$$w = \tilde{f}(z,v) = \tilde{\Pi}(z,v). \quad (28)$$

We claim that the above system is a left inverse of Σ. Indeed, let u be an input with $u(0) \in S(u_o)$. Repeated differentiation of the output with respect to time and the definition of the order yields

$$y_{i_\alpha}^{(\alpha)}(t, x_o, u) = f^\alpha h_{i_\alpha} x(t, x_o, u) = \Pi(x(t, x_o, u), u(t)). \quad (29)$$

Now, if we start $\tilde{\Sigma} x_o, u_o$ from x_o and exercise the control $v = y^{(\alpha)}(t, x_o, u)$, the resulting solution is $x(t, x_o, u)$. This follows easily from uniqueness of solutions. Furthermore, the corresponding output of $\tilde{\Sigma}_{x_o, u_o}$ for all t in a proper interval of R containing the origin is

$$w(t) = \tilde{\Pi}(x(t, x_o, u), \quad \Pi(x(t, x_o, u), u(t)) = u(t)$$

and the proof is complete.

IV. Conclusion

In this paper necessary and sufficient conditions for invertibility of single-input analytic systems have been presented.

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Comments on "Controlled Invariance for Nonlinear Systems"

S. H. MIKHAIL

Abstract—**The sufficient conditions given in Theorem 4.12 of the above paper[1] for controlled invariance are a special case of more general sufficient conditions reported earlier.**

In the above paper[1] conditions are given that are sufficient (and under certain restrictions also necessary) for "controllable invariance." In it, the authors refer to earlier work that I have published [1] on "controlled invariance" for systems of the type $\dot{x} = F(x, u)$ as a "related, but different notion" which is misleading. It seems they have failed to detect that the sufficient conditions in [1, Theorem 2.1] are more general than those in their Theorem 4.12, and were derived to accommodate many situations where $f_*(\Delta_e^0) \cap \dot{D}$ is *not* constant. It could be shown that the conditions $\dim(f_*(\Delta_e^0) \cap \dot{D}) =$ constant, and rank $d_2[df_\alpha F(x,u)] = l =$ constant are exactly equivalent to one another in the respective notations and settings of Theorem 4.12 of the paper[1] and [1, Theorem 2.1], respectively. Similarly, the conditions $f_*(\pi_*^{-1}(D)) \subset \dot{D} + f_*(\Delta_e^0)$ and condition ii) of [1, Theorem 2.1] (as well as [2, condition (2.4)]) could be shown to be equivalent to one another, subject to the former condition being satisfied. It is worth pointing out that conditions iii) (a) and iii) (b) of [1, Theorem 2.1] are automatically satisfied once the above two conditions hold.

The sufficient conditions given in [1, Theorem 2.1], [3, Theorem 4.2], and [4, Theorem 1.2] for "controlled invariance" are the same with minor changes in presentation and format. [3, Theorem 4.1] and [4, Theorem 1.1] give sufficient conditions that are exactly equivalent to those in Theorem 4.12 of the paper,[1] and are shown to be special cases of the more general conditions of Theorem 4.2 of the paper[1] and [4, Theorem 1.2], respectively.

There may be advantages to the setting used in the paper[1] when the problem global controlled invariance is investigated, but that remains to be seen.

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Authors' Reply[2]

HENK NIJMEIJER AND ARJAN VAN DER SCHAFT

S. H. Mikhail incorrectly points out that a result in our paper[1] is already contained in his work [1]. The basic observation is that the notion of controlled invariance used in [1] is related but completely different from the one used in our paper.[1] By working in local coordinates we can take $\mathbb{R}^n$ as the state space, $\mathbb{R}^m$ as the input space, and the system is defined by $\dot{x} = f(x, u)$. Then in [1] an involutive distribution D of fixed dimension is controlled invariant if there exists a map ϕ: $\mathbb{R}^n \to \mathbb{R}^m$, such that

$$[\hat{f}, D] \subset D, \qquad \text{where } \hat{f}(x) = f(x, \phi(x)).$$

In the paper[1] (see also the references in the paper[1]), however, an involutive distribution D of fixed dimension is called controlled invariant if there exists a map

$$\alpha: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m,$$

with the property that $(*)$ $\alpha(x, \cdot)$: $\mathbb{R}^m \to \mathbb{R}^m$ is a diffeomorphism for each $x \in \mathbb{R}^n$, and such that $[\hat{f}(\cdot, \bar{u}), D] \subset D$ for each constant $\bar{u} \in \mathbb{R}^m$, where

$$\hat{f}(x, u) = f(x, \phi(x, u)).$$

The condition $(*)$ expresses the fact that D is *nondegenerate* controlled invariant or controlled invariant with *full control*; see [2]. If $(*)$ is not satisfied, then the distribution D is degenerate controlled invariant, (see Section III, Remark 4 of the paper[1]); see also [3] for some further explanation. In this way the notion of controlled invariance used in [1] is a sort of degenerate controlled invariance of the paper.[1]

The condition ii) of [1, Theorem 2.1] is indeed equivalent to the condition of Theorem 4.12 of the paper,[1] provided that several rank conditions hold. The condition in Theorem 4.12 of the paper[1] really gives

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necessary and sufficient conditions for controlled invariance in the sense of the paper,[1] but this same condition in [1, Theorem 2.1] only gives a sufficient condition for controlled invariance in the sense of [1].

Although the references [3], [4] of Mikhail's comment are not accessible, and they appeared after the paper[1] had been submitted, it is clear that a condition of the type ii) in [1, Theorem 2.1] is far from a necessary condition for controlled invariance in the sense of [1].

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Comments on "Stability of Time-Delay Systems"

T. SASAGAWA

Abstract—**The aim of this paper is to point out that the proofs of the results in the above paper[1] are not correct. A counterexample is constructed for the simplest case.**

In the above paper,[1] the authors give necessary and sufficient conditions for the stability of time-delay systems of the form

$$\dot{x}(t)=A_1x(t)+A_2x(t-h).$$

They insist in the proofs of Lemma and Theorem 1 that $x(t)+P_1(t)*x(t)=0$ iff $x(t)=0$, where $*$ denotes the convolution operator. Moreover, from this insistence they conclude the positive definiteness of a Lyapunov function and the negative definiteness of the time derivative of the Lyapunov function on the space of continuous functions $x\in C([-h,0],R^n)$.

More concretely, they insist that the functional defined on $C([-h,0],R^n)$

$$V(x_t,h)=[x(t)+P_1(t)*x(t)]^TP_0[x(t)+P_1(t)*x(t)] \quad (1)$$

is positive definite, where

$$P_0[A_1+P_1(0)]+[A_1+P_1(0)]^TP_0=-Q, \qquad (Q=Q^T>0) \quad (2)$$

$$\dot{P}_1(\tau)=[A_1+P_1(0)]P_1(\tau), \qquad (0\leqslant\tau\leqslant h) \quad (3)$$

$$P_1(h)=A_2. \quad (4)$$

However, this is an elementary error and, hence, the proofs of Lemma and Theorem 1 are not complete.

To make sure, we construct a counterexample.

Counterexample: Consider the simplest case, i.e., let the system be

$$\dot{x}(t)=-x(t-h) \qquad (0\leqslant t<\infty,\ h>0) \quad (5)$$

with the initial function $\phi(t)=e^{\alpha t}$ $(-h\leqslant t\leqslant 0)$.

Equation (5) has the solution $x(t)=e^{\alpha t}$ $(t\geqslant -h)$ if

$$\alpha+e^{-\alpha h}=0. \quad (6)$$

On the other hand, for (3) and (4) to be satisfied, we have the relation $P_1(\tau)=P_1(0)e^{P_1(0)\tau}$, where

$$1+P_1(0)e^{P_1(0)h}=0. \quad (7)$$

The relation (6) can be transformed to $\alpha e^{\alpha h}+1=0$ and this is the same as (7). From (7), $P_1(0)$ must be a negative constant.

Now, we can calculate with $\alpha=P_1(0)$

$$\begin{aligned}x(t)+P_1(t)*x(t)&=x(t)+\int_0^h P_1(\tau)x(t-\tau)\,d\tau\\&=\left[1+P_1(0)\int_0^h e^{(P_1(0)-\alpha)\tau}d\tau\right]e^{\alpha t}\\&=[1+P_1(0)h]e^{P_1(0)t}.\end{aligned}$$

Hence, if we choose $h=e^{-1}$ (>0), $P_1(0)=-e$ (<0), the relation (7) is valid and $V(x_t,h)=0$ for $x(t)=e^{-et}$ $(\neq 0)$. □

As is clear from this example, Lyapunov functions of the type (1) $(P_0>0)$ are not generally positive definite. However, Sasagawa [2] proved the following lemma for the functional (1) with an additional term and applied for getting a sufficient condition for asymptotic stability for more general systems.

Lemma: Let a functional $V(x_t,h)$ defined on $C([-h,0],R^n)$ be given as follows.

$$\begin{aligned}V(x_t,h)&=[x(t)+H(t)*x(t)]^TP[x(t)+H(t)*x(t)]\\&\quad+\nu\int_0^h|H(\tau)|d\tau\int_0^\tau x^T(t-s)Qx(t-s)\,ds\end{aligned}$$

where $\nu>0$; P,Q are symmetric positive definite matrices and $H(t)$ is a matrix-valued function of bounded variation on $[-h,0]$.

Then there exists a positive constant λ such that

$$\inf_{\|x_t\|\leqslant|x(t)|}V(x_t,h)\geqslant\lambda|x(t)|^2$$

for any $x_t\in C([-h,0],R^n)$. □

In the above lemma, $|\cdot|$ denotes the square root norm of a vector or a matrix and $\|\cdot\|$ denotes the sup norm in the space $C([-h,0],R^n)$.

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Additional Comments on "Stability of Time-Delay Systems"

M. BUSLOWICZ

Abstract—**It is proved by a counterexample that the main result of the above paper[1] is incorrect.**

Recall that in the paper[1] for the system

$$\dot{x}(t)=A_1x(t)+A_2x(t-h) \quad (1)$$

where $x(t)\in R^n$, the following sufficient condition for the stability is given. If for any given positive definite Hermitian matrix Q, there exists a

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