Int J Game Theory (2006) 34:241–258
DOI 10.1007/s00182-006-0014-9

ORIGINAL ARTICLE

Semi-marginalistic values for set games

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Received: 15 December 2005 / Accepted: 7 February2006 /
Published online: 18 May 2006

Abstract Concerning the solution theory for set games, the paper focuses on a family of values, each of which allocates to any player some type of marginalistic contribution with respect to any coalition containing the player. For any value of the relevant family, an axiomatization is given by means of three properties, namely one type of an efficiency property, the equal treatment property and one type of a monotonicity property. We present one proof technique which is based on the decomposition of any arbitrary set game into a union of simple set games, the value of which are much easier to determine. A simple set game is associated with an arbitrary, but fixed item of the universe.

Keywords Set game · Value · Axiomatization

AMS Classification 2000 04A15 · 04A20 · 91A12

1 Introduction

Example 1.1 Let us consider a situation with a finite number of shareholders as well as firms such that the "coalitional control" over firms is given. That is, for each firm it is listed which groups of shareholders, called coalitions, do

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control the relevant firm. For instance, let a particular three-person situation with three shareholders and three firms be described by the control of the first firm through all the supersets of the first shareholder (i.e., all the coalitions containing the first shareholder), while the control of the second firm (the third firm respectively) happens by supersets of the two-person coalition containing both the third shareholder and the second shareholder (the first shareholder respectively). Generally speaking, given the "coalitional control" over firms, we aim to solve the "allocation problem" of "individual control" over firms by shareholders, specifying which of them controls which firms. According to a certain solution rule in the mathematical setting of so-called set games, this particular three-person situation is solved in that the first shareholder controls the first and third firms; the second shareholder controls only the second firm; and the third shareholder controls the second and third firms. We recall the fundamental notions within the field of set game theory.

Let $\mathcal{U}$, called the *universe*, denote an abstract set which is fixed throughout the remainder. Following the introductory papers (Aarts 1994) (chapter 7), (Aarts et al. 1997, 2000; Hoede 1992) a *set game* is a pair (N, v), where N is a nonempty, finite set, called *player set*, and $v : 2^N \to 2^{\mathcal{U}}$ is a *characteristic mapping*, defined on the power set of N, satisfying $v(\emptyset) := \emptyset$. Let $G(\mathcal{U})$ denote the space of all set games with an arbitrary player set, whereas $G^N(\mathcal{U})$ denotes the space of all set games with reference to a player set N which is fixed beforehand. An element of N (notation: $i \in N$) and a nonempty subset S of N (notation: $S \subseteq N$ or $S \in 2^N$ with $S \neq \emptyset$) is called a *player* and *coalition* respectively, and the associated *set* $v(S) \subseteq \mathcal{U}$ is called the *worth* of coalition S, to be interpreted as the (sub)set of items from $\mathcal{U}$ that can be obtained (are needed, preferred, owned) by coalition S if its members cooperate. We do not care about how players feel about what they get, so we do not deal with the notion of an utility function per player with the universe as its domain.

Example 1.2 continued (the shareholder set game). In order to model the shareholder situation as a set game with player set N consisting of all the shareholders, denote the finite set of firms by $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$. With each firm F_j is associated a simple game through its collection $\mathcal{W}_j$ of winning coalitions. That is, $S \in \mathcal{W}_j$ means that coalition $S \subseteq N$ is of type j controlling firm F_j. Combinning these simple games, we may define the set game $(N, v, \mathcal{F})$ with universe $\mathcal{F}$ and its characteristic mapping $v : 2^N \to 2^{\mathcal{F}}$ through $v(S) = \{F_j \in \mathcal{F} \mid S \in \mathcal{W}_j\}$ for all $S \subseteq N$. In words, the worth $v(S)$ of coalition S equals the subset of all firms that are controlled by coalition S.

The particular three-person situation with player set $N = \{1, 2, 3\}$ and universe $\mathcal{F} = \{F_1, F_2, F_3\}$ corresponds with the collection of winning coalitions $\mathcal{W}_1 = \{\{1\}, \{1, 2\}, \{1, 3\}, N\}$, $\mathcal{W}_2 = \{\{2, 3\}, N\}$, $\mathcal{W}_3 = \{\{1, 3\}, N\}$. Then the characteristic mapping v of the corresponding set game is as follows:

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	$\emptyset$	$\{F_1\}$	$\emptyset$	$\emptyset$	$\{F_1\}$	$\{F_1, F_3\}$	$\{F_2\}$	$\{F_1, F_2, F_3\}$

Concerning the solution theory for set games, a *solution* f on $G^N(\mathcal{U})$ associates a so-called allocation $f(N,v) = (f_i(N,v))_{i\in N} \in (2^{\mathcal{U}})^N$ with every set game $(N,v) \in G^N(\mathcal{U})$. The so-called *allocation* $f_i(N,v) \subseteq \mathcal{U}$ to player i in the set game (N,v) represents the items that are given, according to the solution f, to player i for participating in the game. Until further notice, no constrains are imposed upon a solution f on $G^N(\mathcal{U})$. The difference of two sets $A,\ B \subseteq \mathcal{U}$ is denoted by $A \setminus B$ and defined to be $A \setminus B := \{x \mid x \in A \text{ and } x \notin B\}$.

Example 1.3 continued (semi-marginalistic solution of the shareholder set game). Coalition $S \in \mathcal{W}_j$ is a minimal winning coalition of type j if no proper subset is winning, i.e., $S\setminus\{k\} \notin \mathcal{W}_j$ for all $k \in S$. We call the expression $v(S)\setminus\bigcup_{k\in S} v(S\setminus\{k\})$ the semi-marginalistic contribution of coalition S and $F_j \in v(S) \setminus \bigcup_{k\in S} v(S \setminus \{k\})$ means that coalition S is a minimal winning coalition of type j. A semi-marginalistic solution rule, called OIM-value, is defined through $\mathrm{OIM}_i(N,v) = \bigcup_{\substack{S\subseteq N\\ S\ni i}} \left[v(S) \setminus \bigcup_{k\in S} v(S \setminus \{k\})\right]$ for all $i \in N$ and $F_j \in \mathrm{OIM}_i(N,v)$ means that there exists a minimal winning coalition of type j containing shareholder i.

Concerning the particular three-person situation with player set $N = \{1,2,3\}$ and universe $\mathcal{F} = \{F_1,F_2,F_3\}$, the minimal winning coalitions of type 1,2,3 respectively are given by $\mathcal{MW}_1 = \{\{1\}\}$, $\mathcal{MW}_2 = \{\{2,3\}\}$, $\mathcal{MW}_3 = \{\{1,3\}\}$. Thus, the semi-marginalistic solution rule OIM yields $\mathrm{OIM}_1(N,v) = \{F_1,F_3\}$, $\mathrm{OIM}_2(N,v) = \{F_2\}$, $\mathrm{OIM}_3(N,v) = \{F_2,F_3\}$. According to this rule, the first shareholder controls the first and third firms; the second shareholder controls only the second firm; and the third shareholder controls the second and third firms. 1, while the two other firms inclusive of shareholder 3.

In section 2 we introduce a family of solutions called *semi-marginalistic values*. According to a semi-marginalistic value, any player's allocation in a set game is the overall union of appropriately chosen marginalistic contributions of the player with respect to coalitions containing the player. Here the player's marginalistic contribution may be interpreted in various ways to allow for a uniform treatment of semi-marginalistic values (see Definition 2.3). The goal of the paper is to axiomatize semi-marginalistic values by means of three basic axioms. The relevant properties, called global efficiency, equal treatment and semi-marginalistic contribution monotonicity, are discussed in section 3. Section 4 is devoted to the main axiomatization (see Theorem 4.1) and the proof technique is based on the decomposition of any set game into a union of so-called *simple set games*. Each simple set game is associated with an arbitrary, but fixed item, and the worth of a coalition in a simple set game equals either the empty set or the singleton consisting of the underlying item. In the concluding section 5 we discuss the similarities between the two fields of set game theory and cooperative game theory.

2 Semi-marginalistic values for set games

Let $G^N(\mathcal{U})$ denote the space of set games with finite player set N. A *value* f on $G^N(\mathcal{U})$ is a mapping $f : G^N(\mathcal{U}) \to (2^{\mathcal{U}})^N$, which associates with any set game $(N,v) \in G^N(\mathcal{U})$ a single-set-valued vector $f(N,v,\mathcal{U}) = (f_i(N,v,\mathcal{U}))_{i\in N} \in (2^{\mathcal{U}})^N$, or shortly $f(N, v)$. We review five different values, studied throughout the solution theory for set games, before introducing a family of set game-theoretic values containing one type of them. The purpose of the paper is to present a uniform axiomatization of the new family of values under consideration.

Definition 2.1 As of the first category, we review two values for set games.

(i) The *individually marginalistic IM*-value, named marginalistic value by Aarts et al. (1997), is given by

$$\mathrm{IM}_i(N, v) = \bigcup_{\substack{S\subseteq N\\ S\ni i}} [v(S) \setminus v(S \setminus \{i\})], \quad \text{for all } (N, v) \text{ and all } i \in N. \tag{1}$$

(ii) The *overall-coalitionally marginalistic OCM*-value, as introduced by Driessen and Sun (2001), is given by

$$\mathrm{OCM}_i(N, v) = \bigcup_{\substack{S\subseteq N\\ S\ni i}} \left[v(S) \setminus \bigcup_{T\subseteq N\setminus\{i\}} v(T) \right], \quad \text{for all } (N, v) \text{ and all } i \in N. \tag{2}$$

Clearly, the inclusion $\mathrm{IM}_i(N, v) \supseteq \mathrm{OCM}_i(N, v)$ holds for any player i. Given a set game (N, v), we say that an item $x \in \mathcal{U}$ is *attainable* by player i through a certain coalition S containing i whenever the item belongs to the coalition's worth, that is $x \in v(S)$. Also a coalition T cannot *block* an item x whenever the item does not belong to the coalition's worth, that is $x \notin v(T)$. In this terminology, the individually marginalistic IM-value allocates those items that are attainable by player i, but cannot be blocked by the coalition consisting of the remaining members (different from player i). The overall-coalitionally marginalistic OCM-value allocates those items that are attainable by player i, but cannot be blocked by any coalition not containing i. For two values defined by (1) and (2), the underlying expressions $v(S)\setminus v(S\setminus\{i\})$ and $v(S)\setminus\bigcup_{T\subseteq N\setminus\{i\}} v(T)$ respectively, are called the *marginalistic contribution of coalition S* with reference to player i. In this context, the arginalistic contribution depends on both the coalition and the player. A similar type of values, of which the marginalistic contribution will not depend on the player, but only on the coalition, will be studied in the remainder of the paper.

Definition 2.2 As of the second category, we review another three values for set games.

(i) The *individually co-marginalistic ICM*-value, named co-marginalistic contribution value by Sun et al. (2001), is given by

$$\mathrm{ICM}_i(N, v) = \bigcup_{\substack{S\subseteq N\\ S\ni i}} \left[v(S) \setminus \bigcap_{j\in S} v(S \setminus \{j\}) \right], \quad \text{for all } (N, v) \text{ and all } i \in N. \tag{3}$$

(ii) The *overall-individually marginalistic OIM*-value, as introduced by Aarts et al. (1997), is given by

$$\mathrm{OIM}_i(N,v)=\bigcup_{\substack{S\subseteq N\\ S\ni i}}\left[v(S)\setminus\bigcup_{j\in S}v(S\setminus\{j\})\right],\quad \text{for all } (N,v) \text{ and all } i\in N. \tag{4}$$

(iii) The *sub-coalitionally marginalistic SCM*-value, named coalitional power value by Sun et al. (2003), is given by

$$\mathrm{SCM}_i(N,v)=\bigcup_{\substack{S\subseteq N\\ S\ni i}}\left[v(S)\setminus\bigcup_{T\subsetneq S}v(T)\right],\quad \text{for all } (N,v) \text{ and all } i\in N. \tag{5}$$

Given a set game, among these five values, the ICM-value is the largest allocation in that the sequence of inclusions $\mathrm{ICM}_i(N,v)\supseteq\mathrm{IM}_i(N,v)\supseteq\mathrm{OIM}_i(N,v)\supseteq\mathrm{SCM}_i(N,v)$ holds for any player i. The individually co-marginalistic ICM-value allocates those items that are attainable by player i, but cannot be blocked by at least one sub-coalition with one player less. The overall-individually marginalistic OIM-value allocates those items that are attainable by player i, but cannot be blocked by any sub-coalition with one player less. The sub-coalitionally marginalistic SCM-value allocates those items that are attainable by player i, but cannot be blocked by any proper sub-coalition. For these three values defined by (3), (4) and (5), the underlying expressions $v(S)\setminus\bigcap_{j\in S}v(S\setminus\{j\})$, $v(S)\setminus\bigcup_{j\in S}v(S\setminus\{j\})$ and $v(S)\setminus\bigcup_{T\subsetneq S}v(T)$ respectively, are called the *marginalistic contribution of coalition S*. Note that the marginalistic contribution only depends on the coalition S itself and not anymore on a particular player. In the sequel, we adopt the uniform notation SMC_S^v to represent the semi-marginalistic contribution of coalition S in the set game (N,v) and it is supposed to be of the form $v(S)\setminus\nabla_S^v$. Here ∇_S^v is some (yet unspecified) expression which is supposed to depend upon the worths of a certain collection of coalitions, somehow determined by S (for instance, through the unions and/or intersections of a number of (sub)coalitions). The ICM-, OIM-, and SCM-values respectively arise by choosing $\nabla_S^v=\bigcap_{j\in S}v(S\setminus\{j\})$, $\nabla_S^v=\bigcup_{j\in S}v(S\setminus\{j\})$ and $\nabla_S^v=\bigcup_{T\subsetneq S}v(T)$.

Definition 2.3 A *semi-marginalistic value*, or shortly SM-value on the set game space $G^N(\mathcal{U})$ is a member of the family of set game-theoretic values of the following form:

$$\mathrm{SM}_i(N,v)=\bigcup_{\substack{S\subseteq N\\ S\ni i}}\mathrm{SMC}_S^v=\bigcup_{\substack{S\subseteq N\\ S\ni i}}\left[v(S)\setminus\nabla_S^v\right],\quad \text{for all } (N,v) \text{ and all } i\in N. \tag{6}$$

Here the *semi-marginalistic contribution* $\mathrm{SMC}_S^v=v(S)\setminus\nabla_S^v$ of every coalition S is determined by the set difference of the coalition's worth $v(S)$ and some (yet unspecified) expression ∇_S^v satisfying the inclusion

$$\nabla_S^v \subseteq \bigcup_{T \subsetneq S} v(T), \quad \text{for all } S \subseteq N. \tag{7}$$

Condition (7) requires that, for any coalition S, each item of the associated set ∇_S^v is attainable through at least one proper sub-coalition of S. Obviously, (7) is fulfilled in the context of the ICM-, OIM-, and SCM-values, which can be regarded as three specific examples of a semi-marginalistic value. Notice that $\nabla_{\{i\}}^v = \emptyset$ for all $i \in N$.

A set game (N, v) is *monotonic* if the inclusion $v(S) \subseteq v(T)$ holds for all $S \subseteq T \subseteq N$. For monotonic set games, $\text{OCM}_i(N, v) = v(N) \setminus v(N \setminus \{i\})$ for all $i \in N$, whereas the equalities $\text{IM}_i(N, v) = \text{OIM}_i(N, v) = \text{SCM}_i(N, v)$ hold for all $i \in N$. The coincidence of the IM-value and the OIM-value on the class of monotonic set games was shown by means of an inductive proof in Aarts et al. (1997) (Theorem 2.2, pages 110–111). As a minor contribution, we conclude this section with an alternative, but shorter proof of the latter coincidence.

Lemma 2.4 (Aarts et al. 1997) $\text{IM}(N, v) = \text{OIM}(N, v)$, *for all monotonic set games* (N, v).

Proof Let (N, v) be a monotonic set game and $i \in N$. As noted earlier, the inclusion $\text{IM}_i(N, v) \supseteq \text{OIM}_i(N, v)$ is always valid, since $v(S \setminus \{i\}) \subseteq \bigcup_{j \in S} v(S \setminus \{j\})$ for all $S \subseteq N$ with $S \ni i$. In order to prove the reverse inclusion $\text{IM}_i(N, v) \subseteq \text{OIM}_i(N, v)$, it suffices to show that $x \notin \text{OIM}_i(N, v)$ implies $x \notin \text{IM}_i(N, v)$.

Suppose $x \notin \text{OIM}_i(N, v)$. Let $S \subseteq N$ with $S \ni i$. We will show that $x \notin v(S) \setminus v(S \setminus \{i\})$. In case $S = \{i\}$, then $x \notin v(\{i\})$ because $x \notin \text{OIM}_i(N, v)$. In the remainder, let $S \neq \{i\}$. We distinguish two cases.

Case 1 If $x \notin v(S)$, then $x \notin v(S) \setminus v(S \setminus \{i\})$.

Case 2 Let $x \in v(S)$. We show $x \in v(S \setminus \{i\})$. Since $x \notin \text{OIM}_i(N, v) = \bigcup_{\substack{T \subseteq N \\ T \ni i}} \left[v(T) \setminus \bigcup_{j \in T} v(T \setminus \{j\})\right]$, then $x \notin v(S) \setminus \bigcup_{j \in S} v(S \setminus \{j\})$ and, together with the assumption $x \in v(S)$, we arrive at $x \in \bigcup_{j \in S} v(S \setminus \{j\})$. In summary, so far we conclude, from $x \in v(S)$ (where $i \in S$), that $x \in v(S \setminus \{j\})$ for some $j \in S$. By repeating the same procedure, step by step, there exists some $k \in S \setminus \{j\}$ such that $x \in v(S \setminus \{j, k\})$ and so on. Recall that $x \notin v(\{i\})$ because $x \notin \text{OIM}_i(N, v)$. By repeatedly applying the same procedure, we derive the existence of a coalition $R \subseteq S$ not containing player i such that $x \in v(R)$. Finally, from $x \in v(R)$, $R \subseteq S \setminus \{i\}$ and the (tacitly assumed) monotonicity of the set game (N, v), we deduce that $x \in v(S \setminus \{i\})$ as was to be shown. □

3 Properties of values for set games

On the class of monotonic set games, $\bigcup_{i\in N}\mathrm{OCM}_i(N,v) = v(N)\setminus\bigcap_{i\in N} v(N\setminus\{i\})$. According to the next lemma, the OCM-value differs from other values in that another type of efficiency applies.

Definition 3.1 A value f on the set game space $G^N(\mathcal{U})$ possesses the *global efficiency property* if

$$\bigcup_{i\in N} f_i(N,v) = \bigcup_{S\subseteq N} v(S), \quad \text{for all } (N,v). \tag{8}$$

According to global efficiency, each attainable item is allocated to at least one player.

Lemma 3.2 *Any SM-value (satisfying (6) as well as (7)) is globally efficient.*

Proof Clearly, by (6), for any SM-value, global efficiency is equivalent to the following condition:

$$\bigcup_{S\subseteq N} \mathrm{SMC}^v_S = \bigcup_{S\subseteq N} v(S), \quad \text{for all } (N,v). \tag{9}$$

We prove (9) by induction on the number of players. The case $n=1$ is trivial due to $\nabla^v_{\{i\}} = \emptyset$ for one-person set games. Let (N,v) be a set game with $n\geq 2$. Then we obtain the following chain of equalities:

$$\begin{aligned}
&\bigcup_{S\subseteq N} \mathrm{SMC}^v_S \\
&= \mathrm{SMC}^v_N \cup \left[\bigcup_{S\subsetneq N} \mathrm{SMC}^v_S\right] = \mathrm{SMC}^v_N \cup \left[\bigcup_{k\in N}\left[\bigcup_{S\subseteq N\setminus\{k\}} \mathrm{SMC}^v_S\right]\right] \\
&= \mathrm{SMC}^v_N \cup \left[\bigcup_{k\in N}\left[\bigcup_{S\subseteq N\setminus\{k\}} v(S)\right]\right] \\
&= \left[v(N)\setminus \nabla^v_N\right] \cup \left[\bigcup_{S\subsetneq N} v(S)\right] \overset{(7)}{=} \bigcup_{S\subseteq N} v(S),
\end{aligned}$$

where the third equality follows from the induction hypothesis and the last equality holds because of (7). □

In addition to global efficiency, we study the axiom of equal treatment, in order to be able to provide, in the following, an axiomatization of any SM-value satisfying appropriately chosen semi-marginalistic contributions. Let us recall the substitution of a pair of players and the equal treatment property.

Definition 3.3 (Substitutes in a set game and equal treatment property for a value)

(i) Two players $i \in N$, $j \in N$, $i \neq j$, are *substitutes* in the set game (N, v) whenever $v(S \cup \{i\}) = v(S \cup \{j\})$ holds for all $S \subseteq N \setminus \{i, j\}$.
(ii) A value f on the set game space $G^N(\mathcal{U})$ possesses the *equal treatment property* if $f_i(N, v) = f_j(N, v)$ for any pair $i \in N$, $j \in N$, $i \neq j$, of substitutes in the set game (N, v). In words, two substitutes in a set game are allocated the same items.

Lemma 3.4 *Any SM-value possesses the equal treatment property whenever for any pair $i \in N$, $j \in N$, $i \neq j$, of substitutes in the set game (N, v)*

$$\mathrm{SMC}^v_{S\cup\{i\}} = \mathrm{SMC}^v_{S\cup\{j\}} \quad \textit{for all } S \subseteq N \setminus \{i, j\}. \tag{10}$$

Condition (10) expresses that the semi-marginalistic contribution concept inherits the role of substitutes. Further, it was shown in Sun (2003) that the ICM-, OIM-, and SCM-values satisfy the equal treatment property.

Proof For any pair $i \in N$, $j \in N$, $i \neq j$, of substitutes in the set game (N, v), we obtain the following chain of equalities:

$$\begin{aligned}
\mathrm{SM}_i(N,v) &= \bigcup_{\substack{S\subseteq N\\ S\ni i}} \mathrm{SMC}^v_S = \left[\bigcup_{\substack{S\subseteq N\\ S\supseteq\{i,j\}}} \mathrm{SMC}^v_S\right] \cup \left[\bigcup_{\substack{S\subseteq N\\ S\ni i, S\not\ni j}} \mathrm{SMC}^v_S\right] \\
&= \left[\bigcup_{\substack{S\subseteq N\\ S\supseteq\{i,j\}}} \mathrm{SMC}^v_S\right] \cup \left[\bigcup_{S\subseteq N\setminus\{i,j\}} \mathrm{SMC}^v_{S\cup\{i\}}\right] \\
&= \left[\bigcup_{\substack{S\subseteq N\\ S\supseteq\{i,j\}}} \mathrm{SMC}^v_S\right] \cup \left[\bigcup_{S\subseteq N\setminus\{i,j\}} \mathrm{SMC}^v_{S\cup\{j\}}\right] \\
&= \left[\bigcup_{\substack{S\subseteq N\\ S\supseteq\{i,j\}}} \mathrm{SMC}^v_S\right] \cup \left[\bigcup_{\substack{S\subseteq N\\ S\ni j, S\not\ni i}} \mathrm{SMC}^v_S\right] \\
&= \bigcup_{\substack{S\subseteq N\\ S\ni j}} \mathrm{SMC}^v_S = \mathrm{SM}_j(N,v),
\end{aligned}$$

where the fourth equality follows from $\mathrm{SMC}^v_{S\cup\{i\}} = \mathrm{SMC}^v_{S\cup\{j\}}$ for all $S \subseteq N\setminus\{i, j\}$. Hence, the SM-value possesses the equal treatment property whenever (10) holds. □

Definition 3.5 A value f on the set game space $G^N(\mathcal{U})$ possesses the *semi-marginalistic contribution monotonicity property* with respect to the given semi-marginalistic contribution $\{\mathrm{SMC}_S^*\}_{S\subseteq N}$ if for any pair (N,v), (N,w) of set games and all $i \in N$,

$$(\mathrm{SMC}_S^v \subseteq \mathrm{SMC}_S^w \quad \text{for all } S \subseteq N \text{ with } i \in S) \text{ implies } f_i(N,v) \subseteq f_i(N,w) \quad (11)$$

In words, semi-marginalistic contribution monotonicity expresses, with reference to two different set games, that the larger the player's semi-marginalistic contributions in the game, the more items are allocated to the player.

Corollary 3.6 *Any SM-value (satisfying (6) and (7)) possesses global efficiency, the equal treatment property, and the semi-marginalistic contribution monotonicity.*

Remark 3.7 In case $\nabla_S^v = \emptyset$ for all $S \subseteq N$, then the associated semi-marginalistic contribution SMC_S^v agrees with the coalition's worth $v(S)$ for every coalition S and under these circumstances, (10) reduces to the definition of substitutes in the set game (N,v). The associated *maximal semi-marginalistic* MSM-value is given by $\mathrm{MSM}_i(N,v) = \bigcup_{S\subseteq N, S\ni i} v(S)$ for all $i \in N$ and it is easy to verify that this MSM-value satisfies global efficiency, the equal treatment property, and semi-marginalistic contribution monotonicity with respect to $\mathrm{SMC}_S^v = v(S)$ for all $S \subseteq N$. For its additivity, see Remark 4.6.

4 An axiomatization of semi-marginalistic values for set games

The purpose of this section is to present an axiomatic characterization of any semi-marginalistic value. To be exact, we show that such a value is fully determined by global efficiency, the (tacitly assumed) equal treatment property, together with a type of monotonicity. The proof technique is based on the decomposition of any set game into a union of *simple set games*, in which the worth of any coalition equals either the empty set or a singleton consisting of one arbitrary, but fixed item.

Theorem 4.1 Let the player set N of the set game space $G^N(\mathcal{U})$ be fixed. There exists a unique value on $G^N(\mathcal{U})$ satisfying global efficiency, the equal treatment property, and semi-marginalistic contribution monotonicity with respect to a certain semi-marginalistic contribution. This unique value is the SM-value associated with the relevant semi-marginalistic contribution.

The proof of Theorem 4.1 proceeds in three steps. The first preliminary result provides another interpretation of any SM-value as the maximal value satisfying global efficiency and semi-marginalistic contribution monotonicity.

Proposition 4.2 *If a value f on $G^N(\mathcal{U})$ possesses global efficiency and semi-marginalistic contribution monotonicity, then $f_i(N,v) \subseteq \mathrm{SM}_i(N,v)$ for all (N,v) and all $i \in N$.*

Proof Suppose a value f satisfies global efficiency and semi-marginalistic contribution monotonicity. Let (N, v) be a set game and $i \in N$. In order to show $f_i(N, v) \subseteq \mathrm{SM}_i(N, v)$, let $x \in f_i(N, v)$ but $x \notin \mathrm{SM}_i(N, v)$. Define a new set game (N, w) as follows: for any $S \subseteq N$

$$w(S) := \begin{cases} v(S) \setminus \{x\}, & \text{if } x \in v(S), \\ v(S), & \text{otherwise.} \end{cases}$$

Notice that, for all $S \subseteq N$, $x \notin w(S)$. From this observation, together with the global efficiency of f applied to the set game (N, w), we derive the following chain of inclusions:

$$f_i(N, w) \subseteq \bigcup_{j \in N} f_j(N, w) = \bigcup_{S \subseteq N} w(S) \subseteq \mathcal{U} \setminus \{x\}.$$

In particular, $x \notin f_i(N, w)$. Let $S \subseteq N$ with $i \in S$. Since the expression ∇_S^w is supposed to depend on worths of sub-coalitions in the game (N, w), the two set ∇_S^w and ∇_S^v of items do not differ at all, except for the item x itself (to be included or not). Because $x \notin \mathrm{SM}_i(N, v)$, then $x \notin v(S) \setminus \nabla_S^v$, whereas $x \notin w(S)$. From this we conclude that $v(S) \setminus \nabla_S^v = w(S) \setminus \nabla_S^w$. Hence, $\mathrm{SMC}_S^v = \mathrm{SMC}_S^w$ for all $S \subseteq N$ with $i \in S$. Consequently, $f_i(N, v) = f_i(N, w)$ by the semi-marginalistic contribution monotonicity of f, but this equality contradicts the facts that $x \in f_i(N, v)$ and $x \notin f_i(N, w)$. This contradiction completes the proof. □

Further, this proof indicates that the global efficiency $\bigcup_{i \in N} f_i(N, v) = \bigcup_{S \subseteq N} v(S)$ may weakened to the inclusion $\bigcup_{i \in N} f_i(N, v) \subseteq \bigcup_{S \subseteq N} v(S)$ for any set game (N, v). In addition, the definition of the expression ∇_S^w does not matter so much.

The final part of the preliminary results deals with simple set games, which will be treated as the components of a decomposition for any arbitrary set game.

Definition 4.3 With every set game (N, v) and every item $x \in \mathcal{U}$ is associated the *simple set game* (N, v_x) defined to be, for any $S \subseteq N$,

$$v_x(S) := \begin{cases} \{x\}, & \text{if } x \in v(S), \\ \emptyset, & \text{otherwise.} \end{cases} \tag{12}$$

The coalition S is *winning* in the simple set game (N, v_x) if $\{x\} = v_x(S)$ or, equivalently, $x \in v(S)$.

Proposition 4.4 *Decomposition results for set games (N, v) and semi-marginalistic values.*

(i) $$v = \bigcup_{y \in \mathcal{U}} v_y \text{ that is, for all } S \subseteq N,\ v(S) = \bigcup_{y \in \mathcal{U}} v_y(S). \tag{13}$$

(ii) $$\text{For all } x \in \mathcal{U} \text{ and } S \subseteq N, \text{ it holds}: \quad \mathrm{SMC}_S^{v_x} = \{x\} \Leftrightarrow x \in \mathrm{SMC}_S^{v}. \tag{14}$$

(iii) $$\mathrm{SM}_i(N, v) = \bigcup_{y \in \mathcal{U}} \mathrm{SM}_i(N, v_y), \quad \text{for all } i \in N \text{ and all SM} - \text{values}. \tag{15}$$

(iv) *If a value f on $G^N(\mathcal{U})$ satisfies the semi-marginalistic contribution monotonicity, then $f_i(N, v) \supseteq f_i(N, v_x)$, for all $i \in N$ and all $x \in \mathcal{U}$.*

Proof The decomposition statement (13) of the set game (N, v) is trivial since $\mathcal{U} = v(T) \cup (\mathcal{U} \setminus v(T))$ for all $T \subseteq N$. The decomposition statement (15) of the SM-value of the set game (N, v) is a direct consequence of the equivalence (14) because, for all $i \in N$,

$$\bigcup_{y \in \mathcal{U}} \mathrm{SM}_i(N, v_y) = \bigcup_{y \in \mathcal{U}} \bigcup_{\substack{S \subseteq N \\ S \ni i}} \mathrm{SMC}_S^{v_y} = \bigcup_{\substack{S \subseteq N \\ S \ni i}} \bigcup_{y \in \mathcal{U}} \mathrm{SMC}_S^{v_y}$$
$$= \bigcup_{\substack{S \subseteq N \\ S \ni i}} \mathrm{SMC}_S^{v} = \mathrm{SM}_i(N, v).$$

The statement in part (iv) is a direct consequence of the equivalence (14) too, due to the inclusion $\mathrm{SMC}_S^{v_x} \subseteq \mathrm{SMC}_S^{v}$ for all $S \subseteq N$ with $S \ni i$, and all $x \in \mathcal{U}$. It remains to prove, for all $x \in \mathcal{U}$ and all $S \subseteq N$, the equivalence (14). For that purpose, note that $\nabla_S^{v_x} \subseteq \{x\}$, due to (7) and (12). Now we obtain the following chain of equalities:

$$\begin{aligned}
\mathrm{SMC}_S^{v_x} = \{x\} &\Leftrightarrow v_x(S) \setminus \nabla_S^{v_x} = \{x\} \\
&\Leftrightarrow v_x(S) = \{x\} \quad \text{and} \quad \nabla_S^{v_x} = \emptyset \\
&\Leftrightarrow x \in v(S) \quad \text{and} \quad x \notin \nabla_S^{v_x} \\
&\Leftrightarrow x \in v(S) \quad \text{and} \quad x \notin \nabla_S^{v_y}, \quad \text{for all } y \in \mathcal{U} \\
&\Leftrightarrow x \in v(S) \quad \text{and} \quad x \notin \nabla_S^{v} \\
&\Leftrightarrow x \in v(S) \setminus \nabla_S^{v} \\
&\Leftrightarrow x \in \mathrm{SMC}_S^{v}
\end{aligned}$$

Concerning the fourth and fifth equivalence in the above chain, we make use of the relationships $\nabla_S^v = \nabla_S^{\left(\bigcup_{y \in \mathcal{U}} v_y\right)} = \bigcup_{y \in \mathcal{U}} \nabla_S^{v_y}$, while $v_y(T) \cap v_z(T) = \emptyset$ whenever $y \neq z$. □

Proof of the uniqueness part of Theorem 4.1 Suppose a value f on $G^N(\mathcal{U})$ satisfies global efficiency, the equal treatment property, and semi-marginalistic contribution monotonicity with the respect to the given semi-marginalistic contribution $\{\mathrm{SMC}_S^{*}\}_{S\subseteq N}$. Let (N,v) be a set game and $i \in N$. We show that $f_i(N,v) = \mathrm{SM}_i(N,v)$ for all $i \in N$. By Propositions 4.2 and 4.4(iii)–(iv), the following relationships hold:

$$\mathrm{SM}_i(N,v) = \bigcup_{y\in\mathcal{U}} \mathrm{SM}_i(N,v_y)$$

and

$$\mathrm{SM}_i(N,v) \supseteq f_i(N,v) \supseteq \bigcup_{y\in\mathcal{U}} f_i(N,v_y).$$

Fix the set game (N,v), player $i \in N$ and item $x \in \mathcal{U}$. It suffices to show that

$$\mathrm{SM}_i(N,v_x) = f_i(N,v_x), \quad \text{for every simple set game } (N,v_x). \tag{16}$$

The proof of (16) proceeds by induction on the number of winning coalitions in the semi-marginalistic contribution set games (N,SMC^{v_x}), defined to be $\mathrm{SMC}^{v_x}(S) = \mathrm{SMC}_S^{v_x} = v_x(S)\setminus\nabla_S^{v_x}$ for all $S \subseteq N$. Coalition S is said to be winning in the set game (N,SMC^{v_x}) if $\mathrm{SMC}^{v_x}(S) = \{x\}$ or, equivalently, $x \in \mathrm{SMC}_S^{v}$ (see (14)). We distinguish two cases depending upon whether or not there exists a unique winning coalition. *Case one* There exists a unique winning coalition S_1 in the set game (N,SMC^{v_x}), that is $\mathrm{SMC}^{v_x}(S_1) = \{x\}$ and $\mathrm{SMC}^{v_x}(S) = \emptyset$, for all $S \neq S_1$. Our first claim is the following:

$$\mathrm{SM}_j(N,v_x) = f_j(N,v_x) = \emptyset, \quad \text{for all } j \in N\setminus S_1. \tag{17}$$

Indeed, for all $j \in N\setminus S_1$, it follows from the definition of the set game, that $\mathrm{SMC}_S^{v_x} = \emptyset$ for all $S \subseteq N$ with $S \ni j$. From this, together with Proposition 4.2 applied to the simple set game (N,v_x), we deduce the following chain of inclusions: for all $j \in N\setminus S_1$

$$f_j(N,v_x) \subseteq \mathrm{SM}_j(N,v_x) = \bigcup_{\substack{S\subseteq N\\ S\ni j}} \mathrm{SMC}_S^{v_x} = \emptyset,$$

and so, the first claim (17) holds.

Our second claim is the following: $\mathrm{SMC}_S^{v_x} = \mathrm{SMC}_S^{\mathrm{SMC}^{v_x}}$ for all $S \subseteq N$. Indeed, if $S \neq S_1$, then $\mathrm{SMC}^{v_x}(S) = \emptyset$ and so $\mathrm{SMC}_S^{\mathrm{SMC}^{v_x}} = \emptyset$. Further, $\mathrm{SMC}_{S_1}^{\mathrm{SMC}^{v_x}} = \mathrm{SMC}_{S_1}^{v_x} \setminus \nabla_{S_1}^{\mathrm{SMC}^{v_x}} = \mathrm{SMC}_{S_1}^{v_x}$, since $\nabla_{S_1}^{\mathrm{SMC}^{v_x}} = \emptyset$ due to $\mathrm{SMC}^{v_x}(T) = \emptyset$, for all $T \subsetneq S_1$ (see (7)).

From $\mathrm{SMC}_S^{v_x} = \mathrm{SMC}_S^{\mathrm{SMC}^{v_x}}$, for all $S \subseteq N$, together with the semi-marginalistic contribution monotonicity for both f and SM, it follows that

$$f_j(N, \mathrm{SMC}^{v_x}) = f_j(N, v_x) \text{ and}$$
$$\mathrm{SM}_j(N, \mathrm{SMC}^{v_x}) = \mathrm{SM}_j(N, v_x), \quad \text{for all } j \in N. \tag{18}$$

By (17) and (18), we obtain that $f_j(N, \mathrm{SMC}^{v_x}) = \mathrm{SM}_j(N, \mathrm{SMC}^{v_x}) = \emptyset$, for all $j \in N \setminus S_1$. Global efficiency of both f and SM, applied to the set game (N, SMC^{v_x}), yields

$$\bigcup_{k \in N} f_k(N, \mathrm{SMC}^{v_x}) = \bigcup_{k \in N} \mathrm{SM}_k(N, \mathrm{SMC}^{v_x}) = \{x\}, \quad \text{which reduces to}$$
$$\bigcup_{k \in S_1} f_k(N, \mathrm{SMC}^{v_x}) = \bigcup_{k \in S_1} \mathrm{SM}_k(N, \mathrm{SMC}^{v_x}) = \{x\}.$$

Note that any pair of players in S_1 are substitutes in the set game (N, SMC^{v_x}) (since S_1 is the unique winning coalition). From the equal treatment property of both f and SM, applied to the game (N, SMC^{v_x}), we derive

$$f_j(N, \mathrm{SMC}^{v_x}) = f_k(N, \mathrm{SMC}^{v_x}), \quad \text{as well as} \quad \mathrm{SM}_j(N, \mathrm{SMC}^{v_x}) = \mathrm{SM}_k(N, \mathrm{SMC}^{v_x}), \quad \text{for all } j, k \in S_1.$$

Consequently, the latter global efficiency simplifies to

$$f_k(N, \mathrm{SMC}^{v_x}) = \mathrm{SM}_k(N, \mathrm{SMC}^{v_x}) = \{x\}, \quad \text{for all } k \in S_1.$$

From this, together with (17) and (18), we conclude that $\mathrm{SM}_j(N, v_x) = f_j(N, v_x)$ for all $j \in N$. This completes the proof of (16).
Case two There are at least two winning coalitions in the set game (N, SMC^{v_x}), say, among others, coalition S_1. In particular, $\mathrm{SMC}^{v_x}(S_1) = \{x\}$ or equivalently, $x \in \mathrm{SMC}_{S_1}^{v}$. Define two new set games (N, v_1) and (N, v_2), arising from the semi-marginalistic contribution set game (N, SMC^{v}) such that v_1 is almost the semi-marginalistic contribution set game (N, SMC^{v}) and v_2 almost the empty set game. To be exact,

$$v_1(S) := \begin{cases} \mathrm{SMC}_S^v, & \text{if } S \neq S_1, \\ \emptyset, & \text{if } S = S_1, \end{cases} \tag{19}$$

$$v_2(S) := \begin{cases} \mathrm{SMC}_S^v, & \text{if } S = S_1, \\ \emptyset, & \text{if } S \neq S_1. \end{cases} \tag{20}$$

From the descriptions (19), (20) of both set games, together with the equivalence (14), we deduce that their associated simple set games $(N, (v_1)_x)$ and

$(N,(v_2)_x)$ are given by

$$(v_1)_x(S) := \begin{cases} \mathrm{SMC}_S^{v_x}, & \text{if } S \neq S_1, \\ \emptyset, & \text{if } S = S_1, \end{cases} \tag{21}$$

$$(v_2)_x(S) := \begin{cases} \emptyset, & \text{if } S \neq S_1, \\ \mathrm{SMC}_S^{v_x}, & \text{if } S = S_1. \end{cases} \tag{22}$$

Note that the inclusions $(v_1)_x(S) \subseteq v_x(S)$ and $(v_2)_x(S) \subseteq v_x(S)$ hold for all $S \subseteq N$. Concerning the semi-marginalistic contribution in both simple set games, as given by (21), (22), we claim the following:

$$\mathrm{SMC}_{S_1}^{(v_1)_x} = \emptyset \text{ and } \mathrm{SMC}_S^{(v_1)_x} = \mathrm{SMC}_S^{v_x}, \quad \text{for all } S \neq S_1; \tag{23}$$

$$\mathrm{SMC}_{S_1}^{(v_2)_x} = \mathrm{SMC}_{S_1}^{v_x} \text{ and } \mathrm{SMC}_S^{(v_2)_x} = \emptyset, \quad \text{for all } S \neq S_1. \tag{24}$$

In order to verify (23), for all $S \neq S_1$, the following chain of equalities holds:

$$\begin{aligned} \mathrm{SMC}_S^{(v_1)_x} &= (v_1)_x(S) \setminus \nabla_S^{(v_1)_x} \stackrel{(21)}{=} \mathrm{SMC}_S^{v_x} \setminus \nabla_S^{(v_1)_x} \\ &= \left[v_x(S) \setminus \nabla_S^{v_x}\right] \setminus \nabla_S^{(v_1)_x} \\ &= \left[v_x(S) \setminus \nabla_S^{v_x}\right] = \mathrm{SMC}_S^{v_x}, \end{aligned}$$

where the second equality follows from the description (21) of the set game $(N,(v_1)_x)$ and the fourth from $\nabla_S^{(v_1)_x} \subseteq \nabla_S^{v_x}$, because of $T \subsetneq S$, $(v_1)_x(T) \subseteq v_x(T)$. So, (23) holds. In order to verify (24), the following chain of equalities holds:

$$\mathrm{SMC}_{S_1}^{(v_2)_x} = (v_2)_x(S_1) \setminus \nabla_{S_1}^{(v_2)_x} \stackrel{(22)}{=} \mathrm{SMC}_{S_1}^{v_x} \setminus \nabla_{S_1}^{(v_2)_x} = \mathrm{SMC}_{S_1}^{v_x},$$

due to the equality $\nabla_{S_1}^{(v_2)_x} = \emptyset$, because of $(v_2)_x(T) = \emptyset$, for all $T \subsetneq S_1$ (see (7)). So, (24) holds too. Clearly, it concerns a disjoint union so that $\mathrm{SMC}_S^{v_x} = \mathrm{SMC}_S^{(v_1)_x} \cup \mathrm{SMC}_S^{(v_2)_x}$ for all $S \subseteq N$. From this, we deduce the following chain of equalities:

$$\begin{aligned} \mathrm{SM}_i(N,v_x) &\stackrel{(6)}{=} \bigcup_{\substack{S\subseteq N \\ S\ni i}} \mathrm{SMC}_S^{v_x} = \bigcup_{\substack{S\subseteq N \\ S\ni i}} \left[\mathrm{SMC}_S^{(v_1)_x} \cup \mathrm{SMC}_S^{(v_2)_x}\right] \\ &= \left[\bigcup_{\substack{S\subseteq N \\ S\ni i}} \mathrm{SMC}_S^{(v_1)_x}\right] \cup \left[\bigcup_{\substack{S\subseteq N \\ S\ni i}} \mathrm{SMC}_S^{(v_2)_x}\right] \\ &\stackrel{(6)}{=} \mathrm{SM}_i(N,(v_1)_x) \cup \mathrm{SM}_i(N,(v_2)_x). \end{aligned}$$

By (24), the semi-marginalistic contribution set game $(N, \mathrm{SMC}^{(v_2)_x})$ has a unique winning coalition S_1, whereas, by (23), the collection of winning coalitions in the semi-marginalistic contribution set game $(N, \mathrm{SMC}^{(v_1)_x})$ is identical to the ones in the initial semi-marginalistic contribution set game (N, SMC^{v_x}), except for coalition S_1. The induction hypothesis (16) applied to both set games $(N, (v_1)_x$ and $(N, (v_2)_x)$ yields

$$\mathrm{SM}_i(N, (v_1)_x) = f_i(N, (v_1)_x) \text{ as well as } \mathrm{SM}_i(N, (v_2)_x) = f_i(N, (v_2)_x).$$

Further, from the inclusion $\mathrm{SMC}_S^{(v_1)_x} \subseteq \mathrm{SMC}_S^{v_x}$ for all $S \subseteq N$ (see (23)), together with the semi-marginalistic contribution monotonicity for f, we derive $f_i(N, (v_1)_x) \subseteq f_i(N, v_x)$ and, similarly, $f_i(N, (v_2)_x) \subseteq f_i(N, v_x)$. Finally, we conclude that the following chain of inclusion holds:

$$\begin{aligned} \mathrm{SM}_i(N, v_x) &= \mathrm{SM}_i(N, (v_1)_x) \cup \mathrm{SM}_i(N, (v_2)_x) \\ &= f_i(N, (v_1)_x) \cup f_i(N, (v_2)_x) \subseteq f_i(N, v_x). \end{aligned}$$

Hence $\mathrm{SM}_i(N, v_x) \subseteq f_i(N, v_x)$, whereas the reverse inclusion $\mathrm{SM}_i(N, v_x) \supseteq f_i(N, v_x)$ holds by Proposition 4.2. We arrive at the equality $\mathrm{SM}_i(N, v_x) = f_i(N, v_x)$. This completes both the inductive proof of (16) and the full proof of Theorem 4.1. □

Remark 4.5 Throughout the above Theorem 4.1, for any set game (N, v) and any coalition $S \subseteq N$, the associated expression ∇_S^v is supposed to possess the following minor property:

$$\nabla_S^w \subseteq \nabla_S^v \text{ when } w(T) \subseteq v(T), \quad \text{for all } T \subsetneq S. \tag{25}$$

In the context of the empty set game, (25) is meant to be read as $\nabla_S^w = \emptyset$, whenever $w(T) = \emptyset$, for all $T \subsetneq S$.

Remark 4.6 We present the table with rows indexed by set game-theoretic values and columns indexed by properties indicating for each value which one of the properties it satisfies. The abbreviations GEF, ETP, SMC-Mon, and ADD stand for global efficiency, equal treatment property, semi-marginalistic contribution monotonicity and additivity. A value f on the set game space $G^N(\mathcal{U})$ possesses the *additivity property* if $f_i(N, v \cup w) = f_i(N, v) \cup f_i(N, w)$ for any pair (N, v), (N, w) of disjoint set games and all $i \in N$. Here the set game $(N, v \cup w)$ is defined by $(v \cup w)(S) = v(S) \cup w(S)$ for all $S \subseteq N$. The two set games (N, v) and (N, w) are called disjoint if $v(S) \cap w(S) = \emptyset$ for all $S \subseteq N$. Generally speaking, any SM- value (except for the MSM-value) does not satisfy the additivity, although formula (15) states that any $SM-$value is additive with respect to simple set games.

Remark 4.7 We discuss the independence of the three axioms in the main characterization stated in Theorem 4.1. We present a value f on the set game

Value	GEF	ETP	SMC-Mon	ADD	Similar TU-value
IM	Yes	Yes	Yes	No	Shapley value (due to formula)
OCM	No	Yes	Yes	No	Shapley value (due to potential)
ICM	Yes	Yes	Yes	No	Solidarity value
OIM	Yes	Yes	Yes	No	
SCM	Yes	Yes	Yes	No	
MSM	Yes	Yes	Yes	Yes	

space $G(\mathcal{U})$ that will satisfy global efficiency and semi-marginalistic contribution monotonicity with respect to a given marginalistic contribution $\left\{\mathrm{SMC}_S^{*}\right\}_{S\subseteq N}$, but violates the equal treatment property. The value f is defined by $f(N,v) = SM(N,v)$ for all set games (N,v) with $|N| \geq 3$, and $f_i(\{i,j\},v) = v(\{i\})$, while $f_j(\{i,j\},v) = v(\{j\}) \cup [v(\{i,j\}) \setminus \nabla^v_{\{i,j\}}]$. Notice that $f_j(\{i,j\},v) = \mathrm{SM}_j(\{i,j\},v)$. In the framework of two-person set games, this value f satisfies the semi-marginalistic contribution monotonicity because the inclusion $f_i(\{i,j\},v) \subseteq f_i(\{i,j\},w)$ agrees with the assumption $\mathrm{SMC}^v_{\{i\}} \subseteq \mathrm{SMC}^w_{\{i\}}$ for two-person set games $(\{i,j\},v)$ and $(\{i,j\},w)$. Obviously, this value f does not possess the equal treatment property for substitutes in two-person set games.

In case $\nabla^v_N = \emptyset$ and $\nabla^v_S = v(S)$ for all $S \subsetneq N$, then the associated semi-marginalistic contribution SMC^v_S agrees with the empty set for every coalition S, whereas $\mathrm{SMC}^v_N = v(N)$. The associated value f is given by $f_i(N,v) = v(N)$ for all $i \in N$ and it is easy to verify that this value f satisfies equal treatment property and semi-marginalistic contribution monotonicity, but not global efficiency.

5 Concluding remarks

The axiomatization of semi-marginalistic values for set games, as stated in Theorem 4.1, can be considered, more or less, as the counterpart of Young's axiomatization of the Shapley value for cooperative games. In order to elucidate these similarities between the two fields of set game theory and cooperative game theory, let us briefly summarize the basic concepts from the latter field.

A *cooperative game* with transferable utility (TU) is a pair (N,v), where N is a nonempty finite set and $v : 2^N \to \mathbb{R}$ is a *characteristic function*, defined on the power set of N, satisfying $v(\emptyset) := 0$. Let Γ denote the space of all cooperative TU-games with an arbitrary player set. An element of N (notation: $i \in N$) and a nonempty subset S of N (notation: $S \subseteq N$ or $S \in 2^N$ with $S \neq \emptyset$) is called a *player* and *coalition* respectively, and the associated *real number* $v(S)$ is called the *worth* of coalition S, to be interpreted as the earnings (in the utility of money) its members can obtain by mutual cooperation among themselves. Concerning the solution theory for cooperative TU-games, a single-valued *solution* f on Γ associates with every cooperative game $(N,v) \in \Gamma$ a single payoff vector $f(N,v) = (f_i(N,v))_{i\in N} \in \mathbb{R}^N$. The payoff $f_i(N,v)$ to player i in the cooperative game (N,v) represents an assessment by

i of his gains for participating in the game. A single-valued solution f satisfies the *efficiency principle* if $\sum_{i \in N} f_i(N, v) = v(N)$ for all $(N, v) \in \Gamma$. The *equal treatment property* for f on Γ is fully in accordance with Definition 3.3. Further, a single-valued solution f on Γ satisfies the *strong monotonicity property* if $f_i(N, v) \leq f_i(N, w)$ for any pair (N, v), (N, w) of cooperative games and all $i \in N$, satisfying $v(S) - v(S \setminus \{i\}) \leq w(S) - w(S \setminus \{i\})$ for all $S \subseteq N$ with $i \in S$. In Young (1985), it is shown that there exists a unique solution on the cooperative game space Γ^N (with reference to a fixed player set N) satisfying the efficiency, and strong monotonicity, and it is given by the well-known Shapley value $Sh(N, v) = (Sh_i(N, v))_{i \in N} \in \mathbb{R}^N$ as follows (cf. Shapley 1953):

$$\mathrm{Sh}_i(N, v) = \sum_{\substack{S \subseteq N \\ S \ni i}} \frac{(|S| - 1)!(|N| - |S|)!}{|N|!} \Big[v(S) - v(S \setminus \{i\}) \Big] \quad \text{for all } i \in N,$$

where $|S|$ denotes the size (cardinality) of coalition S. For a detailed introduction about cooperative game theory, we refer to Driessen (1988). In summary, the main Theorem 4.1 concerning semi-marginalistic values for set games has been inspired by Young's axiomatization for the Shapley value, although their proofs differ very much. The counterpart of the Shapley value may be stated, at first glance, to be the individually marginalistic IM-value, as given by (1), but from the viewpoint of the potential approach to the solution theory, it is justified to be the overall-coalitionally marginalistic OCM-value (cf. Driessen and Sun 2001). In addition, the semi-marginalistic value f by choosing $\nabla_S^v = \bigcap_{j \in S} v(S \setminus \{j\})$ (cf. Sun et al. 1997) may be interpreted as the counterpart of the solidarity value for cooperative games (cf. Nowak and Radzik 1994).

Acknowledgements The project-sponsored by SRF for ROCS, SEM and by NNSF of China (No. 70571065). The authors are grateful to two anonymous referees and two (associate) editors for helpful suggestions that greatly improved the presentation of the paper.

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