

Static and dynamic properties of fluxons in a zig-zag $0-\pi$ Josephson junction

H. Susanto^{a,*}, Darminto^b, S.A. van Gils^c

^a *Department of Mathematics and Statistics, University of Massachusetts, Amherst, MA 01003-4515, USA*

^b *Jurusan Fisika, FMIPA, Institut Teknologi Sepuluh Nopember, Kampus ITS Sukolilo, Surabaya 60111, Indonesia*

^c *Department of Applied Mathematics, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands*

Received 28 July 2006; received in revised form 16 September 2006; accepted 21 September 2006

Available online 28 September 2006

Communicated by J. Flouquet

Abstract

We consider a long Josephson junction with alternating 0 - and π -facets with different facet lengths between the 0 - and the π -parts. Depending on the combinations between the 0 - and the π -facet lengths, an antiferromagnetically ordered semifluxons array can be the ground state of the system. Due to the fact that in that case there are two independent ground states, an externally introduced 2π fluxon will be splintered or fractionalized. The magnitude of the flux in the fractional fluxons is a function of the difference between the 0 and the π -facet lengths. Here, we present an analytical calculation of the flux of splintered Josephson fluxons for any combination of 0 - and π -facet lengths. In the presence of an applied bias current, we show numerically that only one of the two fractional fluxons can be moved. We also consider the I–V characteristics of the ground state and the one of a 2π -fluxon in a zig-zag junction.

© 2006 Elsevier B.V. All rights reserved.

PACS: 74.50.+r; 85.25.Cp; 05.45.-a

Keywords: Fluxon splintering; Josephson junction; Sine-Gordon equation

1. Introduction

Vortex splintering observed recently in grain boundaries of $\text{YBa}_2\text{Cu}_3\text{O}_{7-x}$ due to d -wave symmetry and faceting of the grain boundary plane has enriched studies on vortex physics of Josephson junctions [1]. The presence of fractionalized flux in a splintered structure is a complement to the quantization of fluxoid generally occurring in the Abrikosov state of the type-II superconductors.

Fluxon fractionalization, or fluxon splintering, was first predicted by Mints [2] who shows that the flux of the individuals of the splintered vortex depends on the critical current density difference between the 0 and the π parts of faceted grain boundary junctions. Later on, Chandran and Kulkarni [3] using an ar-

ray of alternating 0 and π short Josephson junctions show that splintering is due to the existence of two independent ground states of the system, i.e. the ground state is degenerate. In grain boundary junctions, this ground state is a self-generated magnetic flux along the grain boundary line.

In other work, Buzdin and Koshelev [4] consider the properties of a Josephson junction with alternating 0 - and π -facets. They show that even if the junction has a uniform critical current density, depending on the relation between the lengths of the individual junctions, this system can be in the phase-modulated state, i.e. there is a magnetic flux self-created along the junction. Using the fact that splintering is caused by the presence of self-generated magnetic flux [3] and that the average phase of the modulated states can be related immediately to the individual flux of a splintered vortex [2], we can say that Buzdin and Koshelev [4] also study fluxon splintering.

Both Mints [2] and Buzdin and Koshelev [4] have derived an approximation to the dependence of the individual flux of a splintered vortex on the critical current difference and/or the

* Corresponding author.

E-mail addresses: susanto@math.umass.edu, rasahgelo@yahoo.com (H. Susanto).

facet length difference using an averaging method in the case of a small facet length relative to the Josephson penetration depth λ_J .

Essentially, the theoretical results of Mints and Buzdin and Koshelev are the same because a critical current difference can be connected to a facet length difference using a simple transformation. However, the results of Buzdin and Koshelev are still of interest, since there are junctions that can be fabricated with precise control over the facet length, but not the critical current. An example of such a junction is a ramp-type zig-zag $0-\pi$ Josephson junction that has a zig-zag structure when seen from the top [5,6]. In this junction, an antiferromagnetically ordered semifluxon (AFM) array can be the ground state, self-generated at the points where the supercurrent changes sign.

Using a ramp-type structure allows one to fabricate a zig-zag Josephson junction with any particular combination of 0 - and π -facet lengths [6]. One interesting problem is then to see how the dependence of the flux of a fractional fluxon on the facet length combination if the facet lengths is $\propto \lambda_J$. Once fractionalization happens, one might ask whether those splintered vortices can be separated by a bias current. Therefore, it is of interests also to study the mobility of a fractional fluxon as well as an integer 2π fluxon and their interactions with a periodic array of semifluxons that are attached to the points of the shift.

Dynamic interaction between a 2π fluxon and a single semifluxon has been studied numerically by several authors [7–9]. Interactions between a fluxon and periodic small phase-shifts has been studied theoretically by Malomed [10]. However, interaction between a moving fluxon and a semifluxon array, i.e. periodic phase shifts where the shift cannot be considered small anymore, has never been considered before.

Based on interesting problems mentioned above, we consider in this report zig-zag Josephson junctions with uniform critical current density but different 0 - and π -facet lengths. We organize the present Letter in the following way. We begin in Section 2 with discussing the model describing the phase difference of a zig-zag $0-\pi$ Josephson junction and solve it numerically to test the occurrence of splintering. In Section 3 we present a general analytical calculation of the individual flux of a splintered vortex for *any* combination of facet lengths by exploiting the fact that fractionalization can happen when the degenerate AFM state is the ground state. Section 4 discusses zig-zag Josephson junctions biased by an applied current. In the subsections here we consider the possibility of fractional fluxon moving as well as of a 2π fluxon one and calculate numerically the IV characteristics of the AFM state with and without the presence of a 2π fluxon. Section 5 contains the conclusion and open problems for future research.

2. Mathematical model and numerical results

The model that describes the phase difference of $0-\pi$ Josephson junctions is given by

$$\phi_{xx} - \phi_{tt} = \sin(\phi + \theta) + \alpha\phi_t - \gamma, \quad (1)$$

with ϕ is the phase difference of the wavefunction of the superconductors and α is the non-dimensionalized dissipation

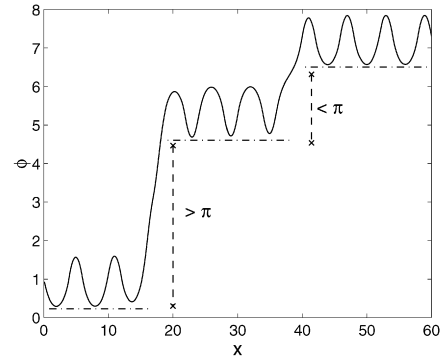


Fig. 1. Plot of a splintered vortex of Eqs. (1) and (2) for $L^0 = 4$ and $L^\pi = 2$. The topological charge of the splintered vortices is determined by measuring the difference of the ‘background’ phase shown in dash-dotted lines.

constant related to the passage of a normal current across the junction. The parameter $\gamma = I_{\text{ext}}/I_c$ represents the external current through the junction where I_c is the critical current of the junction. In this section and the following one, we consider the case of $\gamma = 0$. The alternating phase-shift θ is defined as

$$\theta(x) = \begin{cases} 0, & \text{for } x \in \dots, (-L^0 - L^\pi, -L^\pi), (0, L^0), \dots, \\ \pi, & \text{for } x \in \dots, (-L^\pi, 0), (L^0, L^0 + L^\pi), \dots, \end{cases} \quad (2)$$

where L^0 and L^π are the facet length of the 0 and the π junctions, respectively. Here, the spatial and the temporal variable x and t have been normalized to the Josephson coherent length λ_J and to the inverse of the plasma frequency ω_0^{-1} , respectively.

Eqs. (1) and (2) can have a nonzero ground state which is an antiferromagnetically ordered semifluxon (AFM) array [6]. Representing a semifluxon as \uparrow or \downarrow , where the direction of the arrow denotes the orientation of the magnetic flux, the two nonzero ground states are $\dots\uparrow\downarrow\uparrow\downarrow\dots$ and $\dots\downarrow\uparrow\downarrow\uparrow\dots$.

We have solved Eq. (1) numerically and we found that starting with a fluxon as the initial condition represented by a solution that jumps from 0 to 2π , the fluxon evolves into two fractional Josephson vortices. This splitting process is the so-called splintering [1] or fractionalization. An example of splintering is shown in Fig. 1 for $L^0 = 4$ and $L^\pi = 2$. Besides the spontaneously created semifluxon chain that is attached to the discontinuity points, one can see that there are two vortices with total flux 2π . As is shown in the picture, the flux of the splintered vortex can be calculated numerically by measuring the difference of the ‘background’ phase, i.e. the local minima of the Josephson phase (see Fig. 1). We also observed that there is no splintering in a certain combination range of L^0 and L^π . In that range there is no spontaneously generated vortex chain either, which confirms the result of Buzdin and Koshelev [4].

3. Analytical calculations

Eqs. (1) and (2) can admit two $(\text{mod } 2\pi)$ independent non-constant ground state solutions, i.e. an AFM state. For a zig-zag junction with L^0 and L^π above, i.e. $L^0 = 4$ and $L^\pi = 2$, which will be used through out the Letter, the degenerate ground states $(\text{mod } 2\pi)$ are shown in Fig. 2(a). We denote them as ϕ_{GS}^1 and ϕ_{GS}^2 .

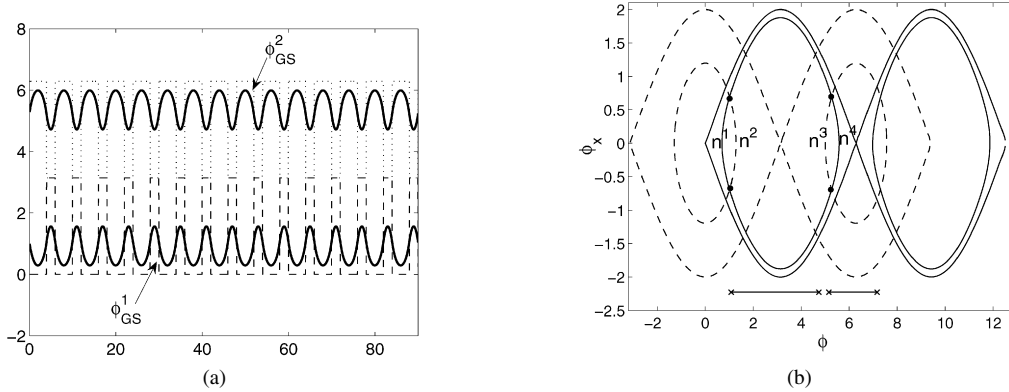


Fig. 2. (a) Two solutions of (1) (mod 2π) that represent the ground states, i.e. a chain of semifluxons in an antiferromagnetic order, that are denoted by ϕ_{GS}^1 and ϕ_{GS}^2 . The phase shift $\theta(x)$ is shown in dashed and dotted. (b) Phase portrait of Eq. (1) where only some trajectories are shown. ϕ_{GS}^1 and ϕ_{GS}^2 in the phase plane are given by lines $\dots -n^1 - n^2 - n^1 - n^2 - \dots$ and $\dots -n^3 - n^4 - n^3 - n^4 - \dots$, respectively. The position of n^i , $i = 1, \dots, 4$ in the phase plane is determined by L^0 and L^π (see the text).

When there is no applied bias current, the ground state is static and can be written in terms of the Jacobi elliptic functions [11–13]. Yet, we will not exploit the functions explicitly. Instead, we will use phase plane analysis [14–16] to do analysis of the splintering such that the calculations only involve the elliptic integral of the first kind [11]. For a more detailed review on the use of phase portrait in the present analysis, we refer to, e.g., [15,16].

The first integral of the time-independent sine-Gordon equation (1) is

$$\frac{1}{2}\phi_x^2 = C - \cos(\phi + \theta), \quad (3)$$

where C is a constant.

In Fig. 2, some trajectories of the first integral (3) with $\theta = 0$ and $\theta = \pi$ are shown respectively in solid and dashed lines in the (ϕ, ϕ_x) -plane. In the phase plane, the ground state ϕ_{GS}^1 and ϕ_{GS}^2 are represented respectively by trajectories $n^1 - n^2 - n^1$ and $n^3 - n^4 - n^3$ where the coordinate of n^i , $i = 1, \dots, 4$ is given by $n^1 = (a_1^0, 0)$, $n^2 = (a_2^\pi, 0)$, $n^3 = (a_2^0, 0)$, and $n^4 = (a_1^\pi, 0)$. The switch from solid to dashed curve and vice versa occur at the black circles. The value of $a_{1,2}^{0,\pi}$ cannot be arbitrary and is determined by the value of L^0 and L^π .

In the phase plane, the ‘background’ phase (see Fig. 1) is the abscissa of n^1 and n^4 , i.e. a_1^0 and a_2^π , respectively. The presence of a fluxon will then try to connect the two backgrounds. Therefore, the fluxon will split and form two separated fractional Josephson vortices with one jumps from $\phi = a_1^0$ to $\phi = a_2^\pi$ and the other from $\phi = a_2^\pi$ to $\phi = 2\pi + a_1^0$. In terms of ϕ_{GS}^1 and ϕ_{GS}^2 , one fractional fluxon connects ϕ_{GS}^1 and ϕ_{GS}^2 and the other connects ϕ_{GS}^2 and $\phi_{\text{GS}}^1 + 2\pi$.

Geometrically, one can directly notice that $a_2^{0,\pi} = 2\pi - a_1^{0,\pi}$. Therefore, the topological charge of the fractional fluxons is $(a_1^0 + a_1^\pi)$ and $2\pi - (a_1^0 + a_1^\pi)$.

Next, we will calculate analytically the topological charge of fractional fluxon for given L^0 and L^π . By now, we already know that we can consider only the trajectory $n^1 - n^2 - n^1$. For simplicity, we will drop out the subscript of $a_1^{0,\pi}$.

First, we will determine the position of the switch. If $(\phi^d, \pm\phi_x^d)$ is the coordinate of the black circles in the trajectory $n^1 - n^2 - n^1$, then

$$\phi^d = \arccos \frac{1}{2}(\cos a^0 + \cos a^\pi). \quad (4)$$

The periodic trajectories through a^0 and a^π in the phase plane are given by

$$\phi_x = \pm\sqrt{2(\cos a^0 - \cos \phi)} \quad (5)$$

and

$$\phi_x = \pm\sqrt{2(\cos \phi - \cos a^\pi)}, \quad (6)$$

respectively. The arclength from a^0 and a^π to ϕ^d then satisfies the following relation

$$\begin{aligned} \frac{1}{2}L^0 &= \int_{a^0}^{\phi^d} \frac{dx}{\sqrt{2(\cos a^0 - \cos \phi)}}, \\ \frac{1}{2}L^\pi &= \int_{\phi^d}^{a^\pi} \frac{dx}{\sqrt{2(\cos \phi - \cos a^\pi)}}. \end{aligned} \quad (7)$$

Let the incomplete Jacobi elliptic integral of the first kind be given by

$$F(z, k) = \int_0^z \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}.$$

After some simple transformations [11,16], Eqs. (7) can then be written as¹

¹ Similar expressions as Eqs. (8)–(9) for the case of a finite $0-\pi$ junction with vanishing magnetic field at the edges, i.e. $\phi_x = 0$, have been derived in [13] (see Eq. (10) of [13]). They differ by the factor one-half at the left-hand side terms. In Fig. 2(b), the ground states are shown by the trajectory connecting n_1 and n_2 , and the one connecting n_4 and n_3 . This implies that the same magnetic field amplitude spontaneously created in a periodic array of $0-\pi$ junctions with the 0 and the π -junction length L_0 and L_π can be obtained in a finite $0-\pi$ junction with the length of the pieces $L_0/2$ and $L_\pi/2$.

$$\frac{1}{2}L^0 = F(1, k^0) - F\left(\frac{\sin((\phi^d - \pi)/2)}{k^0}, k^0\right), \quad (8)$$

$$\frac{1}{2}L^\pi = F(1, k^\pi) - F\left(\frac{\sin(\phi^d/2)}{k^\pi}, k^\pi\right), \quad (9)$$

with $k^0 = \sin((a^0 - \pi)/2)$ and $k^\pi = \sin(a^\pi/2)$.

Solving Eqs. (4), (8), and (9) will give the topology of fractional fluxons for given L^0 and L^π . In Fig. 3, we show the topology of one of the fractional fluxons for a given value of L^0 , but varying L^π . The analytical result is in complete agreement with the numerical one.

Using Eqs. (4), (8), and (9), one can derive that an approximation to the topological charge of splintered vortices in the limit $L^0, L^\pi \rightarrow 0$ is given by

$$(a^0 + a^\pi) = 2 \arccos\left(\frac{12(L^0 - L^\pi)}{L^0 L^\pi (L^0 + L^\pi)}\right). \quad (10)$$

This expression can also be derived from the average phase of the modulated states of a zig-zag Josephson junction calculated by Buzdin and Koshelev [4] as will be shown in the following.

Let φ_0 be the average phase of ϕ_{GS}^1 . Buzdin and Koshelev [4] derive that (see Eq. (9) in [4])

$$\cos \varphi_0 = \left(\frac{12(L^0 - L^\pi)}{L^0 L^\pi (L^0 + L^\pi)}\right). \quad (11)$$

Because we know that one of the fractional fluxons connects ϕ_{GS}^1 and ϕ_{GS}^2 and that the average phase of ϕ_{GS}^2 is $2n\pi - \varphi_0$ where n is an integer, then one can easily re-obtain expression (10).

We found that for given $L^{0,\pi}$, there is a critical value of $L^{\pi,0}$ below which there is no solution to Eqs. (8) and (9). This means that below that critical value, there are no spontaneously created fractional fluxons, and hence no splintering, as we reported previously. This critical value can be obtained by calculating the stability of the constant phase solution of Eq. (1), i.e. $\phi \equiv 0, \pi \pmod{2\pi}$.

Solving the eigenvalue problem, which is nothing else but the linear Schrödinger equation, gives that the instability region of the constant phase solution is bounded by the curves [4]²

$$L^{\pi,0} = \arccos\left(\frac{2e^{L^{0,\pi}}}{1 + e^{2L^{0,\pi}}}\right). \quad (12)$$

To obtain splintering, the combination of L^0 and L^π must then be in the instability region of the constant solution which is in perfect agreement with the above analytical result.

4. Movability of an AFM state, fractional fluxons, and an integer fluxon

After studying the static properties of a splintered Josephson fluxon as well as the generated fractional fluxons, the next natural question is the behavior of those fluxes when a bias current

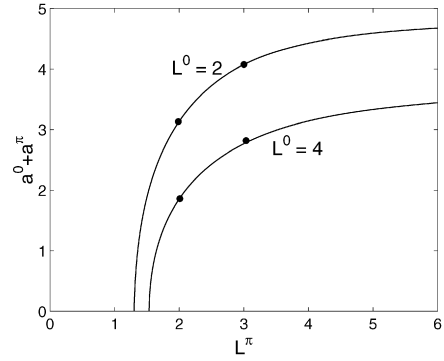


Fig. 3. The topology of one of the fractional fluxons ($a^0 + a^\pi$) obtained by solving Eqs. (8) and (9) for two different values of L^0 , i.e. $L^0 = 2, 4$. The topology of the complementary vortex is $2\pi - (a^0 + a^\pi)$. The curve is in complete agreement with numerical results that are shown in circles.

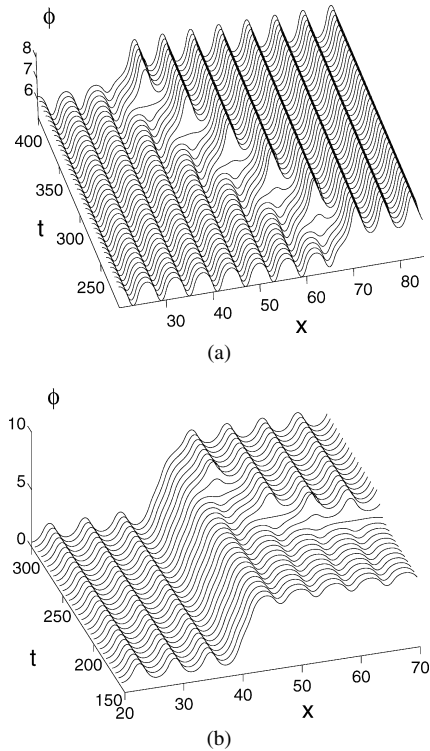


Fig. 4. The magnetic field profiles ϕ_x of (a) ϕ^S and (b) ϕ^L in space–time plane with $\alpha = 0.5$, (a) $\gamma = 0.075$ and (b) $\gamma = 0.085$.

is applied to the system, i.e. $\gamma \neq 0$. If one can control them using an external current, then it might be possible to ‘extract’ an individual fractional fluxon from a splintered 2π fluxon or even to exploit them for technological purposes.

In the following discussions, we set $\alpha = 0.5$, which is not too low nor too high as the majority of $0-\pi$ Josephson junctions has a rather high damping, and the total junction length $L = 90$.

4.1. Movability of fractional fluxons

We have studied numerically the movability of fractional fluxons with topological charge ($a^0 + a^\pi$) and $2\pi - (a^0 + a^\pi)$. Our numerics reveal that between the two fluxes that are com-

² Boundaries of the instability region for the case of finite $0-\pi$ Josephson junctions have been calculated as well by Bulaevskii et al. [13].

plements of each other, only the smaller one can move. In a junction with L^0 and L^π above, it is the fractional fluxon connecting ϕ_{GS}^2 and $2\pi + \phi_{GS}^1$, that for the sake of simplicity will be called ϕ^S . ϕ^S has topological charge ($a^0 + a^\pi$). The critical current for the fluxon to move is $\gamma \approx 0.075$. The dynamics of a moving ϕ^S is shown in Fig. 4(a).

The other fractional fluxon with larger topological charge, that also for simplicity will be called ϕ^L , cannot move as an individual. Applying a bias current, there is a critical value at which its complementary fluxon, i.e. ϕ^S , is created and, hence, results in an integer 2π fluxon. The critical value is $\gamma \approx 0.085$. The appearance of the complementary fluxon is shown in Fig. 4(b).

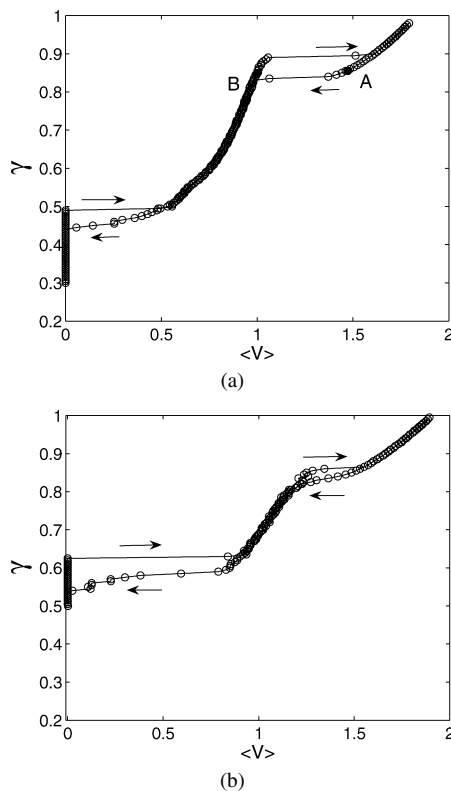


Fig. 5. Numerically calculated IV characteristics of an annular zig-zag junction with (a) the total junction length $L = 90$, $L^0 = 4$, $L^\pi = 2$; and (b) $L = 75$, $L^0 = 4$, $L^\pi = 1$.

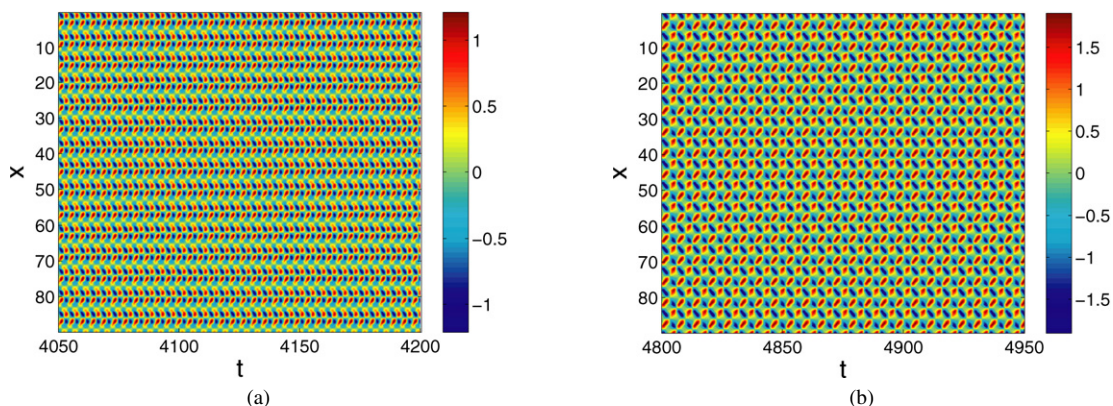


Fig. 6. Space-time contour plot of the magnetic field ϕ_x in two points indicated by A and B of the IV curve presented in Fig. 5(a).

Because ϕ^S can move, it is then of interest to measure its IV characteristics. However, we were not able to do simulations on this because of the problem of boundary conditions. If one uses either periodic or antiperiodic boundary conditions, having ϕ^S will automatically create its counterpart $-\phi^S$, that results in a collision and annihilation between the two fluxons. Using either fixed or free boundary conditions will give a moving ϕ^S , that is eventually trapped at one edge of the junction.

For ϕ^L , the appearance of ϕ^S when we attempt to move it can be explained by studying the ‘robustness’ of the ground state ϕ_{GS}^1 and ϕ_{GS}^2 under the presence of $\gamma > 0$. Our numerics shows that for L^0 and L^π above, the critical current of ϕ_{GS}^1 and ϕ_{GS}^2 is $\gamma_{cr}^1 \approx 0.49$ and $\gamma_{cr}^2 \approx 0.085$, respectively. For $\gamma_{cr}^2 < \gamma \ll 1$, ϕ_{GS}^2 is unstable with respect to ϕ_{GS}^1 , i.e. there is only one ground state. Remembering that ϕ^L connects ϕ_{GS}^1 and ϕ_{GS}^2 , one can then understand why ϕ^L cannot be moved by applying a bias current without creating ϕ^S . We can also conclude that a moving ϕ^S can be observed only for a short range of γ , i.e. $0.075 \lesssim \gamma < \gamma_{cr}^1$.

4.2. IV characteristics of an AFM state with and without a 2π fluxon

In this subsection we present numerical simulations on an annular zig-zag junction that corresponds to Eq. (1) with periodic boundary conditions. One advantage of an annular geometry is that there is no collision of fluxons with boundaries. Even though most of $0-\pi$ Josephson junctions, especially ramp-type zig-zag YBCO/Au/Nb junctions, have overlap geometry [6], an annular $0-\pi$ Josephson junction can be possibly fabricated in experiments using the present technology [17].

First we consider the case of an annular junction with no trapped fluxon. The numerically calculated IV characteristics of the system is presented in Fig. 5. In the first and the second subfigure, we consider the combination of the 0 - and the π -facet lengths where an AFM and a constant flux state is the ground state, respectively.

The important feature of both IV curves is the appearance of a plateau (or step) in V above the critical current γ_{cr}^1 . Two pictures in Fig. 6 show a typical magnetic field profile in space-time contour plot corresponding to the two branches separated by the plateau. One can view the states as a stationary wave of

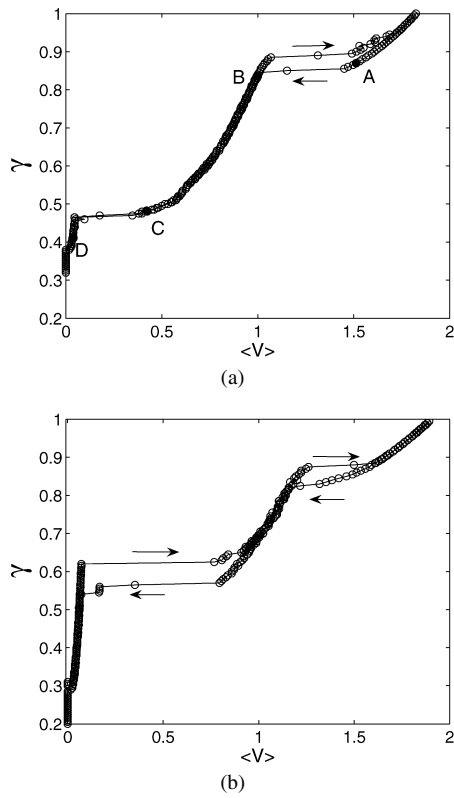


Fig. 7. The same as Fig. 5 but in the presence of an integer fluxon.

breathing semifluxon–antisemifluxon pairs. Similar to an array of discrete Josephson junctions [3], the origin of the plateau might be attributed to the phase locking between the moving (anti)semifluxons and the linear waves emitted by the breathing semiflux–antisemiflux pairs.

Next, we consider the case when there is an integer fluxon in the junction. The current voltage characteristics of the system is shown in Fig. 7. The main different feature between Figs. 5 and 7 is the presence of a curve that corresponds to a traveling fluxon in a static ground state. The depinning current for the fluxon to move is $\gamma \approx 0.38$. Several snapshots of the space–time contour plot of the magnetic field are presented in Fig. 8.

Comparing Fig. 8(a)–(b) and Fig. 6, one should notice that they appear to be similar, but they are actually different in the sense that the ‘pattern’ is tilted when there is a fluxon in the system.

5. Conclusions

To conclude, we have discussed a long Josephson junction containing alternating 0 and π facets. We have shown that in such a system, there can be fluxon splintering due to the presence of spontaneously generated fractional flux quanta. Analytical calculations on the topological charge of splintered vortices have been presented for any given combinations of facet length of the 0 and the π junctions. The analysis is also applicable when the 0 and the π junctions have different critical

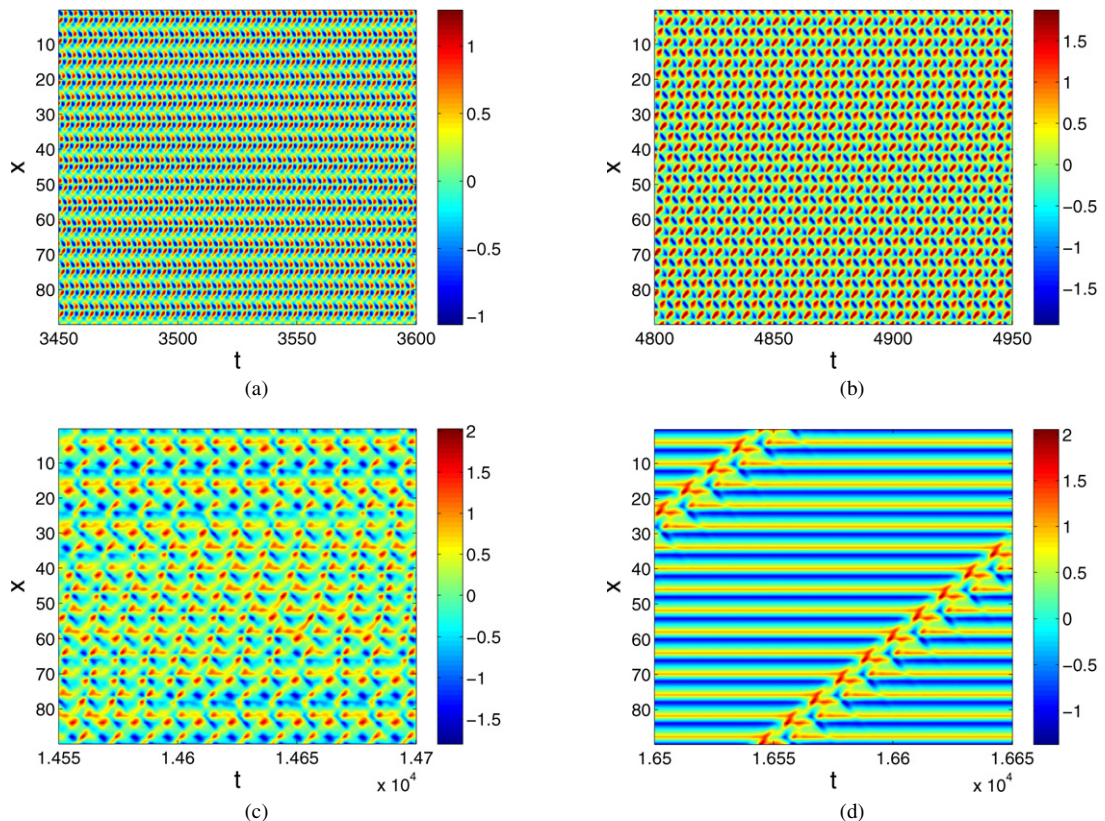


Fig. 8. The same as Fig. 6 but for the IV curve presented in Fig. 7 in the presence of a single 2π fluxon. Panels (a) and (b) of this figure appear to be similar to Fig. 6 (a)–(b), but they are actually different in the sense that the ‘pattern’ is tilted here.

current density as is the case of for instance grain boundary junctions [1].

We also have analyzed the dynamics of a splintered fluxon and its fractional fluxons where we showed that only one of the two fractional fluxons can move in a very short range of γ . The IV characteristics of an AFM state with and without an integer fluxon has also been presented where we found that there is a voltage step in the normal state.

The spectrum of fractional fluxons remains a question. Analytical calculation on their eigenvalues, that involves the so-called Lamé type equations, is addressed for future investigations. A preliminary study on the spectrum of the background only, i.e. an AFM state, has been analyzed numerically in [18].

Recently, Tornes and Stroud [19] consider fluxon splintering in a one-dimensional periodic array of short Josephson junctions in which junction number n is an $n\kappa$ -junction. The Josephson supercurrent of a κ junction is proportional to $\propto \sin(\phi + \kappa)$. Such a junction can be realized in experiments by applying a magnetic field perpendicular to the ladder [20]. Tornes and Stroud show that in this system a fluxon can be fractionalized into q fractional fluxons with each vortex carrying the topological charge of $1/q$.

Acknowledgements

H.S. and S.V.G. acknowledge the support of the Royal Netherlands Academy of Arts and Sciences (KNAW).

References

- [1] R.G. Mints, I. Papiashvili, J.R. Kirtley, H. Hilgenkamp, G. Hammerl, J. Mannhart, Phys. Rev. Lett. 89 (2002) 067004.
- [2] R.G. Mints, Phys. Rev. B 57 (1998) 3221.
- [3] M. Chandran, R.V. Kulkarni, Phys. Rev. B 68 (2003) 104505.
- [4] A. Buzdin, A.E. Koshelev, Phys. Rev. B 67 (2003) 220504(R).
- [5] H.J.H. Smilde, Ariando, D.H.A. Blank, G.J. Gerritsma, H. Hilgenkamp, H. Rogalla, Phys. Rev. Lett. 88 (2002) 057004.
- [6] H. Hilgenkamp, Ariando, H.J.H. Smilde, D.H.A. Blank, H. Rogalla, J.R. Kirtley, C.C. Tsuei, Nature 422 (2003) 50.
- [7] N. Stefanakis, Phys. Rev. B 66 (2002) 214524; N. Stefanakis, cond-mat/0208473.
- [8] N. Lazarides, Phys. Rev. B 69 (2004) 212501.
- [9] E. Goldobin, N. Stefanakis, D. Koelle, R. Kleiner, Phys. Rev. B 70 (2004) 094520.
- [10] B.A. Malomed, Phys. Rev. B 38 (1988) 9242.
- [11] P.F. Byrd, M.D. Friedman, Handbook of Elliptic Integrals, second ed., Springer, Berlin, 1971.
- [12] A. Zenchuk, E. Goldobin, Phys. Rev. B 69 (2004) 024515.
- [13] L.N. Bulaevskii, V.V. Kuzii, A.A. Sobyenin, P.N. Lebedev, Solid State Commun. 25 (1978) 1053.
- [14] D.K. Arrowsmith, C.M. Place, Dynamical Systems: Differential Equations, Maps and Chaotic Behavior, Chapman & Hall, 1992.
- [15] S. Pagano, B. Ruggiero, E. Sarnelli, Phys. Rev. B 43 (1991) 5364; S. Pagano, B. Ruggiero, M. Russo, E. Sarnelli, in: G. Costabile, S. Pagano, N.F. Pedersen, M. Russo (Eds.), Nonlinear Superconductive Electronics and Josephson Devices, Plenum Press, New York, 1991, p. 369.
- [16] H. Susanto, S.A. van Gils, T.P.P. Visser, Ariando, H.J.H. Smilde, H. Hilgenkamp, Phys. Rev. B 68 (2003) 104501.
- [17] E. Goldobin, A. Sterck, T. Gaber, D. Koelle, R. Kleiner, Phys. Rev. Lett. 92 (2004) 057005; A.V. Ustinov, Appl. Phys. Lett. 80 (2002) 3153.
- [18] H. Susanto, E. Goldobin, D. Koelle, R. Kleiner, S.A. van Gils, Phys. Rev. B 71 (2005) 174510.
- [19] I. Tornes, D. Stroud, Phys. Rev. B 72 (2005) 224519.
- [20] S. Watanabe, H.S.J. van der Zant, S.H. Strogatz, T.P. Orlando, Physica D 97 (1996) 429.