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Path-fan Ramsey numbers

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Abstract

For two given graphs F and H, the Ramsey number $R(F, H)$ is the smallest positive integer p such that for every graph G on p vertices the following holds: either G contains F as a subgraph or the complement of G contains H as a subgraph. In this paper, we study the Ramsey numbers $R(P_n, F_m)$, where P_n is a path on n vertices and F_m is the graph obtained from m disjoint triangles by identifying precisely one vertex of every triangle (F_m is the join of K_1 and mK_2). We determine the exact values of $R(P_n, F_m)$ for the following values of n and m: $1 \leqslant n \leqslant 5$ and $m \geqslant 2$; $n \geqslant 6$ and $2 \leqslant m \leqslant (n+1)/2$; $6 \leqslant n \leqslant 7$ and $m \geqslant n-1$; $n \geqslant 8$ and $n-1 \leqslant m \leqslant n$ or $((q \cdot n-2q+1)/2 \leqslant m \leqslant (q \cdot n-q+2)/2$ with $3 \leqslant q \leqslant n-5)$ or $m \geqslant (n-3)^2/2$; odd $n \geqslant 9$ and $((q \cdot n-3q+1)/2 \leqslant m \leqslant (q \cdot n-2q)/2$ with $3 \leqslant q \leqslant (n-3)/2)$ or $((q \cdot n-q-n+4)/2 \leqslant m \leqslant (q \cdot n-2q)/2$ with $(n-1)/2 \leqslant q \leqslant n-5)$. Moreover, we give nontrivial lower bounds and upper bounds for $R(P_n, F_m)$ for the other values of m and n.

Keywords: Fan; Path; Ramsey number

1. Introduction

Throughout this paper, all graphs are finite and simple. Let G be such a graph. We write $V(G)$ or V for the vertex set of G and $E(G)$ or E for the edge set of G. The graph $\overline{G}$ is the *complement* of G, i.e., the graph obtained from the complete graph on $|V(G)|$ vertices by deleting the edges of G. The graph $H=(V', E')$ is a *subgraph* of $G=(V, E)$ if $V' \subseteq V$ and $E' \subseteq E$ (implying that the edges of H have all their end vertices in V').

If $e=\{u, v\} \in E$ (in short, $e=uv$), then u is called *adjacent* to v, and u and v are called *neighbors*. For $x \in V$, define $N(x)=\{y \in V \mid xy \in E\}$ and $N[x]=N(x) \cup \{x\}$. If $S \subset V(G)$, $S \neq V(G)$, then $G-S$ denotes the subgraph of G induced by $V(G) \backslash S$. If $|S|=1$, then we also use $G-z$ for $S=\{z\}$ instead of $G-\{z\}$. If $e \in E(G)$, then $G-e=(V(G), E(G) \backslash \{e\})$.

We denote by P_n, C_n and K_n the *path*, the *cycle* and the *complete graph* on n vertices, respectively. A *fan* F_m is a graph on $2m+1$ vertices obtained from m disjoint triangles (K_3s) by identifying precisely one vertex of every triangle

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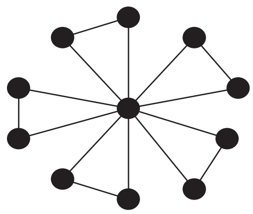

Fig. 1. The fan F_5.

(F_m is the *join* of K_1 and mK_2). The vertex corresponding to K_1 is called the *hub* of the fan. For illustration, consider F_5 in Fig. 1.

Given two graphs F and H, the *Ramsey number* $R(F, H)$ is defined as the smallest positive integer p such that every graph G on p vertices satisfies the following condition: G contains F as a subgraph or $\overline{G}$ contains H as a subgraph.

In 1967, Geréncser and Gyárfás [4] determined all Ramsey numbers for paths versus paths. After that, Ramsey numbers $R(P_n, H)$ for paths versus other graphs H have been investigated in several papers, for example: Parsons [6] when H is a complete graph; Faudree et al. [2] when H is a cycle; Parsons [7] when H is a star; Häggkvist [5] when H is a complete bipartite graph; Faudree et al. [3] when H is a tree; Surahmat and Baskoro [9], Chen et al. [1] and Salman and Broersma [8] when H is a wheel. We study Ramsey numbers for paths versus fans.

2. Main results

In this paper we determine the Ramsey numbers $R(P_n, F_m)$ for the following values of n and m: $1 \leqslant n \leqslant 5$ and $m \geqslant 2$; $n \geqslant 6$ and $2 \leqslant m \leqslant (n+1)/2$; $6 \leqslant n \leqslant 7$ and $m \geqslant n-1$; $n \geqslant 8$ and $n-1 \leqslant m \leqslant n$ or $((q \cdot n - 2q + 1)/2 \leqslant m \leqslant (q \cdot n - q + 2)/2$ with $3 \leqslant q \leqslant n-5)$ or $m \geqslant (n-3)^2/2$; odd $n \geqslant 9$ and $((q \cdot n - 3q + 1)/2 \leqslant m \leqslant (q \cdot n - 2q)/2$ with $3 \leqslant q \leqslant (n-3)/2)$ or $((q \cdot n - q - n + 4)/2 \leqslant m \leqslant (q \cdot n - 2q)/2$ with $(n-1)/2 \leqslant q \leqslant n-5)$. We will present the Ramsey numbers for 'small' paths versus fans in Proposition 1, the Ramsey numbers for paths versus 'small' fans in Theorem 3, and the Ramsey numbers for paths versus 'large' fans in the corollaries based on Lemmas 4 and 6. Moreover, we give nontrivial lower bounds and upper bounds for $R(P_n, F_m)$ for (odd $n \geqslant 11$ and $(q \cdot n - q + 4)/2 \leqslant m \leqslant (q \cdot n - 3q + n - 3)/2$ with $2 \leqslant q \leqslant (n-7)/2)$ or (even $n \geqslant 8$ and $(q \cdot n - q + 3)/2 \leqslant m \leqslant (q \cdot n - 2q + n - 2)/2$ with $2 \leqslant q \leqslant n-5)$ or ($n \geqslant 6$ and $(n+2)/2 \leqslant m \leqslant n-2)$ in Corollaries 8, 9 and Theorem 10.

Proposition 1. *Let $m \geqslant 2$. Then*

$$R(P_n, F_m) = \begin{cases} 1 & \text{for } n = 1, \\ 2m+1 & \text{for } n = 2 \text{ or } 3. \end{cases}$$

Proof. The cases for which $n = 1$ or 2 are (almost) trivial and left to the reader. We only give the proof in case $n = 3$: the graph consisting of m disjoint copies of K_2 shows that $R(P_3, F_m) > 2m$. Now suppose G is a graph on $2m+1$ vertices, and assume G contains no P_3. We will show that $\overline{G}$ contains an F_m. Since $|V(G)|$ is odd and G contains no P_3, there is a vertex $z \in V(G)$ with $|N(z)| = 0$. Since $G - z$ contains no P_3, the vertices of $V(G) \backslash \{z\}$ have degree at least $2m-2$ in $\overline{G - z}$. This implies there exists a cycle C_{2m} in $\overline{G - z}$. Hence $\overline{G}$ contains an F_m (even a wheel on $2m+1$ vertices). □

The next lemma plays a key role in the proofs for the remaining cases.

Lemma 2. *Let $n \geqslant 3$ and G be a graph on at least n vertices containing no P_n. Let the paths $P^1, P^2, \ldots, P^k$ in G be chosen in the following way: $\bigcup_{j=1}^{k} V(P^j) = V(G)$, P^1 is a longest path in G, and, if $k > 1$, P^{i+1} is a longest path in $G - \bigcup_{j=1}^{i} V(P^j)$ for $1 \leqslant i \leqslant k-1$. Denote by ℓ_j the numbers of vertices on the path P^j. Let z be an end vertex of P^k.*

Then:

(i) $\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_k$;
(ii) *If* $\ell_k \geqslant \lfloor n/2 \rfloor$, *then* $N(z) \subset V(P^k)$;
(iii) *If* $\ell_k < \lfloor n/2 \rfloor$, *then* $|N(z)| \leqslant \lfloor n/2 \rfloor - 1$.

Proof. (i) Obviously follows from the choice of the paths. From this choice we can also deduce that for any integer x with $1 \leqslant x < k$, the number of neighbors of z in $V(P^x)$ is

$$\begin{cases} \leqslant \left\lfloor \dfrac{\ell_x + 1 - 2\ell_k}{2} \right\rfloor & \text{if } \ell_x \geqslant 2\ell_k + 1, \\ 0 & \text{if } \ell_x < 2\ell_k + 1. \end{cases} \tag{1}$$

This can be checked easily: first order the neighbors of z on P^x according to the order of their appearance on P^x in a fixed orientation. Then observe that between any two successive neighbors of z on P^x, there is at least one nonneighbor of z, while before the first and after the last neighbor of z on P^x, there are at least ℓ_k nonneighbors of z.

(ii) Assume $\ell_k \geqslant \lfloor n/2 \rfloor$. Then $2\ell_k + 1 \geqslant n > \ell_1$. So by the above observation, we conclude that there is no neighbor of z in $V(G) \backslash V(P^k)$.

(iii) Now assume $\ell_k < \lfloor n/2 \rfloor$. If z has no neighbors in $V(G) \backslash V(P^k)$, we are done. If z has some neighbors in $V(G) \backslash V(P^k)$, similar counting arguments as above yield the desired result: denote by $h_1, \ldots, h_t$ the numbers of vertices on the paths $P^1, \ldots, P^k$ that contain a neighbor of z, chosen in such a way that $h_t \geqslant \cdots \geqslant h_1$, and denote by $d_1, \ldots, d_t$ the numbers of neighbors of z on the corresponding paths. Then, arguing as above, we obtain $h_1 = \ell_k \geqslant d_1 + 1$ and $h_2 \geqslant 2h_1 + 2d_2 - 1$. Similarly, observing that z connects any two of the considered paths, and using the same elementary counting techniques, we get (if $t \geqslant 3$) $h_j \geqslant 2((h_{j-1} - 1)/2 + 2) + 2d_j - 1 = h_{j-1} + 2d_j + 2$ for $3 \leqslant j \leqslant t$. This implies (for $t \geqslant 2$) that $h_t \geqslant 2(d_1 + \cdots + d_t) + 2(t-2) + 1 \geqslant 2|N(z)| + 1$. Since $h_t \leqslant n - 1$ and $|N(z)|$ are integers, this yields the desired result. □

Theorem 3. *Let* $n \geqslant 4$ *and* $2 \leqslant m \leqslant (n+1)/2$. *Then* $R(P_n, F_m) = 2n - 1$.

Proof. The graph $2K_{n-1}$ shows that $R(P_n, F_m) > 2n - 2$. Let G be a graph on $2n - 1$ vertices and assume G contains no P_n. We are going to show that $\overline{G}$ contains an F_m. Choose the paths $P^1, \ldots, P^k$ and the vertex z as in Lemma 2. Since $|V(G)| = 2n - 1$ and G does not contain a P_n, $k \geqslant 3$ and $\ell_k \leqslant (2n-1)/3$. If $\ell_k < \lfloor n/2 \rfloor$ then by Lemma 2(iii) we obtain $|N(z)| \leqslant \lfloor n/2 \rfloor - 1 \leqslant (2n-1)/3 - 1$. If $\lfloor n/2 \rfloor \leqslant \ell_k \leqslant (2n-1)/3$ then by Lemma 2(ii) we obtain $|N(z)| \leqslant \ell_k - 1 \leqslant (2n-1)/3 - 1$. Hence, $|N[z]| \leqslant (2n-1)/3$. We are going to show that there is an F_m in $\overline{G}$ with z as a hub. We distinguish the following two cases.

Case 1: $|N(z)| \leqslant \lfloor n/2 \rfloor - 1$.

Then $|V(G) \backslash N[z]| \geqslant (2n-1) - \lfloor n/2 \rfloor \geqslant n + m - 1$. We can apply the result from [2] that $R(P_n, C_{2m}) = n + m - 1$ for $2 \leqslant m \leqslant \lfloor (n+1)/2 \rfloor$. This implies that $\overline{G - N[z]}$ contains a C_{2m}. So, there is an F_m in $\overline{G}$ with z as a hub (there is even a wheel on $2m+1$ vertices).

Case 2: $|N(z)| \geqslant \lfloor n/2 \rfloor$.

By Lemma 2(ii), we find $N(z) \subset V(P^k)$. Hence, $\ell_k \geqslant \lfloor n/2 \rfloor + 1$. Since $|V(G)| = 2n - 1$, $k = 3$. Take the first m vertices of P^1 (in some fixed orientation) and name them $u_1, \ldots, u_m$, starting at an end vertex. Also take the first m vertices of P^2 (in some fixed orientation) and name them $v_1, \ldots, v_m$, starting at an end vertex. Since P^1 is chosen as a longest path in G, it is obvious that $u_i v_i \notin E(G)$ $(i = 1, \ldots, m)$. So there is an F_m in $\overline{G}$ with z as a hub. □

The following lemma provides upper bounds that yield several exact Ramsey numbers in the sequel.

Lemma 4. *If* $n \geqslant 4$ *and* $m \geqslant n - 1$, *then*

$$R(P_n, F_m) \leqslant \begin{cases} 2m + n - 1 & \textit{for } 2m = 1 \bmod (n-1), \\ 2m + n - 2 & \textit{for other values of } m. \end{cases}$$

Proof. Let G be a graph that contains no P_n and has order

$$|V(G)| = \begin{cases} 2m+n-1 & \text{for } 2m = 1 \bmod (n-1), \\ 2m+n-2 & \text{for other values of } m. \end{cases} \tag{2}$$

Choose the paths $P^1, \ldots, P^k$ and the vertex z in G as in Lemma 2. Because of (2), not all P^i can have $n-1$ vertices, so $\ell_k \leqslant n-2$. By similar arguments as in the proof of Theorem 3, this implies $|N(z)| \leqslant n-3$. We will use the following result that has been proved in [2]: $R(P_t, C_s) = s + \lfloor t/2 \rfloor - 1$ for $s \geqslant \lfloor (3t+1)/2 \rfloor$. We distinguish the following cases.

Case 1: $|N(z)| \leqslant \lfloor n/2 \rfloor - 2$ or n is odd and $|N(z)| = \lfloor n/2 \rfloor - 1$.

Since $|V(G) \backslash N[z]| \geqslant 2m + \lfloor n/2 \rfloor - 1$, we find that $\overline{G - N[z]}$ contains a C_{2m}. So, there is an F_m in $\overline{G}$ with z as a hub.

Case 2: n is even and $|N(z)| = n/2 - 1$.

Since $|V(G) \backslash N[z]| \geqslant (2m+n-2) - n/2 = 2m + n/2 - 2$, we find that $\overline{G - N[z]}$ contains a C_{2m-1}; denote its vertices by $v_1, v_2, v_3, \ldots, v_{2m-1}$ in the order of appearance on the cycle with a fixed orientation. There are $n/2 - 1$ vertices in $U = V(G) \backslash (V(C_{2m-1}) \cup N[z])$, say $u_1, u_2, \ldots, u_{n/2-1}$. If some vertex v_i $(i = 1, \ldots, 2m-1)$ is no neighbor of some vertex u_j $(j = 1, \ldots, n/2 - 1)$, w.l.o.g. assume $v_{2m-1}u_1 \notin E(G)$. Then $\overline{G}$ contains an F_m with z as a hub and additional edges $v_1v_2, v_3v_4, \ldots, v_{2m-3}v_{2m-2}, v_{2m-1}u_1$. Now let us assume each of the v_i is adjacent to all u_j in G. For every choice of a subset of $n/2$ vertices from $V(C_{2m-1})$, there is a path on $n-1$ vertices in G alternating between the vertices of this subset and the vertices of U, starting and terminating in two arbitrary vertices from the subset. Since G contains no P_n, there are no edges $v_iv_j \in E(G)$ $(i, j \in \{1, \ldots, 2m-1\})$. This implies that $V(C_{2m-1}) \cup \{z\}$ induces a K_{2m} in $\overline{G}$. Since G contains no P_n, no v_i is adjacent to a vertex of $N(z)$. This implies that $\overline{G}$ contains a $K_{2m+1} - zw$ for any vertex $w \in N(z)$, and hence $\overline{G}$ contains an F_m with one of the v_i as a hub.

Case 3: Suppose that there is no choice for P^k and z such that one of the former cases applies. Then $|N(w)| \geqslant \lfloor n/2 \rfloor$ for any end vertex w of a path on ℓ_k vertices in $G - \bigcup_{j=1}^{k-1} V(P^j)$. This implies all neighbors of such w are in $V(P^k)$ and $\ell_k \geqslant \lfloor n/2 \rfloor + 1$. So for the two end vertices z_1 and z_2 of P^k we have that $|N(z_i) \cap V(P^k)| \geqslant \lfloor n/2 \rfloor \geqslant \ell_k/2$. Let $P^k: z_1 = v_1v_2 \ldots v_{\ell_k} = z_2$. Then by standard arguments in Hamiltonian graph theory, we can find an index $i \in \{2, \ldots, \ell_k - 1\}$ such that z_1v_{i+1} and z_2v_i are edges of G. It is clear that we can find a cycle on ℓ_k vertices in G. This implies that any vertex of $V(P^k)$ could serve as w. By the assumption of this last case, we conclude that there are no edges in G between $V(P^k)$ and the other vertices. This also implies that all vertices of P^k have degree at least $2m$ in $\overline{G}$.

We now turn to P^{k-1} and consider one of its end vertices w. Since $\ell_{k-1} \geqslant \ell_k \geqslant \lfloor n/2 \rfloor + 1$, similar arguments as in the proof of Lemma 2 show that all neighbors of w are on P^{k-1}. If $|N(w)| < \lfloor n/2 \rfloor$, we get an F_m in $\overline{G}$ as in Case 1 or Case 2. So we may assume $|N(w_i) \cap V(P^{k-1})| \geqslant \lfloor n/2 \rfloor \geqslant \ell_{k-1}/2$ for both end vertices w_1 and w_2 of P^{k-1}. By similar arguments as before we obtain a cycle on ℓ_{k-1} vertices in G. This implies that any vertex of $V(P^{k-1})$ could serve as w. By the assumption of this last case, we conclude that there are no edges in G between $V(P^{k-1})$ and the other vertices. This also implies that all vertices of P^{k-1} have degree at least $2m-1$ in $\overline{G}$. (Note that P^{k-1} can have $n-1$ vertices, whereas $\ell_k \leqslant n-2$.)

Repeating the above arguments for $P^{k-2}, \ldots, P^1$ we eventually conclude that all vertices of G have degree at least $2m-1$ in $\overline{G}$. Now let $H = \overline{G} - V(P^k)$. Then all vertices in $V(H)$ have degree at least $2m - 1 - \ell_k \geqslant m + (n-1) - 1 - \ell_k \geqslant \frac{1}{2}(2m + 2n - 4 - \ell_k - (n-2)) = \frac{1}{2}(2m + n - 2 - \ell_k) \geqslant \frac{1}{2}(|V(H)| - 1)$. This implies there exists a Hamilton path in H. Since $|V(H)| \geqslant 2m$ and z is a neighbor of all vertices in H, it is clear that $\overline{G}$ contains an F_m with z as a hub. This completes the proof of Lemma 4. □

Corollary 5. *If* $(4 \leqslant n \leqslant 7$ *and* $m \geqslant n-1)$ *or* $(n \geqslant 8$ *and* $n - 1 \leqslant m \leqslant n$ *or* $((q \cdot n - 2q + 1)/2 \leqslant m \leqslant (q \cdot n - q + 2)/2$ *for* $3 \leqslant q \leqslant n-5)$ *or* $m \geqslant (n-3)^2/2)$, *then*

$$R(P_n, F_m) = \begin{cases} 2m+n-1 & \textit{for } 2m = 1 \bmod (n-1), \\ 2m+n-2 & \textit{for other values of } m. \end{cases}$$

Proof. Let r denote the remainder of $2m$ divided by $n-1$, so $2m = p(n-1) + r$ for some $0 \leqslant r \leqslant n-2$. Then for $(4 \leqslant n \leqslant 7$ and $m \geqslant n-1)$ or $(n \geqslant 8$ and $n-1 \leqslant m \leqslant n$ or $((q \cdot n - 2q + 1)/2 \leqslant m \leqslant (q \cdot n - q + 2)/2$ for $3 \leqslant q \leqslant n-5)$

or $m \geqslant (n-3)^2/2)$, the graphs

$$\begin{cases} (p-1)K_{n-1} \cup 2K_{n-2} & \text{for } r=0, \\ (p+1)K_{n-1} & \text{for } r=1 \text{ or } 2, \\ (p+r+1-n)K_{n-1} \cup (n+1-r)K_{n-2} & \text{for other values of } r \end{cases}$$

show that

$$R(P_n, F_m) > \begin{cases} 2m+n-2 & \text{for } 2m = 1 \bmod (n-1), \\ 2m+n-3 & \text{for other values of } m. \end{cases}$$

Lemma 4 completes the proof. □

Lemma 6. *If n is odd, $n \geqslant 9$ and $(q \cdot n - q + 3)/2 \leqslant m \leqslant (q \cdot n - 2q + n - 2)/2$ with $2 \leqslant q \leqslant 2\lfloor n/2 \rfloor - 5$, then $R(P_n, F_m) \leqslant 2m + n - 3$.*

Proof. The proof is modelled along the lines of the proof of Lemma 4. Let G be a graph on $2m+n-3$ vertices, and assume G contains no P_n. We will show that $\overline{G}$ contains an F_m. Choose the paths $P^1, \ldots, P^k$ and the vertex z in G as in Lemma 2. Since $|V(G)| = 2m+n-3$ with $n \geqslant 9$ and $(q \cdot n - q + 3)/2 \leqslant m \leqslant (q \cdot n - 2q + n - 2)/2$ with $2 \leqslant q \leqslant 2\lfloor n/2 \rfloor - 5$, $k \geqslant q+2$, and therefore not all P^i can have more than $n-3$ vertices. So $\ell_k \leqslant n-3$. By similar arguments as in the proof of Theorem 3, this implies $|N(z)| \leqslant n-4$. We will use the following result that has been proved in [2]: $R(P_t, C_s) = s + \lfloor t/2 \rfloor - 1$ for $s \geqslant \lfloor (3t+1)/2 \rfloor$. We distinguish the following cases.

Case 1: $|N(z)| \leqslant \lfloor n/2 \rfloor - 2$.

Since $|V(G) \backslash N[z]| \geqslant 2m + \lfloor n/2 \rfloor - 1$, we find that $\overline{G - N[z]}$ contains a C_{2m}. So, there is an F_m in $\overline{G}$ with z as a hub.

Case 2: $|N(z)| = \lfloor n/2 \rfloor - 1$.

Since $|V(G) \backslash N[z]| = (2m+n-3) - \lfloor n/2 \rfloor = 2m + \lfloor n/2 \rfloor - 2$, we find that $\overline{G - N[z]}$ contains a C_{2m-1}; denote its vertices by $v_1, v_2, v_3, \ldots, v_{2m-1}$ in the order of appearance on the cycle with a fixed orientation. There are $\lfloor n/2 \rfloor - 1$ vertices in $U = V(G) \backslash (V(C_{2m-1}) \cup N[z])$, say $u_1, u_2, \ldots, u_{\lfloor n/2 \rfloor - 1}$. If some vertex v_i $(i = 1, \ldots, 2m-1)$ is no neighbor of some vertex u_j $(j = 1, \ldots, \lfloor n/2 \rfloor - 1)$, w.l.o.g. assume $v_{2m-1}u_1 \notin E(G)$. Then $\overline{G}$ contains an F_m with z as a hub and additional edges $v_1v_2, v_3v_4, \ldots, v_{2m-3}v_{2m-2}, v_{2m-1}u_1$. Now let us assume each of the v_i is adjacent to all u_j in G. For every choice of a subset of $\lfloor n/2 \rfloor$ vertices from $V(C_{2m-1})$, there is a path on $n-2$ vertices in G alternating between the vertices of this subset and the vertices of U, starting and terminating in two arbitrary vertices from the subset. Let $z_1 \in N(z)$. Since G contains no P_n, there are no edges $v_iz \in E(G)$ and $v_iz_1 \in E(G)$ $(i \in \{1, \ldots, 2m-1\})$ and there is only (at most) one edge $v_iv_j \in E(G)$ (for some $i, j \in \{1, \ldots, 2m-1\}$). Suppose $v_1v_2 \in E(G)$. This implies $\overline{G}$ contains an F_m with hub v_{2m-1} and additional edges $v_1z, v_2z_1, v_3v_4, \ldots, v_{2m-5}v_{2m-4}, v_{2m-3}v_{2m-2}$. The other cases are similar.

Case 3: Suppose that there is no choice for P^k and z such that one of the former cases applies. Then $|N(w)| \geqslant \lfloor n/2 \rfloor$ for any end vertex w of a path on ℓ_k vertices in $G - \bigcup_{j=1}^{k-1} V(P^j)$. This implies all neighbors of such w are in $V(P^k)$ and $\ell_k \geqslant \lfloor n/2 \rfloor + 1$. So for the two end vertices z_1 and z_2 of P^k we have that $|N(z_i) \cap V(P^k)| \geqslant \lfloor n/2 \rfloor \geqslant \ell_k/2$. By similar arguments as in the proof of Case 3 of Lemma 4 we obtain a cycle on ℓ_k vertices in G. This implies that any vertex of $V(P^k)$ could serve as w. By the assumption of this last case, we conclude that there are no edges in G between $V(P^k)$ and the other vertices. This also implies that all vertices of P^k have degree at least $2m$ in $\overline{G}$.

We now turn to P^{k-1} and consider one of its end vertices w. Since $\ell_{k-1} \geqslant \ell_k \geqslant \lfloor n/2 \rfloor + 1$, similar arguments as in the proof of Lemma 2 show that all neighbors of w are on P^{k-1}. If $|N(w)| < \lfloor n/2 \rfloor$, we get an F_m in $\overline{G}$ as in Case 1 or Case 2. So we may assume $|N(w_i) \cap V(P^{k-1})| \geqslant \lfloor n/2 \rfloor \geqslant \ell_{k-1}/2$ for both end vertices w_1 and w_2 of P^{k-1}. By similar arguments as before we obtain a cycle on ℓ_{k-1} vertices in G. This implies that any vertex of $V(P^{k-1})$ could serve as w. By the assumption of this last case, we conclude that there are no edges in G between $V(P^{k-1})$ and the other vertices. This also implies that all vertices of P^{k-1} have degree at least $2m-2$ in $\overline{G}$. (Note that P^{k-1} can have $n-1$ vertices, whereas $\ell_k \leqslant n-3$.)

Repeating the above arguments for $P^{k-2}, \ldots, P^1$ we eventually conclude that all vertices of G have degree at least $2m-2$ in $\overline{G}$. Now let $H = \overline{G} - V(P^k)$. Then all vertices in $V(H)$ have degree at least $2m - 2 - \ell_k \geqslant m + (n+1) - 2 - \ell_k \geqslant \frac{1}{2}(2m + 2n - 2 - \ell_k - (n-3)) = \frac{1}{2}(2m + n + 1 - \ell_k) = \frac{1}{2}(|V(H)| + 4)$. This implies there exists a Hamilton cycle in H. Since $|V(H)|\geqslant 2m$ and z is a neighbor of all vertices in H, it is clear that $\overline{G}$ contains an F_m with z as a hub. This completes the proof of Lemma 6. □

Corollary 7. *If n is odd, $n\geqslant 9$ and either $((q\cdot n-3q+1)/2\leqslant m\leqslant (q\cdot n-2q)/2$ with $3\leqslant q\leqslant (n-3)/2)$ or $((q\cdot n-q-n+4)/2\leqslant m\leqslant (q\cdot n-2q)/2$ with $(n-1)/2\leqslant q\leqslant n-5)$, then $R(P_n,F_m)=2m+n-3$.*

Proof. For odd $n\geqslant 9$ and $m=(q\cdot n-2q-j)/2$ with either $(3\leqslant q\leqslant (n-3)/2$ and $0\leqslant j\leqslant q-1)$ or $((n-1)/2\leqslant q\leqslant n-5$ and $0\leqslant j\leqslant n-q-4)$, the graph $(q-j-1)K_{n-2}\cup (j+2)K_{n-3}$ shows that $R(P_n,F_m)>2m+n-4$. Lemma 6 completes the proof. □

Corollary 8. *If n is odd, $n\geqslant 11$ and $(q\cdot n-q+4)/2\leqslant m\leqslant (q\cdot n-3q+n-3)/2$ with $2\leqslant q\leqslant (n-7)/2$, then*

$$2m+n-3\geqslant R(P_n,F_m)\geqslant \max\left\{\left\lfloor\frac{2m}{n-1}\right\rfloor(n-1)+n,\ 2m+\left\lfloor\frac{2m-1}{\lceil 2m/(n-1)\rceil}\right\rfloor\right\}.$$

Proof. Let $t=\lceil 2m/(n-1)\rceil$ and s denote the remainder of $2m-1$ divided by t. Then for m and n satisfying $\lfloor 2m/(n-1)\rfloor(n-1)+n\geqslant 2m+\lfloor(2m-1)/t\rfloor$, the graph tK_{n-1} shows that $R(P_n,F_m)>\lfloor 2m/(n-1)\rfloor(n-1)+n-1$.

For other values of m and n, the graph $sK_{\lceil(2m-1)/t\rceil}\cup(t-s+1)K_{\lfloor(2m-1)/t\rfloor}$ shows that $R(P_n,F_m)>2m-1+\lfloor(2m-1)/\lceil 2m/(n-1)\rceil\rfloor$.

The upper bound comes from Lemma 6. □

Corollary 9. *If n is even, $n\geqslant 8$ and $(q\cdot n-q+3)/2\leqslant m\leqslant (q\cdot n-2q+n-2)/2$ with $2\leqslant q\leqslant n-5$, then*

$$2m+n-2\geqslant R(P_n,F_m)\geqslant \max\left\{\left\lfloor\frac{2m}{n-1}\right\rfloor(n-1)+n,\ 2m+\left\lfloor\frac{2m-1}{\lceil 2m/(n-1)\rceil}\right\rfloor\right\}.$$

Proof. Let $t=\lceil 2m/(n-1)\rceil$ and s denote the remainder of $2m-1$ divided by t. Then for m and n satisfying $\lfloor 2m/(n-1)\rfloor(n-1)+n\geqslant 2m+\lfloor(2m-1)/t\rfloor$, the graph tK_{n-1} shows that $R(P_n,F_m)>\lfloor 2m/(n-1)\rfloor(n-1)+n-1$.

For other values of m and n, the graph $sK_{\lceil(2m-1)/t\rceil}\cup(t-s+1)K_{\lfloor(2m-1)/t\rfloor}$ shows that $R(P_n,F_m)>2m-1+\lfloor(2m-1)/\lceil 2m/(n-1)\rceil\rfloor$.

The upper bound comes from Lemma 4. □

Theorem 10. *If $n\geqslant 6$ and $(n+2)/2\leqslant m\leqslant n-2$, then*

$$2m+\left\lfloor\frac{3n}{2}\right\rfloor-2\geqslant R(P_n,F_m)\geqslant\begin{cases}2n-1 & \text{for } \dfrac{n+2}{2}\leqslant m\leqslant\dfrac{n+\lfloor n/3\rfloor}{2},\\ 3m-1 & \text{for } \dfrac{n+\lfloor n/3\rfloor}{2}<m\leqslant n-2.\end{cases}$$

Proof. For $n\geqslant 6$ and $(n+2)/2\leqslant m\leqslant (n+\lfloor n/3\rfloor)/2$, the graph $2K_{n-1}$ shows that $R(P_n,F_m)>2n-2$. For $n\geqslant 6$ and $(n+\lfloor n/3\rfloor)/2<m\leqslant n-2$, the graph $K_m\cup 2K_{m-1}$ shows that $R(P_n,F_m)>3m-2$.

Let G be a graph on $2m+\lfloor 3n/2\rfloor-2$ vertices, and assume G contains no P_n. Choose the paths $P^1,\ldots,P^k$ and the vertex z in G as in Lemma 2. By Lemma 2, $|N(z)|\leqslant n-2$. Hence, $|V(G)\backslash N[z]|\geqslant 2m+\lfloor n/2\rfloor-1$. We can apply the result from [2] that $R(P_n,C_{2m})=2m+\lfloor n/2\rfloor-1$ for $2\leqslant n\leqslant 2m$. This implies that $\overline{G-N[z]}$ contains a C_{2m}. So, there is an F_m in $\overline{G}$ with z as a hub (there is even a wheel on $2m+1$ vertices). □

3. Conclusion

In this paper we determined the exact Ramsey numbers for paths versus fans of varying orders. The numbers are indicated in Table 1. We used different capitals to distinguish the results in the previous section that led to these numbers. The shaded elements indicate open cases. For these cases we established nontrivial lower bounds and upper bounds for $R(P_n,F_m)$. We learned from one of the anonymous referees that Yunqing Zhang et al. established similar results in a recent paper. We are not aware of the present status of this paper.

Table 1
The Ramsey numbers for paths versus fans

F_m \ P_n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	A	D	D	I	I	I	I	I	I	I	I	I	I	I	I
3	A	D	D	N	I	I	I	I	I	I	I	I	I	I	I
4	A	D	D	N	N		I	I	I	I	I	I	I	I	I
5	A	D	D	L	N	N			I	I	I	I	I	I	I
6	A	D	D	N	N	N	N				I	I	I	I	I
7	A	D	D	N	N	N	N	N					I	I	I
8	A	D	D	L	N	L	N	N	N						I
9	A	D	D	N	N	N	N		N	N					
10	A	D	D	N	N	N	N	N	P	N	N				
11	A	D	D	L	N	N	N	L	N		N	N			
12	A	D	D	N	N	N	N		N			N	N		
13	A	D	D	N	N	L	N	N	N	N	P		N	N	
14	A	D	D	L	N	N	N	N	P	L	N			N	N
15	A	D	D	N	N	N	N	N	N		N				N
16	A	D	D	N	N	N	N	N	N		N	N	P		
17	A	D	D	L	N	N	N	N	N	N	P	L	N		
18	A	D	D	N	N	L	N	L	N	N	P		N		
19	A	D	D	N	N	N	N	N	N	N	N		N	N	P
20	A	D	D	L	N	N	N	N	N		N			L	N
21	A	D	D	N	N	N	N	N	N	N	N	N	P		N
22	A	D	D	N	N	N	N	N	N	N	P	N	P		N
23	A	D	D	L	N	L	N	N	N	L	N	N	N		
24	A	D	D	N	N	N	N	N	N		N		N		
25	A	D	D	N	N	N	N	L	N	N	N		N	N	P
26	A	D	D	L	N	N	N	N	N	N	N	N	P	N	P
27	A	D	D	N	N	N	N	N	N	N	P	N	P	N	N
28	A	D	D	N	N	L	N	N	N	N	N	L	N		N
29	A	D	D	L	N	N	N	N	N	N	N		N		N
30	A	D	D	N	N	N	N	N	N	N	N		N		
31	A	D	D	N	N	N	N	N	N	N	N	N	N	N	P
32	A	D	D	L	N	N	N	L	N	L	N	N	P	N	P
33	A	D	D	N	N	L	N	N	N	N	N	N	P	L	N
34	A	D	D	N	N	N	N	N	N	N	N	N	N		N
35	A	D	D	L	N	N	N	N	N	N	N		N		N
36	A	D	D	N	N	N	N	N	N	N	N	N	N		N
37	A	D	D	N	N	N	N	N	N	N	N	N	N	N	P
38	A	D	D	L	N	L	N	N	N	N	N	N	P	N	P
39	A	D	D	N	N	N	N	L	N	N	N	L	N	N	P
40	A	D	D	N	N	N	N	N	N	N	N		N	N	N
41	A	D	D	L	N	N	N	N	N	L	N	N	N		N
42	A	D	D	N	N	N	N	N	N	N	N	N	N		N
43	A	D	D	N	N	L	N	N	N	N	N	N	N	N	N
44	A	D	D	L	N	N	N	N	N	N	N	N	P	N	P
45	A	D	D	N	N	N	N	N	N	N	N	N	N	N	P
46	A	D	D	N	N	N	N	L	N	N	N	N	N	L	N
47	A	D	D	L	N	N	N	N	N	N	N	N	N		N
48	A	D	D	N	N	L	N	N	N	N	N	N	N		N
49	A	D	D	N	N	N	N	N	N	N	N	N	N	N	N
50	A	D	D	L	N	N	N	N	N	L	N	L	N	N	N

By		$R(P_n, F_m)$
Proposition 1	A	1
Proposition 1	D	2m + 1
Theorem 3	I	2n - 1
Corollary 5	L	2m + n - 1
Corollary 5	N	2m + n - 2
Corollary 7	P	2m + n - 3

References

[1] Y. Chen, Y. Zhang, K.M. Zhang, The Ramsey numbers of paths versus wheels, preprint, 2002.
[2] R.J. Faudree, S.L. Lawrence, T.D. Parsons, R.H. Schelp, Path-cycle Ramsey numbers, Discrete Math. 10 (1974) 269–277.
[3] R.J. Faudree, R.H. Schelp, M. Simonovits, On some Ramsey type problems connected with paths, cycles and trees, Ars Combin. 29A (1990) 97–106.

[4] L. Geréncser, A. Gyárfás, On Ramsey-type problems, Annales Universitatis Scientiarum Budapestinensis, Eötvös Sect. Math. 10 (1967) 167–170.
[5] R. Häggkvist, On the path-complete bipartite Ramsey numbers, Discrete Math. 75 (1989) 243–245.
[6] T.D. Parsons, The Ramsey numbers $r(P_m, K_n)$, Discrete Math. 6 (1973) 159–162.
[7] T.D. Parsons, Path-star Ramsey numbers, J. Combin. Theory Ser. B 17 (1974) 51–58.
[8] A.N.M. Salman, H.J. Broersma, On Ramsey numbers for paths versus wheels, Discrete Math., accepted for publication.
[9] Surahmat, E.T. Baskoro, On the Ramsey number of a path or a star versus W_4 or W_5, in: Proceedings of the 12th Australasian Workshop on Combinatorial Algorithms, 2001, pp. 174–178.