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Optimal allocation of MRI scan capacity among competing hospital departments

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ABSTRACT

We consider an MRI scanning facility run by a Radiology department. Several hospital departments compete for capacity and have private information regarding their demand for scans. The fairness of the capacity allocation by the Radiology department depends on the quality of the information provided by the hospital departments. We employ a generic Bayesian game approach that stimulates the disclosure of true demand (truth-telling), so that capacity can be allocated fairly. We derive conditions under which truth-telling is a Bayesian Nash equilibrium. The usefulness of the approach is illustrated with a numerical example.

1. Introduction

We consider an MRI scanning facility run by a Radiology department, that has to distribute MRI scan capacity among several competing hospital departments. The departments have private information regarding their future demands. For a fair allocation, Radiology depends on the information that the departments provide. How can the Radiology department motivate the users to give an honest forecast of their demands in order to ensure a fair allocation?

Various types of MRI scans exist, each used to inspect different parts of the body [12]. Examples are scans of the heart, breasts, nervous system, and bones. It is common practice in most hospitals to dedicate adjacent time slots (blocks) in the appointment schedule to identical MRI types. The demand for MRI scans can vary widely over time, especially in academic institutions. New treatment protocols may result in an in- or decrease of MRI requests; the same holds for the recruitment of new patient cohorts and changes in the hospital's patient mix. This asks for a periodical allocation of MRI capacity. For this it is common that hospital departments provide Radiology with a demand forecast for the next period. Overestimating demand may be tempting, since it is likely that this leads to a larger share of the scarce capacity. The quality of the MRI schedule depends on the quality of the forecast. It is therefore essential for the Radiology department that hospital departments put maximum effort into providing a reliable and honest forecast, and do not over- or underestimate their demand.

1.1. Problem example

We illustrate the necessity of a reliable and honest forecast with an example of a facility with two scanning types. For the first scanning type, a forecast that is lower than the actual demand for the next period is provided. For the other scanning type, a forecast that is higher than the actual demand for the next period is provided. Suppose that the capacity allocated by Radiology equals the forecast of demand. Then for the first scanning type, a waiting list develops because of incorrect allocation (Fig. 1(a)). For the second scanning type, not all allocated capacity is needed and thus the scanner sits idle (Fig. 1(b)). We see that it is very well possible that in the same period, the MRI scanners are idle during certain blocks due to less actual demand for one type of scans, while at the same time the waiting list for another scan type increases caused by a lack of capacity.

1.2. Approach

The problem of capacity allocation to multiple competing users, as sketched above, has several key properties. Namely (i) the users do not cooperate, (ii) the actual demands of the users are private information, and (iii) the resource wants the users to truthfully reveal their actual demand. Relevant models that capture these properties are combinatorial auction models [3], where multiple bidders can place bids on several items at the same time, and Bayesian games [5–7], non-cooperative games where each player has incomplete information about the characteristics of the other users. While in a Bayesian game only the demand of the users is

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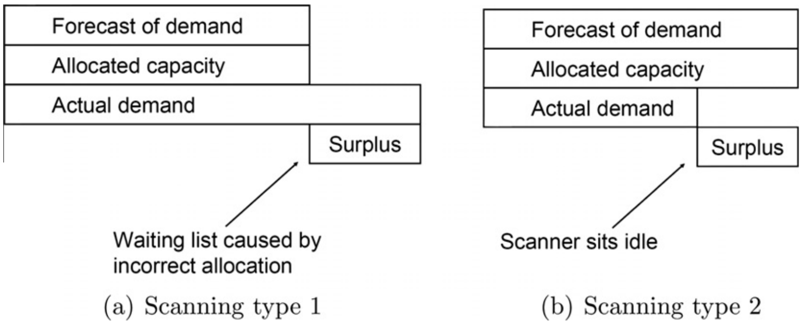


Fig. 1. Example for two scanning types.

needed, in a combinatorial auction also the price the users are willing to pay is required. This, combined with the relatively simple analysis of the Bayesian game model compared to that of the combinatorial auction model, determined our choice for the Bayesian game approach. We are interested in conditions under which the users tell the truth, that is, they provide the resource with their actual demand.

1.3. Literature

Bayesian games are extensively described by Harsanyi [5–7]. For an introduction on this class of games we refer the reader to [4]. In the literature on Bayesian games, two types of models are often studied. In the first type of model a single resource communicates with several users. The users do not cooperate, and the resource has private information. An application of this model is given in [8]. In the second type of model a single resource communicates with a single user. Now, the user has private information. Examples can for instance be found in [14,21]. Unlike these types of models, we consider a single resource and multiple non-cooperative users with private information. To the best of our knowledge, this has hardly been studied so far.

There is a vast body of literature on capacity allocation with truth-telling in the area of supply chain management, see, for example [2,9,20]. The main research questions are how a supplier should allocate his capacity, and how the supplier can induce his buyers to reveal their private information. Furthermore, many papers on capacity and/or resource allocation in health care are available, such as [18], but these do not consider private information and truth-telling. This paper contributes to the literature by studying capacity allocation under private information in a health care setting.

Several other problems in a health care context have been studied using Bayesian games. An application area is the patient–doctor relationship, where either the patient [21] or the doctor [15] has private information. Another example is given in [19], where the authors consider the principle of kidney exchange. Patients waiting for a kidney transplant present one or more potential donors. These donors however are not a match to the patient they are related to. In order to find matching pairs, an exchange group of several patients and their donors is formed. In the paper it is demonstrated with a Bayesian game that it is advantageous in some cases for patients not to reveal all information they possess about their donors. In [1] an economic application is given. Multiple hospitals are regulated by a central authority; hospitals do not cooperate with each other. The regulator has incomplete information on the production information hospitals possess. A Bayesian game is used to study the effect of the information gap on the production contracts the regulator offers the hospitals. We conclude this paragraph with [11], in which the international trading and pricing of pharmaceuticals is studied. The author suggests to introduce asymmetric information with respect to the local demand function of the country the products are sold to. When the problem is modeled as a Bayesian game, it can be shown that in equilibrium parallel imports of pharmaceuticals occur, in contrast to the complete information situation.

1.4. Contents of paper

Since the approach is not limited to the MRI scan example, we use generic terminology (resources and users) in Sections 2–4. First we provide a detailed description of the model. In the Results sections that follows we show that for two allocation mechanisms an optimal strategy for users is to provide an honest forecast of their demand, which enables the resource to make a fair allocation. We demonstrate the approach with a numerical example in Section 5. We conclude with Section 6.

2. Model

In this section we formulate the Bayesian game. An overview of the notation introduced is given in Table 1.

The allocation of capacity goes as follows. Users provide their forecast F_i for the next period. The resource allocates capacity, resulting in an allocated amount A_i per user. During the period the users reveal their actual demand. This process is repeated each period. We make the following assumptions:

(i) All users make rational choices, i.e. they want to maximize benefits and minimize costs.
(ii) The total amount requested by the users exceeds the resource's capacity: $\sum_j F_j > C$.
(iii) The shared resource cannot allocate more capacity than is available: $\sum_j A_j \leqslant C$.
(iv) No user has an actual demand that is higher than the resource's capacity: $0 \leqslant D_i \leqslant C$.
(v) No user has any information about the private demand of any other user. Let $D_{-i} = \{D_j\}_{j \neq i}$ represent the demands of users other than user i. We model the knowledge of user i by the uniform distribution on $[0, C]^{n-1}$,

Table 1
Notation introduced in model section.

Symbol	Description
C	Total amount of capacity available
F_i	Forecast of demand by user i (i.e. request to resource)
A_i	Capacity allocated to user i
D_i	Actual demand of user i
x	Reward per unit of allocated capacity
y	Penalty cost per unit of surplus capacity

$$p_i(D_{-i}) = \begin{cases} \frac{1}{C^{n-1}}, & \text{if } D_{-i} \in [0, C]^{n-1}, \\ 0, & \text{else;} \end{cases}$$

thus all demands are equally likely.

2.1. Utility function

User i has a utility function V_i that measures the immediate happiness or *reward* [17]. The reward is the weighted difference between the allocated amount A_i and a penalty for overestimation. The weights are x per unit of allocated capacity and y per unit that is overestimated, $x, y > 0$. The utility function for user i is given by

$$V_i = xA_i - y\max\{F_i - D_i, 0\}. \tag{1}$$

Each user aims to maximize its utility.

2.2. The allocation mechanism

The resource needs an allocation mechanism to distribute the capacity over the users. Desirable properties of an allocation mechanism are:

(i) Each user receives a nonnegative amount: $A_i \geqslant 0$.
(ii) All capacity is allocated: $\sum_j A_j = C$.
(iii) Each user receives at most the amount it requests: $A_i \leqslant F_i$.
(iv) If the capacity of the resource increases, then all users should obtain more (until they reach their forecast): A_i is increasing in C.

Many allocation mechanisms satisfy these properties. Three mechanisms that are used often in practice are the proportional rule, the constrained equal award rule and the constrained equal loss rule [16]. The proportional rule allocates capacity proportional to the forecasts:

$$A_i = \frac{F_i}{\sum_j F_j} C.$$

The constrained equal award rule divides the capacity equally among the users, with the constraint that a user cannot obtain more than was requested:

$$A_i = \min\{\alpha, F_i\}$$

with α such that $\sum_j A_j = C$. Third, the constrained equal loss rule divides the shortage of capacity equally among the users such that any user receives a nonnegative amount:

$$A_i = \max\{F_i - \beta, 0\}$$

with β such that $\sum_j A_j = C$.

2.3. Bayesian game formulation

Now we formulate the problem as a Bayesian game. Each user provides a forecast F_i, which is a function of his private actual demand D_i. We write $F_i(D_i)$ to denote this dependency. This forecast reflects the claim of user i on the available capacity. The allocated capacity A_i depends on all requests $F_j(D_j)$, $j = 1, \ldots, N$, and hence also on all the private demands.

The goal of each user is to maximize his expected utility by selecting a suitable strategy. A strategy $F_i(D_i)$ of user i specifies which forecast the user should announce as a function of its private information D_i.

The strategies $F^* = (F_1^*(D_1), \ldots, F_N^*(D_N))$ are a so-called Bayesian Nash equilibrium if for each user i and for any private demand D_i the requested number of units $F_i^*(D_i)$ maximizes the expected utility of the user:

$$F_i^*(D_i) = \arg\max_{F_i} \int_{[0,C]^{n-1}} V_i(F_{-i}^*, F_i; D_i) p_i(D_{-i})\, \mathrm{d}D_{-i},$$

where (F_{-i}^*, F_i) denotes the strategies F^* in which the strategy $F_i^*(D_i)$ of user i is replaced by F_i, $D_{-i} = \{D_j\}_{j \neq i}$ is the collection of private demands for users other than i, and $p_i(D_{-i})$ is the prior belief of user i about D_{-i} [13]. Hence, given the uncertainty on the private demands of the other users, it does not pay for user i to deviate from his equilibrium strategy because that will result in lower expected utility.

3. Results for proportional rule

In this section the capacity is allocated according to the proportional rule. Then the utility function of user i is

$$V_i(F; D_i) = x\frac{F_i}{\sum_j F_j} C - y\max\{F_i - D_i, 0\}, \tag{2}$$

which depends on the demands $F = \{F_1, \ldots, F_N\}$ of all users, and on the user's privately known actual demand D_i.

We show that when the number of users exceeds 3, it is optimal for the users to provide an honest forecast. When the number of users is equal to 2 or 3, the same result holds under weak conditions.

3.1. Equal cost and reward parameters

To simplify calculations, we set $x = y = 1$ in the utility function, so $V_i(F; D_i) = \frac{F_i}{\sum_j F_j} C - \max\{F_i - D_i, 0\}$ (we consider other cost and reward parameters in Section 3.2). We investigate when truth-telling, $F_i(D_i) = D_i$, is a Bayesian Nash equilibrium. Without loss of generality we consider user $i = 1$. His expected utility, given that the other users truthfully reveal their demand, equals

$$\mathbb{E}[V_1(F; D_1)] = \int_0^C \cdots \int_0^C \frac{\frac{F_1}{F_1 + \sum_{j=2}^N D_j} C - \max\{F_1 - D_1, 0\}}{C^{N-1}} \mathrm{d}D_2 \cdots \mathrm{d}D_N$$
$$= \frac{1}{C^{N-2}} \int_0^C \cdots \int_0^C \frac{F_1}{F_1 + \sum_{j=2}^N D_j} \mathrm{d}D_2 \cdots \mathrm{d}D_N - \max\{F_1 - D_1, 0\}.$$

To analyze when truth-telling maximizes this expected utility, we calculate the derivative with respect to F_1. The values of F_1 where the derivative equals zero or does not exist, and the boundary values 0 and C are candidate values for a maximum. If the derivative equals zero for some value F_1 then we use the second derivative of the expected utility to check whether this value is indeed a maximum or minimum. These derivatives and their properties are as follows.

Theorem 1. *Consider the situation with N users. The derivative of the expected utility equals*

$$\frac{\partial \mathbb{E}[V_1(F; D_1)]}{\partial F_1} = \frac{1}{C^{N-2}} \int_0^C \cdots \int_0^C \frac{\sum_{j=2}^N D_j}{\left(F_1 + \sum_{j=2}^N D_j\right)^2} \mathrm{d}D_2 \cdots \mathrm{d}D_N - I_{\{F_1 > D_1\}}, \tag{3}$$

where I_E is the indicator function of the event E that takes the value 1 if E is true and 0 otherwise. This derivative is positive if $F_i < D_i$; the expected utility is then increasing in F_i.

The second derivative of the expected utility,

$$\frac{\partial^2 \mathbb{E}[V_1(F; D_1)]}{\partial F_1^2} = \frac{1}{C^{N-2}} \int_0^C \cdots \int_0^C \frac{-2\sum_{j=2}^N D_j}{\left(F_1 + \sum_{j=2}^N D_j\right)^3} \mathrm{d}D_2 \cdots \mathrm{d}D_N, \tag{4}$$

is always negative. So, the derivative of the expected utility is decreasing in F_i, in particular for $F_i > D_i$.

Proof. Without loss of generality let $i = 1$. If $F_1 < D_1$, then the derivative (3) reduces to

$$\frac{1}{C^{N-2}}\int_0^C \cdots \int_0^C \frac{\sum_{j=2}^N D_j}{\left(F_1+\sum_{j=2}^N D_j\right)^2}\,\mathrm{d}D_2\cdots\mathrm{d}D_N,$$

which is always positive; the expected utility is increasing in F_1.

It is easy to see that the second derivative (4) is negative. Hence, the derivative (3) of the expected utility is decreasing in F_1, in particular for $F_1 > D_1$. □

According to this theorem, the expected utility is increasing if $F_1 < D_1$. Therefore, user 1 wants to set F_1 as large as possible. Because $F_1 < D_1$, user 1 sets $F_1 = D_1$ in the limit.

Also by Theorem 1 the derivative of the expected utility is decreasing in F_1. Now if this derivative is negative for all forecasts $F_1 > D_1$, then the expected utility is decreasing in F_1. So, user 1 wants to choose F_1 as small as possible. Because $F_1 > D_1$, user 1 wants to select $F_1 = D_1$ in the limit. In this case we conclude that truth-telling is a Bayesian Nash equilibrium; user 1 always tells the truth. In the next subsections we investigate for several numbers of users when the derivative of the expected utility for $F_i > D_i$ is indeed negative, and under which conditions truth-telling is an equilibrium.

3.1.1. Truth-telling in case of two users

In this section we analyze the allocation problem with two users. Then the derivative (3) of the expected utility for $F_1 > D_1$ equals

$$\int_0^C \frac{D_2}{(F_1+D_2)^2}\,\mathrm{d}D_2 - 1 = \ln\left(\frac{F_1+C}{F_1}\right) + \frac{F_1}{F_1+C} - 2, \tag{5}$$

We want to know for which values of F_1 this derivative is negative. If so, then the expected utility of user 1 is decreasing and this user will select $F_1 = D_1$—the truth-telling outcome—to maximize its expected utility.

Theorem 2. *Consider the situation with two users. Truth-telling is a Bayesian Nash equilibrium if the private demand of any user is at least 18.9% of the total capacity.*

Proof. Without loss of generality consider user $i = 1$. By Theorem 1, the derivative (5) is a decreasing function in F_1. This derivative is negative for all requests $F_1 \in (D_1, C]$ if it is negative for $F_1 = D_1$:

$$\ln\left(\frac{D_1+C}{D_1}\right) + \frac{D_1}{D_1+C} - 2 \leqslant 0.$$

This inequality holds if $D_1 \geqslant b_2C$ with $b_2 \approx 0.189$, where b_2 is such that the derivative (5) is equal to zero for $F_1 = b_2C$. □

In other words, the private demand of either of the two users should be larger than roughly one-fifth of the capacity of the resource. The lower bound of 18.9% on the proportion of privately known demand to the resource's capacity may be too restrictive. What happens if this lower bound is not met for user i, so $D_i < 0.189C$? According to the analysis in the proof of Theorem 2 the expected utility of this user is maximal in forecast $F_i \approx 0.189C$. This forecast is larger than the actual demand D_i; user i overestimates its private demand.

3.1.2. Truth-telling in case of three and more users

For three to six users, the results are as follows.

Theorem 3. *Truth-telling is a Bayesian Nash equilibrium for $N = 3$ users if the private demand of any user is at least 8.0% of the total capacity. For $N = 4$, 5, and 6 users, truth-telling is a Bayesian Nash equilibrium.*

Proof. First consider $N = 3$ users. Without loss of generality focus on user 1 and on the case $F_1 > D_1$. According to (3), the derivative of the expected utility of user 1 equals:

$$\frac{1}{C}\int_0^C\int_0^C \frac{D_2+D_3}{(F_1+D_2+D_3)^2}\,\mathrm{d}D_2\mathrm{d}D_3 - 1$$
$$= \frac{2F_1}{C}\ln\left(\frac{F_1(F_1+2C)}{(F_1+C)^2}\right) + 2\ln\left(\frac{F_1+2C}{F_1+C}\right) - 1.$$

We know from Theorem 1 that this expression is decreasing in F_1. Hence, truth-telling is a Bayesian Nash equilibrium if this expression is non-positive for $F_1 = D_1$,

$$\frac{2D_1}{C}\ln\left(\frac{D_1(D_1+2C)}{(D_1+C)^2}\right) + 2\ln\left(\frac{D_1+2C}{D_1+C}\right) - 1 \leqslant 0.$$

Numerical evaluation reveals that this inequality holds if $D_1 \geqslant b_3C$ with $b_3 \approx 0.080$.

For the situation with more than three users, the complexity of the derivatives (3) increases rapidly. We use the computing software Maple [10] to perform the calculations. Thereafter, we once again use Theorem 1 to establish that truth-telling is a Bayesian Nash equilibrium if the first derivative is non-positive for $F_1 = D_1$. Numerical evaluation by Maple reveals that the inequality is satisfied for N users for all $D_1 \geqslant 0$. Hence, truth-telling is always a Bayesian Nash equilibrium for four till six users. □

Hence, for a Bayesian Nash equilibrium in a situation with three users, we have a lower bound on the demand per user. Note that this bound is smaller than the bound in the situation with two users. The lower bound disappears if we consider at least four users. For situations with seven users, we were not able to perform the necessary calculations within reasonable time limits. We therefore conducted a simulation study. We tested 10 cases, for $I = 7$–10, 12, 15, 20, 30, 50 and 100 departments. In each case, we used a fixed capacity C equal to 2500,[1] and randomly drew from a uniform $(0,C)$ distribution the forecast and demand values for $I - 1$ departments. Then for the remaining department i we checked whether it was optimal, given the utility function (2), to provide a forecast that was equal to the demand. We tested each case 1000 times, and for all $10 \times 1000 = 10{,}000$ instances truth telling was an optimal strategy for the department we studied.

Based on these results, we conjecture the following proposition.

Proposition 1. *If there are more than six users, then truth-telling is a Bayesian Nash equilibrium.*

Note that truth-telling is not a unique Bayesian Nash equilibrium, since there is another (trivial) Bayesian Nash equilibrium, namely $F_i = 0$ for all i. However, this is not of any practical value considering the problem setting.

3.2. Different cost and reward parameters

In this section we return to the general utility function $V_i(F; D_i)$ without the restriction $x = y = 1$. We analyze what happens to the lower bounds on the actual demands of the departments, as stated in the Theorems 2 and 3.

The expected utility for user 1 now equals

$$\mathbb{E}[V_1(F;D_1)] = \frac{x}{C^{N-2}}\int_0^C\cdots\int_0^C \frac{F_1}{F_1+\sum_{j=2}^N D_j}\,\mathrm{d}D_2\cdots\mathrm{d}D_N - y(F_1-D_1)^+.$$

The following theorem generalizes Theorem 1, and is therefore presented without proof.

[1] The value of $C = 2500$ is based on the average capacity of one MRI scanner per year, given that it operates 50 weeks/year for 10 h/working day, processing scans that on average take 1 h.

Theorem 4. *Consider the situation with N users. The derivative of the expected utility,*

$$\frac{\partial \mathbb{E}[V_1(F;D_1)]}{\partial F_1} = \frac{x}{C^{N-2}} \int_0^C \cdots \int_0^C \frac{\sum_{j=2}^N D_j}{(F_1 + \sum_{j=2}^N D_j)^2} \mathrm{d}D_2 \cdots \mathrm{d}D_N - yI_{\{F_1 > D_1\}},$$

is positive if $F_i < D_i$; the expected utility is then increasing in F_i.

The second derivative of the expected utility,

$$\frac{\partial^2 \mathbb{E}[V_1(F;D_1)]}{\partial F_1^2} = \frac{x}{C^{N-2}} \int_0^C \cdots \int_0^C \frac{-2\sum_{j=2}^N D_j}{\left(F_1 + \sum_{j=2}^N D_j\right)^3} \mathrm{d}D_2 \cdots \mathrm{d}D_N,$$

is negative. The derivative of the expected utility is decreasing in F_i, in particular for $F_i > D_i$.

First, consider $N = 2$ users. According to Theorem 4, the expected utility is increasing for $F_1 < D_1$. Hence, user 1 chooses $F_1(D_1) = D_1$ in the limit in case $F_1 < D_1$. If $F_1 > D_1$ then the derivative of the expected utility equals

$$x\left(\ln\left(\frac{F_1 + C}{F_1}\right) + \frac{F_1}{F_1 + C} - 1\right) - y, \tag{6}$$

which is a generalization of (5). Also by Theorem 4, this derivative is decreasing in F_1. Hence, if it is non positive for $F_1 = D_1$ then it takes negative values for all $F_1 > D_1$. This happens if $D_1 \geqslant b_2C$ where the lower bound b_2 is a root of expression (6) after substituting $F_1 = b_2C$. Thus b_2 solves

$$\ln\left(\frac{b_2 + 1}{b_2}\right) + \frac{b_2}{b_2 + 1} - 1 - y/x = 0. \tag{7}$$

This equation shows that lower bound b_2 is a function of y/x, the relative value of the 'cost' parameter y to the 'reward' parameter x (see Fig. 2).

Observe that for $y/x = 1$ the lower bound b_2 agrees with the result in Theorem 2. If y/x increases then the penalty function with weight y becomes more and more important compared to the value of the allocated capacity with weight x. Since the user adds so much relative value to the penalty, truth-telling more and more easily becomes a Bayesian Nash equilibrium. The lower bound b_2 decreases, and in particular, b_2 tends to zero as y/x increases.

We perform the same analysis for situations with three and four users, see the Fig. 3(a) and (b), respectively. For three users, the lower bound b_3 is positive as long as $y/x \leqslant 1.3$. For larger values of y/x there is no positive solution to (7). Thus, if $y/x > 1.3$ then truth-telling is always a Bayesian Nash equilibrium; there is no lower bound on the demand of the users to ensure an equilibrium.

We observe the same for four users. The lower bound b_4 is positive for $y/x < 0.8$. For larger values of y/x truth-telling is always a Bayesian Nash equilibrium. The analysis for five and more users goes along the same lines, and is therefore omitted.

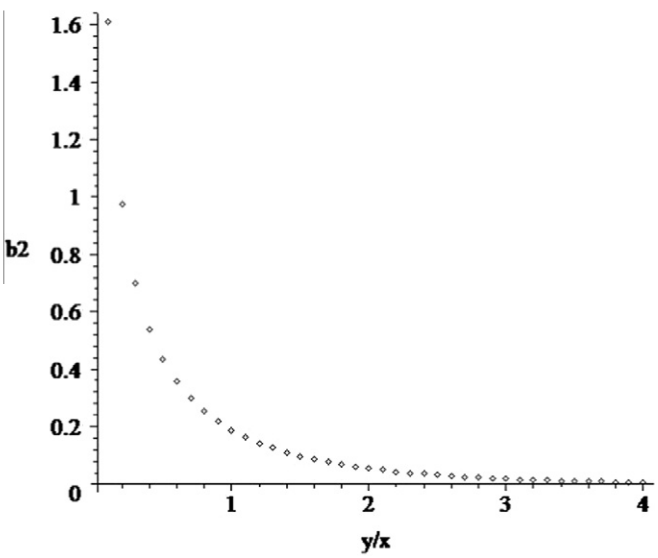


Fig. 2. Lower bound b_2 as a function of y/x.

4. Results for constrained rules

In this section we analyze the effects of capacity allocation when using the constrained equal award rule or the constrained equal loss rule.

4.1. Results for constrained equal award rule

If the capacity is allocated according to the constrained equal award rule, then the utility function of user i is

$$V_i(F;D_i) = x\min\{\alpha, F_i\} - y\max\{F_i - D_i, 0\},$$

with α such that $\sum_j \min\{\alpha, F_j\} = C$. Consider the situation with two users. For simplifications we set $x = y = 1$ in the utility function. Without loss of generality consider user $i = 1$. Assume the second user is truthful, $F_2(D_2) = D_2$. Then

$$\alpha = \begin{cases} C/2, & F_1 \geqslant C/2, \quad D_2 \geqslant C/2, \\ C - F_1, & F_1 < C/2 \leqslant D_2, \\ C - D_2, & D_2 < C/2 \leqslant F_1, \\ C/2, & F_1 < C/2, \quad D_2 < C/2. \end{cases}$$

We investigate if and when truth-telling is a Bayesian Nash equilibrium.

Theorem 5. *Consider the constrained equal award rule and $N = 2$ users. Then truth-telling is a Bayesian Nash equilibrium.*

Proof. First, if $F_1 \leqslant C/2$ then

$$\begin{aligned}\mathbb{E}[V_1(F;D_1)] &= \int_0^C (\min\{\alpha, F_1\} - \max\{F_1 - D_1, 0\}) \frac{1}{C} \mathrm{d}D_2 \\ &= \frac{1}{C}\int_0^C F_1 \mathrm{d}D_2 - \max\{F_1 - D_1, 0\} \\ &= F_1 - \max\{F_1 - D_1, 0\}.\end{aligned}$$

Second, for $F_1 > C/2$

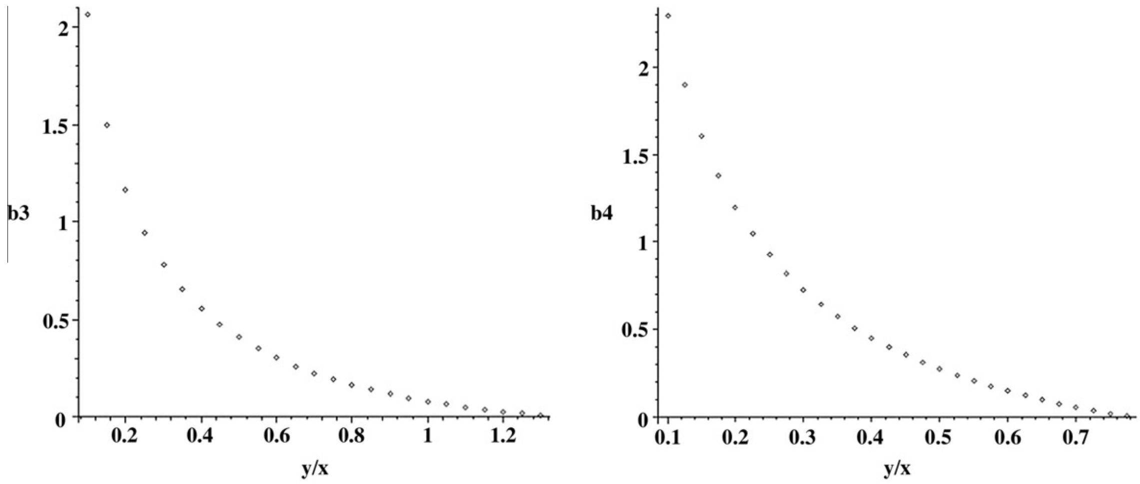
$$\begin{aligned}\mathbb{E}[V_1(F;D_1)] &= \frac{1}{C}\int_0^{C-F_1} F_1 \mathrm{d}D_2 + \frac{1}{C}\int_{C-F_1}^{C/2} (C - D_2)\mathrm{d}D_2 \\ &\quad + \frac{1}{C}\int_{C/2}^C \frac{1}{2}C\mathrm{d}D_2 - \max\{F_1 - D_1, 0\} \\ &= -\frac{F_1^2}{2C} + F_1 + \frac{1}{8}C - \max\{F_1 - D_1, 0\}.\end{aligned}$$

The derivative of the expected utility of user 1 is

$$\frac{\partial \mathbb{E}[V_1(F;D_1)]}{\partial F_1} = \begin{cases} 1 - I_{\{F_1 > D_1\}}, & F_1 < C/2, \\ -\frac{F_1}{C} + 1 - I_{\{F_1 > D_1\}}, & F_1 > C/2. \end{cases}$$

First consider $D_1 \leqslant C/2$. If $F_1 \leqslant C/2$ then the maximal expected utility is D_1 for $F_1 \in [D_1, C/2]$. If $F_1 > C/2$ then the maximal utility in the limit for $F_1 \to C/2$ is also D_1. Hence, there are multiple best replies for user 1. Truth-telling is an equilibrium.

Second, consider $D_1 > C/2$. If $F_1 \leqslant C/2$ then the maximal expected utility is $C/2$ for $F_1 = C/2$. For $F_1 > C/2$ the maximal expected utility is $\frac{5}{8}C$ for $F_1 \in (C/2, D_1]$. Hence, given $F_2 = D_2$ the



(a) Lower bound b_3 for 3 users.
(b) Lower bound b_4 for 4 users.

Fig. 3. The lower bounds b_NC for $N = 3$ and $N = 4$ users as a function of y/x.

best reply of user 1 is to set F_1 such that $C/2 < F_1 \leqslant D_1$. Truth telling is a mutual best reply. Therefore, truth-telling is a Bayesian Nash equilibrium. □

Under the constrained equal award rule, truth-telling is an equilibrium, but it is hard to determine the other equilibria. Furthermore, the analysis of the constrained equal award rule increases in complexity with the number of users. Therefore, the resource might prefer the proportional rule. For this reason, we restrict our analysis of this rule to the case of two users.

4.2. Results for constrained equal loss rule

When using the constrained equal loss rule, the utility function of user i is

$$V_i(F;D_i) = x\max\{F_i - \beta, 0\} - y\max\{F_i - D_i, 0\},$$

with β such that $\sum_j \max\{F_j - \beta, 0\} = C$. Consider a situation with two users. At first, we set $x = y = 1$ in the utility function for simplicity. Assume the second user is truthful, $F_2 = D_2$. Then

$$\beta = \frac{1}{2}(F_1 + D_2 - C)$$

is the equal amount of loss for both users. We investigate if and when truth telling is a Bayesian Nash equilibrium. Without loss of generality consider user 1.

Theorem 6. *Consider the constrained equal loss rule and $N = 2$ users. Then truth-telling is a Bayesian Nash equilibrium.*

Proof. Since

$$\frac{1}{C}\int_0^C \max\{F_1 - \beta, 0\}\,\mathrm{d}D_2 = \frac{1}{C}\int_0^{C-F_1} F_1\,\mathrm{d}D_2 + \frac{1}{C}\int_{C-F_1}^{C} \frac{1}{2}(F_1 - D_2 + C)\,\mathrm{d}D_2 = F_1 - F_1^2/(4C),$$

the expected utility of user 1 equals

$$\mathbb{E}[V_1(F;D_1)] = F_1 - \frac{F_1^2}{4C} - \max\{F_1 - D_1, 0\}.$$

The derivative with respect to F_1 is

$$\frac{\partial \mathbb{E}[V_1(F;D_1)]}{\partial F_1} = 1 - \frac{F_1}{2C} - I_{\{F_1 > D_1\}}.$$

The expected utility is increasing for $F_1 \leqslant D_1$ and decreasing for $F_1 > D_1$. Hence, $F_i(D_i) = D_i$, is the unique best response. Truth-telling is a Bayesian Nash equilibrium. □

The constrained equal loss rule performs better than the proportional rule, since truth-telling is an equilibrium without a lower bound on the private demands of the users.

Next, we consider situations with three users. We consider the expected utility of user 1 and assume that the users 2 and 3 tell the truth, $F_i(D_i) = D_i$, for $i = 2,3$. To determine the value of the loss β that is equally divided, we consider several cases.

First, suppose that all users obtain a positive part of the capacity, $A_i > 0$ for all i. Then $F_1 - \beta + D_2 - \beta + D_3 - \beta = C$, or $\beta = (F_1 + D_2 + D_3 - C)/3$. Thus

$$A_1 = F_1 - \beta = \frac{1}{3}(2F_1 + C - D_2 - D_3). \tag{8}$$

This amount is positive, $A_1 > 0$, if and only if

$$D_2 + D_3 < C + 2F_1. \tag{9}$$

Similarly, $A_2 > 0$ if and only if

$$-2D_2 + D_3 < C - F_1, \tag{10}$$

and $A_3 > 0$ if and only if

$$D_2 - 2D_3 < C - F_1. \tag{11}$$

Notice that at least two users should get a positive amount. If not, then one user gets all capacity, which can only happen if his request exceeds the other requests by more than the capacity C. This cannot occur since $0 \leqslant F_i \leqslant C$ for all users. The table below shows the diverse values of A_1 for the different cases that can occur. For reference below, we numbered the cases from I to IV.

Case	True inequalities	A_1
I	(9)–(11)	$(2F_1 - D_2 - D_3 + C)/3$
II	(10) and (11)	0
III	(9) and (11)	$(C + F_1 - D_3)/2$
IV	(9) and (10)	$(C + F_1 - D_2)/2$

Theorem 7. *Consider the constrained equal loss rule and $N = 3$ users. Then truth-telling is a Bayesian Nash equilibrium.*

Proof. The expected utility for user 1 is

$$\mathbb{E}[V_1(F;D_1)] = \int_0^C\int_0^C (\max\{F_1 - \beta, 0\} - \max\{F_1 - D_1, 0\})\frac{1}{C^2}\mathrm{d}D_2\mathrm{d}D_3 = \frac{1}{C^2}\int_0^C\int_0^C \max\{F_1 - \beta, 0\}\mathrm{d}D_2\mathrm{d}D_3 - \max\{F_1 - D_1, 0\}.$$

We focus on the calculation of the first term for several values of F_1. First, if $F_1 = 0$ then the outcome of the double integral is also 0. Next, consider $0 < F_1 < C/2$. Taking into account the four cases in the table above, we obtain

$$\begin{aligned}\frac{1}{C^2}\int_0^C\int_0^C \max\{F_1-\beta,0\}\mathrm{d}D_2\mathrm{d}D_3 &= \int\int_I (2F_1-D_2-D_3+C)/3\mathrm{d}D_2\mathrm{d}D_3 + \int\int_{II} 0\mathrm{d}D_2\mathrm{d}D_3 \\ &\quad + \int\int_{III}(C+F_1-D_3)/2\mathrm{d}D_2\mathrm{d}D_3 + \int\int_{IV}(C+F_1-D_2)/2\mathrm{d}D_2\mathrm{d}D_3 \\ &= \frac{1}{36C^2}(2C^3+12C^2F_1+24CF_1^2-29F_1^3)+0+\frac{F_1^3}{6C^2}+\frac{F_1^3}{6C^2} \\ &= \frac{1}{36C^2}(2C^3+12C^2F_1+24CF_1^2-17F_1^3).\end{aligned}$$

Similarly, for $C/2 \leqslant F_1 \leqslant C$ we obtain

$$\frac{1}{C^2}\int_0^C\int_0^C \max\{F_1-\beta,0\}\,\mathrm{d}D_2\mathrm{d}D_3 = \frac{1}{36C^2}(24C^2F_1-F_1^3).$$

For $0 \leqslant F_1 < C/2$, the derivative of this expected utility with respect to F_1 is

$$\frac{1}{36C^2}(12C^2+48CF_1-51F_1^2) - I_{\{F_1>D_1\}}.$$

The first term is between 0 and 1 due to $F_1 \in [0,C]$.

For $C/2 < F_1 \leqslant C$, the derivative of the expected utility with respect to F_1 is

$$\frac{1}{12C^2}(8C^2-F_1^2) - I_{\{F_1>D_1\}}.$$

Also here, the first term lies between 0 and 1. So, $F_1 = D_1$ is the unique maximum. In both cases, the expected utility is increasing for $F_1 \leqslant D_1$ and decreasing otherwise. Hence, $F_1 = D_1$ maximizes the expected utility. We conclude that truth-telling is a Bayesian Nash equilibrium. □

Once again, the constrained equal loss rule has truth-telling as an equilibrium. This result is better than the proportional rule, since now we have no lower bound on the private demand of the users.

If there are more than three users, the complexity of the analysis grows rapidly. For N users we have to consider $2^N - N - 1$ special cases. Again, we use simulation to study these cases. In the simulation study for I = 4–10, 12, 15, 20, 30, 50 and 100 departments, truth telling is an optimal strategy.

4.2.1. Different cost and reward parameters

In this paragraph we analyze general utility functions with $x \neq y$. We are interested for which values of x and y truth-telling remains a Bayesian Nash equilibrium. Given the complexity of the analysis we restrict ourselves to the case with $N = 2$ users.

Theorem 8. *Consider the constrained equal loss rule, $N = 2$ users and weights $x \neq y$ in the utility function. Truth-telling is a Bayesian Nash equilibrium if*

$$\frac{y}{x} \geqslant 1 - \frac{1}{2C}\min\{D_1, D_2\}.$$

Proof. From the proof of the previous theorem, the expected utility of user 1 is

$$\mathbb{E}[V_1(F;D_1)] = x\left(F_1 - \frac{F_1^2}{4C}\right) - y(F_1-D_1)^+.$$

The derivative with respect to F_1 is

$$x\left(1-\frac{F_1}{2C}\right) - yI_{\{F_1>D_1\}}.$$

Hence, the expected utility is increasing for $F_1 \leqslant D_1$. User 1 has maximal expected utility in $F_1 = D_1$ if the expected utility is decreasing for $F_1 > D_1$. This occurs if $x(1 - D_1/(2C)) - y \leqslant 0$, or $y/x \geqslant 1 - D_1/(2C)$. The result follows since this inequality should hold for all users.

5. Numerical example

We illustrate the model with a numerical example, which is based on the experience of one of the authors while working as a hospital consultant. We return to the MRI scanner example from the Introduction section, and consider four departments that each make requests for a specific scanning type, namely oncological (O), cardiovascular (C), neurological (N), and musculoskeletal (M). Capacity is distributed proportionally according to the requests, and the cost and reward parameters are both equal to 1, as in Section 3.1. The MRI scan facility has a fixed capacity C of 1000 scans per month. In this example we chose to use the proportional allocation mechanism. Since we consider more than three departments, there is no lower bound on the demand of the users. Also, the proportional allocation rule is intuitive and easy to apply.

We start in month 1, and obtain the estimates of future demand (F_i). Recall that capacity is allocated by the Radiology department, having no knowledge on the actual demand D_i. The demand forecasts F_i and allocated capacities A_i are given in the first two columns of Table 2. At the end of month 1, the actual demand D_i is known. This information can be used to penalize the departments, if necessary. The other columns of Table 2 give the actual demand D_i, the deviation of the allocated amount A_i and forecast F_i from D_i, and the value of the utility function V_i.

We see that in month 1 the waiting list increases with 107 MRI scans (scanning types oncological, neurological, and musculoskeletal), while there is unused capacity of 34 MRI scans (scanning type cardiovascular). Note that there is no penalty on the surplus demand related to the allocated capacity (i.e. $D - A$), since we only focus on the truth-telling element in the problem. In the example, it is implicitly assumed that surplus demand is lost. This lost demand could represent MRI scans that are performed at another institution, or not performed at all, because the physician decides upon another method of diagnostics.

We assume that the departments learn from the penalty given at the end of month 1 and therefore in month 2 provide an honest estimate (i.e. $F_i = D_i$ for all i). Without loss of generality, we assume that the actual demands of the departments in month 2 equal that

Table 2
Month 1: forecast of demand F_i, allocated capacity A_i, actual demand D_i, deviation of D_i from F_i and A_i, and utility V_i.

Department	F_i	A_i	D_i	$F_i - D_i$	% $F_i - D_i$	$A_i - D_i$	V_i
O	137	126	127	10	8	−1	116
C	130	119	85	45	53	34	74
N	630	578	623	7	1	−45	571
M	193	177	238	−45	−19	−61	177
All	1090	1000	1073	17	2	−73	938

Table 3
Month 2: forecast of demand F_i, allocated capacity A_i, actual demand D_i, deviation of D_i from F_i and A_i, and utility V_i.

Department	F_i	A_i	D_i	$F_i - D_i$	% $F_i - D_i$	$A_i - D_i$	V_i
O	127	118	127	0	0	−9	118
C	85	79	85	0	0	−6	79
N	623	581	623	0	0	−42	581
M	238	222	238	0	0	−16	222
All	1073	1000	1073	0	0	−73	1000

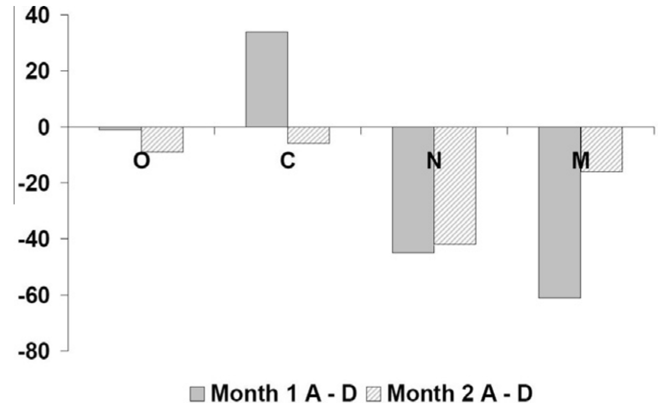


Fig. 4. Difference between allocated capacity A_i and actual demand D_i for both months.

of month 1. Capacity is again allocated proportionally to the requests. See Table 3 for results.

In month 2 the waiting list increases with 73 MRI scans, which equals the capacity shortage of $\sum_j D_j - C$, but there is no unused capacity. Fig. 4 compares the difference between the allocated capacity A_i and actual demand D_i for both months. Furthermore, we see an increase in utility for all departments compared to month 1, while capacity is distributed more fairly.

6. Discussion and conclusions

In this paper we have studied a Radiology department (the resource) that has to distribute MRI scanning capacity among competing hospital departments (the users). Radiology uses forecasts of demand, provided by the hospital departments, to distribute the scanning capacity. The actual value of their demand is only known to the hospital departments. When the departments over- or underestimate the demand it can occur that the actual demand is less than the allocated capacity (i.e. the scanner sits idle) or the actual demand is larger than the allocated capacity. Both situations can arise simultaneously. In order to have a fair allocation, where all available capacity is actually used, Radiology should motivate the departments to provide an honest forecast of their demand.

We have introduced a generic approach to study the allocation of capacity to the users. Using a Bayesian game framework we show that under several capacity allocation mechanisms it is an optimal strategy for each user to provide an honest demand forecast (the truth-telling equilibrium), and as a result the resource can fairly distribute the available capacity. When the number of users is small, certain restrictions on the relative size of the demands apply for the proportional allocation mechanism.

Topics for further research would for instance be the reward users place on allocated capacity. Even though the three capacity allocation mechanisms are intuitively appealing, and satisfy the desired properties of an allocation mechanism as stated in Section 2.2, other allocation mechanisms also might be of interest and may be better related to reality for some practical cases. Also, using a combinatorial auction to model the problem, as mentioned in the introduction section, could be a valuable extension.

We have shown that even with minor restrictions on the behavior of users, it is possible to attain a truth-telling equilibrium, where the shared resource is fairly allocated and all capacity is used.

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