SHORT-PHASE ANOMALIES IN INTERMITTENT BAND SWITCHING

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The distribution of phase lengths t for intermittent band switching is investigated for small t. Some typical deviations from exponential behaviour are reported, in particular the occurrence of a minimal phase length with enhanced probability.

1. Introduction

At present a wide variety of intermittent phenomena is known. One of the most interesting kinds of intermittency occurs in various low-dimensional maps when two chaotic bands merge [1–7].

Just after the merger of the two chaotic bands the orbit of the twice iterated map is found to spend relatively long periods in either of the former band regions, but at irregular times it switches from one region to the other. This phenomenon has been called intermittent hopping or switching and an exponential (geometrical) distribution of the periods t between subsequent switches has been observed.

However, in numerical investigations we find that the phase length t cannot be arbitrarily short. That is, a minimal phase length t_{min} exists, which increases when we get nearer to the merging point. Moreover, the probability distribution is strongly peaked at $t = t_{min}$. Typical distributions of phase lengths will be shown later on in this Letter.

In this Letter we shall deal with the above short-phase phenomena, using the map

$$f(x) = 1 - a|x|^z, \quad z > 1, \tag{1}$$

as an example. In section 2 we discuss the band structure of this map. In section 3 we explain the ex-

istence of a minimal period t_{min} and we derive an expression for it in terms of the small parameter $\varepsilon = a - a_1$, where a_1 denotes the value of a for which the two chaos bands merge into one. Finally, in section 4, we discuss the peak in the distribution and its dependence on ε as well as on the order z of the maximum.

2. Band structure

The map (1) exhibits a period doubling cascade as one increases the parameter a from 0 to $a_\infty(z)$. At this value we have an orbit of period 2^∞, consisting of points which together form a Cantor set. For $a > a_\infty$ the attractor is composed of a collection of chaos bands, which undergoes a sequence of reverse bifurcations, or band mergings, at values ..., a_3, a_2, a_1. Here a_n denotes the value of a at which 2^n chaos bands merge into 2^{n-1} bands, in a pairwise manner. This is illustrated in fig. 1 for the typical case $z = 2$.

For the sake of presentation we shall concentrate on the last band merging at $a = a_1$, although the phenomena to be described occur in exactly the same way at all the other band mergings. Just below $a = a_1$ the orbit of the map alternates between the two separate bands and an orbit of the twice iterated map
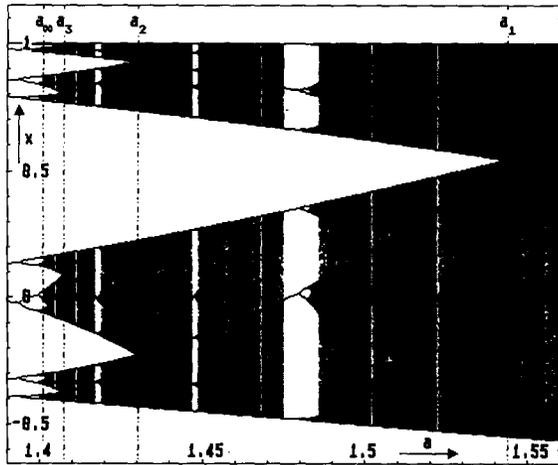
Fig. 1. The structure of the attractor for the map $f(x) = 1 - ax^2$, for $1.39 < a < 1.56$. We have indicated the threshold to chaos a_∞ and the band merging values a_3, a_2 and a_1. (One has $a_1 = 1.5436890$ in this case.)

$f^2(x) = f(f(x))$ remains forever in either one of the bands. It is therefore convenient to consider $f^2(x)$ instead of the original map $f(x)$.

In figs. 2 and 3 we show $f^2(x)$ just before the band merging $(a < a_1)$ and just after it $(a > a_1)$. In both figures the positive fixed point X of $f(x)$ is used for the construction of two square boxes. Each of these boxes contains a (sub)map similar to the original map $f(x)$ but on a smaller scale. The boxes are
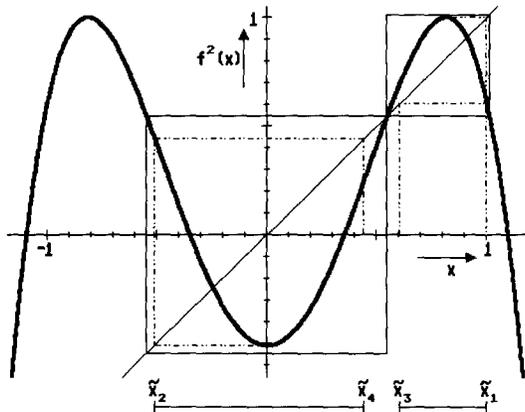

Fig. 2. The map $f^2(x)$ for $z = 2$ at $a = 1.51$, *before* the band merging at $a = a_1 = 1.5436890$. The boundaries of the two bands are indicated by $\tilde{x}_2$, $\tilde{x}_4$ and $\tilde{x}_3$, $\tilde{x}_1$, respectively. Note that the two submaps discussed in the text are completely contained within the square boxes.
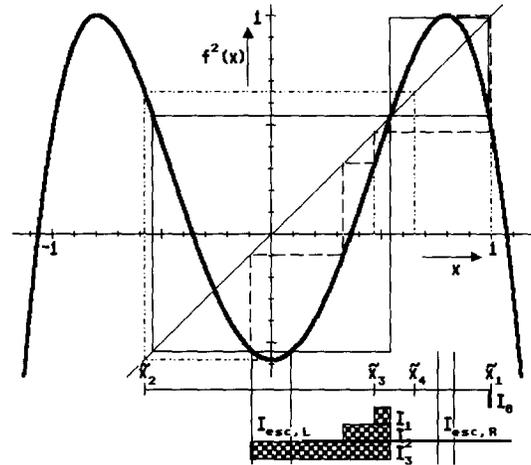

Fig. 3. The map $f^2(x)$ for $z = 2$, at $a = 1.58$, after the band merging. The boundaries of the band are $\tilde{x}_1 = 1$ and $\tilde{x}_2 = 1 - a$. The escape interval $I_{esc,R}$ of the right region is mapped onto the interval I_0, and then successively onto I_1, I_2 and I_3. Note that the interval I_3 overlaps with the escape interval $I_{esc,L}$ of the left region.

bounded by the fixed point X and its pre-images under $f^2(x)$ at the left and the right of X:

$$X_L = -X, \quad X_R = [(1+X)/a]^{1/z}.$$

As long as the extrema of the two submaps lie inside their boxes (as in fig. 2), any orbit of $f^2(x)$ trapped in one of them will remain there forever. The actual chaos bands are given by the intervals $[\tilde{x}_2, \tilde{x}_4]$ and $[\tilde{x}_3, \tilde{x}_1]$, where $\tilde{x}_n = f^n(0)$ denotes the nth iterate of the maximum $\tilde{x}_0 = 0$, i.e. $\tilde{x}_1 = 1$, $\tilde{x}_2 = 1 - a$, $\tilde{x}_3 = 1 - a|1 - a|^z$, and so on. At $a = a_1$ the extrema of the submaps hit their boxes simultaneously, and at this moment we have $\tilde{x}_3 = \tilde{x}_4 = X = a_1 - 1$, which means that the band boundaries touch each other.

For $a > a_1$ we have the situation as shown in fig. 3, after the band merging. The two extrema pierce through the boxes and consequently an orbit can escape from its box. For small values of $\varepsilon = a - a_1$ the escape regions, which are given by the intervals where $f^2(x)$ exceeds the boxes, are quite small, and orbits of $f^2(x)$ will in general remain for long times in one of the former band regions.

3. Minimal phase length

Let us consider the region $x > X$ in fig. 3. An orbit of $f^2(x)$ in the right box will switch to the left region

$x < X$ after it gets into the escape interval $I_{esc,R}$. When this happens the orbit is transferred on the next iteration into the interval $I_0 = [X_R, 1]$. The interval I_0 is mapped by $f^2(x)$ to $I_1 = [\tilde{x}_3, X]$, then to $I_2 = [\tilde{x}_5, X]$, then to $I_3 = [\tilde{x}_7, X]$. In fig. 3 the interval I_3 overlaps with the escape interval of the left box and thus the orbit has a possibility to switch back to the right region on the second next iteration (since the escape interval is first transferred to the interval $[\tilde{x}_2, X_L]$ at the left). So we observe that in this case the orbit must spend at least four iterations of $f^2(x)$ in the left region. We define the minimal phase length in the left region $t_{min,L}$ as the minimal number of iterations spent in that region. In general $t_{min,L}$ will depend on the value of $\varepsilon = a - a_1$ as well as on the order z of the maximum. In the above example we have $t_{min,L} = 4$.

To establish an accurate lower bound for t_{min} as a function of ε we determine the minimal number n of iterations of $f^2(x)$ needed to map $\tilde{x}_3$ into the escape region $I_{esc,L}$ (or to the left of it). For small values of ε the region I_1 is quite narrow, with $\tilde{x}_3$ close to the unstable fixed point X, and the number of iterations can be found from

$$\tilde{x}_{3+2n} - X = [f'(X)]^{2n}(\tilde{x}_3 - X) . \tag{2}$$

The escape interval lies in a small neighbourhood of the minimum at $\tilde{x}_0 = 0$ and as soon as $\tilde{x}_{3+2n}$ reaches this point the orbit has a possibility to escape after one extra iteration. Hence from (2) we can estimate the minimal number of iterations n of $f^2(x)$ to get from $\tilde{x}_3$ to the escape region:

$$n = -\frac{1}{2} \frac{\ln[(X - \tilde{x}_3)/(X - \tilde{x}_0)]}{\ln|f'(X)|} . \tag{3}$$

Since a non-integer number of iterations has no meaning, it is understood that n should be lifted to the next integer, i.e. $n \to [n+1]$.

The minimal phase length for an orbit of $f^2(x)$ in the left region $t_{min,L}$ is then $[n+3]$, since one iteration is needed to get into the interval I_1 and another one is needed to escape to the interval $[\tilde{x}_2, X_L]$ before switching to the right region. For the right region the same arguments can be applied. However, the length of phases of $f^2(x)$ in the right region is *always 1 iteration shorter* than the length of the corresponding phases in the left region. This stems from the fact that the switching of an orbit of $f^2(x)$

amounts to an anomaly of the type LL in the alternating sequence ...LRLRLR... of an orbit of $f(x)$, where we write L when an orbit point belongs to the left region and R when it belongs to the right region. Anomalies of the types RR and LLL do not occur. So the minimal phase lengths in the left and the right region are related by $t_{min,L} = t_{min,R} + 1$. Consequently the minimal phase length t_{min} is equal to $t_{min,R} = [n+2]$.

The minimal phase length can be calculated from eq. (3), using the expressions

$$X - \tilde{x}_3 = \frac{z(2 - a_1)}{a_1(a_1 - 1)} \frac{za_1^2 - 2(z+1)a_1 + 2}{(z-1)a_1 - 2z + 1} \varepsilon$$
$$+ O(\varepsilon^2) , \tag{4}$$

$$X - \tilde{x}_0 = a_1 - 1 + O(\varepsilon) , \tag{5}$$

$$f'(X) = -\frac{z(2 - a_1)}{a_1 - 1} + O(\varepsilon) . \tag{6}$$

For the case $z = 2$ this yields

$$t_{min} = [2 - 0.965 \ln(3.43\varepsilon)] . \tag{7}$$

Applying eq. (7) we find the values $t_{min} = 5, 7, 9, 11, 14$ for $\varepsilon = 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}$, respectively. These values are in perfect agreement with the numerical values obtained by iteration of the interval $I_1 = [\tilde{x}_3, X]$.

Some of the considerations given above are reminiscent of an estimate by Grebogi et al. [8] of the phase length of the transient chaotic phase in the case of intermittent bursting. The interpretation however is quite different and the laminar phase in intermittent bursting does not show a minimal phase length effect. Further, in ref. [2] some plots of distributions of phase lengths were shown, but the minimal phase length phenomenon was not mentioned.

4. The peak

The peak in the distribution $P(t)$ of phase lengths t at or just above t_{min} can be explained as follows. Since the escape regions for small ε are quite narrow, we may assume that the points where the orbit enters the escape region $I_{esc,R}$ are uniformly distributed, that is, $\rho_{esc,R}(x)$ is constant. The escape interval $I_{esc,R}$ is mapped under $f^2(x)$ on the interval $I_0 = [X_R, 1]$.

The (normalized) distribution $\rho_0(x)$ of points mapped on this interval is obtained by applying the Frobenius–Perron operator to the (constant) distribution $\rho_{esc,R}(x)$, that is,

$$\rho_0(x) = \int_{I_{esc,R}} \rho_{esc,R}(x')\delta(x-f^2(x'))\,dx'$$

$$= z^{-1}(1-X_R)^{-1/z}(1-x)^{-1+1/z},$$

$$x \in I_0 . \tag{8}$$

This distribution has a cusp-type singularity at $x=1$ for $z>1$. In the limit $z\downarrow 1$ it straightens out to a uniform distribution. On the next iteration I_0 is mapped to I_1, with a distribution $\rho_1(x)$ having a cusp-type singularity at $\tilde{x}_3$, i.e. at the left boundary of the interval I_1. Starting from $\rho_1(x)$ we find after n more iterations of $f^2(x)$ a distribution $\rho_{n+1}(x)$ with a cusp-type singularity at the left boundary $\tilde{x}_{3+2n}$ of the interval I_{n+1}:

$$\rho_{n+1}(x) = \mathcal{N}(x-\tilde{x}_{3+2n})^{-1+1/z}, \tag{9}$$

$\mathcal{N}$ being the normalizing factor $z^{-1}(X-\tilde{x}_{3+2n})^{-1/z}$.

We now consider the situation that I_{n+1} is the first interval overlapping with the escape region $I_{esc,L}$. We may anticipate that the height of the peak will oscillate as a function of ε, depending on the values of $\rho_{n+1}(x)$ in the escape region. More specifically, when $\varepsilon = a - a_1(z)$ is such that the left bound of I_{n+1}, $\tilde{x}_{3+2n}$ falls inside the left escape region $I_{esc,L}$ we expect the probability $P_L(t_{min,L})$ for a phase of length $t_{min,L}$ in the left region to be most pronounced, yielding a maximal value for the height of the peak.

The probability $P_L(t)$ for a phase of length t in the left region equals the probability $P_R(t+1)$ for a phase of length $t+1$ in the right region, so we have

$$P(t_{min}) = \tfrac{1}{2}P_L(t_{min}+1) . \tag{10}$$

We shall now calculate the maximal height of the peak at t_{min}. It should be attained for values of ε for which the left boundaries of I_n and $I_{esc,L}$ coincide. In that case the normalized distribution on I_n can be expressed as

$$\rho_{t_{min}}(x) = z^{-1}(X+\tfrac{1}{2}L)^{-1/z}(x+\tfrac{1}{2}L)^{-1+1/z}. \tag{11}$$

So the maximal probability for the minimal phase length t_{min} to occur equals

$$P_{max}(t_{min}) = \tfrac{1}{2}P_{max,L}(t_{min}+1)$$

$$= \tfrac{1}{2}\int_{-L/2}^{L/2} z^{-1}(X+\tfrac{1}{2}L)^{-1/z}(x+\tfrac{1}{2}L)^{-1+1/z}\,dx$$

$$= \frac{1}{2}\left(\frac{L}{X+\tfrac{1}{2}L}\right)^{1/z} \sim \varepsilon^{1/z^2}. \tag{12}$$

In the last step we have used that $L \sim \varepsilon^{1/z}$. On the other hand, a minimal probability $P_{min}(t_{min})$ for t_{min} is expected in a situation where the left boundary of the interval I_n is just right of the escape region. From this it can be shown that $P_{min}(t_{min}) \sim \varepsilon^{1/z}$, i.e. $P_{min}(t_{min})$ follows the same power law as the inverse of the average phase length [1]; the maximal peak height in eq. (12), however, goes to zero with a smaller power of ε.

In fig. 4 we have given the numerical measurements of phase length distributions for $z=1.1$, $z=2$ and $z=3$, for values of ε where a maximal value of $P(t_{min})$ occurs for $t_{min}=6$, 7 and 9, respectively. From fig. 4 it is seen that the maximal height of the peak as compared to the inverse of the average phase length grows with increasing z, in agreement with the smaller power of ε. For actual systems such a feature may be used to gain information on the z value of the underlying map. In table 1 the measured $P_{max}(t_{min})$ is compared to the theoretical value according to eq. (12) and it can be concluded that the agreement is quite good.

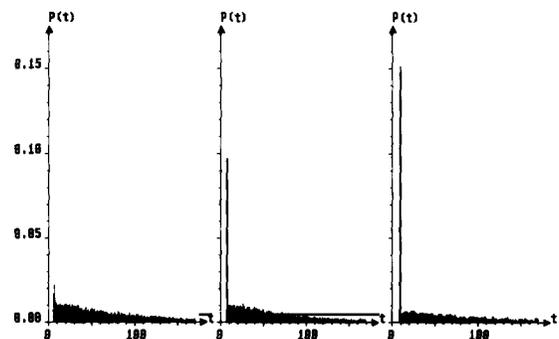A special situation occurs when the left boundary of I_{n+1} in fig. 3 falls exactly in the middle of the left

Fig. 4. Numerically obtained phase length distributions for (a) $z=1.1$ and $a=1.4377955$, (b) $z=2$ and $a=1.54424004$, (c) $z=3$ and $a=1.61804789$.

Table 1
Comparison between prediction and measurement of $P_{max}(t_{min})$ for some values of z and ε (see text and fig. 4).

z	a	ε	t_{mean}	t_{min}	$P_{max}(t_{min})$	
					pred.	meas.
1.1	1.43779550	6.08×10^{-3}	91	6	0.0169	0.0172
2	1.54424004	5.51×10^{-4}	68	7	0.101	0.097
3	1.61804789	1.49×10^{-5}	82	9	0.170	0.148

escape region. Then we have $\tilde{x}_{3+2n} = \tilde{x}_0$, so at this value of ε we have a superstable cycle of period $2n+3$, and consequently $P(t_{min}) = P(t_{min}+1) = \frac{1}{2}$, with $t_{min} = n+1$, $P(t)$ being zero for all other values of t. Such a cycle occurs for every value of n, inside a narrow window of ε values. For ε values corresponding to the interior part of a window one has only the two periods t_{min} and $t_{min}+1$ occurring both with probability $\frac{1}{2}$. For ε values very close to the opening and the closing of a window one may except intermittency of the Pomeau–Manneville type [9] and intermittent bursting [8,10], respectively. For ε values sufficiently far away from a window only the intermittent switching behaviour is left. The estimate of the minimal phase length which we have given is valid independently of the details of the windows. In the narrow subregime of intermittent bursting an analogy with earlier considerations in ref. [8] appears, because our minimal phase length in that case can be regarded as the (minimal) transient time of the bursts between the remnants of the attractor at the boundaries of the window. The interplay of the various types of intermittency will be subject of future investigation.

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