frac{s!(n-s-1)!}{n!} [b_{s+1}\bar{v}(S\cup i) - b_s \bar{v}(S)]$$
$$= \sum_{s=k}^{n-1} \sum_{S\subset N\setminus i} \frac{s!(n-s-1)!}{n!} (b_{s+1} - b_s)$$
$$= \sum_{s=k}^{n-1} \binom{n-1}{s} \frac{s!(n-s-1)!}{n!} (b_{s+1} - b_s)$$
$$= \frac{1}{n} \sum_{s=k}^{n-1} (b_{s+1} - b_s) = \frac{1}{n}(b_n - b_k) = \frac{1}{n}(1 - b_k).$$

Since the game $\bar{v}$ is monotonic, therefore, by the monotonocity property, $\Phi_i(\bar{v}) \ge 0$, and consequently $b_k \le 1$. This, in view of the arbitrarity of k, ends the proof of part ($\Rightarrow$) of the theorem.

($\Leftarrow$) Assume now that a value Φ is of the form (9) with constants $b_1, \ldots, b_n$ satisfying (12). (We recall that $b_0 \equiv 0$ in (9)).

Let us arbitrarily fix $i \in N$ and choose any monotonic game v on N. In view of Theorem 1, we need to show only that Φ verifies the monotonicity, that is, $\Phi_i(v) \geq 0$.

To begin with, for $q = 0, 1, \ldots, n-2$, we will show the following inequalities:

$$\Phi_i(v) \geq \frac{q!}{n(n-1)\ldots(n-q)} \sum_{\substack{S\subset N\setminus i\\ |S|=n-(q+1)}} (1-b_{n-(q+1)})v(S\cup i) + \sum_{\substack{S\subset N\setminus i\\ |S|\leq n-(q+2)}} \frac{s!(n-s-1)!}{n!}(b_{s+1}-b_s)v(S\cup i). \quad (27)$$

The proof of the inequality (27) will be carried out with the help of induction. First we will show it for $q = 0$.

Since v is monotonic, therefore (9) with (12) imply the following:

$$\Phi_i(v) \geq \sum_{S\subset N\setminus i} \frac{s!(n-s-1)!}{n!}(b_{s+1}-b_s)v(S\cup i)$$

which, in turn, is equivalent to the inequality

$$\Phi_i(v) \geq \frac{1}{n}v(N)(1-b_{n-1}) + \sum_{\substack{S\subset N\setminus i\\ |S|\leq n-2}} \frac{s!(n-s-1)!}{n!}(b_{s+1}-b_s)v(S\cup i).$$

One can easily see that the right-hand side of the last inequality coincides with the same for (27) with $q = 0$, and thereby, inequality (27) holds for $q = 0$.

Assume now that inequality (27) holds for some $0 \leq q < n-2$. We will show that this inequality also holds after replacing q by $q+1$.

We easily state that

$$\sum_{\substack{S\subset N\setminus i\\ |S|=n-(q+1)}} \sum_{k\in S} v(S\cup i\setminus k) = (q+1) \sum_{\substack{S\subset N\setminus i\\ |S|=n-(q+2)}} v(S\cup i).$$

The monotonicity of the game v and the above equality imply the following

$$\sum_{\substack{S\subset N\setminus i\\ |S|=n-(q+1)}} v(S\cup i) \geq \frac{1}{n-(q+1)} \sum_{\substack{S\subset N\setminus i\\ |S|=n-(q+1)}} \sum_{k\in S} v(S\cup i\setminus k) = \frac{q+1}{n-(q+1)} \sum_{\substack{S\subset N\setminus i\\ |S|=n-(q+2)}} v(S\cup i).$$

Therefore

$$\frac{q!}{n(n-1)\ldots(n-q)} \sum_{\substack{S\subset N\setminus i\\ |S|=n-(q+1)}} (1-b_{n-(q+1)})v(S\cup i) \geq \frac{(q+1)!}{n(n-1)\ldots(n-(q+1))} \sum_{\substack{S\subset N\setminus i\\ |S|=n-(q+2)}} (1-b_{n-(q+1)})v(S\cup i).$$

Hence, using the above inequality and (27) (inductive assumption), we can conclude as follows:

$$\begin{aligned}
\Phi_i(v) &\geq \frac{(q+1)!}{n(n-1)\ldots(n-(q+1))} \times \sum_{\substack{S\subset N\setminus i\\ |S|=n-(q+2)}} (1-b_{n-(q+1)})v(S\cup i) + \sum_{\substack{S\subset N\setminus i\\ |S|\leq n-(q+2)}} \frac{s!(n-s-1)!}{n!}(b_{s+1}-b_s)v(S\cup i)\\
&= \frac{(q+1)!}{n(n-1)\ldots(n-(q+1))} \times \sum_{\substack{S\subset N\setminus i\\ |S|=n-(q+2)}} (1-b_{n-(q+1)})v(S\cup i)\\
&\quad + \sum_{\substack{S\subset N\setminus i\\ |S|=n-(q+2)}} \frac{(q+1)!}{n(n-1)\ldots(n-(q+1))} \times (b_{n-(q+1)}-b_{n-(q+2)})v(S\cup i)\\
&\quad + \sum_{\substack{S\subset N\setminus i\\ |S|\leq n-(q+3)}} \frac{s!(n-s-1)!}{n!}(b_{s+1}-b_s)v(S\cup i)\\
&= \frac{(q+1)!}{n(n-1)\ldots(n-(q+1))} \times \sum_{\substack{S\subset N\setminus i\\ |S|=n-(q+2)}} (1-b_{n-(q+2)})v(S\cup i) + \sum_{\substack{S\subset N\setminus i\\ |S|\leq n-(q+3)}} \frac{s!(n-s-1)!}{n!}(b_{s+1}-b_s)v(S\cup i).
\end{aligned}$$

This implies that inequality (27), after replacing q by $q+1$, holds. Therefore, by the induction principle, it proves the validity of (27) for $q = 0, 1, \ldots, n-2$.

Putting now $q = n-2$ in (27), we get

$$\Phi_i(v) \geq \frac{(n-2)!}{n(n-1)\ldots 2} \sum_{\substack{S\subset N\setminus i\\ |S|=1}} (1-b_1)v(S\cup i) + \sum_{\substack{S\subset N\setminus i\\ |S|=0}} \frac{(n-2)!}{n(n-1)\ldots 1}(b_1-b_0)v(S\cup i),$$

which is nonnegative, because $0 \leq b_1 \leq 1$ and $b_0 = 0$. Thus we have proved that $\Phi_i(v) \geq 0$, ending the proof of Theorem 2. □

Proof of Theorem 3. Let $\emptyset \neq T \subsetneq N$, $i \in T$ and $j \in N \setminus T$. Let Φ be an *ESL*-value of the form (9). Therefore, it verifies efficiency and anonymity. However, it implies that for the unanimity game u_T, $t\Phi_i(u_T) + (n-t)\Phi_j(u_T) = 1$. Hence, one can easily deduce that Φ is socially acceptable, that is, it satisfies the double inequality in (18), if and only if

$$\frac{1}{n} \leq \Phi_i(u_T) \leq \frac{1}{t} \quad \text{for all } T \text{ and } i\in T,\ \emptyset\neq T\subsetneq N. \quad (28)$$

But, using the definition of the game u_T and (9), we can conclude as follows:

$$\begin{aligned}
\Phi_i(u_T) &= \sum_{S\subset N\setminus i} \frac{s!(n-s-1)!}{n!}\Big[b_{s+1}u_T(S\cup i) - b_s u_T(S)\Big]\\
&= \sum_{S\subset N\setminus i} \frac{s!(n-s-1)!}{n!} b_{s+1}u_T(S\cup i)\\
&= \sum_{T\setminus i\subset S\subset N\setminus i} \frac{s!(n-s-1)!}{n!}\, b_{s+1}\\
&= \sum_{s=t-1}^{n-1} \binom{n-t}{s-t+1} \cdot \frac{s!(n-s-1)!}{n!}\, b_{s+1}\\
&= \sum_{s=t-1}^{n-1} \binom{s+1}{t} \cdot \frac{b_{s+1}}{s+1} \Big/ \binom{n}{t},
\end{aligned}$$

$$= \sum_{k=t}^{n} \binom{k}{t} \cdot \frac{b_k}{k} \Big/ \binom{n}{t},$$

whence one can finally get

$$\Phi_i(u_T) = \frac{1}{n} + \sum_{k=t}^{n-1} \left\{ \binom{k}{t} \cdot \frac{b_k}{k} \right\} \Big/ \binom{n}{t}. \quad (29)$$

Now, using (29), one can easily deduce that (28) is equivalent to (19). Thus the proof is completed. □

Proof of Theorem 4. (⇐) By assumption, Φ is of the form (9). Obviously, it suffices to show the inequality (21) for $\emptyset \neq S \subset T \subset N$ with $|T| = |S| + 1$. Therefore, considering the result of the sequence of equalities before (29), for $s = 1, \ldots, n-1$ and $i \in S$, we can conclude as follows:

$$\begin{aligned} &\Phi_i(u_S) - \Phi_i(u_T) \\ &= \sum_{k=s}^{n} \left\{ \binom{k}{s} \cdot \frac{b_k}{k} \Big/ \binom{n}{s} \right\} \\ &\quad - \sum_{k=s+1}^{n} \left\{ \binom{k}{s+1} \cdot \frac{b_k}{k} \Big/ \binom{n}{s+1} \right\} \\ &= \frac{b_s}{s} \Big/ \binom{n}{s} + \sum_{k=s+1}^{n} \left\{ \binom{k}{s} \Big/ \binom{n}{s} \right. \\ &\quad \left. - \binom{k}{s+1} \Big/ \binom{n}{s+1} \right\} \cdot \frac{b_k}{k} \\ &= \frac{b_s}{s} \Big/ \binom{n}{s} + \sum_{k=s+1}^{n} \left\{ \frac{n-k}{n-s} \cdot \frac{b_k}{k} \cdot \binom{k}{s} \Big/ \binom{n}{s} \right\} \\ &= \sum_{k=s}^{n-1} \left\{ \frac{n-k}{k} \binom{k}{s} b_k \right\} \Big/ \left\{ (n-s) \binom{n}{s} \right\} \geq 0, \end{aligned}$$

where the last inequality is a consequence of (22).

(⇒) In view of the equalities in part (⇐), the proof is obvious. □

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