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Stability and robustness of planar switching linear systems ☆

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ABSTRACT

This paper presents a decision algorithm for the analysis of the stability of a class of planar switched linear systems, modeled by hybrid automata. The dynamics in each location of the hybrid automaton is assumed to be linear and asymptotically stable; the guards on the transitions are hyperplanes in the state space. We show that for every pair of an ingoing and an outgoing transition related to a location, the exact gain in the norm of the vector induced by the dynamics in that location can be computed. These exact gains are used in defining a gain automaton which forms the basis of an algorithmic criterion to determine if a planar hybrid automaton is stable or not.

1. Introduction

A hybrid automaton [1,2] is an automaton with locations and transitions between the locations, together with continuous dynamics in the locations, usually described by differential equations, and constraints on both locations and transitions. This is a prominent model for the study of hybrid systems [3].

The analysis of the stability of a hybrid system is an important and interesting problem. Even in the case of switched linear systems with asymptotically stable dynamics in each location, it is possible that the switching regime gives rise to a global behavior of the system that is unstable (see e.g. [4]). For an overview of results on hybrid stability see [5–10]. Some results assume arbitrary switching between locations [9,11]; it is then possible to look for a Lyapunov function common to all locations [8,12]. The arbitrary switching assumption would be unsuitable in general for hybrid automata, since there the possible switchings are restricted by the guards of the transitions and the invariants of the locations.

Another stability criterion is that of multiple Lyapunov functions [4,13,8]. Each location is assumed to have a Lyapunov function. Then all behaviors of the system should satisfy the so-called non-increasing sequence property: when a location is visited for the second time, the value of its Lyapunov function should be less than what it was the last time the location was visited. This is a sufficient condition for the stability of a hybrid automaton. In general, checking the non-increasing sequence property may be difficult, as checking all possible behaviors of a hybrid system is clearly not an option.

A different approach relies on the construction of Lyapunov functions that are either piecewise linear [14] or piecewise quadratic [15–17]. In the latter case the piecewise quadratic function should be continuous on the switching boundaries, which can be checked efficiently by solving a linear matrix inequality. The approach has originally been formulated for piecewise affine systems, where the state space is divided into regions, and to each region corresponds a dynamics. It is not so easy to adapt the approach to the more general model of hybrid automata.

For a restricted class of hybrid systems, [18] proposes a sufficient condition for stability that relies on the construction of optimal quadratic Lyapunov functions to provide estimates of the maximal ratio between the norm of an outgoing and an incoming vector induced by the dynamics of a location.

The only source of instability is the switching regime of a cycle. It is therefore sufficient to prove that for every cycle the product of the gains of consecutive transitions is less than one to guarantee stability. However, this technique is inconclusive for systems of three or more dimensions, or when there is a cycle with a gain of more than one.

In this paper we present a decision algorithm for the stability of planar switching linear systems. Switching is enabled by linear guards. This yields a homogeneous switching system in the sense that scalar multiples of admissible trajectories are again admissible trajectories. By restricting ourselves to this case, we are able to show that the maximal gains can be computed exactly by transforming the dynamics to its real Jordan form instead of using a quadratic Lyapunov function to provide an upper estimate of the gain.

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By providing an upper and a lower bound to this gain, we characterize a cycle of the automaton as being (strictly) contractive if the upper bound is (strictly) less than 1, expanding if the lower bound is larger than 1. The absence of (non-strict) non-contractive cycles is a sufficient condition for the (asymptotic) stability of the hybrid automaton. The presence of expanding cycles is a sufficient condition for the instability of the hybrid automaton. In the planar case, the absence of non-contractive (or expanding) cycles is a *necessary and sufficient* condition for the stability of the hybrid automaton.

The remainder of the paper is organized as follows. Section 2 gives basic definitions about hybrid automata and defines the class of hybrid automata considered in this paper. In Section 3 we give the definition of stability of a hybrid automaton and provide sufficient conditions for the stability or instability based on the absence or presence of non-contractive cycles. Section 4 introduces the notion of gain automaton. This is an automaton that is associated to the hybrid automaton under consideration and is used to detect, in a systematic way, the presence of non-contractive or expanding cycles. Section 5 shows how to compute maximal gains exactly for planar systems using the real Jordan form of the matrices. Finally, in Section 8, we give conclusions and directions for future research. For basic notions of systems theory used in this paper we refer to [19].

2. Planar LCH

A planar Linear Continuous Hyperplane hybrid automaton [18] is a hybrid automaton [1,2] such that the continuous dynamics are linear in $\mathbb{R}^2$, the guards of the transitions are lines through the origin, and there are no resets associated with a transition.

Definition 2.1 (*Hybrid Automaton*)**.** A *Planar Linear Continuous Hyperplane* hybrid automaton is a tuple $H = (X, L, \text{Init}, f, E, \text{Guard}, \Sigma)$ where:

- $X = \mathbb{R}^2$ is the *state space* ranged over by the state vector x.
- L is a finite set of *locations*.
- $\text{Init} = L' \times \mathbb{R}^2$ is the set of *initial* location–state pairs for a set $L' \subseteq L$ of locations. So for a given initial location we can start from any state, which is technically convenient when studying stability.
- $f : L \to \mathbb{R}^{2\times 2}$ assigns to each location ℓ a matrix $A_\ell \in \mathbb{R}^{2\times 2}$.
- $E \subseteq L \times L$ is a finite set of *transitions*, also called *switches* between locations.
- $\text{Guard} : E \to \text{R}^2 \setminus \{0\}$ assigns to each transition a nonzero vector $v_e \in \mathbb{R}^2$. The corresponding *guard* is the line through the origin perpendicular to $v_e \in \mathbb{R}^2$: $\{x \mid v_e^{\text{T}} x = 0\}$.
- Σ is a set of *transition labels*. We assume a labeling function $\text{lab} : E \to \Sigma$ and refer to transitions by their labels (assuming uniqueness).

Remark 2.2. The intuitive interpretation of the formal definition above is as follows. The automaton has a finite number of locations. In each location the continuous state evolves according to the dynamics $\frac{d}{dt}x = A_\ell x$. Furthermore, for all pairs of locations for which there exists a transition from the first to the second location, the transition is enabled if and only if the continuous state passes through a line specific to that transition. The automaton contains no invariants and the continuous state is never reset.

Definition 2.3 (*Hybrid Trace*)**.** A *hybrid trace* of an LCH hybrid automaton is a finite or infinite sequence of the form $\sigma = x_1 e_1 x_2 e_2 \cdots x_{m-1} e_{m-1} x_m, \ldots$, with an associated monotonically increasing timing sequence $\tau_0 \tau_1 \cdots \tau_m, \ldots$ (with $\tau_0 = 0$, $\tau_i \in \mathbb{R} \cup \{\infty\}$), such that

- each e_i is a transition from location ℓ_i to location ℓ_{i+1}
- each x_i is a mapping from $[\tau_{i-1}, \tau_i]$ to $\mathbb{R}^n$ satisfying $\frac{d}{dt}x_i = A_{\ell_i} x_i$
- initial and switching constraints and assignments are respected, so $(\ell_1, x_1(0)) \in \text{Init}$, and for all $1 \le i \le m-1$: $v_{e_i}^{\text{T}} x_i(\tau_i) = 0$ and $x_i(\tau_i) = x_{i+1}(\tau_i)$.

Assumption 2.4. To avoid Zeno behavior we impose that for every location, the guards of any incoming transition and any outgoing transition are different. That means that transitions are delayed by a fixed minimal dwell time. As a consequence, for any infinite trace we have $\lim_{i\to\infty} \tau_i = \infty$.

3. Stability

Definition 3.1 (*Stability*)**.** An LCH hybrid automaton is *stable* if $\forall \epsilon > 0\ \exists \delta > 0$ such that for all hybrid traces $x_1 e_1 x_2 e_2 \cdots$ with $\|x_1(0)\| < \delta$ and $\forall i \forall t \in [\tau_{i-1}, \tau_i] : \|x_i(t)\| < \epsilon$. An automaton that is not stable is called *unstable*.

Definition 3.2 (*Asymptotic Stability*)**.** An LCH hybrid automaton is *asymptotically stable* if it is stable and for any infinite hybrid trace $x_1 e_1 x_2 e_2 \cdots$ and for all sequences $\{t_i\}$ with $\tau_i \le t_i \le \tau_{i+1}$ we have $\lim_{i\to\infty} \|x_i(t_i)\| = 0$.

It is well known that even if for each location ℓ the dynamics is asymptotically stable, so the matrix A_ℓ is Hurwitz (i.e., all eigenvalues have negative real part, see [19]), still the hybrid automaton can be unstable (see e.g. [4] for a simple example) because of the switching. We say that a hybrid automaton has *Hurwitz locations* (with some abuse of terminology) if for each location ℓ the matrix A_ℓ is Hurwitz.

Our problem consists in analyzing the stability of an LCH hybrid automaton with Hurwitz locations. We first introduce the concept of gain associated with a pair of an incoming and an outgoing transition from a given location. We then show how gains can be used to give sufficient conditions for the stability or instability of an LCH hybrid automaton. The analysis of the stability is based on the analysis of the cycles of the automaton. Because we consider Hurwitz locations, instability can arise only from executions with infinite switching. Indeed, in case of only a finite number of switches the continuous state remains in a single location from a certain time instant on and hence converges to zero.

3.1. Gains

Suppose a location ℓ is entered via a transition a with a state vector x_a and is left via a transition b with a state vector x_b. An indication as to how the location contributes to the stability or instability is the ratio of the norm of the outbound state and the inbound state. A ratio below one is in favor of stability whereas a ratio above one points at instability.

Since the actual ratio depends on the trace and the state trajectory (and in particular on the dwell time in a location) we consider the maximal gain that only depends on the pair of inbound and outbound transitions of a given location.

Definition 3.3 (*Maximal Gain*)**.** The maximal gain when entering location ℓ via transition a and leaving it via transition b is $\gamma_{ab} \in \mathbb{R}^+ \cup \{\perp\}$ such that, for any solution $x(t)$ of $\frac{d}{dt}x = A_\ell x$ with $v_a^{\text{T}} x(0) = 0$ we have:

- $\gamma_{ab} = \perp$ if $\nexists t$ s.t. $v_b^{\text{T}} x(t) = 0$
- $\gamma_{ab} > 0$ if $\exists t^*$ s.t. $v_b^{\text{T}} x(t^*) = 0$ and $\frac{\|x(t^*)\|}{\|x(0)\|} = \gamma_{ab}$
- $\forall t$ if $v_b^{\text{T}} x(t) = 0$ then $\frac{\|x(t)\|}{\|x(0)\|} \le \gamma_{ab}$.

A maximal gain equal to $\perp$ means that location ℓ entered via a will never be left via b. A gain strictly greater than 0 means that the location can be left via b and that the corresponding gain in the norms of the vectors will be γ_{ab} in the worst case. The existence of the maximum is ensured because we consider stable locations and linear dynamics. It is easy to see that for planar systems the maximal gain is attained when the system leaves the location at the first possible occasion.

In the following, we assume that we know a lower and an upper bound of the maximal gain, i.e. $\alpha_{ab}, \beta_{ab} \in \mathbb{R}^+$ such that $0 \leq \beta_{ab} \leq \gamma_{ab} \leq \alpha_{ab}$.

3.2. Contractive cycles

Suppose we have a hybrid automaton with Hurwitz locations. If for each location that can be visited infinitely often the gain is ≤ 1, then it can be seen that the hybrid automaton is stable: since the number of locations in a trace that are visited only once (seen as a function of a trace) is bounded, there is a bound to the gains corresponding to the traces. However, such a condition is unnecessarily restrictive as it does not take into account situations where a higher gain in one location is compensated by a lower gain in another location. So we need a more global condition.

Definition 3.4. Let H be a hybrid automaton, then a *(strictly) contractive cycle* of H is a sequence of transitions $C = e_1e_2 \cdots e_m$ such that each e_i is a transition from ℓ_i to ℓ_{i+1}, with $\ell_1 = \ell_{m+1}$, and $\alpha(C) = \alpha_{e_1e_2}\alpha_{e_2e_3} \cdots \alpha_{e_me_1} \leq 1 (< 1)$. The scalar $\alpha(C)$ is called the upper estimate of the cycle gain.

Theorem 3.5 provides a sufficient condition for the (asymptotic) stability of an LCH hybrid system based on the absence of (non-strict) non-contractive cycles.

Theorem 3.5. *Let H be an LCH hybrid automaton with Hurwitz locations. If all cycles in H are (strictly) contractive then H is (asymptotically) stable.*

Proof. Let σ be an infinite trace and σ_n the first n steps of σ, say:

$$\sigma_n = x_1e_1 \cdots e_{n-1}x_n.$$

Denote the number of locations of the hybrid automaton by N. Every sub-trace of length $N + 1$ contains a cycle C:

$$\sigma_n = x_1e_1 \cdots C \cdots e_{n-1}x_n.$$

The cycle C gives a contribution $\alpha(C)$ to the gain of σ_n. Remove C from σ_n and proceed inductively on the remainder to conclude that:

$$\alpha(\sigma_n) \leq \alpha_0\alpha^k,$$

where $\alpha \leq 1$ is the maximum over all cycle gains and α_0 the maximum over all possible finite traces of length smaller than $N + 1$. Furthermore, $k > \frac{N-1}{n}$. The latter lower bound on k follows from a reduction step on σ_n that can be repeated as long as the length after reduction exceeds $N + 1$. As a consequence we have that:

$$\|x_n(\tau_n)\| \leq \alpha_0\alpha^k\|x_0(0)\|.$$

Notice furthermore that there exist positive constants d_0, d_1 such that within any location we have that

$$d_0\|x(\tau_i)\| \leq \|x(t)\| \leq d_1x(\tau_{i+1})\|.$$

Therefore (asymptotic) stability can be concluded from the behavior at the switching times τ_i. The conclusion is that if all cycles are contractive, implying that $\alpha \leq 1$, we have at least stability, and if all cycles are strictly contractive we have asymptotic stability. □

3.3. Expanding cycles

Definition 3.6. Let H be a hybrid automaton, then a *(strictly) expanding cycle* of H is a sequence of transitions $C = e_1e_2 \cdots e_m$ such that each e_i is a transition from ℓ_i to ℓ_{i+1}, with $\ell_1 = \ell_{m+1}$, and $\beta(C) = \beta_{e_1e_2}\beta_{e_2e_3} \cdots \beta_{e_me_1} > 1$. The scalar $\beta(C)$ is called the lower estimate of the cycle gain.

Theorem 3.7 provides a sufficient condition for the instability of an LCH hybrid system based on the detection of (strictly) expanding cycles.

Theorem 3.7. *Let H be an LCH hybrid automaton with Hurwitz locations. If H has a strictly expanding cycle then H is unstable.*

Proof. If H has an expanding cycle, then there exists a trace that keeps cycling through that cycle. Obviously such a trace will grow without bound if the initial continuous state is nonzero. □

4. Interval gain automata and cycle analysis

Theorem 3.5 provides us with a sufficient condition for stability, namely the absence of non-contractive cycles, and Theorem 3.7 with a sufficient condition for instability, namely the presence of expanding cycles. In order to check for non-contractive or expanding cycles we first transform a hybrid automaton into what we call *gain automaton*.

Definition 4.1. An *interval gain automaton* is a tuple $GA = (S, S^0, G)$ where

- S is the set of nodes,
- S^0 is the set of initial nodes,
- $G \subseteq S \times (\mathbb{R}^+ \times \mathbb{R}^+) \times S$ is the set of edges labeled with intervals of gains.

Definition 4.2. Let H be a planar LCH, then the gain automaton for H is defined by $GA(H) = (S_H, S_H^0, G_H)$ where

- The nodes of the gain automaton are the transitions of H, i.e. $S_H = E$.
- The initial nodes S_H^0 are the transitions from an initial location of H.
- For each pair of adjacent transitions a and b in H such that $\xrightarrow{a} l \xrightarrow{b}$ and $\alpha_{ab} \neq \perp$ there is an edge $a \xrightarrow{\beta_{ab},\alpha_{ab}} b$ in G_H.

It must be noted that there is an edge in the interval gain automaton only if the maximal gain corresponding to the pair of transitions in H is well defined.

We present an algorithm on the gain automaton of a hybrid automaton for the detection of non-contractive and expanding cycles. This algorithm is inspired by the well-known algorithm for transforming an automaton into an equivalent regular expression (see e.g. [20,21]). It works by successively deleting nodes of the gain automaton, while transforming the edges. The basic steps of the algorithm are:

- *Node elimination:* a node is eliminated, as illustrated in Fig. 1(a). Each possible pair of an incoming and outgoing edge of this node leads to a new edge, labeled with the product of the interval gains defined as $(\beta_1, \alpha_1) \otimes (\beta_2, \alpha_2) = (\beta_1\beta_2, \alpha_1\alpha_2)$.
- *Double edge elimination:* as illustrated in Fig. 1(c), if two edges have the same initial and final node they are transformed into a single edge, labeled with the union of the interval gains defined as $(\beta_1, \alpha_1) \oplus (\beta_2, \alpha_2) = (\min(\beta_1, \beta_2), \max(\alpha_1, \alpha_2))$.

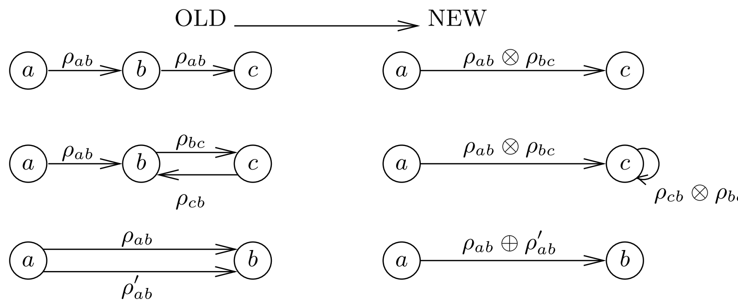

Fig. 1. Basic steps of the algorithm.

- *Loop edge analysis:* it is possible that deleting a node creates a loop edge, as illustrated in Fig. 1(b). The algorithm analyzes the gain of a loop edge and then removes it. If the lower bound of the gain of such a loop edge is >1 (i.e. an *expanding* loop edge) then the algorithm terminates and the system is unstable. If the upper bound of the gain of such a loop edge is >1 (i.e. a *non-contractive* loop edge) then the algorithm marks the system as non-stable.

The algorithm terminates when an expanding cycle is detected, and the system is unstable, or when all nodes have been removed, in which case the system is stable if no non-contractive cycle has been detected.

Algorithm 4.3.

```
Input: a gain automaton GA.
stable = True
unstable = False
repeat
    check all loop edges;
    if a non-contractive loop edge is found
        then stable = False;
    if an expanding loop edge is found
        then unstable = True;
    remove all loop edges
    eliminate a node;
    eliminate all resulting double edges;
until there is only one node or unstable
return unstable, stable
```

Theorem 4.4. *Let H be an LCH hybrid automaton with Hurwitz locations. If Algorithm 4.3 detects a non-contractive (resp. expanding) loop edge in $GA(H)$ then H contains a non-contractive (resp. expanding) cycle.*

Proof. See [18, Theorem VI.5]. □

The number of nodes in $GA(H)$ is quadratic in the number of nodes of H, and the complexity of Algorithm 4.3 is linear in the number of nodes of GA, so the complexity of Algorithm 4.3 is quadratic in the number of nodes of H. This means we have a computationally efficient way of checking the sufficiency condition for stability and instability.

5. Exact gain computation for planar systems

In this section we show that for *planar* LCH systems, the exact maximal gain for any pair of incoming and outgoing transitions can be obtained by computing the real Jordan form of the dynamics matrix. A case by case analysis of the different types of Jordan form shows that there is an analytic solution to the problem of finding, for any incoming state, the outgoing state with maximal gain, which corresponds in the planar case to leaving the location at the first possible occasion.

With the exact computation of maximal gains we obtain a necessary and sufficient condition for the stability of planar LCH from Theorems 3.5 and 3.7. The interval gain automaton of a planar LCH is such that the lower and upper bounds in every edge are equal to the corresponding maximal gain. In this case, Algorithm 4.3 becomes a decision procedure for the stability (or instability) of the system.

5.1. Gain in real Jordan form

The real Jordan form for a 2×2 matrix is of one of the following forms:

$$\text{(a)} \begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_1 \end{pmatrix}, \quad \lambda_0 \neq \lambda_1 \qquad \text{(b)} \begin{pmatrix} \lambda_0 & 1 \\ 0 & \lambda_0 \end{pmatrix}$$

$$\text{(c)} \begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_0 \end{pmatrix} \qquad \text{(d)} \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \quad \beta \neq 0$$

where $\lambda_0, \lambda_1, \alpha$ and β are real. The different types of Jordan forms correspond to the different possibilities of eigenvalues and eigenvectors. Case (a) corresponds to the case of two different real eigenvalues, case (b) to the case of one real eigenvalue of algebraic multiplicity two and geometric multiplicity one, and case (d) is for two different complex eigenvalues. Case (c) is the case of one real eigenvalue of algebraic and geometric multiplicity two; it is easy to deal with.

For every location ℓ with an incoming transition a and an outgoing transition b determined by vectors v_a and v_b respectively, we first compute the eigenvalues and eigenvectors of A_ℓ to determine its real Jordan form J_ℓ such that $A_\ell = MJ_\ell M^{-1}$, M being the matrix of the change of basis from J_ℓ to A_ℓ.

We show how to compute the maximal gain γ'_{ab} for a matrix in real Jordan form. For that, we assume that the guard of a is given by $a_1 y = a_2 x$ and the guard of b is $b_1 y = b_2 x$ in the basis of J_ℓ. To determine the maximal gain, we need to find if and when the solution of the system $\dot{x} = Jx$, with $x(0) = \binom{x_0}{y_0} \neq 0$ an initial state in the incoming line (i.e. $a_1 y_0 = a_2 x_0$), intersects the switching line $b_1 y = b_2 x$.

Case (a): two *different real eigenvalues*. Let $J = \begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_1 \end{pmatrix}$ be a stable matrix in diagonal form with $\lambda_0, \lambda_1 < 0$. The trajectory is given by

$$x(t) = x_0 e^{\lambda_0 t} \qquad y(t) = y_0 e^{\lambda_1 t}.$$

We can determine exactly if and when the trajectory intersects the outgoing switching line $b_1 y = b_2 x$. It is easy to see that there is no intersection if any of a_1, a_2, b_1 or b_2 is equal to 0. Otherwise, let $a = a_1/a_2$ and $b = b_1/b_2$. The intersection of the trajectory with the guard happens when $y_0 e^{\lambda_1 t} = b x_0 e^{\lambda_0 t}$, that is, when $a e^{\lambda_1 t} = b e^{\lambda_0 t}$. So the intersection happens at

$$t^* = \frac{\log(b/a)}{\lambda_1 - \lambda_0} \quad \text{iff} \quad t^* > 0.$$

The exact maximal gain is then given by $\perp$ if $t^* < 0, a = 0$ or $b = 0$, otherwise $\gamma'_{ab} = \frac{\|\mathbf{x}(t^*)\|}{\|\mathbf{x}(0)\|}$ which, after simplification, yields

$$\gamma'_{ab} = \left(\left|\frac{a}{b}\right|\right)^{\frac{\lambda_0}{\lambda_0-\lambda_1}} \sqrt{\frac{1+b^2}{1+a^2}}. \tag{1}$$

Case (b): two *equal real eigenvalues*, one *eigenvector*. Let $J = \begin{pmatrix} \lambda_0 & 1 \\ 0 & \lambda_0 \end{pmatrix}$ be a stable matrix with $\lambda_0 < 0$. The solution of this system is

$$x(t) = (1+at)x_0 e^{\lambda_0 t} \qquad y(t) = a x_0 e^{\lambda_0 t}.$$

If $a = 0$ or $b = 0$ there is no intersection of the trajectory with $y = bx$. Otherwise, the trajectory intersects the switching line when $x(t) = (\frac{1}{a} + t)y(t) = \frac{1}{b}y(t)$, that is for

$$t^* = \frac{1}{b} - \frac{1}{a} \quad \text{iff} \quad t^* > 0.$$

The exact maximal gain is then $\perp$ if $t^* < 0, a = 0$ or $b = 0$, otherwise $\gamma'_{ab} = \frac{\|\mathbf{x}(t^*)\|}{\|\mathbf{x}(0)\|}$ yields, after simplification,

$$\gamma'_{ab} = \frac{a}{b} e^{\lambda_0 \frac{a-b}{ab}} \sqrt{\frac{1+b^2}{1+a^2}}. \tag{2}$$

Case (d): two *different complex eigenvalues*. Let $J = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$ be a stable matrix with $\alpha \neq 0$. Using polar coordinates (r, θ) with $r = \sqrt{x^2 + y^2}$ and $\tan(\theta) = y/x$, it can be shown that the trajectories must satisfy $\dot{r} = \alpha r$ and $\dot{\theta} = \beta$ and the solution is

$$r(t) = r_0 e^{\alpha t} \qquad \theta(t) = \beta t + \theta_a$$

where $r_0 = \sqrt{x_0^2 + y_0^2}$ and $\theta_a = \operatorname{atan2}(a, 1)$.

We denote by $\theta_b = \operatorname{atan2}(b, 1)$ the angle of the switching line $y = bx$. The first possible switching occurs when $\theta(t) = \theta^*$ such that, if the trajectory is anti-clockwise (i.e. $\beta > 0$),

$$\theta^* = \begin{cases} \theta_b & \text{if } \theta_b - \pi < \theta_a < \theta_b \\ \theta_b + \pi & \text{if } \theta_b < \theta_a < \theta_b + \pi \\ \theta_b + 2\pi & \text{if } \theta_b + \pi < \theta_a < \theta_b + 2\pi \end{cases}$$

and if the trajectory is clockwise (i.e. $\beta < 0$),

$$\theta^* = \begin{cases} \theta_b - \pi & \text{if } \theta_b - \pi < \theta_a < \theta_b \\ \theta_b & \text{if } \theta_b < \theta_a < \theta_b + \pi \\ \theta_b + \pi & \text{if } \theta_b + \pi < \theta_a < \theta_b + 2\pi. \end{cases}$$

This intersection will *always* happen for

$$t^* = \frac{\theta^* - \theta_a}{\beta}$$

and the exact maximal gain is then $\gamma'_{ab} = \frac{r(t^*)}{r_0} = e^{\alpha t^*}$ so

$$\gamma'_{ab} = e^{\frac{\alpha}{\beta}(\theta^* - \theta_a)}. \tag{3}$$

5.2. Gain from a change of basis

For planar systems we can associate a constant gain to a change of basis. Let $\mathbf{i}_a$ be a unit vector in the incoming line and $\mathbf{i}_b$ a unit vector in the outgoing line ($\|\mathbf{i}_a\| = \|\mathbf{b}_b\| = 1$). Let M be a non-singular matrix representing a change of basis and $\mathbf{y} = M\mathbf{x} = \lambda M \mathbf{i}_a$ the image of an incoming vector $\mathbf{x}$ by M. Then,

$$\left(\frac{\|\mathbf{y}\|}{\|\mathbf{x}\|}\right)^2 = \frac{\lambda^2 \mathbf{i}_a^T \cdot M^T M \cdot \mathbf{i}_a}{\lambda^2 \mathbf{i}_a^T \cdot \mathbf{i}_a} = \mathbf{i}_a^T \cdot M^T M \cdot \mathbf{i}_a.$$

We define then the incoming gain γ_a^M and the outgoing gain γ_b^M for the change of basis M as:

$$\gamma_a^M = \left(\mathbf{i}_a^T \cdot M^T M \cdot \mathbf{i}_a\right)^{\frac{1}{2}} \qquad \gamma_b^M = \left(\mathbf{i}_b^T \cdot M^T M \cdot \mathbf{i}_b\right)^{-\frac{1}{2}}.$$

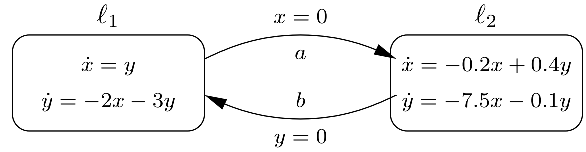

Fig. 2. A simple planar LCH.

5.3. Gain in original basis

The gain in the basis of the original matrix is then the product of the gain in the basis of the Jordan form times the gain for the change of basis:

$$\gamma_{ab} = \gamma_a^M \gamma'_{ab} \gamma_b^M.$$

6. Continuity and robustness

Although for the calculation of the gains for the planar case we distinguish several cases, depending on the location of the eigenvalues, the gains depend continuously on the matrices in the locations.

Theorem 6.1. *The gain as defined in Definition 3.3 depends analytically on A.*

Proof. Choose $\bar{A}$ such that $\gamma_{ab} > 0$ and $v_a^{\mathrm{T}} x_0 = 0$ and $\|x_0\| = 1$. Then t^* is the smallest positive scalar such that

$$v_b^{\mathrm{T}} e^{\bar{A}t^*} x_0 = 0.$$

It follows from the implicit function theorem that there exist an open neighborhood Ω of $\bar{A}$ and an analytic function $t^*(A)$, such that for $A \in \Omega$, we have that $t^*(A)$ is the smallest positive scalar with the property that

$$v_b^{\mathrm{T}} e^{t^*(A)} x_0 = 0.$$

It follows that

$$\gamma_{ab} = \frac{1}{\|e^{t^*(A)} x_0\|}.$$

Therefore the gain is the composition of two analytic functions and hence the gain is also analytic. □

7. Example

Let us illustrate the computation of the exact maximal gain with the simple planar LCH hybrid system of Fig. 2. The system has two locations ℓ_1 and ℓ_2, a transition a from ℓ_1 to ℓ_2, and a transition b from ℓ_2 back to ℓ_1. The dynamics are given by matrix A_1 and A_2 respectively with

$$A_1 = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \qquad A_2 = \begin{pmatrix} -0.2 & 0.4 \\ -7.5 & -0.1 \end{pmatrix}$$

and the guards are orthogonal to $v_a = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $v_b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

We compute the exact gain of the cycle with the method of Section 5 and show that the system is (asymptotically) stable, and that stability cannot be determined using optimal Quadratic Lyapunov Functions as proposed in [18]. Indeed, in [18] only a sufficient condition for stability based on optimally selected quadratic Lyapunov functions is derived. Our example shows once more the well known fact that quadratic Lyapunov functions are restrictive in the stability analysis of hybrid systems.

7.1. Gain in ℓ_1

Let's consider location ℓ_1 with incoming transition a with guard $x = 0$, and outgoing transition b with guard $y = 0$, and dynamics given by $\dot{\mathbf{x}} = A_1\mathbf{x}$. The eigenvalues of A_1 are $\lambda_0 = -1$ and $\lambda_1 = -2$. Matrix A_1 can be put in diagonal form J with the change of basis M corresponding to the eigenvectors, such that $A_1 = MJM^{-1}$ where

$$J = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \qquad M = \begin{pmatrix} 0.707 & -0.447 \\ -0.707 & 0.894 \end{pmatrix}.$$

The gains from the change of basis M^{-1} are computed for the unit vectors $\mathbf{i}_a = \binom{0}{1}$ and $\mathbf{i}_b = \binom{1}{0}$. We obtain $\gamma'_a = \sqrt{7} = 2.645$ and $\gamma'_b = 1/\sqrt{13} = 0.277$.

The gain for the diagonal matrix J is computed using Eq. (1). In this case we have

$$M^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ \sqrt{5} \end{pmatrix}$$

so $a = \sqrt{5/2}$, and

$$M^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2\sqrt{2} \\ \sqrt{5} \end{pmatrix}$$

so $b = \frac{1}{2}\sqrt{5/2}$. Therefore, the gain for J is $\gamma'_{ab} = \frac{1}{4}\sqrt{\frac{13}{7}}$.

The maximal gain in the original basis is

$$\gamma_{ab} = \gamma'_a \gamma'_{ab} \gamma'_b = \frac{1}{4}.$$

On the other hand, the upper bound on the maximal gain obtained with the optimal Quadratic Lyapunov Function as in [18] is $\rho_{ab} = 1/\sqrt{12.68} = 0.28$.

7.2. Gain in ℓ_2

Let's consider location ℓ_2 with incoming transition b with guard $y = 0$, and outgoing transition c with guard $x = 0$, and dynamics given by $\dot{\mathbf{x}} = A_2\mathbf{x}$. The conjugate complex eigenvalues of A_2 are $\lambda_0 = -0.15 + 1.731j$ and $\lambda_1 = -0.15 - 1.731j$. Matrix A_2 is similar to matrix J with a change of basis M such that $A_2 = MJM^{-1}$ where

$$J = \begin{pmatrix} -0.15 & -1.731 \\ 1.731 & -0.15 \end{pmatrix} \qquad M = \begin{pmatrix} -0.35 & -1.731 \\ -7.5 & 0 \end{pmatrix}.$$

The gains from the change of basis M^{-1} are computed for the unit vectors $\mathbf{i}_b = \binom{1}{0}$ and $\mathbf{i}_a = \binom{0}{1}$. We obtain $\gamma'_b = 0.577$ and $\gamma'_a = 7.497$.

The gain for matrix J is computed using Eq. (3). In this case we have

$$M^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -0.577 \end{pmatrix}$$

so $\theta_b = -\pi/2$, and

$$M^{-1}\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0.133 \\ -0.004 \end{pmatrix}$$

so $\theta_a = -0.029$. Since $\beta > 0$ and $\theta_a - \pi < \theta_b < \theta_a$, then $\theta^* = \theta_a$. Therefore, the gain for J is $\gamma'_{ba} = e^{\frac{\alpha}{\beta}(\theta_a - \theta_b)} = 0.875$.

The maximal gain for A is

$$\gamma_{ba} = \gamma'_b \gamma'_{ba} \gamma'_a = 3.78.$$

On the other hand, the upper bound on the maximal gain obtained with the optimal Quadratic Lyapunov Function as in [18] is $\rho_{ba} = \sqrt{15.77} = 3.97$.

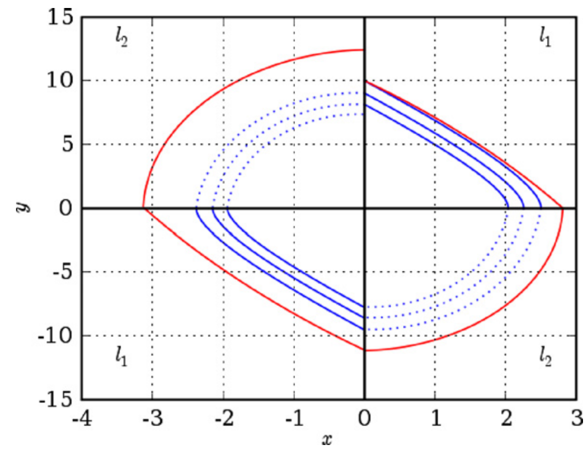

Fig. 3. Trajectory of the planar LCH starting at $(x, y) = (0, 10)$ with the corresponding optimal Quadratic Lyapunov Functions.

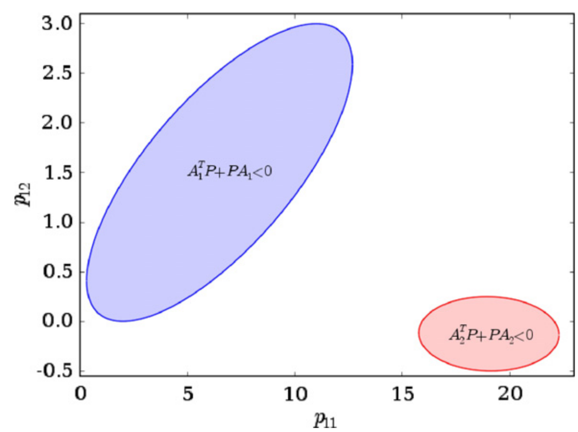

Fig. 4. No common Quadratic Lyapunov Function.

7.3. Stability

We conclude that the system is (asymptotically) stable because the exact maximal gain of the cycle is $\gamma_{ab}\gamma_{ba} < 1$. On the other hand, using the upper bounds obtained with optimal Quadratic Lyapunov Function we obtain $\rho_{ab}\rho_{ba} > 1$ and therefore we cannot conclude on the stability of the system. Fig. 3 shows the first cycles of a trajectory of the system which approaches the equilibrium, and depicts the optimal Quadratic Lyapunov Functions for this trajectory. As expected, Fig. 4 shows that there is no common Quadratic Lyapunov Function for the two locations.

8. Conclusions

We have derived a *necessary and sufficient* condition for the *stability* of a planar LCH hybrid automaton, namely the absence of expanding cycles (i.e. all cycles are contractive), together with an algorithm for efficiently checking this condition. We have made use of both systems theoretic concepts (in calculating the estimated gains) and computer science concepts (in checking the cycles in the gain automaton), thereby doing justice to both the continuous and the discrete aspects of hybrid systems.

As future work, we are interested in widening the class of hybrid systems to which this approach can be applied, for instance considering a more general type of guards. If this approach cannot directly be applied to systems of dimension three or more, we plan to study if it would be possible to do so by making further assumptions. Finally, by computing gains for unstable locations the approach might be applied to study the stabilization of unstable dynamics within the hybrid automaton framework.

References

[1] R. Alur, C. Courcoubetis, T.A. Henzinger, P.H. Ho, Hybrid automata: an algorithmic approach to the specification and verification of hybrid systems, in: R. Grossman, A. Nerode, A. Ravn, H. Rischel (Eds.), Hybrid Systems I, in: Lecture Notes in Computer Science, vol. 736, Springer Verlag, 1993, pp. 209–229.

[2] T.A. Henzinger, The theory of hybrid automata, in: Proceedings LICS'96, 1996, pp. 278–292.

[3] A.J. van der Schaft, J.M. Schumacher, An Introduction to Hybrid Dynamical Systems, in: Lecture Notes in Control and Information Sciences, vol. 251, Springer-Verlag, London, 2000.

[4] M.S. Branicky, Stability of switched and hybrid systems, in: Proc. 33rd IEEE Conf. Decision and Control, Orlando, FL, 1994, pp. 3498–3503.

[5] M. Balde, U. Boscai, P. Mason, A note on stability conditions for planar switched systems, Internat. J. Control 82 (2002) 1882–1888.

[6] U. Boscain, Stability of planar switched systems: the linear single input case, SIAM J. Control Optim. 41 (2002) 89–112.

[7] R.A. DeCarlo, M.S. Branicky, S. Pettersson, B. Lennartson, Perspectives and results on the stability and stabilizability of hybrid systems, Proc. IEEE 88 (2000) 1069–1082.

[8] D. Liberzon, A.S. Morse, Basic problems in stability and design of switched systems, IEEE Control Syst. Mag. 19 (1999) 59–70.

[9] A.N. Michel, Recent trends in the stability analysis of hybrid dynamical systems, IEEE Trans. Circuits Syst. I 45 (1999) 120–134.

[10] M. Margaliot, Stability analysis of switched systems using variational principles: an introduction, Automatica 42 (2006) 2059–2077.

[11] A.N. Michel, Hui Ye, L. Hou, Stability theory for hybrid dynamical systems, IEEE Trans. Automat. Control 43 (1998) 461–474.

[12] A.A. Agrachev, D. Liberzon, Lie-algebraic stability criteria for switched systems, SIAM J. Control Optim. 40 (2001) 253–269.

[13] M.S. Branicky, Multiple Lyapunov functions and other analysis tools for switched and hybrid systems, IEEE Trans. Automat. Control 43 (1998) 475–482.

[14] X.D. Koutsoukos, P.J. Antsaklis, Design of stabilizing switching control laws for discrete and continuous-time linear systems using piecewise-linear Lyapunov functions, Technical Report ISIS-2001-002, ISIS, 2001.

[15] S. Pettersson, B. Lennartson, Stability and robustness for hybrid systems, in: Proc. 35th IEEE Conf. Decision and Control, Kobe, Japan, 1996, pp. 1202–1207.

[16] M. Johansson, A. Rantzer, Computation of piecewise quadratic Lyapunov functions for hybrid systems, IEEE Trans. Automat. Control 43 (1998) 555–559.

[17] D. Mignone, G. Ferrari-Trecate, M. Morari, Stability and stabilization of piecewise affine and hybrid systems: an LMI approach, in: Proceedings 39th IEEE CED, 2000, pp. 504–509.

[18] R. Langerak, J.W. Polderman, Tools for stability of switching linear systems: gain automata and delay compensation, in: 44th IEEE Conference on Decision and Control and European Control Conference ECC 2005, IEEE Computer Society, Sevilla, Spain, 2005, pp. 4867–4872.

[19] J.W. Polderman, J.C. Willems, Introduction to Mathematical Systems Theory: A Behavioral Approach, in: Texts in Applied Mathematics, vol. 26, Springer, New York NY, USA, 1998.

[20] R. Motwani, John E. Hopcroft, J.D. Ullman, Introduction to Automata Theory, Languages, and Computation, second ed., Adisson-Wesley, 2001.

[21] P. Linz, An Introduction to Formal Languages and Automata, third ed., Jones and Bartlett Publishers, 2001.

[22] C.F. Daws, R. Langerak, J.W. Polderman, Decision algorithm for the stability of planar switching linear systems, in: Proceedings of the 18th International Symposium on Mathematical Theory of Networks & Systems, Blacksburg, Virginia, USA, Virginia Tech., USA, 2008.