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ON MAXIMUM CRITICALLY *h*-CONNECTED GRAPHS

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Let h be an integer with $h \geq 2$. A graph G is called *critically h-connected* or *h-critical* if G is h-connected while, for every vertex v of G, the graph $G-v$ is not h-connected. $\mathscr{C}$ denotes the class of all h-critical graphs and $\mathscr{A}$ the class of all graphs of $\mathscr{C}$ in which every vertex is adjacent to a vertex of degree h. $\hat{\mathscr{C}}$ and $\hat{\mathscr{A}}$ are the classes of maximum graphs in $\mathscr{C}$ and $\mathscr{A}$, respectively. Entringer's characterization of $\hat{\mathscr{C}}$ for $h=2$ shows that $\hat{\mathscr{C}} \neq \hat{\mathscr{A}}$ in case $h=2$. Here $\hat{\mathscr{A}}$ is determined for each $h \geq 2$. Then it is shown that $\hat{\mathscr{C}} = \hat{\mathscr{A}}$ for $h=3$ and it is conjectured that $\hat{\mathscr{C}} = \hat{\mathscr{A}}$ for each $h \geq 3$.

Terminology

We use [2] for basic terminology and notations, but speak of vertices and edges instead of points and lines. Accordingly we denote the edge set of a graph G by $E(G)$.

If G is a connected graph, then by a *cut* of G we mean a set of vertices of G whose deletion results in a disconnected graph. If T_1 and T_2 are cuts of G, then T_1 *interferes with* T_2 if at least two components of $G-T_1$ contain vertices of T_2. An *h-cut* is a cut of h elements. A vertex v of G is *critical* if $\kappa(G-v) < \kappa(G)$. G is called *critically h-connected*, or briefly *h-critical*, if $\kappa(G)=h$ and every vertex of G is critical.

If $\mathscr{G}$ is a class of graphs, then the elements of $\mathscr{G}$ are called *$\mathscr{G}$-graphs*. The set of graphs in $\mathscr{G}$ with n vertices is denoted by $\mathscr{G}_n$. G is a *maximum* $\mathscr{G}$-graph if no $\mathscr{G}$-graph with $|V(G)|$ vertices has more edges than G. The set of maximum $\mathscr{G}$-graphs is denoted by $\hat{\mathscr{G}}$. $\mu_{\mathscr{G}}(n)$ is the number of edges of graphs in $\hat{\mathscr{G}}_n$.

Let h be a fixed integer with $h \geq 2$. By $\mathscr{C}$ we denote the set of all h-critical graphs. $\mathscr{A}$ is the subset of $\mathscr{C}$ consisting of all h-connected graphs in which every vertex is adjacent to a vertex of degree h. The set $\mathscr{B}$ is defined by $\mathscr{B} = \mathscr{C} - \mathscr{A}$. For a graph G, $M(G) = \{v \in V(G) \mid \deg_G v = h\}$, $K(G) = V(G) - M(G)$, $\rho(G) = \sum_{v \in M(G)} \deg_{\langle M(G)\rangle} v$ and $B(G)$ is the set of edges of G with one end in $K(G)$ and the other in $M(G)$. For the sake of notational simplicity we have chosen not to express the fact that the above notions depend on h; unless h is specified, propositions involving the relevant notions will hold for each $h \geq 2$.

We use $[x]$ to denote the greatest integer less than or equal to x.

1. Introduction

Entringer [1] characterized $\hat{\mathscr{C}}_n$-graphs for $h=2$ and $n \geq 3$. In the proof of his characterization, which is by induction on n, he uses an upper bound for $\mu_{\mathscr{A}}(n)$ [1, Lemma 2]. It appears that, for $h=2$, there are infinitely many $\hat{\mathscr{C}}$-graphs which are not $\hat{\mathscr{A}}$-graphs.

Here we first determine $\hat{\mathscr{A}}$ for each $h \geq 2$ (Section 2). Then for $h=3$ it is proved, also by induction on n, that $\mu_{\mathscr{B}}(n) < \mu_{\mathscr{A}}(n)$ for all n, so that, in consequence, $\hat{\mathscr{C}} = \hat{\mathscr{A}}$ (Section 3). Finally it is conjectured that $\hat{\mathscr{C}} = \hat{\mathscr{A}}$ for each $h \geq 3$ (Section 4).

2. Characterization of $\hat{\mathscr{A}}$-graphs

Throughout this section h will be a fixed integer with $h \geq 2$.

Noting that no h-connected graph with less than $h+1$ vertices exists, we first determine $\hat{\mathscr{A}}_n$ for $h+1 \leq n \leq 2h$. Define, for $h+1 \leq n \leq 2h$, the graph H_n as follows:

(a) $V(H_n) = \{v_1, v_2, \ldots, v_n\}$;

(b) $N(v_1) = \{v_2, v_3, \ldots, v_{h+1}\}$;

(c) $N(v_2) = \{v_1, v_{n-h+2}, v_{n-h+3}, \ldots, v_n\}$;

(d) $\langle v_3, v_4, \ldots, v_n \rangle$ is complete.

Clearly, $H_n \in \mathscr{A}_n$.

Lemma 1. *If $h+1 \leq n \leq 2h$, then $\mu_{\mathscr{A}}(n) = \frac{1}{2}(n^2-5n+4h+4)$ and $\hat{\mathscr{A}}_n = \{H_n\}$.*

Proof. Let G be an $\mathscr{A}_n$-graph with $h+1 \leq n \leq 2h$. Then G contains, by definition of $\mathscr{A}$, two adjacent vertices v_1 and v_2 with $\deg v_1 = \deg v_2 = h$. Hence

$$|E(G)| = 2h-1+|E(G-\{v_1, v_2\})|$$
$$\leq 2h-1+\binom{n-2}{2} = \tfrac{1}{2}(n^2-5n+4h+4). \tag{1}$$

Suppose equality holds in (1). Then $G-\{v_1, v_2\}$ is complete; furthermore, since G is h-connected and every vertex of G is adjacent to a vertex of degree h, one easily deduces that $N(v_1) \cup N(v_2) = V(G)$. These properties determine G up to isomorphism: $G \cong H_n$. $\square$

We proceed by deriving (for $n \geq 2h+1$) an upper bound for the number of edges of an $\mathscr{A}_n$-graph G in case $|K(G)|$ has a prescribed value. Let the function f_n be defined by

$$f_n(x) = \tfrac{1}{2}(x^2-2nx+(2h-1)n).$$

Lemma 2. *Let G be an $\mathscr{A}_n$-graph with $|K(G)|=k$. Then $|E(G)| \leq f_n(k)$. Moreover, if $n \geq 2h+1$ and $k \leq h-1$, then $|E(G)| \leq f_n(h)-1$ unless $h=2$ and $n=5$.*

Proof. Let G be an $\mathscr{A}_n$-graph with $|K(G)|=k$. Then

$$\begin{aligned}|E(G)| &= |E(\langle K(G)\rangle)|+|E(\langle M(G)\rangle)|+|B(G)|\\ &\leqslant \binom{k}{2}+\tfrac{1}{2}\rho(G)+\sum_{v\in M(G)}(h-\deg_{\langle M(G)\rangle}v)\\ &= \tfrac{1}{2}k(k-1)+h(n-k)-\tfrac{1}{2}\rho(G).\end{aligned} \tag{2}$$

Since $G\in\mathscr{A}$, $\rho(G)\geqslant|M(G)|=n-k$. Thus

$$|E(G)|\leqslant\tfrac{1}{2}k(k-1)+h(n-k)-\tfrac{1}{2}(n-k)=f_n(k),$$

proving the first part of the lemma.

Now let $n\geqslant 2h+1$ and assume first that $k\leqslant h-2$. Then

$$\begin{aligned}|E(G)| &= \tfrac{1}{2}\Big(\sum_{v\in K(G)}\deg_G v+\sum_{v\in M(G)}\deg_G v\Big)\\ &\leqslant \tfrac{1}{2}(k(n-1)+h(n-k))=\tfrac{1}{2}((n-1-h)k+hn)\\ &\leqslant \tfrac{1}{2}((n-1-h)(h-2)+hn)=f_n(h)-\tfrac{1}{2}(n-h-2)\\ &\leqslant f_n(h)-1 \quad \text{unless } h=2 \text{ and } n=5.\end{aligned}$$

Assume next that $k=h-1$ (and $n\geqslant 2h+1$). Then, since G is h-connected, $G-K(G)$ is connected, implying that $|E(\langle M(G)\rangle)|\geqslant|V(\langle M(G)\rangle)|-1$, or, equivalently, $\rho(G)\geqslant 2(n-h)$. From (2) we deduce that

$$\begin{aligned}|E(G)| &\leqslant \tfrac{1}{2}(h-1)(h-2)+h(n-h+1)-(n-h)\\ &= f_n(h)-\tfrac{1}{2}(n-h-2)\\ &\leqslant f_n(h)-1 \quad \text{unless } h=2 \text{ and } n=5. \quad \square\end{aligned}$$

In the following lemma an upper bound for the cardinality of $|K(G)|$ in an $\mathscr{A}_n$-graph G is obtained. Define

$$k_n=\begin{cases}\left[\dfrac{h-1}{h}n\right] & \text{if } n\not\equiv h \bmod 2h,\\[2ex] \dfrac{h-1}{h}n-1 & \text{if } n\equiv h \bmod 2h.\end{cases}$$

Lemma 3. *If G is an $\mathscr{A}_n$-graph, then $|K(G)|\leqslant k_n$.*

Proof. Let G be an $\mathscr{A}_n$-graph. Every vertex of $K(G)$ has a neighbour in $M(G)$, so

$$|B(G)|\geqslant|K(G)|. \tag{3}$$

On the other hand, every vertex of $M(G)$ has at most $h-1$ neighbours in $K(G)$, since each vertex of $M(G)$ also has at least one neighbour in $M(G)$. Hence

$$|B(G)|\leqslant(h-1)\,|M(G)|=(h-1)(n-|K(G)|). \tag{4}$$

From (3) and (4) it follows that $|K(G)| \leqslant (h-1)(n-|K(G)|)$, or, equivalently,

$$|K(G)| \leqslant \frac{h-1}{h} n. \tag{5}$$

To complete the proof we show that the inequality (5) is strict if $n \equiv h \bmod 2h$. Assume that $n = 2hi + h$ and (5) holds with equality. Then $|M(G)| = 2i+1$. Since (4) also holds with equality, the graph $\langle M(G) \rangle$ is regular of degree 1, implying that $|M(G)|$ is even, a contradiction. □

Lemmas 2 and 3 enable us to determine an upper bound for $\mu_{\mathscr{A}}(n)$ in case $n \geqslant 2h+1$. Define

$$a(n) = [f_n(k_n)].$$

Lemma 4. *Let G be an* $\mathscr{A}_n$*-graph such that* $n \geqslant 2h+1$ *and either* $h \neq 2$ *or* $n \neq 5$. *Then* $|E(G)| \leqslant a(n)$. *Moreover, unless* $h = 3$ *and* $n = 7$, $|E(G)| = a(n)$ *only if* $|K(G)| = k_n$; *if* $h = 3$ *and* $n = 7$, *then* $|E(G)| = a(n)$ *only if* $|K(G)| \in \{k_n - 1, k_n\}$.

Proof. Let G satisfy the conditions of the lemma. For $x > h$, $f_n(x)$ is an increasing function of x. Since $n \geqslant 2h+1$, $k_n \geqslant h$. By using Lemmas 2 and 3 it follows that

$$|E(G)| \leqslant [f_n(k_n)]$$

and, if $k_n = h$,

$$|E(G)| = [f_n(k_n)] \quad \text{only if } |K(G)| = k_n.$$

Now assume $k_n \geqslant h+1$. For every x we have

$$f_n(x) - f_n(x-1) = x - h - \tfrac{1}{2},$$

implying that $[f_n(k_n)] = [f_n(k_n - 1)]$ if and only if $k_n = h+1$ and $f_n(k_n)$ is not integer-valued, i.e., if and only if $h = 3$ and $n = 7$, as is easily checked. The result follows. □

We finally show that $\mu_{\mathscr{A}}(n) = a(n)$ for all $n \geqslant h+1$ and characterize $\hat{\mathscr{A}}$.

Let T_7 and T_7' be the graphs depicted in Fig. 1 and define a class $\mathscr{H}$ of graphs

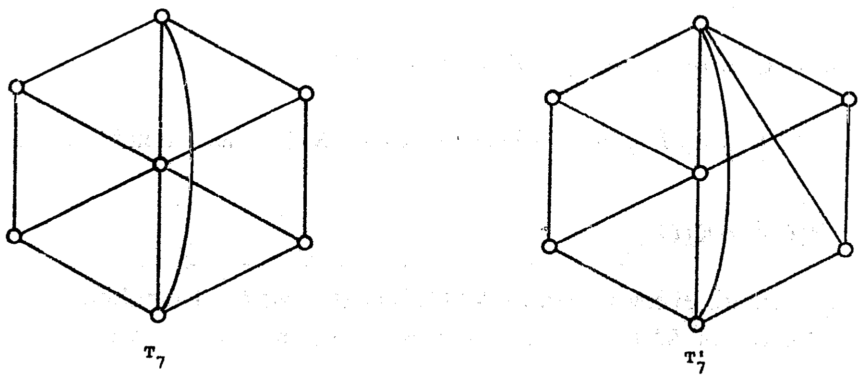

Fig. 1.

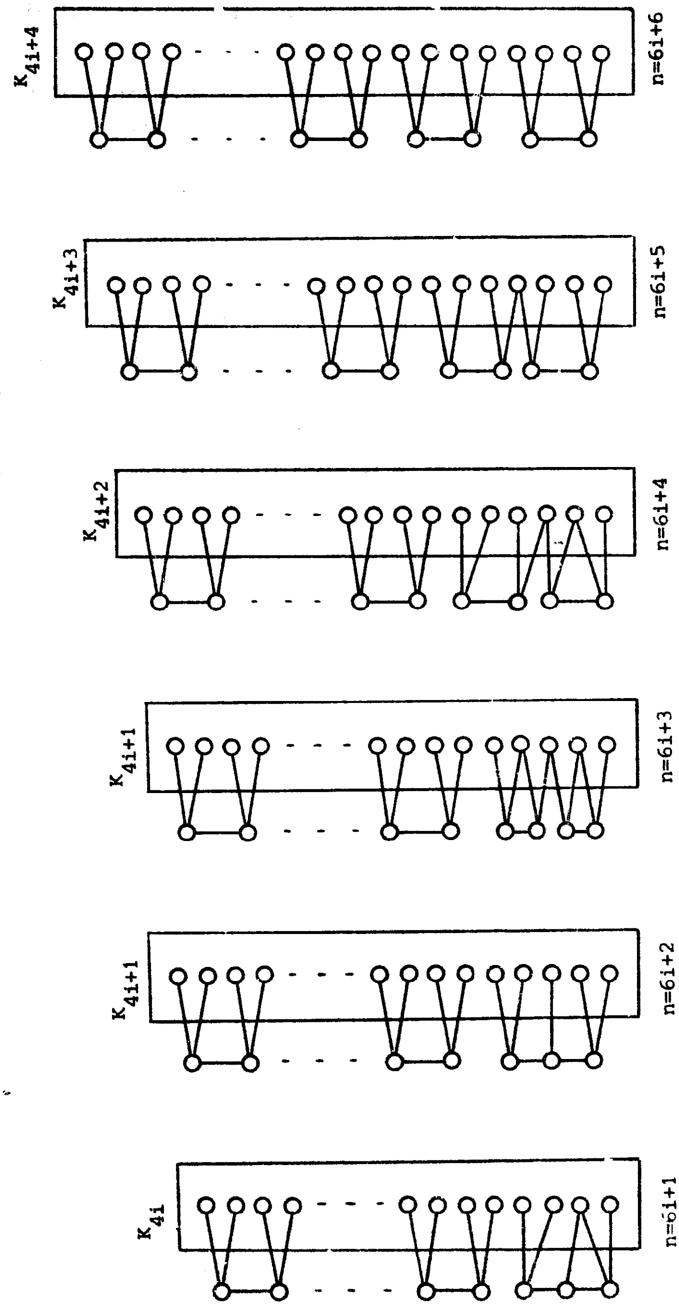

Fig. 2.

by the assertion that a graph G with n vertices belongs to $\mathscr{H}$ if and only if the following requirements are met:

(i) $n \geq 2h+1$;

(ii) $|K(G)| = k_n$ and $\langle K(G)\rangle$ is complete;

(iii) if $n-k_n$ is even, then $\langle M(G)\rangle \cong \frac{1}{2}(n-k_n)P_2$; if $n-k_n$ is odd, then $\langle M(G)\rangle \cong P_3 \cup \frac{1}{2}(n-k_n-3)P_2$;

(iv) every vertex of $K(G)$ is incident with at least one edge of $B(G)$;

(v) if v_1 and v_2 are the vertices of a component of $\langle M(G)\rangle$ isomorphic to P_2, then $|(N(v_1) \cup N(v_2)) \cap K(G)| \geq h$.

Note that $\mathscr{H}_n = \emptyset$ if $h=2$ and $n=5$; if $h \neq 2$ or $n \neq 5$, then $\mathscr{H}_n \neq \emptyset$ for all $n \geq 2h+1$. In Fig. 2 an element of $\mathscr{H}_{2hi+j}$ is sketched for $h=3$ and $j \in \{1,2,\ldots,2h\}$; i is an arbitrary positive integer.

For $h=2$ $\mathscr{H}_n$-graphs are unique up to isomorphism unless $n \equiv 2 \bmod 4$ and $n \neq 6$. For $h \geq 3$ $\mathscr{H}_n$-graphs are unique up to isomorphism if and only if $n \equiv 0 \bmod 2h$ or $n \equiv (h-1) \bmod 2h$. For relevant values of h and n, nonisomorphic $\mathscr{H}_n$-graphs can be obtained from one another by repeatedly applying the following operation: find two vertices u_1 and u_2 of degree greater than h such that u_1 has at least two neighbours of degree h, v_1 and v_2 say; replace the edge u_1v_1 by the edge u_2v_1.

Define

$$\mathscr{H}' = \mathscr{H} \cup \{H_n \mid h+1 \leq n \leq 2h\}.$$

Theorem 5. *$\mu_{\mathscr{A}}(n) = a(n)$ for $n \geq h+1$ and*

$$\hat{\mathscr{A}} = \begin{cases} \mathscr{H}' & \text{if } h \neq 2,3, \\ \mathscr{H}' \cup \{C_5\} & \text{if } h=2, \\ \mathscr{H}' \cup \{T_7, T_7'\} & \text{if } h=3. \end{cases}$$

Proof. For $h+1 \leq n \leq 2h$ we are through by Lemma 1 and the observation that

$$f_n(k_n) = f_n(n-2) = \tfrac{1}{2}(n^2-5n+4h+4).$$

Now let $n \geq 2h+1$. We distinguish three cases.

Case 1. ($h \neq 2$ or $n \neq 5$) and ($h \neq 3$ or $n \neq 7$). Then $\mathscr{H}_n \neq \emptyset$. Since every $\mathscr{H}_n$-graph is an $\mathscr{A}_n$-graph with $a(n)$ edges and, by Lemma 4, $\mu_{\mathscr{A}}(n) \leq a(n)$, it follows that $\mu_{\mathscr{A}}(n) = a(n)$. It remains to be shown that $\hat{\mathscr{A}}_n \subset \mathscr{H}_n$.

Let G be an $\hat{\mathscr{A}}_n$-graph. Then $\langle K(G)\rangle$ is complete, otherwise an $\mathscr{A}_n$-graph with more edges than G would be obtained by joining two nonadjacent vertices of $K(G)$ by an edge. Now inequality (2) holds with equality:

$$|E(G)| = \tfrac{1}{2}|K(G)|\,(|K(G)|-1) + h(n-|K(G)|) - \tfrac{1}{2}\rho(G). \tag{6}$$

By Lemma 4, $|K(G)| = k_n$. Substituting $|K(G)|$ by k_n and $|E(G)|$ by $a(n)$, one deduces from (6) that $\rho(G) = n-k_n = |M(G)|$ if $n-k_n$ is even and $\rho(G) = n-k_n+1 = |M(G)|+1$ if $n-k_n$ is odd. Since $\delta(\langle M(G)\rangle) \geq 1$ by definition of $\mathscr{A}$, it follows that $\langle M(G)\rangle \cong \frac{1}{2}(n-k_n)P_2$ if n is even and $\langle M(G)\rangle \cong P_3 \cup \frac{1}{2}(n-k_n-3)P_2$ if n is odd. Using the definition of $\mathscr{A}$ once more, we conclude that $G \in \mathscr{H}_n$.

Case 2. $h=2$ and $n=5$. Clearly, C_5 is the only $\mathscr{A}$-graph with five vertices. Hence $\mu_{\mathscr{A}}(5)=5=a(5)$.

Case 3. $h=3$ and $n=7$. By Lemma 4, $\mu_{\mathscr{A}}(7) \leqslant a(7)=13$. All graphs in $\mathscr{H}_7 \cup \{T_7, T_7'\}$ are $\mathscr{A}_7$-graphs with 13 edges. Conversely, suppose G is an $\mathscr{A}_7$-graph with 13 edges. By Lemma 4, $|K(G)|=k_7=4$ or $|K(G)|=k_7-1=3$. If $|K(G)|=4$, then, like in Case 1, $G \in \mathscr{H}_7$. If $|K(G)|=3$, then (6) implies that $\rho(G)=4=|M(G)|$, so that $\langle M(G)\rangle \cong 2P_2$. Since $G \in \mathscr{A}$, it follows that $G \cong T_7$ or $G \cong T_7'$. □

Theorem 5 contains [1, Lemma 2].

3. Characterization of $\mathscr{C}$-graphs for $h=3$

Assume throughout this section that $h=3$. We shall present some evidence for the following result.

Theorem 6. *$\mu_{\mathscr{C}}(n)=a(n)$ for $n \geqslant 4$ and $\mathscr{C}=\hat{\mathscr{A}}$.*

Theorem 6 is equivalent to the assertion that $\mu_{\mathscr{B}}(n)<a(n)$ for $n \geqslant 4$. It is, however, convenient to prove the following slightly stronger statement.

Lemma 7.

$$\mu_{\mathscr{B}}(n) \leqslant \begin{cases} a(n)-1 & \text{if } n \not\equiv 0 \bmod 6, \\ a(n)-2 & \text{if } n \equiv 0 \bmod 6. \end{cases}$$

To get an impression of the proof of Lemma 7, which is by induction on n, let G be a $\mathscr{B}$-graph. Then G contains a vertex p with $N(p) \subset K(G)$. Let S be a 3-cut of G containing p. In the proof several cases with respect to the structure of $\langle S\rangle$ are distinguished. In each case two smaller $\mathscr{C}$-graphs are constructed from G. Thereby an upper bound for $|E(G)|$ is obtained via the induction hypothesis. For the proof in full detail, which is quite long, we refer to [3]. Here we only treat the case that $\langle S\rangle$ is complete. More precisely, we shall prove the following lemma.

Lemma 8. *Let G be a $\mathscr{B}_n$-graph which contains a 3-cut $S=\{p, q_1, q_2\}$ such that $N(p) \subset K(G)$ and $\langle S\rangle$ is complete. If, for all $m<n$,*

$$\mu_{\mathscr{B}}(m) \leqslant \begin{cases} a(m)-1 & \text{if } m \not\equiv 0 \bmod 6, \\ a(m)-2 & \text{if } m \equiv 0 \bmod 6, \end{cases}$$

then

$$|E(C)| \leqslant \begin{cases} a(n)-1 & \text{if } n \not\equiv 0 \bmod 6, \\ a(n)-2 & \text{if } n \equiv 0 \bmod 6. \end{cases}$$

Before proving Lemma 8 we state four additional lemmas, two of which are adopted from [4].

Lemma 9 (Veldman [4]). *If T_1 and T_2 are distinct minimum cuts of a graph, then T_1 interferes with T_2 if and only if T_2 interferes with T_1.*

The following lemma is a special case of [4, Lemma 1].

Lemma 10 (Veldman [4]). *If v is a vertex of degree 3 in a 3-connected graph G, then $N(v)$ is the only 3-cut of G contained in $\{v\} \cup N(v)$.*

Lemma 10 is applied in the proof of the next lemma.

Lemma 11. *If v is a vertex of degree 3 in a $\mathscr{C}$-graph, then $\langle N(v) \rangle$ is not complete.*

Proof. Let G be a $\mathscr{C}$-graph, v a vertex of G of degree 3 and U a 3-cut of G containing v. By Lemma 10, U contains a vertex which is not in $\{v\} \cup N(v)$. Hence $N(v)$ interferes with U. By Lemma 9, U also interferes with $N(v)$. In particular, $N(v)$ contains a pair of nonadjacent vertices. □

Lemma 12. *If some vertex of an $\mathscr{A}_{6k}$-graph $G (k \geq 2)$ has at least two neighbours in $M(G)$, then $|E(G)| \leq a(6k) - 2$.*

Proof. Let G satisfy the conditions of the lemma. From Lemma 3 and its proof it is apparent that $|K(G)| \leq 4k - 1$. Hence

$$|E(G)| \leq \binom{|K(G)|}{2} + 3\,|M(G)| \leq \tfrac{1}{2}(4k-1)(4k-2) + 3(2k+1)$$
$$= 8k^2 + 4 = a(6k) - 3k + 4 \leq a(6k) - 2. \quad \square$$

Although the upper bound in Lemma 12 is far from sharp, it is all we need in the proof of Lemma 8 (and Lemma 7).

Proof of Lemma 8. Assume that all conditions of Lemma 8 are satisfied. Let $\{Q_1, Q_2\}$ be a partition of $V(G) - S$ such that $\langle Q_i \rangle$ is a disjoint union of one or more components of $G - S$ $(i = 1, 2)$. Construct from G the graphs G_1 and G_2 as depicted in Fig. 3. It is easily seen that G_1 and G_2 are 3-connected. Since $\langle S \rangle$ is complete, no 3-cut of G interferes with S, so that, by Lemma 9, S interferes with no 3-cut of G. Hence if U is a 3-cut of G with $U \cap Q_i \neq \emptyset$, then $U \subset Q_i \cup S$, implying that U is a 3-cut of G_i too $(i = 1, 2)$. Thus all vertices of Q_i, being critical in G, are also critical in G_i $(i = 1, 2)$. The remaining vertices of G_i, having a neighbour of degree 3, are critical too $(i = 1, 2)$. Hence G_1 and G_2 are $\mathscr{C}$-graphs. From Lemma 11 one easily deduces that $|Q_i| \geq 3$, so that $|V(G_i)| < |V(G)|$ $(i = 1, 2)$. If $G_i \in \mathscr{A}$, then $|E(G_i)| \leq a(|V(G_i)|)$ by Theorem 5; if $G_i \in \mathscr{B}$, then

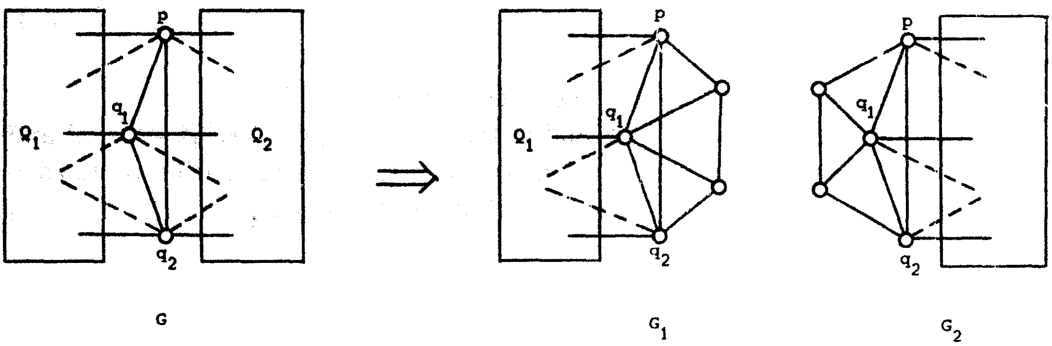

Fig. 3.

$|E(G_i)| \leq a(|V(G_i)|)$ by the conditions of Lemma 8. Looking at Fig. 3 we now deduce that

$$\begin{aligned} |E(G)| &\leq |E(G_1)| + |E(G_2)| - 13 \\ &\leq a((|Q_1|-2)+7) + a(n-(|Q_1|-2)) - 13 \\ &\leq \max_{1\leq x\leq n-8} \{a(x+7)+a(n-x)-13\} \\ &= \max_{1\leq x\leq [(n-1)/2]-3} \{a(x+7)+a(n-x)-13\}. \end{aligned}$$

Let $\phi_n(x) = a(x+7)+a(n-x)-13$. It is easily checked that, if $1 \leq 6i+j \leq [(n-1)/2]-3$, $\phi_n(6i+j)$ is a decreasing function of i for each j with $0 \leq j \leq 5$. Hence

$$|E(G)| \leq \max_{1\leq x\leq \min\{6,[(n-1)/2]-3\}} \phi_n(x).$$

Straightforward checking yields that, for $1 \leq x \leq \min\{6, [(n-1)/2]-3\}$,

$$\phi_n(x) \leq a(n)-2 \quad \text{if } n \equiv 0 \bmod 6;$$

furthermore, for $1 \leq x \leq \min\{6, [(n-1)/2]-3\}$,

$$\phi_n(x) \leq a(n)-1 \quad \text{if } n \not\equiv 0 \bmod 6,$$

except in three cases. We show that $|E(G)| \leq a(n)-1$ in each of these cases.

Case 1. $n = 6k+1$, $x = 1$ $(k \geq 2)$: $\phi_{6k+1}(1) = a(6k+1)+2$.

In Fig. 3 there are two analogous possibilities corresponding to $x = 1$: either $|V(G_1)| = 8$ and $|V(G_2)| = 6k$, or $|V(G_1)| = 6k$ and $|V(G_2)| = 8$. We proceed with the first possibility. $G_1 \notin \hat{\mathscr{A}}_8$, since $K(G_1)$ contains a vertex with two neighbours of degree 3. Since $\mu_{\mathscr{B}}(8) \leq a(8)-1$, it follows that $|E(G_1)| \leq a(8)-1$. From Lemma 12 and the fact that $\mu_{\mathscr{B}}(6k) \leq a(6k)-2$ we deduce that $|E(G_2)| \leq a(6k)-2$. Thus instead of $|E(G)| \leq \phi_{6k+1}(1)$ we reach the stronger conclusion that

$$|E(G)| \leq \phi_{6k+1}(1) - 3 = a(6k+1)-1.$$

Case 2. $n = 6k+1$, $x = 5$ $(k \geq 2)$: $\phi_{6k+1}(5) = a(6k+1) - 12k + 26$.

$\phi_{6k+1}(5) > a(6k+1)-1$ only if $k=2$. Then, however, we are back in Case 1, since $\phi_{13}(5) = \phi_{13}(1)$.

Case 3. $n=6k+3$, $x=1$ $(k \geq 1)$: $\phi_{6k+3}(1) = a(6k+3)+1$.

Then in Fig. 3 either $|V(G_1)|=8$ and $|V(G_2)|=6k+2$, or $|V(G_1)|=6k+2$ and $|V(G_2)|=8$. In particular, $|V(G_i)| \equiv 2 \bmod 6$ $(i=1,2)$. Since $K(G_i)$ contains a vertex with two neighbours of degree 3, it follows that $G_i \notin \hat{\mathscr{A}}$ $(i=1,2)$. Thus, in fact,

$$|E(G)| \leq \phi_{6k+3}(1)-2 = a(6k+3)-1.$$

The proof is completed by verifying the following inequalities:

$$\phi(6k+1,7) \leq a(6k+1)-1 \quad (k \geq 3),$$

$$\phi(6k+1,11) \leq a(6k+1)-1 \quad (k \geq 3),$$

$$\phi(6k+3,7) \leq a(6k+3)-1 \quad (k \geq 3). \quad \square$$

4. Discussion

In Section 3 it appeared that $\mu_{\mathscr{B}}(n) < \mu_{\mathscr{A}}(n)$ for $h=3$. For large values of h, $\hat{\mathscr{A}}_n$-graphs have a very high edge density. We expect that, for increasing values of h, $\mu_{\mathscr{A}}(n)$ will grow more rapidly than $\mu_{\mathscr{B}}(n)$, leading us to the following conjecture.

Conjecture 13. For all $h \geq 3$, $\mu_{\mathscr{C}}(n) = a(n)$ $(n \geq h+1)$ and $\hat{\mathscr{C}} = \hat{\mathscr{A}}$.

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