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A P- and T-invariant characterization of product form and decomposition in stochastic Petri nets

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ABSTRACT

Structural product form and decomposition results for stochastic Petri nets are surveyed, unified and extended. The contribution is threefold. First, the literature on structural results for product form over the number of tokens at the places is surveyed and rephrased completely in terms of T-invariants. Second, based on the underlying concept of group-local-balance, the product form results for stochastic Petri nets are demarcated and an intuitive explanation is provided of these results based on T-invariants, only. Third, a decomposition result is provided that is completely formulated in terms of both T-invariants *and* P-invariants.

1. Introduction

Competition over resources is an important issue in many practical systems. Examples of such systems are computer systems, telecommunication networks, flexible manufacturing systems and hospitals, which typically consist of many departments and serve a wide variety of patient types. Pathways of patients are generally stochastic and various patient flows share different resources, of which operating rooms and diagnostic testing facilities are the most apparent. Typical questions arising are identification of bottlenecks, achievable throughput and maximization of resource utilization. Therefore, performance analysis is an important issue in the design and implementation of such real life systems.

Several approaches exist for performance analysis of complex systems, such as discrete-event simulation, numerical approximations or exact analytical results. Obtaining analytical results has two main advantages. First, it provides vital insight in the qualitative behaviour of involved systems, so that the key characteristics of a system can be detected. In particular, qualitative results related to the structure of the system are often of great importance. Second, it enables efficient computation of relevant performance measures. In many theoretical and practical studies of performance models involving stochastic effects, the statistical distribution of items (customers, jobs, etc.) over places (workstations, queues, etc.) is of great interest, since various performance measures can be computed from this distribution.

Three main formalisms exist for obtaining analytical closed form results for networks: queueing networks, stochastic process algebras and stochastic Petri nets. The selection of a specific formalism when studying a system preferably depends on the characteristics under investigation. Queueing networks are most suitable when the queueing structure at different locations in the network is the key aspect of the system. When a system consists of building blocks of different processes that are composed into a network, stochastic process algebras may be preferred. Stochastic Petri nets are appropriate when the flow of items and information through the network is the main feature of the system. When a specific formalism is applied, all network characteristics and all results are preferably formulated in the semantics of that formalism. In this paper we

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focus on Stochastic Petri nets, since we are interested in the interaction of flows within the system, such as naturally occur in hospital environments. All results are formulated in terms of the Petri net structure given by the P- and T-invariants, the central concepts in Petri Nets.

Composition and decomposition of closed form results contribute to less computational effort requirements and greater understanding of network behavior and performance. They allow for studying a system by analysing the characteristics of separate components. In this paper, we study closed form results for the equilibrium distribution of the number of tokens at the places of a stochastic Petri net and the decomposition of this equilibrium distribution into several components corresponding to subnets of the stochastic Petri net. Exact analytical results for the distribution of the number of items at places in performance models are in general very difficult to obtain. One of the most important analytical results for the equilibrium distribution describing the number of items at places in a performance model is the so-called *product form* equilibrium distribution found for a fairly wide class of theoretical queueing models. However, practical performance models seldom satisfy the product form conditions. Still, results obtained via the theoretical product form distributions are used for practical problems since these results are found to be robust, that is models which violate the product form conditions are often found to behave in a way very similar to a product form counterpart. The obvious advantages of these product form distributions are their simplicity, since the network behavior is captured in closed form in only a limited set of parameters. This makes product form solutions easy and powerful to use for computational reasons as well as for theoretical reflections for performance models involving congestion. Another important advantage of product form solutions is that it enables us to break down the analysis of a network in the analysis of separate components of the network.

It is widely believed that a form of *local balance* is the common element for all performance models with a product form equilibrium distribution. In this paper, *group-local-balance* is shown to be the concept identifying that the equilibrium distribution of a stochastic Petri net is of product-form nature. Boucherie and Van Dijk [1] presented the group-local-balance concept as the basis for the analysis of batch routing queueing networks. This paper provides a translation of these results into Petri net terminology. The results on the Markov chain level will then provide the foundation to discuss and further investigate structural Petri net implications. We survey the various structural results that are known for stochastic Petri nets with a product form equilibrium distribution over the number of tokens at the places [2–8]. The product form results for stochastic Petri nets known from the literature will shown to be unified by group-local-balance, as it forms the connecting principle between these results and the results known for batch routing queueing networks [1,9]. The results are derived and presented step-by-step to provide an intuitive understanding of the Petri net structure underlying the product form results.

The first structural product form results for stochastic Petri nets were presented by Henderson et al. [7]. These results are based on the assumption that a positive solution exists for a linear set of equations similar to the traffic equations for queueing networks. It will be shown that group-local-balance implies a positive solution to this linear set of equations, known as the *routing chain*, to exist. A characterization of the structure of the Petri net that is necessary and sufficient for the existence of a positive solution to the routing chain was provided by Boucherie and Sereno [2]. We show that this characterization implies that group-local-balance requires the stochastic Petri net to be an $\mathcal{S}\Pi$-net [6], a stochastic Petri net in which each transition is covered by a minimal support T-invariant. Taking group-local-balance as a starting point enables us to provide additional structural implications and a more intuitive explanation of the known results. By formulating every result in terms of the Petri net structure given by the T-invariants, we also provide structural insights for results known at an algebraic level.

Finally, from the detailed understanding of the structure behind product form results, we are able to establish a decomposition result. This decomposition result is a generalization of the results obtained by Frosch and Natarajan [10,11] for closed synchronized systems of stochastic sequential processes, a class of Petri nets in which state machines are synchronized via buffer places. The decomposition result is completely formulated in terms of P- and T-invariants. Similar to buffer places, we define conflict places, which are places that are shared by different minimal closed support T-invariants. Using the P-invariants to assign conflict places as surplus places, places that can be omitted in characterizing the marking of the Petri net, we obtain an algorithmic procedure to verify whether product form holds and for decomposition of the stochastic Petri net into subnets. These subnets correspond to one or more common input bag classes, equivalence classes of T-invariants of the stochastic Petri nets that share an input bag.

Statement of contribution. Our contribution is threefold:

1. We survey the various structural results that are known for stochastic Petri nets with a product form equilibrium distribution over the number of tokens at the places and rephrases all these results in terms of T-invariants.
2. We unify and extend the product form results for stochastic Petri nets by showing that *group-local-balance* can be identified as the concept underlying all these structural results and we provide additional structural implications and an intuitive explanation of the known and new results, all based on T-invariants only.
3. We provide a decomposition result that is completely formulated in terms of both P- and T-invariants and their derivatives as defined in the paper: common input bag classes, conflict places and surplus places.

Outline. This paper is organized as follows. In Section 2, a detailed literature survey of product form results and decomposition is provided. For insight and self-containedness, a thorough introduction into the (stochastic) Petri net formalism is provided in Section 3. In Section 4, product form results for batch routing queueing networks based on the group-local-balance concept are translated into Petri net terminology. These results, presented on the Markov chain level, provide the basis for Section 5, in which structural Petri net implications are discussed. This section is concluded by an algorithm to verify whether a specific stochastic Petri net possesses a product form equilibrium distribution, and if so, to construct this product form. Section 6 presents the new decomposition result and is ended with an algorithm by which all possible decompositions of a product form stochastic Petri can be generated. In the closing Section 7, the results are summarized and directions for future research are discussed.

2. Literature

Product form results exist on different levels. In the classical product form result the equilibrium distribution of a network can be expressed as a product over the nodes of the network. In this section we provide a survey of such results for stochastic Petri nets in Section 2.1. A more general product form result is when the equilibrium distribution of a network is a (normalized) product over the marginal distribution of subnets. A survey of such decomposition results will be provided in Section 2.2.

2.1. Product form results for stochastic Petri nets

The first product form results date back to Jackson's [12] and Gordon and Newell's [13] results for the equilibrium distribution of the number of customers at the stations in a queueing network. These results were generalized to Kelly–Whittle networks (see e.g. [14,15]), networks with job-types and various service disciplines (see e.g. [16–18]), to batch routing (see e.g. [1,19,9]) and discrete-time networks (see e.g. [20]). For stochastic Petri nets, the first product form results for the number of tokens at the places were obtained by Lazar and Robertazzi [21] for the class of stochastic Petri nets consisting of 'linear task sequences', a number of tasks that must be executed consecutively. Since these first results, considerable extensions have been derived by several authors. In a series of papers, Henderson et al. [7,22,23] translated and extended product form results for batch routing queueing networks to stochastic Petri nets, which are equivalent to batch routing queueing networks at the level of the underlying stochastic process.

The starting point for the analysis of product form stochastic Petri nets is the assumption that a solution exists for the 'routing chain', a set of linear equations similar to the traffic equations for queueing networks. The product form results for stochastic Petri nets obtained in [7,22,23] were based on the assumption that a positive solution exists for the routing chain. Necessary conditions for such a solution to exist were provided in [7].

A full characterization of the structure of stochastic Petri nets necessary and sufficient for the existence of a positive solution for the routing chain was obtained in [2,5]: all transitions of the Petri net should be covered by 'closed support T-invariants'. This new type of T-invariant was also introduced in [2,5] and is a T-invariant that closely resembles the 'task sequences' used by Lazar and Robertazzi [21]. As such, the existence of a solution for the routing chain was completely characterized on the basis of the structure of the Petri net. This class of stochastic Petri nets was later denoted as $\mathcal{S}\Pi$-nets by Haddad et al. [6].

For an $\mathcal{S}\Pi$-net, Coleman et al. [24] were the first to formulate an additional requirement sufficient for product form in stochastic Petri net by a numerical condition on the transition rates. Haddad et al. [6] and Mairesse and Nguyen [8] established characterizations of $\mathcal{S}\Pi$-nets possessing a product form solution irrespective of the values of the transition rates. Haddad et al. [6] achieved this via the concept of $\mathcal{S}\Pi^2$-nets and Mairesse and Nguyen [8] via the concept of 'zero-deficiency' $\mathcal{S}\Pi$-nets. The conditions of Coleman et al. [24], Haddad et al. [6] and Mairesse and Nguyen [8] are algebraic conditions which lack intuition in terms of Petri net structure. *The present paper unifies these results via the concept of group-local-balance and extends these results by formulating all product form results in terms of T-invariants.*

2.2. Decomposition

A network can be decomposed if its stationary distribution factorizes into the stationary distributions of the nodes of which the network is comprised; the network is then of product form. Apart from the theoretical interest, decomposition results are also of substantial practical importance: finding the stationary distribution of an entire network usually requires an enormous computational effort, whereas the stationary distribution of a single node can be found relatively easily. The first, and perhaps most famous, decomposition results for queueing networks were reported by Jackson [12]: the classical Jackson product form result. Decomposition of networks into subnetworks have been a topic of research for queueing networks. Two streams of literature developed in parallel: results based on partial balance (e.g. [25–29]) and results based on quasi-reversibility (e.g. [30–33]). Recently, in a setting of general stochastic processes, these results have been unified and extended in [34,35].

For stochastic Petri nets decomposition results were initialized by Lazar and Robertazzi [36] for connected subnets of task sequences and extended by Boucherie [37] in the framework of competing Markov chains. Frosch and Natarajan [10,11] derived product form results for so-called closed synchronized systems of stochastic sequential processes, a class of Petri nets in which state machines are synchronized via buffer places. The results in these references may also be interpreted as composition results since the networks are essentially obtained by composing subnets in to a larger net, similar to the composition structure of stochastic process algebras. As such, no procedure is provided in the literature to algorithmically characterize subnets in a given stochastic Petri net and to verify whether product form holds. *In this paper, decomposition results will be presented based on the structure of a Petri net formulated exclusively in terms of P- and T-invariants.*

3. Preliminaries

The aim of this section is to provide a general introduction into the formal Petri net language and the Petri net concepts that will be relevant for the analysis in subsequent sections. First, basic definitions of Petri nets and stochastic Petri nets are presented. Next, structural and behavioural properties are introduced. Also, some results derived from these properties of a Petri net that will be used in subsequent sections are listed.

3.1. Petri nets

Definitions, properties and results will be presented schematically to provide the reader a convenient reference to the numerous concepts. More elaborate overviews of definitions, properties and results can be found in the survey of Murata [38] and the book of Peterson [39].

3.1.1. Definitions

Definition 3.1 (*Petri Net*)**.** A Petri net is a weighted bipartite graph with nodes being either places or transitions and is defined by the 4-tuple $\mathcal{PN} = (P, T, I, O)$, where

- $P = \{p_1, \ldots, p_N\}$ is a finite set of places,
- $T = \{t_1, \ldots, t_M\}$ is a finite set of transitions,
- $I, O : P \times T \to \mathbb{N}$ are the input and output functions identifying the relation between the places and the transitions.

Definition 3.2 (*Marking*)**.** A *marking* $\boldsymbol{m} = (m(n),\ n = 1, \ldots, N)$ of a Petri net is a vector in $\mathbb{N}_0^N$, where $m(n)$ represents the number of *tokens* at place p_n.

Definition 3.3 (*Marked Petri Net*)**.** A marked Petri net is a Petri net defined by the 5-tuple $(\mathcal{PN}, \boldsymbol{m}_0) = (P, T, I, O, \boldsymbol{m}_0)$, where $\boldsymbol{m}_0$ is the initial marking.

Definition 3.4 (*Input Bag—Output Bag*)**.** $I(\cdot,\cdot)$ and $O(\cdot,\cdot)$ give the vectors $\boldsymbol{I}(t) = (I_1(t), \ldots, I_N(t))$ and $\boldsymbol{O}(t) = (O_1(t), \ldots, O_N(t))$, where $I_n(t) = I(p_n, t)$, and $O_n(t) = O(p_n, t)$. The vectors $\boldsymbol{I}(t)$ and $\boldsymbol{O}(t)$ are called the *input* and *output bags* of transition $t \in T$, respectively representing the number of tokens needed at the places to fire transition t, and the number of tokens released to the places after firing transition t.

Definition 3.5 (*Transition Enabling and Firing*)**.** A necessary and sufficient condition for transition t to be *enabled* in marking $\boldsymbol{m}$ is that $m(n) \geq I_n(t)$. When transition t *fires*, then the next state of the Petri net is $\boldsymbol{m}' = \boldsymbol{m} - \boldsymbol{I}(t) + \boldsymbol{O}(t)$. Symbolically this is denoted as $\boldsymbol{m}[t > \boldsymbol{m}'$.

Definition 3.6 (*Firing Sequence*)**.** A finite sequence of transitions $\sigma = t_{\sigma_1} t_{\sigma_2} \cdots t_{\sigma_k}$ is a finite *firing sequence* of the Petri net if there exists a sequence of markings $\boldsymbol{m} = \boldsymbol{m}_{\sigma_1}, \ldots, \boldsymbol{m}_{\sigma_{k+1}} = \boldsymbol{m}'$ for which $\boldsymbol{m}_{\sigma_i}[t_{\sigma_i} > \boldsymbol{m}_{\sigma_{i+1}},\ i = 1, \ldots, k$. Symbolically this will be denoted as $\boldsymbol{m}[\sigma > \boldsymbol{m}'$.

Definition 3.7 (*Incidence Matrix*)**.** The *incidence matrix* $\boldsymbol{A}$ with entries $A(p, t) = O(p, t) - I(p, t)$ describes the change in the number of tokens in place p when transition t fires.

Definition 3.8 (*Firing Count Vector*)**.** A vector $\bar{\sigma}$ is the *firing count vector* of the firing sequence σ if $\bar{\sigma}(t)$ equals the number of times transition t occurs in the firing sequence σ.

Definition 3.9 (*State Equation*)**.** If $\boldsymbol{m}_0[\sigma > \boldsymbol{m}$, then $\boldsymbol{m} = \boldsymbol{m}_0 + \boldsymbol{A}\bar{\sigma}$. This equation is referred to as the *state equation* for the Petri net.

Definition 3.10 (*Closed Set*)**.** For $\mathcal{T} \subseteq T$ define $\mathcal{R}(\mathcal{T})$, the set of input and output bags for the transitions in $\mathcal{T}$, as $\mathcal{R}(\mathcal{T}) = \bigcup_{t\in\mathcal{T}}\{\boldsymbol{I}(t) \cup \boldsymbol{O}(t)\}$. $\mathcal{R}(\mathcal{T})$ is a closed set if for all $\boldsymbol{g} \in \mathcal{R}(\mathcal{T})$ there exist $t, t' \in \mathcal{T}$ such that $\boldsymbol{g} = \boldsymbol{I}(t)$, as well as $\boldsymbol{g} = \boldsymbol{O}(t')$, that is if each output bag is also an input bag, and each input bag is also an output bag for a transition in $\mathcal{T}$.

Definition 3.11 (*(Cyclic) State Machine*)**.** A Petri net $\mathcal{PN}$ is a *state machine* if and only if $\sum_p I_p(t) = 1$ and $\sum_p O_p(t) = 1$ for all transitions. $\mathcal{PN}$ is a *cyclic* state machine if and only if $\mathcal{PN}$ is strongly connected.

3.1.2. Properties

Two types of properties are distinguished. Properties which depend on the initial marking are referred to as *behavioural* and those which are independent on the initial marking as *structural*. Behavioural and structural properties will respectively be marked by the labels [B] and [S].

Definition 3.12 (*Reachability [B]*). A marking $\boldsymbol{m}'$ is *reachable* from marking $\boldsymbol{m}_0$ if a firing sequence σ exists such that $\boldsymbol{m}_0[\sigma > \boldsymbol{m}'$.

Definition 3.13 (*Reachability Set [B]*). The *reachability set* $\mathcal{M}(\mathcal{PN}, \boldsymbol{m}_0)$ is a subset of $\mathbb{N}^N$ and gives all reachable markings of the Petri net with initial making $\boldsymbol{m}_0$.

Definition 3.14 (*T-Invariant [S]*). A vector $\boldsymbol{x} \in \mathbb{N}_0^M$ is a *T-invariant* if $\boldsymbol{x} \neq 0$, and $\boldsymbol{Ax} = 0$. From the state equation we obtain that a T-invariant represents a firing sequence that brings a marking back to itself [38]. So T-invariants define potential cycles in the reachability set.

Definition 3.15 (*P-Invariant [S]*). A vector $\boldsymbol{y} \in \mathbb{N}_0^N$ is a *P-invariant* (sometimes called S-invariant) if $\boldsymbol{y} \neq 0$, and $\boldsymbol{yA} = 0$. P-invariants correspond to the conservation of tokens in subsets of places. A P-invariant identifies a set of places such that the weighted sum of the number of tokens distributed over these places remains constant for all markings in the reachability set.

Definition 3.16 (*Support [S]*). The *support* of a T-invariant $\boldsymbol{x}$ or P-invariant $\boldsymbol{y}$ is the set of transitions or places respectively corresponding to non-zero entries of $\boldsymbol{x}$ and $\boldsymbol{y}$, and are denoted by $\|\boldsymbol{x}\|$ and $\|\boldsymbol{y}\|$, i.e., $\|\boldsymbol{x}\| = \{t \in T \mid x(t) > 0\}$ and $\|\boldsymbol{y}\| = \{p \in \mathcal{P} \mid y(p) > 0\}$.

Definitions 3.17 and 3.18 are stated in terms of T-invariants. The definitions are analogous for P-invariants.

Definition 3.17 (*Minimal Invariant [S]*). A T-invariant is a *minimal T-invariant* if there is no other T-invariant $\boldsymbol{x}'$ such that $x'(t) \leq x(t)$ for all t.

Definition 3.18 (*Minimal Support Invariant [S]*). A support is minimal if no proper nonempty subset of the support is also the support of a T-invariant. An invariant with minimal support is a *minimal support invariant*.

Definition 3.19 (*Closed T-Invariant [S]*). A T-invariant is *closed* if the set of input and output bags for the transitions in its support, $\mathcal{R}(\|\boldsymbol{x}\|)$, is a closed set.

Definition 3.20 (*Minimal Closed Support T-Invariant [S]*). A T-invariant is a *minimal closed support T-invariant* if it is closed and has minimal support.

Definition 3.21 (*Liveness [B]*). A transition is $t \in T$ is *live* if no matter what marking has been reached from $\boldsymbol{m}_0$ it is possible to ultimately fire transition t again. A Petri net is *live* under initial marking $\boldsymbol{m}_0$ if every transition is live under $\boldsymbol{m}_0$. An extensive discussion of liveness and related concepts is given in [38].

Definition 3.22 (*Structural Liveness [S]*). A Petri net is *structurally live* if there exists an initial marking $\boldsymbol{m}_0$ for which the net is live.

Definition 3.23 (*Home State [B]*). A marking $\boldsymbol{m}$ is a *home state* if for each marking in $\boldsymbol{m}' \in \mathcal{M}(\mathcal{PN}, \boldsymbol{m}_0)$, $\boldsymbol{m}$ is reachable from $\boldsymbol{m}'$, i.e., $\forall \boldsymbol{m}' \in \mathcal{M}(\mathcal{PN}, \boldsymbol{m}_0) : \boldsymbol{m} \in \mathcal{M}(\mathcal{PN}, \boldsymbol{m}')$.

Definition 3.24 (*Boundedness [B]*). A Petri net is *k-bounded* or simply *bounded* if the number of tokens in each place does not exceed a finite number k for any marking in the reachability set $\mathcal{M}(\mathcal{PN}, \boldsymbol{m}_0)$.

Definition 3.25 (*Structural Boundedness [S]*). A Petri net is *structurally bounded* if it is bounded for all initial markings.

3.1.3. Results

Result 3.26 (*Murata [38]*). A structurally bounded and structurally live Petri net is covered by both P-invariants and T-invariants.

Result 3.27 (*Memmi and Roucairol [40]*). There is a unique minimal T-invariant corresponding to a minimal support (*minimal support T-invariant*). Let $\boldsymbol{x}^1, \ldots, \boldsymbol{x}^k$ denote the minimal support T-invariants. Any T-invariant $\boldsymbol{x}$ can be written as a linear combination of minimal support T-invariants:

$$\boldsymbol{x} = \sum_{i=1}^{k} \lambda_i \boldsymbol{x}^i$$

where $\lambda_i \in \mathbb{Q}^+,\ i = 1, \ldots, k$. The equivalent result holds for P-invariants.

Remark 3.28. Two remarks with respect to the decomposition Result 3.27 of Memmi and Roucairol can be made. First, since the elements of minimal invariants are required to be non-negative, the minimal support invariants may be linearly dependent, so that there may exist more invariants than the dimension of the null space. Second, for the decomposition to be in minimal support invariants it is essential that the weight factors λ_i are allowed to be rational numbers. If one restricts to integral weight factors, additional invariants may need to be added to the set of minimal support T-invariants to obtain a decomposition result. An extensive discussion on different decomposition results is provided by Krückeberg and Jaxy [41]. In this reference, efficient algorithms are also presented to obtain the sets of minimal T- and P-invariants from the incidence matrix $\boldsymbol{A}$.

Result 3.29 (*Boucherie and Sereno [3]*)**.** A T-invariant $\boldsymbol{x}$ is a minimal closed support T-invariant if the firing sequence of $\boldsymbol{x}$ is *linear*, that is for each $t \in \|\boldsymbol{x}\|$ there is a unique $t' \in \|\boldsymbol{x}\|$ such that $\boldsymbol{O}(t) = \boldsymbol{I}(t')$. As a consequence $x_i \leq 1,\ i = 1, \ldots, M$. Conversely, if the firing sequence of a T-invariant $\boldsymbol{x}$ is linear, then $\boldsymbol{x}$ is a closed support T-invariant.

3.2. Stochastic Petri nets

Definition 3.30 (*Stochastic Petri Net*)**.** A stochastic Petri net is a Petri net defined by the 5-tuple $\mathcal{SPN} = (P, T, I, O, Q)$, where (P, T, I, O) is a Petri net, and $Q = (q(t_1), \ldots, q(t_M))$ is a set of exponential firing rates associated with the set of transitions $T = \{t_1, \ldots, t_M\}$. Distributions associated with different transitions are independent. The firing execution policy of the stochastic Petri net is the race model.

Definition 3.31 (*Marked Stochastic Petri Net*)**.** A marked stochastic Petri net is a stochastic Petri net defined by the 6-tuple $(\mathcal{SPN}, \boldsymbol{m}_0) = (P, T, I, O, Q, \boldsymbol{m}_o)$, where $\boldsymbol{m}_0$ is the initial marking.

Definition 3.32 (*$\mathcal{S}\Pi$-Net*)**.** A Π-net is a Petri net in which all transitions $t \in T$ are covered by minimal closed support T-invariants $\boldsymbol{x}^i,\ i = 1, \ldots, k$, that is for all $t \in T$ there exists an $i \in \{1, \ldots, k\}$ such that $t \in \|\boldsymbol{x}^i\|$ and $\|\boldsymbol{x}^i\|$ is a closed set. A $\mathcal{S}\Pi$-net is a stochastic Π-net.

There exist various firing execution policies for stochastic Petri nets. For an extensive discussion on these policies, see [42]. We assume that the firing execution policy follows a race model. As a consequence of the exponential firing times, the stochastic process describing the evolution of the Petri net is a time-homogeneous continuous-time Markov chain $\mathbf{X}$ at state space $\mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)$. Denote the transition rates of $\mathbf{X}$ by $Q = (q(\boldsymbol{m}, \boldsymbol{m}'), \boldsymbol{m}, \boldsymbol{m}' \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0))$. To avoid anomalies, we assume the process is regular, that is, at most finitely many transitions can fire in finite time [43, Chapter 2]. It will be assumed that each transition of the Markov chain representing the Petri net is due to exactly one transition $t \in T$ that fires. Note that the firing of multiple transitions can be incorporated by adding extra transitions representing the combination of several transitions that fire with suitable firing rates.

The evolution of the Markov chain describing the stochastic Petri net is as follows. A transition t in marking $\boldsymbol{m}$ can be enabled only if $\boldsymbol{m} - \boldsymbol{I}(t) \in \mathbb{N}_0^N$. Furthermore, we will allow multiple transitions to have the same enabling condition, i.e., for $t_i \neq t_j$ it is allowed that $\boldsymbol{I}(t_i) = \boldsymbol{I}(t_j)$. Of course, the output bag will not be the same, otherwise these two transitions could be represented by only one. The rate

$$q(\boldsymbol{I}(t), \boldsymbol{O}(t); \boldsymbol{m} - \boldsymbol{I}(t)) \tag{1}$$

is associated with transition t bringing $\boldsymbol{m}$ to $\boldsymbol{m}' = \boldsymbol{m} - \boldsymbol{I}(t) + \boldsymbol{O}(t)$. Note that a transition from marking $\boldsymbol{m}$ to marking $\boldsymbol{m} - \boldsymbol{I}(t) + \boldsymbol{O}(t)$ may occur due to other transitions too. The total transition rate from marking $\boldsymbol{m}$ to marking $\boldsymbol{m}'$ is therefore

$$q(\boldsymbol{m}, \boldsymbol{m}') = \sum_{\{\boldsymbol{n} \in \mathbb{N}_0^N,\ t \in T:\ \boldsymbol{n}+\boldsymbol{I}(t)=\boldsymbol{m},\ \boldsymbol{n}+\boldsymbol{O}(t)=\boldsymbol{m}'\}} q(\boldsymbol{I}(t), \boldsymbol{O}(t); \boldsymbol{n}). \tag{2}$$

When analysing the Markov chain $\mathbf{X}$ describing the behavior of a stochastic Petri net, it will be convenient to aggregate transitions with identical input bag to one transition with a probabilistic output bag. In that case, all transitions, say $t_{i_1}, \ldots, t_{i_k}$ with identical input bag are aggregated into a single transition t. The output bag of this new transition is probabilistic, with the probability that output bag $\boldsymbol{O}(t_{i_j})$ occurs determined by the original firing rates, so that:

$$q(\boldsymbol{I}(t), \boldsymbol{O}(t); \boldsymbol{m} - \boldsymbol{I}(t)) = \mu(t; \boldsymbol{m} - \boldsymbol{I}(t)) p(\boldsymbol{I}(t), \boldsymbol{O}(t); \boldsymbol{m} - \boldsymbol{I}(t)) \tag{3}$$

where $\mu(t; \boldsymbol{m} - \boldsymbol{I}(t)) = \sum_{j=1}^{k} q(\boldsymbol{I}(t_{i_j}), \boldsymbol{O}(t_{i_j}); \boldsymbol{m} - \boldsymbol{I}(t_{i_j}))$ is the total firing rate and $p(\boldsymbol{I}(t), \boldsymbol{O}(t_{i_j}); \boldsymbol{m} - \boldsymbol{I}(t)) = q(\boldsymbol{I}(t), \boldsymbol{O}(t_{i_j}); \boldsymbol{m} - \boldsymbol{I}(t))/\mu(t; \boldsymbol{m} - \boldsymbol{I}(t))$ is the probability of selecting a specific output bag $\boldsymbol{O}(t_{i_j})$.

We are interested in calculating the steady-state behavior of the continuous-time Markov chain $\mathbf{X}$ modelling the marked stochastic Petri net $(\mathcal{SPN}, \boldsymbol{m}_0)$. From standard Markov theory we know that $\mathbf{X}$ is irreducible and positive recurrent if and only if a unique collection of positive numbers $\pi = (\pi(\boldsymbol{m}), \boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0))$ summing to unity, exists satisfying the *global balance equations*,

$$\sum_{\boldsymbol{m}' \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)} \left\{ \pi(\boldsymbol{m}) q(\boldsymbol{m}, \boldsymbol{m}') - \pi(\boldsymbol{m}') q(\boldsymbol{m}', \boldsymbol{m}) \right\} = 0, \quad \boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0). \tag{4}$$

This $\pi = (\pi(\boldsymbol{m}), \boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0))$ is called the *equilibrium distribution*.

As the Markov chain is chosen such that it describes the evolution of the stochastic Petri net under consideration, irreducibility and positive recurrence properties necessary to obtain a unique equilibrium distribution for the Markov chain should preferably be characterized directly from the Petri net structure.

The state space of a Markov chain **X** partitions in communicating classes [44]. As we are interested in the steady state behavior of **X** we can analyse the process at each class separately. Moreover, we are not interested in transient classes, as transient states will vanish in the equilibrium distribution of the stochastic Petri net. Thus, we will focus on stochastic Petri nets of which the corresponding Markov chain **X** is irreducible.

To prevent the presence of transient classes, we restrict ourselves to bounded Petri nets that are live and therefore covered by T-invariants. If the Petri net is live and has a home state, then **X** is irreducible. (Note that irreducibility of the Markov chain is called reversibility in the Petri net literature [38]. The notion of reversibility for Petri nets should not be confused with the notion of reversibility for Markov chains [14].)

If the reachability set is finite, positive recurrence follows from irreducibility. Otherwise, for **X** to be stable additional assumptions on the transition rates are required to ensure that the rate at which tokens are created is smaller then the rate at which they are destroyed. This problem is for example addressed in [45]. To avoid non-regularity, we restrict our attention to stochastic Petri nets with a finite reachability set, thus to structurally bounded nets. By Result 3.26, for a live net to be structurally bounded, the net must be covered by P-invariants.

A live Petri net is structurally live. A complete characterization of structural liveness for a general Petri net is unknown [38]. Liveness and boundedness are not related to the existence of a home state [38] for general net structures. It is beyond the scope of this paper to provide a complete overview for general Petri nets (see [46,38] for elaborate discussions). For $\mathcal{S}\Pi$-nets (see Definition 3.32), in Theorem 5.6 we will provide a complete characterization of structurally liveness and existence of a home state. Note that also in this case, for a specific initial marking liveness still needs to be checked, which may be a cumbersome problem (see [6] for some exploratory results).

4. The Markov chain and group-local-balance

In this section, we first analyse the Markov chain **X** of an $\mathcal{SPN}$. Boucherie and Van Dijk [1] presented the group-local-balance concept as the basis for the analysis of product form batch routing queueing networks. Here, we translate the definitions and results of Boucherie and Van Dijk into Petri net terminology. It is showed that group-local-balance allows us to calculate the steady state distribution of an $\mathcal{SPN}$. This will serve as the foundation to investigate the structural Petri net implications of group-local-balance in Section 5.

Inserting (2) into the global balance equations (4) yields that a distribution π at $\mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)$ is the unique equilibrium distribution if for all $\boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)$:

$$\sum_{\{\boldsymbol{n},\, t,\, t' \in T : \boldsymbol{n}+\boldsymbol{I}(t)=\boldsymbol{n}+\boldsymbol{O}(t')=\boldsymbol{m}\}} \left\{\pi(\boldsymbol{m})q(\boldsymbol{I}(t), \boldsymbol{O}(t); \boldsymbol{n}) - \pi(\boldsymbol{n}+\boldsymbol{I}(t'))q(\boldsymbol{I}(t'), \boldsymbol{O}(t'); \boldsymbol{n})\right\} = 0. \tag{5}$$

A distribution satisfying these equations for fixed combinations of residual marking $\boldsymbol{n}$ and input bag $\boldsymbol{I}(t)$ is the unique equilibrium distribution. This form of *local balance* is introduced in [1] as *group-local-balance*.

Definition 4.1 (*Group-Local-Balance*)**.** A measure ϕ satisfies *group-local-balance* (GLB) if, for all fixed residual markings $\boldsymbol{n}$ and for all fixed input bags $\boldsymbol{I}(t)$, such that $\boldsymbol{n} + \boldsymbol{I}(t) \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)$:

$$\sum_{\{t' \in T:\, \boldsymbol{I}(t')=\boldsymbol{I}(t)\}} \phi(\boldsymbol{n}+\boldsymbol{I}(t'))q(\boldsymbol{I}(t'), \boldsymbol{O}(t'); \boldsymbol{n}) = \sum_{\{t' \in T:\, \boldsymbol{O}(t')=\boldsymbol{I}(t)\}} \phi(\boldsymbol{n}+\boldsymbol{I}(t'))q(\boldsymbol{I}(t'), \boldsymbol{O}(t'); \boldsymbol{n}). \tag{6}$$

Summation of the group-local-balance equations over all $\boldsymbol{n}, \boldsymbol{I}(t)$ such that $\boldsymbol{n} + \boldsymbol{I}(t) = \boldsymbol{m}$ gives the global balance equations. The Markov chain **X** has the GLB-property if the equilibrium distribution π satisfies (6).

GLB expresses that under a given residual marking the rate at which input bag $\boldsymbol{I}(t)$ is absorbed is balanced by the rate at which exactly $\boldsymbol{I}(t)$ is formed. Obviously, the group-local-balance equations (6) are generally more restrictive than the global balance equations (5). GLB requires that $\boldsymbol{I}(t)$ is an output bag of a transition t'. Also, GLB requires that the output bag of a transition t, is an input bag for another transition t'.

Lemma 4.2. *If the Markov chain* **X** *of an* $\mathcal{SPN}$ *satisfies GLB, then* $\mathcal{R}(T)$ *is a closed set.*

Proof. From the group-local-balance equations (6) it is seen that if $\boldsymbol{I}(t)$ is an input bag of a transition that is enabled in an arbitrary marking $\boldsymbol{m}$, then, if GLB holds, $\boldsymbol{I}(t)$ must also be an output bag of a transition t'. If there is no such transition t', the left hand side of (6) would be positive while the right hand side is zero, which contradicts GLB.

Similarly, if $\boldsymbol{O}(t')$ is an output bag of a transition that is enabled in an arbitrary marking $\boldsymbol{m}$, then, if GLB holds, $\boldsymbol{O}(t')$ must also be an input bag of a transition t. If there is no such transition t, the right hand side of (6) would be positive while the left hand side is zero, which contradicts GLB. □

Following [1], let us introduce the concepts of the *local state space* and the *local irreducible sets*. For a fixed $\boldsymbol{n}$ the local state space $V(\boldsymbol{n})$ is the state space of the Markov chain with transition rates $q(\boldsymbol{I}(t), \boldsymbol{O}(t); \boldsymbol{n})$ restricted to $\mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)$. So $V(\boldsymbol{n})$

consists of all states $\boldsymbol{n}+\boldsymbol{I}(t)$ and $\boldsymbol{n}+\boldsymbol{O}(t)$, for which $q(\boldsymbol{I}(t),\boldsymbol{O}(t);\boldsymbol{n})>0$. Let $V_i(\boldsymbol{n})$ denote the local irreducible sets in $V(\boldsymbol{n})$ with respect to the Markov chain with transition rates $q(\boldsymbol{I}(t),\boldsymbol{O}(t);\boldsymbol{n})$ for fixed $\boldsymbol{n}$. A state $\boldsymbol{m}$ may be element of different local state spaces $V(\boldsymbol{n})$, so that transitions from one local state space to another are possible. It is not uncommon that $V(\boldsymbol{n})$ consists of multiple local irreducible sets $V_i(\boldsymbol{n})$, $i\in\{1,\dots,k(\boldsymbol{n})\}$, which is shown in [1] via an example. In addition, it is shown that if a Markov chain satisfies GLB, the local state spaces $V(\boldsymbol{n})$ consist only of irreducible sets, which guarantees:

$$V(\boldsymbol{n})=\bigcup_{i=1}^{k(\boldsymbol{n})}V_i(\boldsymbol{n}).$$

Now, it follows that, if the Markov chain $\mathbf{X}$ of an $\mathcal{SPN}$ net has the GLB property, then for any fixed $\boldsymbol{n}$ for which $V(\boldsymbol{n})\neq\emptyset$ and $i\in\{1,\dots,k(\boldsymbol{n})\}$ the following set of equations has a unique positive solution up to a multiplicative constant:

$$x(\boldsymbol{I}(t);\boldsymbol{n})\sum_{t'\in T}q(\boldsymbol{I}(t),\boldsymbol{I}(t');\boldsymbol{n})=\sum_{t'\in T}x(\boldsymbol{I}(t');\boldsymbol{n})q(\boldsymbol{I}(t'),\boldsymbol{I}(t);\boldsymbol{n}),\quad \boldsymbol{n}+\boldsymbol{I}(t)\in V_i(\boldsymbol{n}). \tag{7}$$

These local solutions per communicating class can be used to characterize the equilibrium distribution π, by translating these solutions to the global state space. To this end, an additional process with transition rate $\bar{q}$ is defined. For any Markov chain $\mathbf{X}$ at $\mathcal{M}(\mathcal{SPN},\boldsymbol{m}_0)$ that satisfies the Eqs. (7) the $\bar{q}$-process can be defined. However, such a Markov chain does not necessarily satisfy the GLB property. To point out in when this relation does hold, [1] introduces the concept of strong reversibility.

Definition 4.3 ($\bar{q}$-*Process*)**.** If for any fixed $\boldsymbol{n}$ for which $V(\boldsymbol{n})\neq\emptyset$, for $i\in\{1,\dots,k(\boldsymbol{n})\}$ the system (7) has a unique positive solution $\{x(\boldsymbol{I}(t);\boldsymbol{n})\mid \boldsymbol{n}+\boldsymbol{I}(t)\in V_i(\boldsymbol{n})\}$ up to a multiplicative constant, then the following process, called the $\bar{q}$-*process*, can be defined.
For any $\boldsymbol{n}$, $i\in\{1,\dots,k(\boldsymbol{n})\}$, and $\boldsymbol{n}+\boldsymbol{I}(t),\boldsymbol{n}+\boldsymbol{I}(t')\in V_i(\boldsymbol{n})$, for which $q(\boldsymbol{I}(t),\boldsymbol{I}(t');\boldsymbol{n})>0$ or $q(\boldsymbol{I}(t'),\boldsymbol{I}(t);\boldsymbol{n})>0$

$$\frac{\bar{q}(\boldsymbol{I}(t),\boldsymbol{I}(t');\boldsymbol{n})}{\bar{q}(\boldsymbol{I}(t'),\boldsymbol{I}(t);\boldsymbol{n})}=\frac{x(\boldsymbol{I}(t'),\boldsymbol{n})}{x(\boldsymbol{I}(t),\boldsymbol{n})}, \tag{8}$$

and otherwise

$$\bar{q}(\boldsymbol{I}(t),\boldsymbol{I}(t');\boldsymbol{n})=0.$$

Definition 4.4 (*Strong Reversibility*)**.** The $\bar{q}$-process is *strongly reversible* at $\mathcal{M}(\mathcal{SPN},\boldsymbol{m}_0)$ if for all $\boldsymbol{n}$ for which $V(\boldsymbol{n})\neq\emptyset$ and $i\in\{1,\dots,k(\boldsymbol{n})\}$, the equilibrium distribution $\bar{\pi}$ satisfies

$$\bar{\pi}(\boldsymbol{n}+\boldsymbol{I}(t))\bar{q}(\boldsymbol{I}(t),\boldsymbol{I}(t');\boldsymbol{n})=\bar{\pi}(\boldsymbol{n}+\boldsymbol{I}(t'))\bar{q}(\boldsymbol{I}(t'),\boldsymbol{I}(t);\boldsymbol{n}),\quad \boldsymbol{n}+\boldsymbol{I}(t),\boldsymbol{n}+\boldsymbol{I}(t')\in V_i(\boldsymbol{n}). \tag{9}$$

Theorem 4.5 ([1])**.** *The equilibrium distribution of a Markov chain* $\mathbf{X}$ *at* $\mathcal{M}(\mathcal{SPN},\boldsymbol{m}_0)$ *satisfies GLB if and only if the* $\bar{q}$*-process is defined and is strongly reversible at* $\mathcal{M}(\mathcal{SPN},\boldsymbol{m}_0)$*. Moreover, with* $\bar{\pi}$ *its equilibrium distribution, for all* $\boldsymbol{m}\in\mathcal{M}(\mathcal{SPN},\boldsymbol{m}_0)$*:* $\pi(\boldsymbol{m})=\bar{\pi}(\boldsymbol{m})$*. Finally,* π *satisfies GLB if and only if for an arbitrary reference state* $\boldsymbol{m}_0$*, and all* $\boldsymbol{m}\in\mathcal{M}(\mathcal{SPN},\boldsymbol{m}_0)$

$$\pi(\boldsymbol{m})=\pi(\boldsymbol{m}_0)\prod_{k=0}^{s}\frac{\bar{q}(\boldsymbol{I}(t_k),\boldsymbol{I}(t'_k);\boldsymbol{n}_k)}{\bar{q}(\boldsymbol{I}(t'_k),\boldsymbol{I}(t_k);\boldsymbol{n}_k)}, \tag{10}$$

for all firing sequences of the form (such that the denominator of (10) *is positive)*

$$\begin{aligned}\boldsymbol{m}_0&=\boldsymbol{n}_0+I(t_0)\to\boldsymbol{n}_0+I(t'_0)=\boldsymbol{n}_1+\boldsymbol{I}(t_1)\to\boldsymbol{n}_1+I(t'_1)=\cdots\to\cdots\\&=\boldsymbol{n}_s+\boldsymbol{I}(t_s)\to\boldsymbol{n}_s+\boldsymbol{I}(t'_s)=\boldsymbol{n}_{s+1}+\boldsymbol{I}(t_{s+1})=\boldsymbol{m}.\end{aligned} \tag{11}$$

Corollary 4.6. *The equilibrium distribution* π *satisfies GLB if and only if for* $\boldsymbol{n},\boldsymbol{I}(t)$ *and* $\boldsymbol{I}(t')$ *such that* $\boldsymbol{n}+\boldsymbol{I}(t),\boldsymbol{n}+\boldsymbol{I}(t')\in\mathcal{M}(\mathcal{SPN},\boldsymbol{m}_0)$*, for which* $q(\boldsymbol{I}(t),\boldsymbol{I}(t');\boldsymbol{n})>0$

$$\frac{\pi(\boldsymbol{n}+\boldsymbol{I}(t))}{\pi(\boldsymbol{n}+\boldsymbol{I}(t'))}=\frac{x(\boldsymbol{I}(t);\boldsymbol{n})}{x(\boldsymbol{I}(t');\boldsymbol{n})}. \tag{12}$$

Corollary 4.6 provides the relation between the equilibrium distribution π and the local solutions $x(\boldsymbol{n};\boldsymbol{I}(t))$. Note that (12) is a condition for $\boldsymbol{n},\boldsymbol{I}(t)$ and $\boldsymbol{I}(t')$ such that $\boldsymbol{n}+\boldsymbol{I}(t)$ and $\boldsymbol{n}+\boldsymbol{I}(t')$ are within a single local irreducible set $V_i(\boldsymbol{n})$, and it relates the ratio $x(\boldsymbol{I}(t);\boldsymbol{n})/x(\boldsymbol{I}(t');\boldsymbol{n})$ to the ratio $\pi(\boldsymbol{n}+\boldsymbol{I}(t))/\pi(\boldsymbol{n}+\boldsymbol{I}(t'))$. For a firing sequence from marking $\boldsymbol{m}$ to $\boldsymbol{m}'$ that traverses multiple local irreducible sets $V_j(\boldsymbol{n}_j)$, $j=1,\dots,s$, for each transition in this firing sequence (12) is imposed. The latter implies that if there exist multiple firing sequences from $\boldsymbol{m}$ to $\boldsymbol{m}'$ additional restrictions on the ratios $\bar{q}(\boldsymbol{I}(t_k),\boldsymbol{I}(t'_k);\boldsymbol{n}_k)/\bar{q}(\boldsymbol{I}(t'_k),\boldsymbol{I}(t_k);\boldsymbol{n}_k)$ in (10) are implied to obtain consistency in the ratio $\pi(\boldsymbol{m})/\pi(\boldsymbol{m}')$ in (10). In Section 5, the impact of these conditions at the Petri net level will be studied in detail.

This section has described results on the Markov chain level. Reversibility of the $\bar{q}$-process provides a way to 'build' the solution $\bar{\pi}(\boldsymbol{m})$, following any path to $\boldsymbol{m}$ from the initial marking $\boldsymbol{m}_0$. To understand and exploit the results on the Petri net level, in the next section, we will investigate the translation of these characteristics to the stochastic Petri nets and in particular present the implications for the stochastic Petri net structure. The key ingredients of that analysis will be the local irreducible sets and ratio condition of Corollary 4.6.

5. The stochastic Petri net and group-local-balance

In this section, we will show that stochastic Petri nets with marking-independent firing rates for which group-local-balance holds have a steady state distribution that is a product over the places of the network. Therefore, we are interested in the necessary and sufficient structural properties of Petri nets that are required to obtain group-local-balance.

The first structural condition was already presented in Lemma 4.2: the set of input and output bags $\mathcal{R}(T)$ is a closed set. In Section 5.1, this condition is extended to 'each transition has to be covered by a minimal closed support T-invariant', i.e., the $\mathcal{SPN}$ has to be an $\mathcal{S}\Pi$-net. To this end, it is shown that the local irreducible sets defined in Section 4 are sets of minimal closed support T-invariants. Section 5.2 shows that an $\mathcal{S}\Pi$-net does not necessarily have a product form solution. The additional relation between states can be found by tracing closed support T-invariants. This observation forms the key to formulate the additional requirements to obtain a characterization of product form stochastic Petri nets. Section 5.3 identifies the structural characteristics of $\mathcal{S}\Pi$-nets for which a product form equilibrium distribution can be concluded without considering the numerical values of the transition rates and nets for which these values have to satisfy specific conditions. This subsection concludes with an algorithm to verify whether a specific $\mathcal{SPN}$ possesses a product form equilibrium distribution, and if so, to construct this product form. Section 5.4 provides several insightful examples of product form $\mathcal{SPN}$s.

The Markov chain $\mathbf{X}$ on state space $\mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)$ modelling the Petri net with marking-independent firing rates has transition rates

$$q(\boldsymbol{I}(t), \boldsymbol{O}(t); \boldsymbol{m} - \boldsymbol{I}(t)) = \mu(t)p(\boldsymbol{I}(t), \boldsymbol{O}(t))\mathbb{1}(m(n) \geq I_n(t),\ n = 1, \dots, N). \tag{13}$$

Observe that for the nets with transition rates (13) the condition $m(n) \geq I_n(t),\ n = 1, \dots, N$, is necessary and sufficient for transition t to be enabled in marking $\boldsymbol{m}$.

5.1. The routing chain and minimal closed support T-invariants

Under marking independent transition rates the Eq. (7) are equivalent for all $\boldsymbol{n} + \boldsymbol{I}(t) \in V_i(\boldsymbol{n})$, which can be seen from inserting (13) in (7), for all $\boldsymbol{n} + \boldsymbol{I}(t) \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)$:

$$\begin{aligned} &x(\boldsymbol{I}(t); \boldsymbol{n}) \sum_{t' \in T} \mu(t)p(\boldsymbol{I}(t), \boldsymbol{I}(t'))\mathbb{1}(m(n) \geq I_n(t),\ n = 1, \dots, N) \\ &\quad = \sum_{t' \in T} x(\boldsymbol{I}(t'); \boldsymbol{n})\mu(t')p(\boldsymbol{I}(t'), \boldsymbol{I}(t))\mathbb{1}(m(n) \geq I_n(t'),\ n = 1, \dots, N). \end{aligned} \tag{14}$$

Considering (14) for all residual markings $\boldsymbol{n}$ and input bags $\boldsymbol{I}(t)$ and local irreducible sets $V_i(\boldsymbol{n})$ such that $\boldsymbol{n} + \boldsymbol{I}(t) \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)$, exposes that the set of equations of the form (14) only differ in the local irreducible sets $V_i(\boldsymbol{n})$ ($i \in 1, \dots, k(\boldsymbol{n})$) being enabled or disabled. Therefore, if the equilibrium distribution π satisfies GLB, then for each $\boldsymbol{n} + \boldsymbol{I}(t) \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)$ Eq. (14) has a unique positive solution $x(\boldsymbol{I}(t); \boldsymbol{n}) := y(\boldsymbol{I}(t))$.

This implies that a positive solution can be found to the global balance equations of a Markov chain which is defined by Henderson et al. as the *routing chain* [7]. Define the Markov chain $\mathbf{Y} = (Y(t),\ t \geq 0)$ on finite state space $S = \{\boldsymbol{I}(t), t \in T\}$ with transition rates $q_{\mathbf{Y}}(\boldsymbol{I}(t), \boldsymbol{I}(t')) = \mu(t)p(\boldsymbol{I}(t), \boldsymbol{I}(t'))$. The global balance equations for $\mathbf{Y}$ are, for $t \in T$,

$$\sum_{t' \in T} \{y(\boldsymbol{I}(t))\mu(t)p(\boldsymbol{I}(t), \boldsymbol{I}(t')) - y(\boldsymbol{I}(t'))\mu(t')p(\boldsymbol{I}(t'), \boldsymbol{I}(t))\} = 0. \tag{15}$$

These global balance equations for Markov chain $\mathbf{Y}$ are state independent versions of the group-local-balance equations (7). The definition of the routing chain relies on the condition that $\mathcal{R}(T)$ is a closed set, so that for all $t \in T$, $\boldsymbol{I}(t) = \boldsymbol{O}(t')$ for some t' and therefore $p(\boldsymbol{I}(t), \boldsymbol{I}(t')) = p(\boldsymbol{I}(t), \boldsymbol{O}(t))$ is well-defined.

Observe that GLB cannot hold if no positive solution for the routing chain can be found. Therefore, in the following, we first investigate the structural conditions under which a positive solution for the routing chain exists. The condition that $\mathcal{R}(T)$ is a closed set is necessary for a solution $\mathbf{Y}$ to exist. This condition is exactly the condition that Henderson et al. impose in Corollary 1 of [7] on the $\mathcal{SPN}$s they consider. In their further analysis, they assume a positive solution for the routing chain exists; an assumption which is usually made in the literature. The following example, taken from [2], shows that the closedness of $\mathcal{R}(T)$ is not a sufficient condition for GLB to hold.

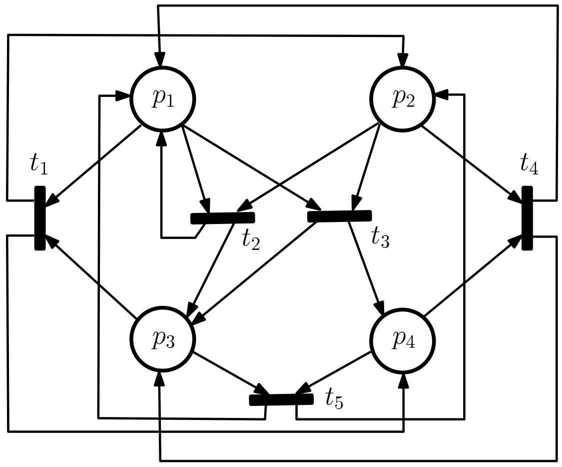

Fig. 1. Petri net for which $\mathcal{R}(T)$ is a closed set.

Example 5.1. Consider the $\mathcal{SPN}$ depicted in Fig. 1. $\boldsymbol{I}(t_1) = (1, 0, 1, 0), \boldsymbol{I}(t_2) = (1, 1, 0, 0), \boldsymbol{I}(t_3) = (1, 1, 0, 0), \boldsymbol{I}(t_4) = (0, 1, 0, 1), \boldsymbol{I}(t_5) = (0, 0, 1, 1)$ and $\boldsymbol{O}(t_1) = (0, 1, 0, 1), \boldsymbol{O}(t_2) = (1, 0, 1, 0), \boldsymbol{O}(t_3) = (0, 0, 1, 1), \boldsymbol{O}(t_4) = (1, 0, 1, 0), \boldsymbol{O}(t_5) = (1, 1, 0, 0)$, which shows that $\mathcal{R}(T)$ is a closed set. Since $\boldsymbol{I}(t_2) = \boldsymbol{I}(t_3)$, the state space of the routing chain is

$$S = \{\boldsymbol{I}(t_1), \boldsymbol{I}(t_2), \boldsymbol{I}(t_4), \boldsymbol{I}(t_5)\}$$

and the solution for the routing chain (15) is (up to a multiplicative constant)

$$y(\boldsymbol{I}(t_1)) = 1/\mu_1, y(\boldsymbol{I}(t_4)) = 1/\mu_4, y(\boldsymbol{I}(t_2)) = y(\boldsymbol{I}(t_3)) = y(\boldsymbol{I}(t_5)) = 0$$

which shows that closedness of $\mathcal{R}(T)$ is not sufficient for a *positive* solution for the routing chain. $\square$

In Example 5.1, **Y** does not partition in irreducible classes, since $S_1 = \{\boldsymbol{I}(t_2), \boldsymbol{I}(t_5)\}$ is a transient class. Boucherie and Sereno [3] present a necessary and sufficient condition: for an $\mathcal{SPN}$ a positive solution for the routing chain exists if and only if all transitions $t \in T$ are covered by minimal closed support T-invariants, i.e., it is an $\mathcal{S}\Pi$-net. They prove this by showing that only in this case does the state space of the Markov chain **Y** partition into irreducible sets.

Obviously, the condition of the $\mathcal{SPN}$ to be an $\mathcal{S}\Pi$-net implies that $\mathcal{R}(T)$ is a closed set. In addition to the closedness condition, in an $\mathcal{S}\Pi$-net transitions t, s with $\boldsymbol{O}(t) = \boldsymbol{I}(s)$ are elements of the support of a single minimal closed support T-invariant. Returning to Example 5.1 illustrates this essential extension.

Example 5.1 revisited. From the incidence matrix

$$\boldsymbol{A} = \begin{pmatrix} -1 & 0 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 \\ -1 & 1 & 1 & 1 & -1 \\ 1 & 0 & 1 & -1 & -1 \end{pmatrix}$$

we obtain that this net has 3 minimal support T-invariants: $\boldsymbol{x}^1 = (10010), \boldsymbol{x}^2 = (00101), \boldsymbol{x}^3 = (12001)$, of which $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ have closed support, but $\boldsymbol{x}^3$ does not have closed support. Since transition t_2 is contained in $\|\boldsymbol{x}^3\|$ only, t_2 is not covered by a minimal closed support T-invariant, which contradicts the definition of an $\mathcal{S}\Pi$-net. This explains why no positive solution for the routing chain exists. $\square$

Observe that the essential characteristic of an $\mathcal{S}\Pi$-net is that all transitions are contained in a *closed* support T-invariant. The condition that all transitions are covered by minimal support T-invariants (closed or not closed) is a natural assumption if one is interested in the equilibrium or stationary distribution of a stochastic Petri net (see Section 3.2).

To obtain the partitioning of **Y** into irreducible classes, Boucherie and Sereno [3] provide a decomposition of the transitions of the Petri net into equivalence classes based on the characterization of minimal closed support T-invariants that are connected by having an input bag in common. By this equivalence class decomposition, the global balance equations of the routing chain (15) decompose into disjoint sets of equations, one set of equations for each equivalence class of connected T-invariants. The equivalence relation is defined by analogy with a similar equivalence relation introduced in [11] for cyclic state machines.

Assume that the minimal support T-invariants $\boldsymbol{x}^1, \ldots, \boldsymbol{x}^h$ are numbered such that $C\ell T \stackrel{\text{def}}{=} \{\boldsymbol{x}^1, \ldots, \boldsymbol{x}^k\}$ is the set of minimal closed support T-invariants ($k \leq h$).

Definition 5.2 (*Common Input Bag Relation [3]*). Let $\boldsymbol{x}, \boldsymbol{x}' \in C\ell T$. The T-invariants $\boldsymbol{x}, \boldsymbol{x}'$ are in common input bag relation (notation: $\boldsymbol{x}\, CI\, \boldsymbol{x}'$) if there exist $t \in \|\boldsymbol{x}\|$, $t' \in \|\boldsymbol{x}'\|$ such that $\boldsymbol{I}(t) = \boldsymbol{I}(t')$. The relation CI^* is the transitive closure of CI.[1]

Definition 5.3 (*Common Input Bag Class [3]*). The common input bag class $C\ell(\boldsymbol{x})$ is the equivalence class of $\boldsymbol{x} \in C\ell T$, that is $CI\,(\boldsymbol{x}) = \{\boldsymbol{x}' | \boldsymbol{x}\, CI^*\, \boldsymbol{x}'\}$.

The common input bag relation characterizes the irreducible sets of the routing chain. The equivalence classes partition $C\ell T$: each $\boldsymbol{x} \in C\ell T$ belongs to exactly one equivalence class. Let $\boldsymbol{x} \in C\ell T$ with equivalence class $CI(\boldsymbol{x})$. Define $S(\boldsymbol{x}) \subset S$, the input bags corresponding to $CI(\boldsymbol{x})$, as

$$S(\boldsymbol{x}) = \{\boldsymbol{I}(t) \mid \exists\, \boldsymbol{x}' \in CI(\boldsymbol{x}) \text{ such that } x'(t) > 0\}.$$

The partitioning of $C\ell T$ into equivalence classes $\{CI(\boldsymbol{x})\}_{\boldsymbol{x}\in C\ell T}$ induces a partition $\{S(\boldsymbol{x})\}_{\boldsymbol{x}\in C\ell T}$ of S into irreducible sets of the Markov chain $\mathbf{Y}$ if and only if all transitions are covered by minimal closed support T-invariants [3]. To this end, note that first $S(\boldsymbol{x}') = S(\boldsymbol{x})$ if $CI(\boldsymbol{x}') = CI(\boldsymbol{x})$, and $S(\boldsymbol{x}') \cap S(\boldsymbol{x}) = \emptyset$ if $CI(\boldsymbol{x}') \cap CI(\boldsymbol{x}) = \emptyset$. Second, by the definition of $S(x)$, the input bags $\boldsymbol{I}(t)$ in a set $S(\boldsymbol{x})$ are communicating states. Third, when every transition is covered by a minimal closed support T-invariant, each transition is contained in a set $S(\boldsymbol{x}) \in S$.

Theorem 5.4 (*[3]*). *For the stochastic Petri net $\mathcal{SPN}$ a positive solution for the routing chain (15) exists if and only if $\mathcal{SPN}$ is an $\mathcal{S}\Pi$-net.*

In the next corollary, Theorem 5.4 is expanded to the reachability set level. A proof is omitted, as it follows exactly the lines as the proof of Theorem 5.4.

Corollary 5.5. *For an $\mathcal{S}\Pi$-net, there is a one-to-one mapping between the partitioning of S into irreducible sets $\{S(\boldsymbol{x})\}_{\boldsymbol{x}\in CIT}$ that is induced by the partitioning of CIT into equivalence classes $\{CI(\boldsymbol{x})\}_{\boldsymbol{x}\in CIT}$ and the partitioning of local state spaces $V(\boldsymbol{n})$ into the local irreducible sets $V_i(\boldsymbol{n})$.*

The next theorem shows that an $\mathcal{S}\Pi$-net not only guarantees a positive solution for the global balance equations for the routing chain (15), but for live initial markings also for the global balance equations (4) for the Markov chain $\mathbf{X}$ of the stochastic Petri net.

Theorem 5.6 (*[47]*). *The marked Π-net $\mathcal{PN} = (P, T, I, O, \boldsymbol{m}_0)$ underlying a marked $\mathcal{S}\Pi$-net $(\mathcal{SPN}, \boldsymbol{m}_0)$ has home state $\boldsymbol{m}_0$ and is structurally live.*

If the net is covered by P-invariants, it is structurally bounded (Result 3.26). Positive recurrence then follows and thus a positive solution solution summing to unity exists. Furthermore, Theorem 5.6 shows that there exists an initial marking for which the net is live. The proof indicates that if each common input bag is initially marked, the net is live. If it is not the case that each common input bag is initially marked, checking liveness may be cumbersome (see [6]).

Remark 5.7. When the equilibrium behaviour of stochastic Petri nets is of interest, a natural condition is that all transitions are covered by minimal support T-invariants. For bounded nets this condition is necessary for liveness (see Result 3.26). If this condition is not satisfied, there exists a transition, say t_0, that is enabled in a reachable marking $\boldsymbol{m}$, and $\boldsymbol{x}(t_0) = 0$ for all minimal support T-invariants (if t_0 is never enabled, then we can delete t_0 from T). Let t_0 fire in marking $\boldsymbol{m}$. Then there exists no firing sequence from $\boldsymbol{m} - \boldsymbol{I}(t_0) + \boldsymbol{O}(t_0)$ back to $\boldsymbol{m}$ (otherwise t_0 would be contained in a T-invariant). Thus $\boldsymbol{m}$ is a transient state and does not appear in the equilibrium description of the stochastic Petri net. As a consequence, both $\boldsymbol{m}$ and t_0 can be deleted from the equilibrium description of the Petri net.

Observe the Petri nets in Fig. 2(a)–(c), which are not $\mathcal{S}\Pi$-nets. As can be seen from the Petri net of Fig. 2(b), the condition that all transitions are covered by T-invariants is necessary, but not sufficient for liveness of the Petri net. For liveness additional conditions are required.

An $\mathcal{S}\Pi$-net does guarantee structural liveness of the Petri net. As can be seen from Fig. 2(a) and (c), the condition of an $\mathcal{SPN}$ being an $\mathcal{S}\Pi$-net is sufficient, but not necessary. Comparison of Fig. 2(b) and (c), however, shows that the property of liveness is cumbersome since Petri nets that are almost identical may show completely different behaviour. Therefore, a characterization of liveness for $\mathcal{S}\Pi$-nets is of interest on its own. □

[1] The transitive closure of a relation is defined as follows: if $\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{x}'' \in C\ell T$, and $\boldsymbol{x}\; CI\; \boldsymbol{x}'$, $\boldsymbol{x}'\; CI\; \boldsymbol{x}''$, then we define $\boldsymbol{x}\; CI^*\; \boldsymbol{x}'$, $\boldsymbol{x}'\; CI^*\; \boldsymbol{x}''$, and $\boldsymbol{x}\; CI^*\; \boldsymbol{x}''$. This reflects the property that we can go from $\boldsymbol{x}$ to $\boldsymbol{x}''$ via $\boldsymbol{x}'$. This makes the common input bag relation CI^* an equivalence relation on CIT.

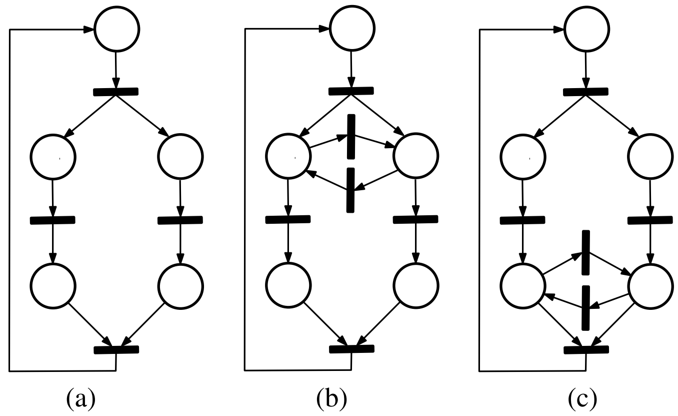

Fig. 2. The illustrative Petri nets of Remark 5.7.

5.2. Group-local-balance and product form

In Section 5.1, we have first seen that if GLB holds, a positive solution to the routing chain (15) and thus to the local balance equations (7) is guaranteed. Second, a positive solution to the routing chain exists if and only if the stochastic Petri net is an $\mathcal{S}\Pi$-net. In this section, we investigate the equivalence of GLB and a product form solution over the places of the Petri net. As can be seen from Corollary 4.6, a positive solution to the routing chain does not yet imply GLB and thus a product from solution. The additional condition to be satisfied is also formulated in this section, of which the structural implications are discussed in Section 5.3.

From Corollary 4.6 we obtain the key idea that under GLB the marking independent solution $y(\cdot)$ of the routing chain can be translated into a marking dependent solution with the same properties. This is reflected by the ratio condition (12). Also, from the analysis in Section 5.1 we know that $x(\boldsymbol{I}(t); \boldsymbol{n}) = y(\boldsymbol{I}(t))$ is a solution to the local balance equations (7). For state independent firing rates this leads to the following corollary, which is similar to Theorem 1 of [9].

Corollary 5.8. *The equilibrium distribution π of an $\mathcal{SPN}$ with state independent firing rates satisfies GLB if and only if it is an $\mathcal{S}\Pi$-net and a function $\pi_y : \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0) \to \mathbb{R}^+$ exists such that for all $\boldsymbol{n} + \boldsymbol{I}(t) \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)$, $t, t' \in T$ with $p(\boldsymbol{I}(t), \boldsymbol{I}(t')) > 0$,*

$$\frac{\pi_y(\boldsymbol{n} + \boldsymbol{I}(t))}{\pi_y(\boldsymbol{n} + \boldsymbol{I}(t'))} = \frac{y(\boldsymbol{I}(t))}{y(\boldsymbol{I}(t'))} \tag{16}$$

and $\pi(\boldsymbol{m}) = B\pi_y(\boldsymbol{m})$, $\boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)$ with $B^{-1} = \sum_{\boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)} \pi_y(\boldsymbol{m})$ is the unique equilibrium distribution of the Markov chain describing $\mathcal{SPN}$.

Note that Condition (16) is a condition on y and *not* on the structure of the Petri net. If a solution $y(\cdot)$ for the routing chain is found, a function $\pi_y(\cdot)$ satisfying (16) cannot always be found without additional assumptions on the $\mathcal{SPN}$. Theorem 5.12 provides a product form solution for π_y under additional conditions on the Petri net. To formulate and understand the structural characterization of the $\mathcal{SPN}$s guaranteeing the ratio condition (16), first Lemmas 5.9 and 5.11 and Corollary 5.10 are presented.

Corollary 5.8 implies that the equilibrium distribution π of an $\mathcal{S}\Pi$-net with state independent firing rates satisfies GLB if and only if for an arbitrary reference state $\boldsymbol{m}_0$, and all $\boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)$

$$\pi(\boldsymbol{m}) = \pi(\boldsymbol{m}_0) \prod_{k=0}^{s} \frac{y(\boldsymbol{I}(t_k))}{y(\boldsymbol{I}(t'_k))}, \tag{17}$$

for all firing sequences of the form

$$\begin{aligned} \boldsymbol{m}_0 &= \boldsymbol{n}_0 + I(t_0) \to \boldsymbol{n}_0 + I(t'_0) = \boldsymbol{n}_1 + \boldsymbol{I}(t_1) \to \boldsymbol{n}_1 + I(t'_1) = \cdots \to \cdots \\ &= \boldsymbol{n}_s + \boldsymbol{I}(t_s) \to \boldsymbol{n}_s + \boldsymbol{I}(t'_s) = \boldsymbol{n}_{s+1} + \boldsymbol{I}(t_{s+1}) = \boldsymbol{m}. \end{aligned}$$

This is seen by first observing that for state independent firing rates $x(\boldsymbol{I}(t); \boldsymbol{n}) = y(\boldsymbol{I}(t))$ is a solution of the local balance equations (7) and then substituting (8) in (10) of Theorem 4.5. Applying (17) to a cyclic firing sequence, so for $\boldsymbol{m}_0 = \boldsymbol{m}$, yields the following lemma.

Lemma 5.9. *The equilibrium distribution π of an $\mathcal{S}\Pi$-net with state independent firing rates (13) satisfies GLB if and only if for each T-invariant $\boldsymbol{x} = (x_1, \ldots, x_M)$*

$$\prod_{t=1}^{M} \left(\frac{y(\boldsymbol{I}(t))}{y(\boldsymbol{O}(t))} \right)^{x_t} = 1. \tag{18}$$

In Section 5.3, we will investigate which structural Petri net conditions Lemma 5.9 imposes. First, we will use Lemma 5.9 in showing that a solution π_y satisfying the ratio condition (16) must be a product form over the places of the network.

Following Coleman et al. [24], we introduce the row vector $\boldsymbol{C}(y)$, defined as $\boldsymbol{C}(y)_t = \log(y(\boldsymbol{I}(t))/y(\boldsymbol{O}(t)))$. As $y(\cdot)$ is determined up to a multiplicative constant, and $\boldsymbol{C}(y)$ is determined by the ratios of y's, the vector $\boldsymbol{C}(y)$ is unique, so that is can safely be denoted by $\boldsymbol{C}$. Taking logarithms on both sides in Eq. (18), Lemma 5.9 can now be reformulated as follows.

Corollary 5.10. *The equilibrium distribution π of an $\mathcal{S}\Pi$-net with state independent firing rates (13) satisfies GLB if and only if $\boldsymbol{Cx} = 0$ for every T-invariant $\boldsymbol{x}$.*

Coleman [4] presents the following equivalent statements.

Lemma 5.11 *([4]). The following statements are equivalent*

(i) *$\boldsymbol{Cx} = 0$ for each T-invariant $\boldsymbol{x}$*
(ii) $\mathrm{Rank}[\boldsymbol{A}] = \mathrm{Rank}[\boldsymbol{A}|\boldsymbol{C}]$, *where $[\boldsymbol{A}|\boldsymbol{C}]$ is the matrix augmented with the row vector $\boldsymbol{C}$.*
(iii) *Equation $\boldsymbol{zA} = \boldsymbol{C}$ has a solution $\boldsymbol{z}$.*

The following key-result identifies the equivalence between GLB and a product form solution over the places of the network. The solution $\boldsymbol{z}$ of the condition (iii) is used to express the product form. Section 5.3 investigates the intuition behind this theorem and provides an explanation in terms of T-invariants.

Theorem 5.12. *Consider an $\mathcal{SPN}$ with state independent firing rates (13). The equilibrium distribution π satisfies GLB if and only if the $\mathcal{SPN}$ is an $\mathcal{S}\Pi$-net, $\boldsymbol{zA} = \boldsymbol{C}$ has a solution and π is a product form over the places of the network*

$$\pi_y(\boldsymbol{m}) = \prod_{p=1}^{N} (f_p)^{m_p}, \quad \boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0) \tag{19}$$

where $f_p = e^{-z_p}$ and $\pi(\boldsymbol{m}) = B\pi_y(\boldsymbol{m})$ with $B^{-1} = \sum_{\boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)} \pi_y(\boldsymbol{m})$.

Proof. Under GLB, by Corollary 5.10, $\boldsymbol{Cx} = 0$ for each minimal support T-invariant. This implies by Lemma 5.11 that the equation $\boldsymbol{zA} = \boldsymbol{C}$ has a solution. Thus we obtain for each transition $t \in T$

$$\sum_{p=1}^{N} z_p A(p, t) = \log \left(\frac{y(\boldsymbol{I}(t))}{y(\boldsymbol{O}(t))} \right). \tag{20}$$

Taking exponentials gives

$$\prod_{p=1}^{N} e^{z_p A(p,t)} = \left(\frac{y(\boldsymbol{I}(t))}{y(\boldsymbol{O}(t))} \right).$$

By Corollary 5.8, we then have for all $\boldsymbol{n} + \boldsymbol{I}(t) \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)$, $t, t' \in T$ with $p(\boldsymbol{I}(t), \boldsymbol{I}(t')) > 0$

$$\frac{\pi_y(\boldsymbol{n} + \boldsymbol{I}(t))}{\pi_y(\boldsymbol{n} + \boldsymbol{I}(t'))} = \frac{y(\boldsymbol{I}(t))}{y(\boldsymbol{I}(t'))} = \prod_{p=1}^{N} e^{z_p A(p,t)}.$$

By (17), for all markings $\boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)$, $\pi(\boldsymbol{m})$ can be expressed in terms of the reference state $\boldsymbol{m}_0$

$$\begin{aligned} \pi(\boldsymbol{m}) &= \pi(\boldsymbol{m}_0) \prod_{k=0}^{s} \prod_{p=1}^{N} e^{z_i A(i, t_k)} = \pi(\boldsymbol{m}_0) \prod_{p=1}^{N} e^{z_p (m_0(p) - m(p))} \\ &= \pi(\boldsymbol{m}_0) \left\{ \prod_{p=1}^{N} e^{z_p m_0(p)} \right\} \left\{ \prod_{p=1}^{N} e^{-z_p m(p)} \right\} = B \prod_{p=1}^{N} (f_p)^{m(p)} = B\pi_y(\boldsymbol{m}). \end{aligned}$$

Conversely, if an $\mathcal{S}\Pi$-net has an equilibrium distribution $\pi(\boldsymbol{m}) = B\prod_{p=1}^{N} f_p^{m(p)}$, then GLB is satisfied, since for a $\mathcal{S}\Pi$-net the GLB equations (6) reduce to

$$\pi(\boldsymbol{n}+\boldsymbol{I}(t))\sum_{t'\in T} q(\boldsymbol{I}(t),\boldsymbol{I}(t');\boldsymbol{n}) = \sum_{t'\in T}\pi(\boldsymbol{n}+\boldsymbol{I}(t'))q(\boldsymbol{I}(t'),\boldsymbol{I}(t);\boldsymbol{n}) \tag{21}$$

for all $\boldsymbol{n}, \boldsymbol{I}(t)$ such that $\boldsymbol{n}+\boldsymbol{I}(t) \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)$. Substituting $\pi(\boldsymbol{m}) = B\prod_{p=1}^{N} f_p^{m(p)}$ into (21) and dividing by $B\prod_{p=1}^{N} f_p^{n_p}$ yields

$$\prod_{p=1}^{N} f_p^{(\boldsymbol{I}_p(t))}\sum_{t'\in T}\mu(t)p(\boldsymbol{I}(t),\boldsymbol{I}(t')) = \sum_{t'\in T}\prod_{p=1}^{N} f_p^{(\boldsymbol{I}_p(t'))}\mu(t')p(\boldsymbol{I}(t'),\boldsymbol{I}(t)).$$

We recognize the routing chain equations (15). The solution $y(\cdot)$ to the routing chain is unique. So for the GLB-equations to be verified, it remains to show that, for all $t \in T$

$$\prod_{p=1}^{N} f_p^{(\boldsymbol{I}_p(t))} = y(\boldsymbol{I}(t)). \tag{22}$$

To this end, note that by the definition of the f_p's

$$\log\left(\frac{y(\boldsymbol{I}(t))}{y(\boldsymbol{O}(t))}\right) = \sum_{p=1}^{N} A(p,t)z_p = \sum_{p=1}^{N}\boldsymbol{I}_p(t)\log(f_p) - \boldsymbol{O}_p(t)\log(f_p) = \sum_{p=1}^{N}\log\left(\frac{f_p^{(\boldsymbol{I}_p(t))}}{f_p^{(\boldsymbol{O}_p(t))}}\right)$$

and thus

$$\frac{y(\boldsymbol{I}(t))}{y(\boldsymbol{O}(t))} = \prod_{p=1}^{N}\frac{f_p^{(\boldsymbol{I}_p(t))}}{f_p^{(\boldsymbol{O}_p(t))}},$$

which shows that (22) is satisfied. □

Under the condition that a solution to the routing chain exists, equivalence of condition (ii) of Lemma 5.11 and product form π_y satisfying (16), was obtained by Coleman et al. [24]. The solution $\boldsymbol{z}$ of the alternative condition (iii) was used to express the explicit solution of the product form. The contribution of Theorem 5.12 is the explicit relation between GLB and product form.

Theorem 5.12 characterizes product forms for $\mathcal{SPN}$s based on the incidence matrix. The product form (19) is of the Jackson-type since it is a product over the places similar to the result of Jackson [12]. Note that Petri nets are substantially more complex than Jackson networks. The product form distribution (19) contains one term for each token in the Petri net. Therefore, under GLB the only dependence between tokens lies in the normalising constant, as is the case in closed Jackson networks. Observe that Theorem 5.12 does not state that an arbitrary $\mathcal{SPN}$ with product form equilibrium distribution satisfies GLB.

Remark 5.13. Each T-invariant can be written as a linear combination of minimal support T-invariants (Result 3.27). Therefore, it can readily be seen that in Lemma 5.9, Corollary 5.10 and Lemma 5.11 the statement 'for each T-invariant', can be replaced by 'for each minimal support T-invariant'. This observation will be convenient when studying the structural implications of the results presented in this section.

5.3. *Structural implications of product form $\mathcal{SPN}$s*

In this section, we study the structural implication of Theorem 5.12 on the Petri net. The condition Rank$[\boldsymbol{A}]$ = Rank$[\boldsymbol{A}|\boldsymbol{C}]$ was presented in [24] as a necessary and sufficient condition for product form. Three comments can be placed regarding their results: (1) they assumed that a solution of the routing chain exists, (2) the condition Rank$[\boldsymbol{A}]$ = Rank$[\boldsymbol{A}|\boldsymbol{C}]$ generally depends on the numerical values of the transition rates, and (3) Rank$[\boldsymbol{A}]$ = Rank$[\boldsymbol{A}|\boldsymbol{C}]$ is a technical condition without intuitive interpretation.

The first comment is addressed in Theorem 5.4; for a solution of the routing chain to exist the Petri net must be an $\mathcal{S}\Pi$-net. The second comment was already observed by Coleman et al. [24], where it is shown that in some cases conditions on the numerical values of the firing rates must be imposed and in some cases not. To this end, Haddad et al. [6] introduced $\mathcal{S}\Pi^2$-nets, a subclass of $\mathcal{S}\Pi$-nets that have product form irrespective of the numerical values of the firing rates. Mairesse and Nguyen [8] relate the Deficiency Zero Theorem of Feinberg [48], developed for chemical reaction networks, to product form results for stochastic Petri nets. They show that the concept of $\mathcal{S}\Pi^2$-nets coincides with $\mathcal{S}\Pi$-nets that have 'deficiency zero'. However, neither the characterization of $\mathcal{S}\Pi^2$-nets or deficiency-zero $\mathcal{S}\Pi$-nets do intuitively explain why no restrictions on the numerical values of the firing rates are imposed. The structural implications of the product form results of Theorem 5.12, are based on the *minimal support* T-invariants (see Remark 5.13). First, we will show that $\mathcal{S}\Pi$-nets in which all minimal support T-invariants are minimal *closed* support T-invariants have product form without additional conditions on the firing rates. Second, we will show that this characterization exactly corresponds to the definition of $\mathcal{S}\Pi^2$-nets provided by Haddad et al. [6] and deficiency-zero $\mathcal{S}\Pi$-nets provided by Mairesse and Nguyen [8]. Third, via this characterization in terms of the minimal support T-invariants we are able to provide an explanation in terms of T-invariants of the condition $\text{Rank}[\boldsymbol{A}] = \text{Rank}[\boldsymbol{A}|\boldsymbol{C}]$ of the $\mathcal{SPN}$. The condition is shown to be required only for $\mathcal{S}\Pi$-nets that are not $\mathcal{S}\Pi^2$-nets.

Theorem 5.14. *For an $\mathcal{SPN}$, (18) is satisfied for each minimal* closed *support T-invariant $\boldsymbol{x}$. For an $\mathcal{S}\Pi$-net in which each minimal support T-invariant is a minimal* closed *support T-invariant, the equivalent conditions* (i)–(iii) *of Lemma 5.11 are satisfied.*

Proof. The firing sequence of a minimal closed support T-invariant is linear (Result 3.29). Thus, $x_t \le 1,\ t = 1, \ldots, T$, and within this T-invariant every output bag is an input bag of a unique next transition. Therefore, in (18) the denominator of each fraction $y(\boldsymbol{I}(t))/y(\boldsymbol{O}(t))$ is cancelled by the numerator of the fraction of the subsequent transition in this T-invariant. As a consequence, conditions (i)–(iii) of Lemma 5.11 are satisfied irrespective of of the numerical values of the firing rates. □

By means of Theorem 5.14, in the case that there exists a minimal T-invariant that is *not* closed, additional conditions are required on the numerical values of the firing rates to ensure a product form solution. Below, we will provide an intuitive explanation of these additional conditions. First, the definition of $\mathcal{S}\Pi^2$-nets, as introduced by Haddad et al. [6], is presented.

Definition 5.15 ($\mathcal{S}\Pi^2$*-Net [6]*)**.** A Π^2-net is a Π-net such that for every $\boldsymbol{g} \in \mathcal{R}(\mathcal{T})$, there is an $\boldsymbol{a}_g \in \mathbb{Q}^N$ such that

$$\boldsymbol{a}_g \boldsymbol{A} = \boldsymbol{b}_g$$

in which for $t = 1, \ldots, N$

$$\boldsymbol{b}_g(t) = \begin{cases} -1 & \text{if } \boldsymbol{g} = \boldsymbol{I}(t), \\ 1 & \text{if } \boldsymbol{g} = \boldsymbol{O}(t), \\ 0 & \text{otherwise.} \end{cases}$$

An $\mathcal{S}\Pi^2$-net is a stochastic Π^2-net.

Although not defined as such by Haddad et al. [6], and not recognized before, the characterization of an $\mathcal{S}\Pi^2$-net can be provided via the minimal support T-invariants of the $\mathcal{S}\Pi$-net, as is shown in the next theorem.

Theorem 5.16. *An $\mathcal{S}\Pi$-net is an $\mathcal{S}\Pi^2$-net if and only if* all *minimal support T-invariants are minimal* closed *support T-invariants.*

Proof. Consider an $\mathcal{S}\Pi$-net. We must show that $\boldsymbol{a}_g\boldsymbol{A} = \boldsymbol{b}_g$ has a solution if and only if all minimal support T-invariants are minimal closed support T-invariants. First observe that $\boldsymbol{a}_g\boldsymbol{A} = \boldsymbol{b}_g$ has a solution if and only if the row vector $\boldsymbol{b}_g$ is a linear combination of the rows of $\boldsymbol{A}$, i.e., $\boldsymbol{b}_g\boldsymbol{x} = 0$ for every $\boldsymbol{x}$ such that $\boldsymbol{A}\boldsymbol{x} = 0$, that is $\boldsymbol{b}_g\boldsymbol{x} = 0$ for all T-invariants. Second, if a solution $\boldsymbol{a}_g$ exists, it is rational since $\boldsymbol{A}$ is an integer matrix and $\boldsymbol{b}_g$ an integer vector.

Now, assume that all minimal support T-invariants are minimal closed support. Consider a minimal closed support T-invariant $\boldsymbol{x}$ and a bag $\boldsymbol{g} \in \mathcal{R}(T)$ with $\boldsymbol{O}(t_i) = \boldsymbol{I}(t_j)$, then $\boldsymbol{b}_g\boldsymbol{x} = x_{t_i} - x_{t_j}$, since the firing sequence of $\boldsymbol{x}$ is linear (Result 3.29). Either $\boldsymbol{g}$ is both an input bag and an output bag of transitions in the firing sequence of $\boldsymbol{x}$ (i.e., $x_{t_i} = x_{t_j} = 1$), or $\boldsymbol{g}$ is neither an input bag nor an output bag of any transition in the firing sequence of $\boldsymbol{x}$ (i.e., $x_{t_i} = x_{t_j} = 0$). By assumption all minimal support T-invariants are minimal closed support, which completes the first part of the proof.

Conversely, if there is a minimal support T-invariant $\boldsymbol{x}$ of which the support is not closed, then $\exists \boldsymbol{g} \in \mathcal{R}(T), t \in \|\boldsymbol{x}\|$, such that $\boldsymbol{b}$ is the output of t, but there is no $t' \in \|\boldsymbol{x}\|$ such that $\boldsymbol{g}$ is the input bag of t'. For such $\boldsymbol{x}$ we have $\boldsymbol{b}_g\boldsymbol{x} \neq 0$ and this completes the proof of the second part. □

Corollary 5.17. *For an $\mathcal{S}\Pi^2$-net the equivalent conditions* (i)–(iii) *of Lemma 5.11 are satisfied irrespective of the firing rates. Therefore, GLB and a product form solution of the form (19) can be verified without checking one of these conditions.*

Proof. By Theorems 5.14 and 5.16, for an $\mathcal{S}\Pi^2$-net the equivalent conditions (i)–(iii) of Lemma 5.11 are satisfied irrespective of the transition rates. Applying Theorem 5.12 concludes the proof. □

Now, we give the definition of the deficiency of a Petri net. Mairesse and Nguyen [8] show that $\mathcal{S}\Pi$-nets that have deficiency zero have a product form equilibrium distribution irrespective of the numerical values of the transition rates. They also observe that the class of zero-deficiency $\mathcal{S}\Pi$-nets coincides with that of $\mathcal{S}\Pi^2$-nets.

Definition 5.18 (*Deficiency [8]*)**.** The deficiency δ of a Petri net $\mathcal{PN}$ is:

$$\delta = |\mathcal{R}(\mathcal{T})| - \ell - \text{rank}(\boldsymbol{A}),$$

where $|\mathcal{R}(\mathcal{T})|$ represents the number of bags $\boldsymbol{g} \in \mathcal{R}(\mathcal{T})$ and ℓ is the number of common input bag classes of $\mathcal{PN}$.

Lemma 5.19 (*[8]*)**.** *Consider an $\mathcal{S}\Pi$-net $\mathcal{SPN}$. $\mathcal{SPN}$ is an $\mathcal{S}\Pi^2$-net if and only if it has deficiency $\delta = 0$.*

Theorem 5.16 and Lemma 5.19 imply that for $\mathcal{S}\Pi$-net deficiency zero is a property that can also be identified via its minimal support T-invariants. Deficiency is directly related to the number of linearly independent minimal *non*-closed support T-invariants.

To conclude, Theorem 5.12 states that the equilibrium distribution of an $\mathcal{S}\Pi$-net is characterized by the solution of the routing chain $y(\cdot)$, characterized by the probability flow through classes of minimal closed support T-invariants. In $\mathcal{S}\Pi$-nets, all transitions are covered by minimal closed support T-invariants. Therefore, every minimal support T-invariant that is not closed support is built up by transitions of different minimal closed support T-invariants. The conditions (i)–(iii) of Lemma 5.11 imply that the total probability flow through a minimal non-closed support T-invariant should be equal to the probability flow imposed by the minimal closed support T-invariants. Examples 5.22 and 5.23 in the next subsection will provide an illustration.

From the results presented above, it is clear that characterization of product form results for $\mathcal{SPN}$s with transition rates (13) can be done at the structural level. The steps that have to be performed to this end are summarized in the following algorithm.

Algorithm 5.20 (*Structural Characterization of Product Form*)**.**

Step 1. Obtain the incidence matrix $\boldsymbol{A}$ of the $\mathcal{SPN}$ and compute the minimal support T-invariants $\boldsymbol{x}^1, \ldots, \boldsymbol{x}^h$ and the minimal support P-invariants $\boldsymbol{y}^1, \ldots, \boldsymbol{y}^j$.

Step 2. Obtain the minimal closed support T-invariants from the minimal support T-invariants, and renumber the T-invariants such that $\{\boldsymbol{x}^1, \ldots, \boldsymbol{x}^k\}$ is the set of minimal closed support T-invariants ($k \le h$).

Step 3. Verify that all transitions are covered by minimal closed support T-invariants and minimal support P-invariants. If not: stop, we cannot conclude that the $\mathcal{SPN}$ has a product form equilibrium distribution, else: go to step 4.

Step 4. Determine from $\{\boldsymbol{x}^1, \ldots, \boldsymbol{x}^k\}$ the set of common input bag classes $\{CI(\boldsymbol{x}^1), \ldots, CI(\boldsymbol{x}^\ell)\}$. Compute per common input bag class i the solution to the routing chain $y^i(\cdot)$. If all minimal support T-invariants are minimal closed support T-invariants, i.e., $k = h$, then proceed to step 6, else go to step 5.

Step 5. Determine $\boldsymbol{C}$ and verify that $\boldsymbol{C}\boldsymbol{x}^i = 0$, for the minimal non-closed support T-invariants $\boldsymbol{x}^{k+1}, \ldots, \boldsymbol{x}^h$. If not: stop, the $\mathcal{SPN}$ does not have a product form equilibrium distribution, else go to step 6.

Step 6. Solve $\boldsymbol{zA} = \boldsymbol{C}$. The equilibrium distribution is $\pi(\boldsymbol{m}) = B\pi_y(\boldsymbol{m})$ with π_y given in (19).

5.4. *Examples of product form $\mathcal{SPN}$s*

This section presents some examples illustrating the structural characterization of product form presented above. First, in Example 5.21 we present an example of an $\mathcal{S}\Pi^2$-net. Examples 5.22 and 5.23 present $\mathcal{S}\Pi$-nets that are not $\mathcal{S}\Pi^2$-nets, which means that they posses a product form equilibrium distribution only for specific choices of the firing rates. Finally, in Example 5.24, we illustrate the importance of the boundedness assumption, by presenting a net that may not possess an equilibrium distribution, due to a possibly unbounded number of tokens. Examples 5.21, 5.22 and 5.24 are obtained from [2].

Example 5.21. Consider the $\mathcal{SPN}$ depicted in Fig. 3(a) and execute the steps of the algorithm of Section 5.3.

Step 1–3. From the incidence matrix

$$\boldsymbol{A} = \begin{pmatrix} -1 & -1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 2 & 1 & -2 & 2 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix},$$

we obtain that this net has two minimal support T-invariants $\boldsymbol{x}^1 = (10100)$, $\boldsymbol{x}^2 = (01111)$, which are both minimal closed support T-invariants, and two minimal support P-invariants $\boldsymbol{y}^1 = (11011)$, $\boldsymbol{y}^2 = (20112)$. $\mathcal{SPN}$ is covered by both minimal support T-invariants and P-invariants.

Step 4. Since the T-invariants share $\boldsymbol{I}(t_1)$ they are in a common input bag relation, which implies that the routing chain has one irreducible set:

$$S = \{\boldsymbol{I}(t_1), \boldsymbol{I}(t_3), \boldsymbol{I}(t_4), \boldsymbol{I}(t_5)\} \quad (\boldsymbol{I}(t_1) = \boldsymbol{I}(t_2)).$$

Amalgamate transition t_1 and t_2 into a single transition t_{12} with $\mu(t_{12}) = \mu(t_1) + \mu(t_2)$, $p(\boldsymbol{I}(t_1), \boldsymbol{O}(t_1)) = \mu(t_1)/\mu(t_{12})$ and $p(\boldsymbol{I}(t_1), \boldsymbol{O}(t_2)) = \mu(t_2)/\mu(t_{12})$. The solution of the routing chain is (up to normalisation):

$$y(\boldsymbol{I}(t_1))\mu(t_{12}) = y(\boldsymbol{I}(t_3))\mu(t_3) = 1, \qquad y(\boldsymbol{I}(t_4))\mu(t_4) = y(\boldsymbol{I}(t_5))\mu(t_5) = p(\boldsymbol{I}(t_1), \boldsymbol{O}(t_2)).$$

The $\mathcal{SPN}$ is an $\mathcal{S}\Pi^2$-net, so we may proceed to step 6.

Step 6. The vector $\boldsymbol{C}$ is obtained from the solution of the routing chain:

$$\boldsymbol{C} = \left(\log\left[\frac{\mu(t_3)}{\mu(t_{12})}\right], \log\left[\frac{\mu(t_5)}{\mu(t_2)}\right], \log\left[\frac{\mu(t_{12})}{\mu(t_3)}\right], \log\left[\frac{\mu(t_2)\mu(t_3)}{\mu(t_{12})\mu(t_4)}\right], \log\left[\frac{\mu(t_4)}{\mu(t_5)}\right]\right).$$

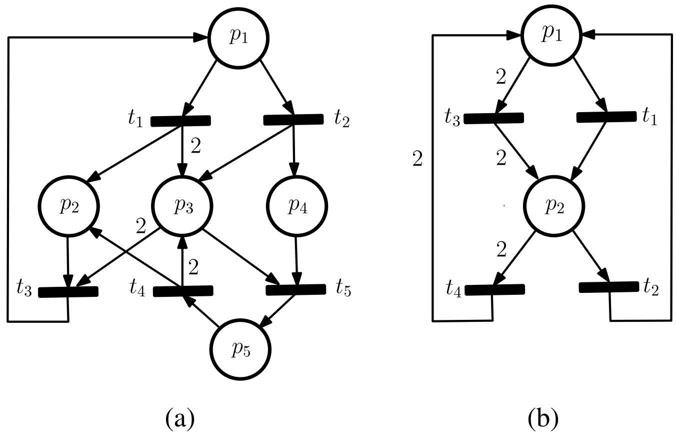

Fig. 3. The stochastic Petri nets of Examples 5.21 and 5.22.

A solution $\boldsymbol{z}$ of $\boldsymbol{zA} = \boldsymbol{C}$ is:

$$z_1 = 0, \qquad z_2 = \log\left(\frac{\mu(t_3)}{\mu(t_{12})}\right), \qquad z_3 = 0, \qquad z_4 = \log\left(\frac{\mu(t_5)}{\mu(t_2)}\right), \qquad z_5 = \log\left(\frac{\mu(t_4)}{\mu(t_2)}\right)$$

and the equilibrium distribution is

$$\pi(\boldsymbol{m}) = B\left(\frac{\mu(t_{12})}{\mu(t_3)}\right)^{m(2)}\left(\frac{\mu(t_2)}{\mu(t_5)}\right)^{m(4)}\left(\frac{\mu(t_2)}{\mu(t_4)}\right)^{m(5)}$$

for any marking $\boldsymbol{m}$ in the reachability set

$$\mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0) = \{\boldsymbol{m} : \boldsymbol{y}^1(\boldsymbol{m} - \boldsymbol{m}_0) = 0,\ \boldsymbol{y}^2(\boldsymbol{m} - \boldsymbol{m}_0) = 0\},$$

where $\boldsymbol{y}^1 = (11011)$, $\boldsymbol{y}^2 = (20112)$ are the two minimal support P-invariants of the net. □

Example 5.22. Consider the $\mathcal{SPN}$ depicted in Fig. 3(b). This is an example of an $\mathcal{S}\Pi$-net which is not an $\mathcal{S}\Pi^2$-net so that additional conditions on the firing rates have to be satisfied.

Step 1–3. This $\mathcal{SPN}$ has incidence matrix

$$\boldsymbol{A} = \begin{pmatrix} -1 & 1 & -2 & 2 \\ 1 & -1 & 2 & -2 \end{pmatrix}.$$

Observe that each transition is covered by the minimal closed support T-invariants $\boldsymbol{x}^1 = (1100)$, $\boldsymbol{x}^2 = (0011)$, but that $\boldsymbol{x}^3 = (2001)$ and $\boldsymbol{x}^4 = (0210)$ are also minimal support T-invariants that do not have closed support. The $\mathcal{SPN}$ is covered by its one minimal support P-invariant $\boldsymbol{y}^1 = (11)$.

Step 4. The routing chain has two irreducible sets $S(\boldsymbol{x}^1) = \{\boldsymbol{I}(t_1), \boldsymbol{I}(t_2)\}$, and $S(\boldsymbol{x}^2) = \{\boldsymbol{I}(t_3), \boldsymbol{I}(t_4)\}$. The solution of the routing chain is:

$$\frac{y^1(\boldsymbol{I}(t_2))}{y^1(\boldsymbol{I}(t_1))} = \frac{\mu(t_1)}{\mu(t_2)}, \qquad \frac{y^2(\boldsymbol{I}(t_4))}{y^2(\boldsymbol{I}(t_3))} = \frac{\mu(t_3)}{\mu(t_4)},$$

with corresponding vector $\boldsymbol{C}$

$$\boldsymbol{C} = \left(\log\left[\frac{\mu(t_2)}{\mu(t_1)}\right], \log\left[\frac{\mu(t_1)}{\mu(t_2)}\right], \log\left[\frac{\mu(t_4)}{\mu(t_3)}\right], \log\left[\frac{\mu(t_3)}{\mu(t_4)}\right]\right).$$

Step 5. $\boldsymbol{Cx}^i = 0$ for the minimal non-closed support T-invariants $\boldsymbol{x}^3 = (2001)$ and $\boldsymbol{x}^4 = (0210)$, if $2C_1 + C_4 = 0$ and $2C_2 + C_3 = 0$, thus if

$$\left(\frac{\mu(t_2)}{\mu(t_1)}\right)^2 = \frac{\mu(t_4)}{\mu(t_3)}. \tag{23}$$

Step 6. If (23) is satisfied, this $\mathcal{SPN}$ has an equilibrium distribution

$$\pi(\boldsymbol{m}) = B\left(\frac{\mu(t_2)}{\mu(t_1)}\right)^{\boldsymbol{m}(1)},$$

for any marking $\boldsymbol{m}$ in the reachability set

$$\mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0) = \{\boldsymbol{m} : \boldsymbol{m}(1) + \boldsymbol{m}(2) = \boldsymbol{m}_0(1) + \boldsymbol{m}_0(2)\}.$$

This example provides insight in the intuition for the conditions of Lemma 5.11. As can be seen from Fig. 3(b), there are two possibilities for the movement of two tokens from place 1 to place 2. In the first case (via t_1) the tokens jump one after the other, in the second case (via t_3) the tokens jump simultaneously. The probability flow for these two possibilities must be the same. This is reflected in the condition (23) on the firing rates: two transitions with rate $\mu(t_1)$ must be proportional to one transition at rate $\mu(t_3)$. □

Example 5.23. Consider the $\mathcal{SPN}$ of Fig. 4(a). This example indicates that minimal *non-closed* support T-invariants can also exist in $\mathcal{S}\Pi$-nets where in the minimal support T-invariants no transition fires more than once, i.e., $x_t \leq 1, \forall t \in T$ is not sufficient for a T-invariant to be closed support.

Step 1–3. The minimal closed support T-invariants are $\boldsymbol{x}^1 = (110000), \boldsymbol{x}^2 = (001100)$ and $\boldsymbol{x}^3 = (000011)$, and the minimal non-closed support T-invariants $\boldsymbol{x}^4 = (100101)$ and $\boldsymbol{x}^5 = (011010)$. $\mathcal{SPN}$ is covered by its one minimal support P-invariant $\boldsymbol{y}^1 = (111)$.

Step 4–6. This $\mathcal{SPN}$ has a product form equilibrium distribution if $C_1 = C_4 + C_6$ and $C_2 = C_3 + C_5$, so if

$$\frac{\mu(t_2)}{\mu(t_1)} = \frac{\mu(t_3)}{\mu(t_4)}\frac{\mu(t_5)}{\mu(t_6)}. \quad \square$$

Example 5.24. Consider the $\mathcal{SPN}$ of Fig. 4(b).

Step 1–3. The net has one T-invariant $\boldsymbol{x} = (1111)$ covering all transitions, and $\boldsymbol{x}$ has closed support. It has no P-invariants.

Note that without additional conditions the algorithm stops here. Yet we proceed to provide an illustration of such conditions that prevents the creation of an unbounded number of tokens.

Step 4. The solution of the routing chain is (up to a multiplicative constant)

$$y(\boldsymbol{I}(t_1)) = 1/\mu(t_1), \qquad y(\boldsymbol{I}(t_2)) = 1/\mu(t_2), \qquad y(\boldsymbol{I}(t_3)) = 1/\mu(t_3), \qquad y(\boldsymbol{I}(t_4)) = 1/\mu(t_4).$$

Step 6. The $\mathcal{SPN}$ has an invariant measure

$$\pi_y(\boldsymbol{m}) = \left(\frac{\mu(t_2)\mu(t_4)}{\mu(t_1)\mu(t_3)}\right)^{\boldsymbol{m}(1)} \left(\frac{\mu(t_2)}{\mu(t_3)}\right)^{\boldsymbol{m}(2)} \left(\frac{\mu(t_4)}{\mu(t_3)}\right)^{\boldsymbol{m}(3)}.$$

From Fig. 4(b) we can see that the number of tokens in the net is unbounded (repetitive firing of transitions t_1 and t_4 increases the number of tokens by 1), but that for every marking a firing sequence to $\boldsymbol{m}_0 = (100)$ exists. Under the *additional* conditions $\mu(t_2)\mu(t_4) < \mu(t_1)\mu(t_3)$, $\mu(t_2) < \mu(t_3)$, $\mu(t_4) < \mu(t_3)$ the $\mathcal{SPN}$ has an equilibrium distribution

$$\pi(\boldsymbol{m}) = B\pi_y(\boldsymbol{m}), \quad \boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0) = \mathbb{N}_0^3 \setminus \{0\}. \quad \square$$

6. Decomposing the stochastic Petri net

The analysis of the previous sections enables us to formulate a new decomposition result. This result uses the T- and P-invariants to decompose an $\mathcal{SPN}$ in subnets, consisting of one or more common input bag classes as defined in Section 5. It is a generalization of the decomposition result formulated by Frosch and Natarajan [11] for Closed Synchronized Systems of Stochastic Sequential Processes (*CS*) that consist of state machines (see Definition 3.11) connected by so-called buffer places. A formal definition of a *CS* is given below in Definition 6.13. By removing these buffer places from the network, the equilibrium (product-form) distribution of a *CS* is shown to be a product over the product-form equilibrium distributions of the separate state machines. This section generalizes the results of Frosch and Natarajan to decomposition results for product form $\mathcal{S}\Pi$-nets.

We will formulate sufficient conditions for decomposition of an arbitrary $\mathcal{S}\Pi$-net into subnetworks so that the equilibrium distribution is a product over the invariant measures of the subnetworks defined by common input bag classes. The decomposition is based on conflict places, the generalization of buffer places. First, we define three different place sets: the sufficient place set, the surplus place set, and the conflict place set. Next, these sets are used to formulate in Theorem 6.10 the class of decomposable nets to $\mathcal{S}\Pi$-nets that can be decomposed into subnets each corresponding to one or more common input bag classes. Finally, an algorithm is presented by which all possible decompositions of an $\mathcal{S}\Pi$-net are generated.

The *sufficient place set* was introduced by Florin and Natkin [49]. The places not contained in the sufficient place set will be the places at which we decompose the $\mathcal{SPN}$. We define this complementary set of places as the *surplus place set*.

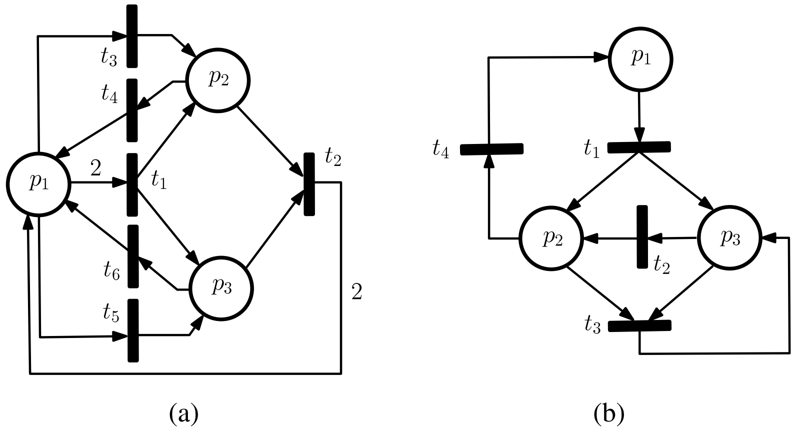

Fig. 4. The stochastic Petri nets of Examples 5.23 and 5.24.

Definition 6.1 (*Sufficient Place Set—Surplus Place Set*)**.** A subset of places $\mathcal{P}^{\text{suf}} \subseteq P$ is a *sufficient place set* if the marking of each place in $\mathcal{P}^{\text{suf}}$ provides sufficient information to define uniquely the marking of all places. A subset of places $\mathcal{P}^{\text{sur}} \subseteq P$ is a *surplus place set* if the subset of places $P \setminus \mathcal{P}^{\text{sur}}$ is a sufficient place set. A place contained in a surplus place set will be referred to as a *surplus place*.

Lemma 6.2. *Consider a structurally live and structurally bounded Petri net. A set of places $\mathcal{P} \subseteq P$ is a sufficient place set if and only if all the rows of $\boldsymbol{A}$ can be written as linear combinations of the rows of $\boldsymbol{A}$ corresponding to places in $\mathcal{P}$, i.e., for all $j \in P$*

$$\boldsymbol{A}_j = \sum_{i \in \mathcal{P}} \lambda_{ij} \boldsymbol{A}_i, \tag{24}$$

where $\boldsymbol{A}_p$ is the row of $\boldsymbol{A}$ corresponding to place p and $\lambda_{ij} \in \mathbb{Q}$.

Proof. For every $\boldsymbol{m} \in \mathcal{M}(\mathcal{PN}, \boldsymbol{m}_0)$, $\exists\sigma$ such that $\boldsymbol{m}_0|\sigma > \boldsymbol{m}$, which implies $\boldsymbol{m} = \boldsymbol{m}_0 + \boldsymbol{A}\bar{\sigma}$. From (24), for all $j \in P$:

$$m(j) = m_0(j) + \boldsymbol{A}_j\bar{\sigma} = m_0(j) + \sum_{i \in \mathcal{P}} \lambda_{ij}\boldsymbol{A}_i\bar{\sigma} = m_0(j) + \sum_{i \in \mathcal{P}} \lambda_{ij}(m(i) - m_0(i)). \tag{25}$$

Conversely, assume $\exists j \in P \setminus \mathcal{P}$ such that (24) does not hold. Then, there exists a vector $\boldsymbol{v}$ which is perpendicular to the rows $\boldsymbol{A}_i$, $i \in \mathcal{P}$, but not to $\boldsymbol{A}_j$, i.e., $\exists \boldsymbol{v} \in \mathbb{Q}^N$ with $\boldsymbol{A}_i\boldsymbol{v} = 0$, $\forall i \in \mathcal{P}$, and $\boldsymbol{A}_j\boldsymbol{v} = 1$. For such $\boldsymbol{v}$, consider the firing sequence σ with firing count vector $\bar{\sigma} = c\boldsymbol{v} + \sum_{i=1}^{h} \alpha_i \boldsymbol{x}^i$, with $c \in \mathbb{Z}/\{0\}$, $\boldsymbol{x}^1, \ldots, \boldsymbol{x}^h$ the T-invariants of the net and $\alpha_i \in \mathbb{N}$. Consider the initial marking $\boldsymbol{m}_0^\sigma$ from which firing σ yields $\boldsymbol{m}^\sigma$. We have $m^\sigma(i) = m_0^\sigma(i) + \boldsymbol{A}_i\bar{\sigma} = m_0^\sigma(i)$ for all $i \in \mathcal{P}$, while the markings $\boldsymbol{m}^\sigma$ and $\boldsymbol{m}_0^\sigma$ are different because $m^\sigma(j) = m_0^\sigma(j) + \boldsymbol{A}_j\bar{\sigma} = m_0^\sigma(j) + \boldsymbol{A}_j(c\boldsymbol{x} + \sum_{i=1}^{h} \alpha_i \boldsymbol{x}^i) = m_0^\sigma(j) + c$. Therefore, if (24) does not hold, $\mathcal{P}$ cannot be a sufficient place set. $\square$

The sufficient place set of a Petri net (and the corresponding surplus place set) is in general not unique. Sufficient places sets, and thus surplus place sets, can be characterized from the P-invariants, since the linear relations between the rows of $\boldsymbol{A}$ are described by its P-invariants. This is also intuitive, because P-invariants characterize a constant weighted marking over a subset of places (see Definition 3.15).

Lemma 6.3. *Consider a structurally live and structurally bounded Petri net. Let the set of its minimal support P-invariants be $\{\boldsymbol{y}^1, \ldots, \boldsymbol{y}^p\}$ and choose a place set $\mathcal{P} \subseteq P$. Whether $\mathcal{P}$ is a surplus place set can be characterized as follows:*

Step 1. Obtain a basis $\{\bar{\boldsymbol{y}}^1, \ldots, \bar{\boldsymbol{y}}^r\}$ composed of elements from $\{\boldsymbol{y}^1, \ldots, \boldsymbol{y}^p\}$. Define matrix $\boldsymbol{Y}$ consisting of the rows $\{\bar{\boldsymbol{y}}^1, \ldots, \bar{\boldsymbol{y}}^r\}$.
Step 2. Order the columns of $\boldsymbol{Y}$ such that the columns according to places $p \in \mathcal{P}$ are in front. Denote the obtained matrix by $\widehat{\boldsymbol{Y}}$.
Step 3. Apply Gauss–Jordan elimination on matrix $\widehat{\boldsymbol{Y}}$ to obtain its reduced row echelon form $\text{rref}(\widehat{\boldsymbol{Y}})$.
Step 4. $\mathcal{P}$ is a surplus place set if and only if $\text{rref}(\widehat{\boldsymbol{Y}})$ contains leading ones in columns $1, \ldots, |\mathcal{P}|$.

Now, if $\mathcal{P}$ is a surplus place set, the marking of the places $j \in \mathcal{P}$ is expressed by the marking of the places $\mathcal{P}^{\text{suf}} = P \setminus \mathcal{P}$ as follows:

$$m(j) = m_0(j) - \sum_{i \in \mathcal{P}^{\text{suf}}} \text{rref}(\widehat{\boldsymbol{Y}})_{ji}(m(i) - m_0(i)). \tag{26}$$

Proof. Let $\widehat{\boldsymbol{A}}$ be the permutation of $\boldsymbol{A}$ corresponding to the permutation applied to obtain $\widehat{\boldsymbol{Y}}$. Since $\boldsymbol{YA} = 0$, also $\widehat{\boldsymbol{Y}}\widehat{\boldsymbol{A}} = 0$ and $\text{rref}(\widehat{\boldsymbol{Y}})\widehat{\boldsymbol{A}} = 0$. If $\text{rref}(\widehat{\boldsymbol{Y}})$ has leading ones in the first $|\mathcal{P}|$ columns, setting $\lambda_{ij} = -\text{rref}(\widehat{\boldsymbol{Y}})_{ji}$ in (24) implies by Lemma 6.2 that $\mathcal{P}$ is a surplus place set. In addition, (26) follows from (25).

Conversely, if $\mathcal{P}$ is a surplus place set, from (24) we can find a $\boldsymbol{w}_i \in \mathbb{Q}^N$ for every $i \in \mathcal{P}$ such that $\boldsymbol{w}_i\widehat{\boldsymbol{A}} = 0$ by taking $w_i(i) = 1$, $w_i(p) = 0$ for all $p \in \mathcal{P} \setminus \{i\}$, and $w_i(p) = \lambda_{ij}$ for all $p \in P \setminus \mathcal{P}$. From Result 3.27 follows that each such $\boldsymbol{w}_i$ is a linear combination of minimal support P-invariants. This implies $\boldsymbol{w}_i \in \text{rowspan}(\boldsymbol{Y}) = \text{rowspan}(\widehat{\boldsymbol{Y}})$ and thus $\boldsymbol{w}_i \in \text{rowspan}(\text{rref}(\widehat{\boldsymbol{Y}}))$. Now assume that $\text{rref}(\widehat{\boldsymbol{Y}})$ does not have leading ones in the first $|\mathcal{P}|$ columns. Let j be the first column without a leading one and $\text{rref}(\widehat{\boldsymbol{Y}})_j$ the j-th row of $\text{rref}(\widehat{\boldsymbol{Y}})$. By showing that the equation

$$\boldsymbol{w}_j = \sum_{i=1}^{r} \alpha_i \text{rref}(\widehat{\boldsymbol{Y}})_j \tag{27}$$

has no solution, we obtain the contradiction $\boldsymbol{w}_j \notin \text{rowspan}(\text{rref}(\widehat{\boldsymbol{Y}}))$, from which we conclude that $\text{rref}(\widehat{\boldsymbol{Y}})$ must have leading ones in the first $|\mathcal{P}|$ columns. $w_j(i) = 0$ for $i < j$ implies $\alpha_i = 0$ which reduces (27) to $\boldsymbol{w}_j = \sum_{i=j}^{r} \alpha_i \text{rref}(\widehat{\boldsymbol{Y}})_j$. Since $w_j(j) = 1$ the latter equation has no solution, because otherwise column j is a pivot column during the Gauss Jordan elimination, which would have resulted in a leading one in column j. □

Remark 6.4. Lemma 6.3 provides a test to check for a given candidate place set whether or not it is a surplus place set, since the columns of $\boldsymbol{Y}$ are pre-ordered. This test be used in the decomposition algorithm that we present at the end of this section. Observe that by starting from $\boldsymbol{Y}$ and applying Gauss–Jordan elimination while allowing swapping of columns, it is also possible to trace surplus place sets.

The minimal number of places a sufficient place set was already expressed (and defined as the *dimension of the marking process*) by Florin and Natkin [49]. From each additional linearly independent P-invariant an additional surplus place can be selected. The number of linearly independent minimal support P-invariants is equal to $\dim(\text{Ker}(\boldsymbol{A}^T))$. Recall that this number can be smaller than the number of minimal support P-invariants (see Remark 3.28).

Lemma 6.5 (*[49]*). *For each sufficient place set* $\mathcal{P}^{\text{suf}}$*:*

$$\left|\mathcal{P}^{\text{suf}}\right| \geq N - \dim(\text{Ker}(\boldsymbol{A}^T)).$$

Remark 6.6. Note that the minimal number of places in a sufficient place set $\min\{\left|\mathcal{P}^{\text{suf}}\right|\}$ is directly connected to the notion of deficiency (discussed in Section 5.3): $\delta = |\mathcal{R}(\mathcal{T})| - \ell - \text{Rank}(\boldsymbol{A}) = |\mathcal{R}(\mathcal{T})| - \ell - (N - \dim(\text{Ker}(\boldsymbol{A}^T))) = |\mathcal{R}(\mathcal{T})| - \ell - \min\{\left|\mathcal{P}^{\text{suf}}\right|\}$.

In Theorem 5.12, there may be solutions to the matrix equation $\boldsymbol{z}\boldsymbol{A} = \boldsymbol{C}$ with $\boldsymbol{z}_p = 0$ for some places p. Such a place has $f_p = 1$ and no term involving place p appears in the product form (19). The following lemma shows that such places are uniquely related to places contained in a surplus place set.

Lemma 6.7. *Assume a solution to the matrix equation* $\boldsymbol{z}\boldsymbol{A} = \boldsymbol{C}$ *exists. If* $\mathcal{P}'$ *is a surplus place set, then there exists a solution to* $z\boldsymbol{A} = \boldsymbol{C}$*, where* $z_p = 0$*, for all* $p \in \mathcal{P}'(\mathcal{P}' \subseteq P)$.

Proof. Consider a surplus set $\mathcal{P}'$. By Lemma 6.2, the row vectors $\boldsymbol{A}_j$ of $\boldsymbol{A}$ corresponding to the places $j \in \mathcal{P}'$ can be written as linear combination of the rows $\boldsymbol{A}_i$, $i \in P \setminus \mathcal{P}'$. Therefore, under the assumption that a solution $\boldsymbol{z}$ to $\boldsymbol{z}\boldsymbol{A} = \boldsymbol{C}$ exists, there exists a solution where $z_p = 0, \forall p \in \mathcal{P}'$. □

For each common input bag class $CI(\boldsymbol{x})$, denote the set of places that are elements of the closed support T-invariants in $CI(\boldsymbol{x})$ by $P(CI(\boldsymbol{x}))$:

$$P(CI(\boldsymbol{x})) = \left\{p \in P \mid \exists \boldsymbol{x} \in CI(x) \wedge \exists t \in \|\boldsymbol{x}\| \text{ with } I_p(t) \geq 0\right\}.$$

Firing of transitions of T-invariants of different common input bag classes interacts and conflicts in the places that are shared among the common input bag classes. Focussing on such places will enable us to formulate decomposition results. Therefore, we formally define *conflict places* and the set of all conflict places among all common input bag classes.

Definition 6.8 (*Conflict Place—Conflict Place Set*). Let $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ be minimal closed support T-invariants such that $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ are not in common input bag relation, i.e., $CI(\boldsymbol{x}^1) \neq CI(\boldsymbol{x}^2)$. Let p be a place that is an element of both $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$, i.e., $p \in \left(P(CI(\boldsymbol{x}^1)) \cap P(CI(\boldsymbol{x}^2))\right)$. Then p is called a *conflict place* of $CI(\boldsymbol{x}^1)$ and $CI(\boldsymbol{x}^2)$. The *conflict place set* is the subset $\mathcal{P}^{\text{con}} \subseteq P$, of places that are a conflict place between any two common input bag classes:

$$\mathcal{P}^{\text{con}} = \left\{p \in P \mid \exists i, j \text{ with } CI(\boldsymbol{x}^i) \neq CI(\boldsymbol{x}^j) \text{ and } p \in \left(P(CI(\boldsymbol{x}^i)) \cap P(CI(\boldsymbol{x}^j))\right)\right\}.$$

Our decomposition result will be obtained by removing conflict places. Therefore, before stating the decomposition result, the following lemma is presented.

Lemma 6.9. *If in an* $\mathcal{S}\Pi$*-net* $\mathcal{SPN}$ *the places and all arcs incident to all the places* $p \in \mathcal{P} \subset P$ *can be removed so that no complete input bag is removed, then the remaining net is an* $\mathcal{S}\Pi$*-net, possibly consisting of several separated components.*

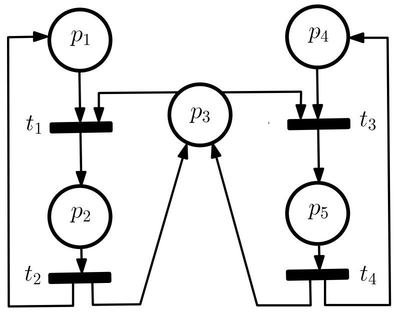

Fig. 5. An $\mathcal{SPN}$ decomposing into all common input bag classes.

Proof. Remove from $\mathcal{SPN}$ a place $p' \in \mathcal{P}$ and the arcs incident to this place. There is no transition for which has $\boldsymbol{I}_p(t) = 0$ for all $p \in P \setminus p'$, since by removing all places $p \in \mathcal{P}$ no complete input bag is removed. Denote the remaining net by $\mathcal{SPN}'$. $\mathcal{SPN}'$ only differs from $\mathcal{SPN}$ in the transitions incident to place p'. We need to show that these transitions are still covered by minimal closed support T-invariants. Consider the set of minimal closed support T-invariants in $\mathcal{SPN}$ that visit place p', i.e., $\{\boldsymbol{x} \mid \exists t \in \|\boldsymbol{x}\| \text{ with } \boldsymbol{I}_{p'}(t) \geq 0 \vee \boldsymbol{O}_{p'}(t) \geq 0\}$. Now consider the consecutive transitions $t, t' \in \|\boldsymbol{x}\|$ for which $\boldsymbol{O}(t) = \boldsymbol{I}(t')$ and $\boldsymbol{O}_{p'}(t) \geq 0$ in the original net $\mathcal{SPN}$. In net $\mathcal{SPN}'$, $\boldsymbol{O}(t) = \boldsymbol{I}(t')$ still holds, since both in $\boldsymbol{O}(t)$ and $\boldsymbol{I}(t')$ place p' is removed. Therefore, each minimal closed support T-invariant $\boldsymbol{x}$ in $\mathcal{SPN}$ is still a minimal closed support T-invariant in $\mathcal{SPN}'$. Since it may be that for two minimal closed support T-invariants $\boldsymbol{x}^1, \boldsymbol{x}^2$ that both visit place p', place p' is the only conflict place of $CI(\boldsymbol{x}^1)$ and $CI(\boldsymbol{x}^2)$, i.e., $CI(\boldsymbol{x}^1) \cap CI(\boldsymbol{x}^2) = p'$, $\mathcal{SPN}'$ may consist of two separate $\mathcal{S}\Pi$-nets. The proof is completed by repeating this argument until all places $p \in \mathcal{P}$ are removed. □

Theorem 6.10. *Consider a product form $\mathcal{SPN}$ and a surplus place set $\mathcal{P}^{\text{sur}}$ with corresponding sufficient place set $\mathcal{P}^{\text{suf}}$. If $\nexists t \in T$ for which $\{p \in P \mid I_p(t) > 0\} \subseteq \mathcal{P}^{\text{int}} = \{p \in P \mid p \in (\mathcal{P}^{\text{con}} \cap \mathcal{P}^{\text{sur}})\}$, then*

- *removing all places $p \in \mathcal{P}^{\text{int}}$ and all arcs incident to the places $p \in \mathcal{P}^{\text{int}}$ yields s product form $\mathcal{S}\Pi$-nets: $\mathcal{SPN}^1, \ldots, \mathcal{SPN}^s$; each $\mathcal{SPN}^i$ corresponding of one or more connected common input bag classes,*
- *the equilibrium distribution π of $\mathcal{SPN}$ is a product over the invariant measures of the subnets:*

$$\pi(\boldsymbol{m}) = B \prod_{i=1}^{s} \pi_y^{\mathcal{SPN}^i}(\boldsymbol{m}^i), \quad \boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0),$$

where $\boldsymbol{m}^i$ is the submarking in places that belong to subnet $\mathcal{SPN}^i$, $\pi_y^{\mathcal{SPN}^i}(\boldsymbol{m}^i)$ is the invariant measure of subnet $\mathcal{SPN}^i$ with

$$\pi_y^{\mathcal{SPN}^i}(\boldsymbol{m}^i) = \prod_{\{p \in \cap_{j=1}^{J^i} P(CI^i(\boldsymbol{x}^j)) \setminus \mathcal{P}^{\text{con}}\}} f_p^{m_p}, \tag{28}$$

where $CI^i(\boldsymbol{x}^j)$, $j = 1, \ldots, J^i$, denote the J^i common input bag classes contained in subnet $\mathcal{SPN}^i$, and B is a normalizing constant such that $B^{-1} = \sum_{\boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)} \pi_y(\boldsymbol{m})$.

Proof. When the places $p \in \mathcal{P}^{\text{int}}$ and all arcs connected to these places are removed from $\mathcal{SPN}$, by Lemma 6.9, $\mathcal{SPN}$ falls apart in subnets $\mathcal{SPN}^1, \ldots, \mathcal{SPN}^s$ that are again $\mathcal{S}\Pi$-nets. Since in general not all conflict places are contained in $\mathcal{P}^{\text{int}}$, common input bag classes that share a conflict place that is not contained in $\mathcal{P}$ are contained in the same subnet $\mathcal{SPN}^i$.

For the second part, by Lemma 6.7, for $\mathcal{SPN}$ there exists a solution to $\boldsymbol{zA} = \boldsymbol{C}$, in which $z_p = 0, \forall p \in \mathcal{P}^{\text{con}}$. The product form stationary distribution (19) can thus be rewritten as

$$\pi_y(\boldsymbol{m}) = \prod_{i=1}^{s} \left\{ \prod_{\{p \in \cap_{j=1}^{J^i} P(CI^i(\boldsymbol{x}^j)) \setminus \mathcal{P}^{\text{con}}\}} f_p^{m_p} \right\}.$$

We are left to show that the f_p values are the same for the subnets as for the original net. This can be seen as follows. Introduce matrix $\boldsymbol{A}'$, which is the modified incidence matrix $\boldsymbol{A}$ so that the rows corresponding to the places of the conflict place set are set to zero, i.e., $\boldsymbol{a}_p = 0$ for all $p \in \mathcal{P}^{\text{con}}$. Then we have $\boldsymbol{zA} = \boldsymbol{zA}'$. The system of equations $\boldsymbol{zA}' = \boldsymbol{C}$ can be permuted such that the conflict places are grouped and the places of each $\mathcal{SPN}^i$ class are grouped:

$$\widetilde{\boldsymbol{z}}\widehat{\boldsymbol{A}'} = \widetilde{\boldsymbol{z}} \begin{pmatrix} \boldsymbol{A}^1 & 0 & \cdots & 0 \\ 0 & \boldsymbol{A}^2 & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ \vdots & \cdots & 0 & \boldsymbol{A}^s \\ 0 & \cdots & \cdots & 0 \end{pmatrix} = \widetilde{\boldsymbol{C}} = \begin{pmatrix} \boldsymbol{C}^1 & \cdots & \boldsymbol{C}^s \end{pmatrix}.$$

The proof is concluded by observing that the matrices $\boldsymbol{A}^i$ and vectors $\boldsymbol{C}^i$, $i = 1, \dots, s$ correspond exactly to the incidence matrices and the $\boldsymbol{C}$-vectors of the subnets $\mathcal{SPN}^1, \dots, \mathcal{SPN}^s$. $\square$

To illustrate Theorem 6.10, we present three examples. First, in Example 6.11, all conflict places can be removed, which implies a decomposition that separates all common input bag classes. Second, Example 6.12 presents a net with a decomposition where several common input bag classes stay connected, because it is not allowed that a complete input bag is contained in $\mathcal{P}^{\text{int}}$. Otherwise, at least one of the minimal closed support T-invariants would be removed. As will be discussed, both Examples 6.11 and 6.12 can be formulated as *CS* class of decomposable $\mathcal{SPN}$s according to Frosch and Natarajan [11]. Example 6.15 shows that Theorem 6.10 is a generalization of Frosch and Natarajan, by presenting a decomposable $\mathcal{S}\Pi$-net which is not a *CS*.

Example 6.11 (*Complete Decomposition in Common Input Bag Classes*)**.** Consider the Petri net depicted in Fig. 5. From the incidence matrix

$$A = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & -1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

we obtain that this net has two T-invariants $\boldsymbol{x}^1 = (1100)$ and $\boldsymbol{x}^2 = (0011)$ and three minimal support P-invariants $\boldsymbol{y}^1 = (11000)$, $\boldsymbol{y}^2 = (00011)$ and $\boldsymbol{y}^3 = (01101)$, which are linearly independent. The number of places in a sufficient place set is thus $N - 3 = 2$. The two minimal support T-invariants both have a closed support, so that it is an $\mathcal{S}\Pi^2$-net, and $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ are not in common input bag relation, so that we have common input bag classes $CI(\boldsymbol{x}^1)$ and $CI(\boldsymbol{x}^2)$, with one conflict place p_3.

Consider the sufficient place set $\mathcal{P}^{\text{suf}} = \{p_1, p_4\}$, with corresponding surplus place set $\mathcal{P}^{\text{sur}} = \{p_2, p_3, p_5\}$. Then, the conditions of Theorem 6.10 are satisfied, and by removing place p_3 the net decomposes into two subnets: $\mathcal{SPN}^1$ related to $CI(\boldsymbol{x}^1)$ and $\mathcal{SPN}^2$ related to $CI(\boldsymbol{x}^2)$, with invariant measures

$$\pi_y^{\mathcal{SPN}^1}(\boldsymbol{m}^1) = \left(\frac{\mu_2}{\mu_1}\right)^{m_1} \quad \text{and} \quad \pi_y^{\mathcal{SPN}^2}(\boldsymbol{m}^2) = \left(\frac{\mu_4}{\mu_3}\right)^{m_4}.$$

The equilibrium distribution of $\mathcal{SPN}$ is

$$\pi(\boldsymbol{m}) = B\pi_y^{\mathcal{SPN}^1}(\boldsymbol{m}^1)\pi_y^{\mathcal{SPN}^2}(\boldsymbol{m}^2), \quad \boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0).$$

This example is an illustration of a special case of Theorem 6.10. To observe this, let $\{CI(\boldsymbol{x}^1), \dots, CI(\boldsymbol{x}^\ell)\}$ be the set of common input bag classes of a certain $\mathcal{S}\Pi$-net $\mathcal{SPN}$. When for $\mathcal{SPN}$ there exists a surplus place set $\mathcal{P}^{\text{sur}}$ and corresponding sufficient place set $\mathcal{P}^{\text{suf}}$, such that $\mathcal{P}^{\text{con}} \subseteq \mathcal{P}^{\text{sur}}$ and $\nexists t \in T$ for which $\{p \in P \mid \boldsymbol{I}_p(t) \geq 0\} \subseteq \mathcal{P}^{\text{con}}$, $\mathcal{SPN}$ decomposes in ℓ subnets $\mathcal{SPN}^1, \dots, \mathcal{SPN}^\ell$ with each $\mathcal{SPN}^i$ corresponding to one common input bag class $CI(\boldsymbol{x}^i)$. $\square$

Example 6.12 (*Decomposition in Connected Common Input Bag Classes*)**.** Consider the Petri net depicted in Fig. 6. From the incidence matrix

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix},$$

we obtain that this net has three T-invariants $\boldsymbol{x}^1 = (110000)$, $\boldsymbol{x}^2 = (001100)$ and $\boldsymbol{x}^3 = (000011)$ and four minimal support P-invariants $\boldsymbol{y}^1 = (0001100)$, $\boldsymbol{y}^2 = (0000011)$ and $\boldsymbol{y}^3 = (1101000)$ $\boldsymbol{y}^4 = (1010010)$, which are linearly independent. The number of places in a minimal sufficient place set is thus $N - 4 = 3$. The three minimal support T-invariants all have a closed support, so that it is an $\mathcal{S}\Pi^2$-net, and $\boldsymbol{x}^1$, $\boldsymbol{x}^2$ and $\boldsymbol{x}^3$ are not in common input bag relation, so that we have common input bag classes $CI(\boldsymbol{x}^1) = \{\boldsymbol{x}^1\}$, $CI(\boldsymbol{x}^2) = \{\boldsymbol{x}^2\}$ and $CI(\boldsymbol{x}^3) = \{\boldsymbol{x}^3\}$. The conflict place set is $\mathcal{P}^{\text{con}} = \{p_2, p_3\}$. The complete input bag of transition t_1 is contained in the conflict set, so that not all conflict places can be removed.

However, the connection of common input bag class $CI(\boldsymbol{x}^3)$ with the rest of the network is such that it can be decomposed from the network. Note that for a given sufficient place set $\mathcal{P}^{\text{suf}}$ and corresponding surplus place set $\mathcal{P}^{\text{sur}}$, $\mathcal{P}' = (\mathcal{P}^{\text{suf}} \cup p)$ with $p \in \mathcal{P}^{\text{sur}}$ is also a sufficient place set. Therefore, choose $\mathcal{P}^{\text{sur}} = \{p_3, p_5, p_7\}$, so that $(\mathcal{P}^{\text{sur}} \cap \mathcal{P}^{\text{con}}) = \{p_3\}$. By Theorem 6.10 the network decomposes into $\mathcal{SPN}^1 = \{CI(\boldsymbol{x}^1), CI(\boldsymbol{x}^2)\}$ and $\mathcal{SPN}^2 = \{CI(\boldsymbol{x}^3)\}$, with invariant measures

$$\pi_y^{\mathcal{SPN}^1}(\boldsymbol{m}^1) = \left(\frac{\mu_1}{\mu_2}\right)^{m_1}\left(\frac{\mu_4}{\mu_3}\right)^{m_4} \quad \text{and} \quad \pi_y^{\mathcal{SPN}^2}(\boldsymbol{m}^2) = \left(\frac{\mu_6}{\mu_5}\right)^{m_6}.$$

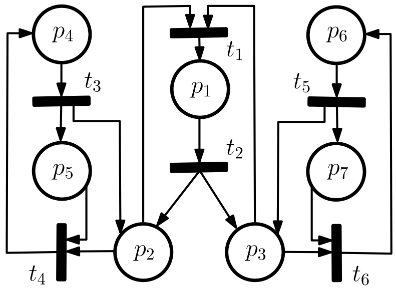

Fig. 6. An $\mathcal{SPN}$ that decomposes into two components.

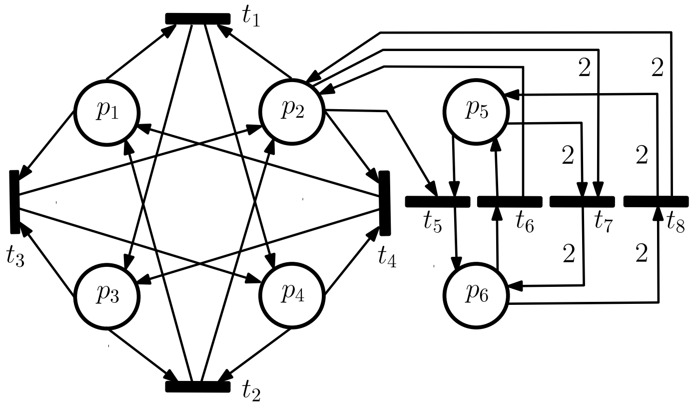

Fig. 7. A decomposable $\mathcal{SPN}$ which neither a CS nor an $\mathcal{S}\Pi^2$-net.

The equilibrium distribution of $\mathcal{SPN}$ is

$$\pi(\boldsymbol{m}) = B\pi_y^{\mathcal{SPN}^1}(\boldsymbol{m}^1)\pi_y^{\mathcal{SPN}^2}(\boldsymbol{m}^2), \quad \boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0). \quad \square$$

Below, we argue that Theorem 6.10 is a generalization of Frosch and Natarajan [11]. Let us first provide the formal definition of a *CS* and provide the theorem of Frosch and Natarajan.

Definition 6.13 (*Closed Synchronized Systems of Stochastic Sequential Processes (CS)*). A structurally bounded stochastic Petri net $\mathcal{SPN} = (P_1 \cup \cdots \cup P_m \cup \mathcal{B}, T_1 \cup \cdots \cup T_m, I, O, Q)$ is a closed synchronized system of stochastic sequential processes if and only if:

1. $\forall i, j \in \{1, \ldots, m\}$ such that $i \neq j : P_i \cap P_j = \emptyset, T_i \cap T_j = \emptyset, P_i \cap \mathcal{B} = \emptyset$
2. $\forall i \in \{1, \ldots, m\} : \mathcal{M}_i = (P_i, T_i, I|_i, O|_i, Q|_i)$ are cyclic state machines (where $I|_i, O|_i, Q|_i$ are the restrictions of I, O and Q to P_i and T_i).

Theorem 6.14 (*[11]*). *Let $(\mathcal{SPN}, \boldsymbol{m}_0)$ be a live marked CS. Consider the following assumption:*

$\mathcal{A}$: Let $\boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)$ and t_0 a transition in state machine $\mathcal{M}_i$, which is enabled in $\boldsymbol{m}$. Further, let $\boldsymbol{x}$ be a minimal support T-invariant of $\mathcal{M}_i$ such that $t_0 \in \|\boldsymbol{x}\|$. Then the sequential transition sequence $\sigma = (t_0, t_1, \ldots, t_n)$ in $\mathcal{M}_i$ corresponding to $\boldsymbol{x}$ has to be a firing sequence in $\boldsymbol{m}$, i.e. $\boldsymbol{m}[\sigma > \boldsymbol{m}' \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)$.

Let $(\mathcal{SPN}, \boldsymbol{m}_0)$ satisfy $\mathcal{A}$. Then the equilibrium distribution π of $(\mathcal{SPN}, \boldsymbol{m}_0)$ is given by

$$\pi(\boldsymbol{m}) = B\prod_{i=1}^{m} \pi_y^{\mathcal{SPN}^i}(\boldsymbol{m}^i), \quad \boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0),$$

where B is a normalizing constant and $\pi_y^{\mathcal{SPN}^i}(\boldsymbol{m}^i)$ is the invariant measure of state machine i.

Both Petri nets from Examples 6.11 and 6.12 can be regarded as *CS*s, when the buffer places in $\mathcal{B}$ are respectively chosen as $\{p_3\}$ and $\{p_3, p_5\}$. A *CS* is obtained by starting from separate state machines and linking these by buffer places, so that the buffer places are defined beforehand. Therefore, Theorem 6.14 can be interpreted as a composition result rather than a decomposition result. In addition, note that it not a structural decomposition result, but a behavioural one.

Assumption $\mathcal{A}$ ensures that the connection of the state machines is such that the state machines are synchronized by the buffer places in a way that the transitions of the state machines are expanded with arcs to the buffer places so that only minimal closed support T-invariants are formed from the T-invariants of the state machines. As a consequence, a *CS* that satisfies assumption $\mathcal{A}$ is an $\mathcal{S}\Pi^2$-net.

Example 6.15 (*Non-CS, Non-$\mathcal{S}\Pi^2$*)**.** Consider the stochastic Petri net $\mathcal{SPN}$ depicted in Fig. 7. This is an example of an $\mathcal{S}\Pi$-net, which is neither a *CS*, the class of decomposable $\mathcal{SPN}$s defined by Frosch and Natarajan [11], nor an $\mathcal{S}\Pi^2$-net. From the incidence matrix

$$A = \begin{pmatrix} -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 & -1 & 1 & -2 & 2 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 & 2 & -2 \end{pmatrix}$$

we obtain that this net has six minimal support T-invariants $\boldsymbol{x}^1 = (11000000)$, $\boldsymbol{x}^2 = (00110000)$, $\boldsymbol{x}^3 = (00001100)$, $\boldsymbol{x}^4 = (00000011)$, $\boldsymbol{x}^5 = (00000210)$ and $\boldsymbol{x}^6 = (00002001)$, of which $\boldsymbol{x}^1, \boldsymbol{x}^2, \boldsymbol{x}^3$ and $\boldsymbol{x}^4$ have a closed support. It has three minimal support P-invariants $\boldsymbol{y}^1 = (100100)$, $\boldsymbol{y}^2 = (011001)$ and $\boldsymbol{y}^3 = (000011)$, which are linearly independent. The number of places in a sufficient place set is thus $N - 3 = 3$.

The minimal closed support T-invariants $\boldsymbol{x}^1, \boldsymbol{x}^2, \boldsymbol{x}^3, \boldsymbol{x}^4$ are not in a common input bag relation, so that we have common input bag classes $CI(\boldsymbol{x}^1)$, $CI(\boldsymbol{x}^2)$, $CI(\boldsymbol{x}^3)$ and $CI(\boldsymbol{x}^4)$ with conflict place set $\mathcal{P}^{\text{con}} = \{p_1, p_2, p_3, p_4, p_5, p_6\}$. Since $\mathcal{SPN}$ is not an $\mathcal{S}\Pi^2$-net, for product form an additional condition on the numerical values of the transition rates is imposed, which is $(\mu_5/\mu_6)^2 = \mu_7/\mu_8$.

$CI(\boldsymbol{x}^1)$ and $CI(\boldsymbol{x}^2)$ cannot be disconnected according to Theorem 6.10, since it would require removal of a complete output bag. The same holds for $CI(\boldsymbol{x}^3)$ and $CI(\boldsymbol{x}^4)$. Therefore, consider the surplus place set $\mathcal{P}^{\text{sur}} = \{p_2\}$, with corresponding sufficient place set $\mathcal{P}^{\text{suf}} = \{p_1, p_3, p_4, p_5, p_6\}$. Then the conditions of Theorem 6.10 are satisfied, and by removing place p_2 the net decomposes in two subnets: $\mathcal{SPN}^1$ related to $CI(\boldsymbol{x}^1)$ and $CI(\boldsymbol{x}^2)$, and $\mathcal{SPN}^2$ related to $CI(\boldsymbol{x}^3)$ and $CI(\boldsymbol{x}^4)$, with invariant measures

$$\pi_y^{\mathcal{SPN}^1}(\boldsymbol{m}^1) = \left(\frac{\mu_1\mu_4}{\mu_2\mu_3}\right)^{\frac{1}{2}(m_1+m_3)} \left(\frac{\mu_1}{\mu_2}\right)^{m_4} \quad \text{and} \quad \pi_y^{\mathcal{SPN}^2}(\boldsymbol{m}^2) = \left(\frac{1}{\mu_5}\right)^{m_5} \left(\frac{1}{\mu_6}\right)^{m_6}.$$

The equilibrium distribution of $\mathcal{SPN}$ is

$$\pi(\boldsymbol{m}) = B\pi_y^{\mathcal{SPN}^1}(\boldsymbol{m}^1)\pi_y^{\mathcal{SPN}^2}(\boldsymbol{m}^2), \quad \boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0).$$

To conclude, observe that an example of a decomposable $\mathcal{S}\Pi$-net which is not a *CS*, but which is an $\mathcal{S}\Pi^2$-net, would be the $\mathcal{SPN}$ from this example without transitions t_7 and t_8. □

Since a sufficient place set is in general not unique, the decomposition according to Theorem 6.10 is not unique. For instance, in Example 6.12, a decomposition in $\mathcal{SPN}^1 = \{CI(\boldsymbol{x}^1), CI(\boldsymbol{x}^3)\}$, $\mathcal{SPN}^2 = \{CI(\boldsymbol{x}^2)\}$ is possible too. Removing the places in $\mathcal{P}^{\text{int}}$ in Theorem 6.10 either removes a complete input bag or implies a decomposition. To conclude, we will present an algorithm that exploits Lemma 6.3 and Theorem 6.10 to find all possible decompositions. Observe that decomposition according to Theorem 6.10 is realized by identifying places that are both conflict places and surplus places. In the algorithm below we exploit this property, by generating surplus place sets that are contained in the conflict place set. Each surplus place set that provides a decomposition, provides a specific decomposition. However, different surplus place sets may lead to the same decomposition if they have an identical intersection with the conflict place set.

Algorithm 6.16 (*Generating All Decompositions*)**.**

Step 1. Consider a product form $\mathcal{SPN}$. Execute the following initialization steps:
  (a) determine from the set of common input bag classes $\{CI(\boldsymbol{x}^1), \dots, CI(\boldsymbol{x}^\ell)\}$, the set of conflict places: $\mathcal{P}^{\text{con}} = \left\{p \in P \mid p \in \left(P(CI(\boldsymbol{x}^i)) \cap P(CI(\boldsymbol{x}^j))\right), \forall i, j \text{ with } CI(\boldsymbol{x}^i) \neq CI(\boldsymbol{x}^j)\right\}$.
  (b) obtain the powerset $\mathcal{P}^{\text{con}}_{\text{all}} = \text{Power}(\mathcal{P}^{\text{con}})$ of the set $\mathcal{P}^{\text{con}}$. Remove from $\mathcal{P}^{\text{con}}_{\text{all}}$ all sets that contain a complete input bag.
  (c) define the set of surplus place sets that provide a decomposition $\mathcal{P}^{\text{dec}}_{\text{all}}$ and set $\mathcal{P}^{\text{dec}}_{\text{all}} = \emptyset$.

Step 2. Take an element $\mathcal{P} \in \mathcal{P}^{\text{con}}_{\text{all}}$ and apply the procedure from Lemma 6.3 to check whether $\mathcal{P}$ is a surplus place set. If yes, go to step 3, else go to step 4.

Step 3. $\mathcal{P}^{\text{con}}_{\text{all}} := \mathcal{P}^{\text{con}}_{\text{all}} \setminus \text{Power}(\mathcal{P})$ and $\mathcal{P}^{\text{dec}}_{\text{all}} := \mathcal{P}^{\text{dec}}_{\text{all}} \cup \text{Power}(\mathcal{P})$. Go to step 5.

Step 4. Remove $\mathcal{P}$ and all its supersets from $\mathcal{P}^{\text{con}}_{\text{all}}$, i.e. $\mathcal{P}^{\text{con}}_{\text{all}} := \mathcal{P}^{\text{con}}_{\text{all}} \setminus \{\mathcal{P}' | \mathcal{P}' \in \mathcal{P}^{\text{con}}_{\text{all}}, \mathcal{P} \subseteq \mathcal{P}'\}$.

Step 5. If $\mathcal{P}^{\text{con}}_{\text{all}} \neq \emptyset$ go back to step 2, else go to step 6.

Step 6. For each surplus place set $\mathcal{P} \in \mathcal{P}^{\text{dec}}_{\text{all}}$, solving $z\boldsymbol{A} = \boldsymbol{C}$ with $z_p = 0$ for $p \in \mathcal{P}$, yields a unique decomposition of the equilibrium distribution of $\mathcal{SPN}$:
$\pi(\boldsymbol{m}) = B\prod_{i=1}^{s} \pi_y^{\mathcal{SPN}^i}(\boldsymbol{m}^i)$, with $\pi_y^{\mathcal{SPN}^i}(\boldsymbol{m}^i)$ given in (28).

Example 6.12 revisited. To illustrate the application of Algorithm 6.16 let us return to the simple and insightful Example 6.12 once more and execute the algorithm.

Step 1. The conflict place set is $\mathcal{P}^{\text{con}} = \{p_2, p_3\}$. Therefore, the candidate decomposition place sets are $\{p_2\}$, $\{p_3\}$ and $\{p_2, p_3\}$, from which $\{p_2, p_3\}$ is removed as it contains a complete input bag. Thus, $\mathcal{P}^{\text{con}}_{\text{all}} = \{\{p_2\}, \{p_3\}\}$.

Step 2–5. Both $\{p_2\}$ and $\{p_3\}$ are surplus place sets. As a consequence, there are two options to decompose the $\mathcal{SPN}$: $\mathcal{P}^{\text{dec}}_{\text{all}} = \{\{p_2\}, \{p_3\}\}$.

Step 6. The two possible decompositions both divide the $\mathcal{SPN}$ in two subnetworks such that

$$\pi(\boldsymbol{m}) = B\pi_y^{\mathcal{SPN}^1}(\boldsymbol{m}^1)\pi_y^{\mathcal{SPN}^2}(\boldsymbol{m}^2), \quad \boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0),$$

where for the first surplus place set $\{p_2\}$ the two subnetworks are $\mathcal{SPN}^1 = \{CI(\boldsymbol{x}^1, \boldsymbol{x}^3)\}$ and $\mathcal{SPN}^2 = \{CI(\boldsymbol{x}^2)\}$ and for the second surplus place set $\{p_3\}$ these are $\mathcal{SPN}^1 = \{CE(\boldsymbol{x}^1, \boldsymbol{x}^2)\}$ and $\mathcal{SPN}^2 = \{CI(\boldsymbol{x}^3)\}$. □

7. Discussion

Structural product form and decomposition results for stochastic Petri nets have been surveyed, unified and extended. Group-local-balance has been shown to be the unifying concept between known product form results for stochastic Petri nets and has provided the ground to formulate necessary and sufficient structural conditions for product form and decomposition and to obtain a structural and intuitive explanation of these conditions, completely in terms of P- and T-invariants. Product form has been discussed in Section 5 and decomposition was the topic of Section 6. Below, we provide an overview of the main results of this paper.

Theorem 4.5 opens the batch-routing queueing network literature for stochastic Petri nets as it provides the translation of product form results for batch routing queueing networks based on group-local-balance to stochastic Petri nets. Group-local–local balance implies that for product form a positive solution is required to the routing chain (15). Theorem 5.4 states that for a stochastic Petri net a positive solution for the routing chain exists if and only if it is an $\mathcal{S}\Pi$-net. Theorem 5.12 states that an $\mathcal{S}\Pi$-net has an equilibrium distribution that is a product form over the places of the network if and only if it satisfies group-local-balance. As such, Theorem 5.12 closes the cycle to batch-routing queueing networks. This brings us in the position to investigate the Petri net structure behind group-local-balance.

From Theorem 5.12 it appears that, in general, for group-local-balance to hold in an $\mathcal{S}\Pi$-net, an additional condition on the numerical values of the transition rates is required to be satisfied (see Lemma 5.11). Theorem 5.14 shows that for each minimal *closed* support T-invariant this numerical condition is satisfied irrespective of the numerical values of the transition rates. Therefore, for an $\mathcal{S}\Pi$-net in which each minimal support T-invariant is a minimal *closed* support T-invariant, group-local-balance is satisfied, and thus product form holds.

In this way, we have unified the key steps presented in literature with respect to structural results for product form stochastic Petri nets. Henderson et al. [7] introduced the routing chain. Assuming that a positive solution exists to the global balance equations of the routing chain, they showed that if a closed form solution to ratio condition (16) on the solution of the routing chain can be found, this is the equilibrium distribution. Coleman et al. [24] identified the numerical condition, which is in this paper stated in Lemma 5.11, under which such a closed form solution exists and is of product form. We have shown that both the results of Henderson et al. and Coleman et al. can be explained as originating from group-local-balance. The last step was to unify Theorem 5.14 with the characterization by Haddad et al. [6] and Mairesse and Nguyen [8] of rate-insensitive product form stochastic Petri nets. Their algebraic definitions of respectively $\mathcal{S}\Pi^2$-nets and deficiency zero $\mathcal{S}\Pi$-nets, subclasses of $\mathcal{S}\Pi$-nets, were in Theorem 5.16 shown to be equivalent with our characterization of rate-insensitive product form stochastic Petri nets; Theorem 5.16 states that an $\mathcal{S}\Pi$-net is an $\mathcal{S}\Pi^2$-net if and only if *all* minimal support T-invariants are minimal *closed* support T-invariants.

Product form results for network structures often allow for hierarchical composition and decomposition of subnetworks. When interested in global characteristics of a network it is convenient to decompose the network so that local characteristics can be investigated without considering the complete network in detail. Section 6 introduced decomposition results by which subnetworks can be identified in which a given product form stochastic Petri net can be decomposed. These subnetworks correspond to one or more common input bag classes, equivalence classes of minimal closed support T-invariants connected by having an input bag in common. Essential in achieving the decomposition is the notion of the sufficient place set of a Petri net, the set of places sufficient for uniquely characterizing the marking of a Petri net at all its places. The complement of the sufficient place set is the surplus place set, places that can be omitted in characterizing the marking of the Petri net. A procedure to characterize surplus place sets of a Petri net from its P-invariants is provided in Lemma 6.3. Removing conflict places that can be assigned as a surplus place yields decomposition. The restriction is that no complete input bag may be removed. To be specific, Theorem 6.10 states that if a sufficient place set can be found so that there is no input bag of which all places are both surplus and conflict places, a product form stochastic Petri net decomposes into subnets each corresponding to one or more common input bag classes. The steps that have to be performed to verify and construct product form and to obtain all possible decompositions are summarized in Algorithms 5.20 and 6.16.

Finally, observe that characterizing product form for a stochastic Petri net can be done completely in terms of its T-invariants, while decomposition of the network into subnetworks not only requires the T-invariants, but also its

P-invariants. The results presented in this paper suggest several directions for future research. A first extension would be to include state dependent firing and enabling, similar to Henderson et al. [7], Boucherie and Sereno [3] and Haddad et al. [6]. Also, colouring of tokens such as included in [23] can be incorporated in the model by enlarging the state space in a way very similar to the inclusion of multiple customer types in Markov chain models for product form queueing networks (e.g. [1,50]). In addition, we have a particular interest in extending the decomposition results. First, decomposition results seem possible not by removing places, but by assigning conflict places to a unique common input bag class. Second, such a decomposition result may be an opening to a decomposition result in which a stochastic Petri net completely decomposes into its T-invariants. Finally, such exact decomposition results could provide a starting point for deriving approximate results for non-product form stochastic Petri nets which may also be useful in developing a method to algorithmically identify subnets in the framework of competing Markov chains as introduced in [37].

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