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Values for games with two-level communication structures☆

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A B S T R A C T

We consider a new model of a TU game endowed with both coalition and two-level communication structures that applies to various network situations. The approach to the value is close to that of both Myerson (1977) and Aumann and Drèze (1974): it is based on ideas of component efficiency and of one or another deletion link property, and it treats an a priori union as a self-contained unit; moreover, our approach incorporates also the idea of the Owen's quotient game property (1977). The axiomatically introduced values possess an explicit formula representation and in many cases can be quite simply computed. The results obtained are applied to the problem of sharing an international river, possibly with a delta or multiple sources, among multiple users without international firms.

1. Introduction

The study of TU games with coalition structures was initiated first by Aumann and Drèze [2], then Owen [12]. Another model of a game with limited cooperation presented by means of a communication graph was introduced in Myerson [11]. Various studies in both directions were done during the last three decades but mostly either within one model or another. The generalization of the Owen and Myerson values, applied to the combination of both models that resulted in a TU game with both independent coalition and communication structures, was investigated by Vázquez-Brage et al. [16].

In the paper we study TU games endowed with both coalition and communication structures. Different from [16], in our case a communication structure is a two-level communication structure that relates fundamentally to the given coalition structure. It is assumed that communication (via bilateral agreements among participants) is only possible either among the entire coalitions of a coalition structure, called a priori unions, or among single players within a priori unions. No communication and therefore no cooperation is allowed between proper subcoalitions, in particular single players, of different a priori unions. This approach allows to model different network situations, in particular, telecommunication problems, distribution of goods among different cities (countries) along highway networks connecting the cities and local road networks within the cities, or sharing an international river with multiple users but without international firms, i.e., when no cooperation is possible among single users located at different levels along the river, and so on. A two-level communication structure is introduced by means of graphs of two types, first, presenting links between a priori unions of a coalition structure and second,

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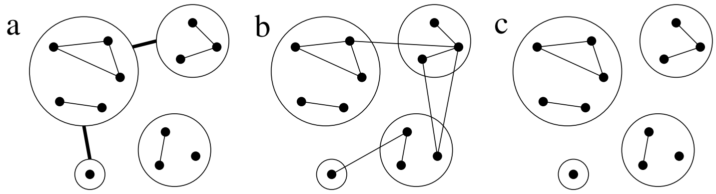

Fig. 1. (a) A model of the paper; (b) a model of Vázquez-Brage et al.; (c) a case of coincidence of both models.

presenting links between players within each a priori union. We consider communication structures given by combinations of graphs of different types both undirected—arbitrary graphs and cycle-free graphs, and directed—line-graphs with linearly ordered players, rooted forests and sink forests. Fig. 1(a) illustrates one of the possible situations within the model while Fig. 1(b) provides an example of a possible situation within the model of Vázquez-Brage et al. with the same set of players, the same coalition structure, and even the same links connecting players within a priori unions. In general, the introduced model of a game with two-level communication structure cannot be reduced to the model of Vázquez-Brage et al. Consider for example negotiations between two countries held on the level of prime ministers who in turn are citizens of their countries. The communication link between countries can be replaced neither by communication link connecting the prime ministers as single persons and therefore presenting only their personal interests, nor by all communication links connecting citizens of one country with citizens of another country that also present links only on the personal level. The two models coincide only if a communication graph between a priori unions in our model is empty and components of a communication graph in the model of Vázquez-Brage et al. are subsets of a priori unions. An example illustrating this situation with the same player set, the same coalition structure, and the same graphs within a priori unions, as on Fig. 1(a) is given on Fig. 1(c).

Our main concern is to provide a theoretical justification of solution concepts reflecting the two-stage distribution procedure and also to reveal the conditions when such a procedure is feasible. It is assumed that at first, a priori unions through upper level bargaining based only on cumulative interests of all members of each involved entire a priori union when nobody's personal interests are taken into account collect their total shares. Thereafter, via bargaining within a priori unions based only on personal interests of participants, the collected shares are distributed to single players. As a bargaining output on both levels one or another value for games with communication structures, in other terms graph games, can be applied. Following Myerson [11] we assume that cooperation possible only among connected players or connected groups of players and, therefore, we concentrate on component efficient values. Different component efficient values for graph games with graphs of various types, both undirected and directed, are known in the literature. We introduce a unified approach to a number of component efficient values for graph games that allows application of various combinations of known solution concepts, first at the level of entire a priori unions and then at the level within a priori unions, within a single framework. Our approach to values for games with two-level graph structures is close to that of both Myerson [11] and Aumann and Drèze [2]: it is based on ideas of component efficiency and one or another deletion link property, and it treats an a priori union as a self-contained unit. Moreover, to link both communication levels between and within a priori unions we incorporate the idea of the Owen's quotient game property [12]. This approach generates two-stage solution concepts that provide consistent application of values for graph games on both levels. The incorporation of different solutions for graph games aims not only to enrich the solution concept for games with two-level graph structures. It also opens a broad diversity of applications impossible otherwise because there exists no universal solution concept for graph games that is applicable to the full variety of possible undirected and directed graph structures. Furthermore, it allows to chose, depending on types of graph structures under scrutiny, the most preferable, in particular, the most computationally efficient combination of values among others suitable. The idea of a two-stage solution concept is not new. The well known example is the Owen value [12] for games with coalition structures that can be equivalently defined by applying the Shapley value [13] twice, first, the Shapley value is employed at the level of a priori unions to define a new game within each one of them and then, the Shapley value is applied to these new games. As a practical application we consider the problem of sharing of an international river, possibly with a delta or multiple sources, among multiple users without international firms.

The paper has the following structure. Basic definitions and notation along with the formal definition of a game with two-level communication structure and its core are introduced in Section 2. Section 3 provides the uniform approach to several known component efficient values for games with communication structures. In Section 4 we introduce values for games with two-level communication structures axiomatically and present an explicit formula representation, we also investigate stability and distribution of Harsanyi dividends. Section 5 discusses application to the water distribution problem of an international river among multiple users.

2. Preliminaries

2.1. TU games and values

Recall some definitions and notation. A *cooperative game with transferable utility* (*TU game*) is a pair $\langle N, v\rangle$, where $N \subset \mathbb{N}$ is a finite set of $n \geq 2$ players and $v\colon 2^N \to \mathbb{R}$ is a *characteristic function*, defined on the power set of N such that $v(\emptyset) = 0$.

A subset $S \subseteq N$ (or $S \in 2^N$) of s players is called a *coalition*, and the associated real number $v(S)$ presents the *worth* of S. The set of all games with fixed N we denote by $\mathcal{G}_N$. For simplicity of notation and if no ambiguity appears, we write v instead of $\langle N, v\rangle$ when refer to a game. A *value* is a mapping that assigns for every $N \subset \mathbb{N}$ and every $v \in \mathcal{G}_N$ a vector $\xi(v) \in \mathbb{R}^N$; the real number $\xi_i(v)$ represents the *payoff* to player i in v. A *subgame* of v with a player set $T \subseteq N$, $T \neq \emptyset$, is a game $v|_T$ defined as $v|_T(S) = v(S)$, for all $S \subseteq T$. A game v is *superadditive*, if $v(S \cup T) \geq v(S) + v(T)$, for all $S, T \subseteq N$, such that $S \cap T = \emptyset$. A game v is *convex*, if $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$, for all $S, T \subseteq N$. In what follows for all $x \in \mathbb{R}^N$ and $S \subseteq N$, we use standard notation $x(S) = \sum_{i \in S} x_i$. The cardinality of a given set A we denote by $|A|$ along with lower case letters like $n = |N|,\ m = |M|,\ n_k = |N_k|$, and so on.

It is well known (cf. Shapley [13]) that *unanimity games* $\{u_T\}_{\substack{T \subseteq N \\ T \neq \emptyset}}$, defined as $u_T(S) = 1$, if $T \subseteq S$, and $u_T(S) = 0$ otherwise, create a basis in $\mathcal{G}_N$, i.e., every $v \in \mathcal{G}_N$ can be uniquely presented in the linear form $v = \sum_{T \subseteq N, T \neq \emptyset} \lambda_T^v u_T$, where $\lambda_T^v = \sum_{S \subseteq T} (-1)^{t-s} v(S)$, for all $T \subseteq N$, $T \neq \emptyset$. Following Harsanyi [6] the coefficient λ_T^v is referred to as a *dividend* of coalition T in game v.

The *core* (cf. Gilles [5]) of $v \in \mathcal{G}_N$ is defined as

$$C(v) = \{x \in \mathbb{R}^N \mid x(N) = v(N),\ x(S) \geq v(S),\ \text{for all } S \subseteq N\}.$$

A value ξ is *stable*, if for any $v \in \mathcal{G}_N$ with nonempty core $C(v)$, $\xi(v) \in C(v)$.

2.2. Games with coalition structures

A *coalition structure*, or in other terms a *system of a priori unions*, on $N \subset \mathbb{N}$ is given by a partition $\mathcal{P} = \{N_1, \ldots, N_m\}$ of N, i.e., $N_1 \cup \cdots \cup N_m = N$ and $N_k \cap N_l = \emptyset$ for $k \neq l$. Let $\mathfrak{P}_N$ denote the set of all coalition structures on N, and let $\mathcal{G}_N^P = \mathcal{G}_N \times \mathfrak{P}_N$. A pair $\langle v, \mathcal{P}\rangle \in \mathcal{G}_N^P$ constitutes a *game with coalition structure*, or simply *P-game*, on N. A *P-value* is a mapping that assigns for every $N \subset \mathbb{N}$ and every $\langle v, \mathcal{P}\rangle \in \mathcal{G}_N^P$ a payoff vector $\xi(v, \mathcal{P}) \in \mathbb{R}^N$. Given $\langle v, \mathcal{P}\rangle \in \mathcal{G}_N^P$, Owen [12] defines a game $v_{\mathcal{P}}$, called a *quotient game*, on $M = \{1, \ldots, m\}$ in which each a priori union N_k acts as a player:

$$v_{\mathcal{P}}(Q) = v\left(\bigcup_{k \in Q} N_k\right), \quad \text{for all } Q \subseteq M.$$

Note that $\langle v, \{N\}\rangle$ represents the same situation as v itself. In what follows by $\langle N\rangle$ we denote the coalition structure composed by singletons, i.e., $\langle N\rangle = \{\{1\}, \ldots, \{n\}\}$. Furthermore, for every $i \in N$, let $k(i)$ be defined by the relation $i \in N_{k(i)}$, and for any $x \in \mathbb{R}^N$, let $x^{\mathcal{P}} = \big(x(N_k)\big)_{k \in M} \in \mathbb{R}^M$ be the corresponding vector of total payoffs to a priori unions.

2.3. Games with communication structures

A *communication structure* on N is specified by a graph Γ, undirected or directed. An *undirected/directed graph* is a collection of unordered/ordered pairs of nodes (players) $\Gamma \subseteq \Gamma_N^c = \{\{i,j\} \mid i,j \in N,\ i \neq j\}$ or $\Gamma \subseteq \bar{\Gamma}_N^c = \{(i,j) \mid i,j \in N, i \neq j\}$ respectively, where an unordered pair $\{i,j\}$ or correspondingly ordered pair (i,j) presents a *undirected/directed link* between $i, j \in N$. Let $\mathfrak{G}_N$ denote the set of all communication structures on N, and let $\mathcal{G}_N^\Gamma = \mathcal{G}_N \times \mathfrak{G}_N$. A pair $\langle v, \Gamma\rangle \in \mathcal{G}_N^\Gamma$ constitutes a *game with graph (communication) structure*, or simply *graph game* or *Γ-game*, on N. A *Γ-value* is a mapping that assigns for every $N \subset \mathbb{N}$ and every $\langle v, \Gamma\rangle \in \mathcal{G}_N^\Gamma$ a payoff vector $\xi(v, \Gamma) \in \mathbb{R}^N$.

In a graph Γ a sequence of different nodes $(i_1, \ldots, i_r)$, $r \geq 2$, is a *path* in Γ from node i_1 to node i_r if for $h = 1, \ldots, r-1$ it holds that $\{i_h, i_{h+1}\} \in \Gamma$ when Γ is undirected and $\{(i_h, i_{h+1}), (i_{h+1}, i_h)\} \cap \Gamma \neq \emptyset$ when Γ is directed. In a directed graph (digraph) Γ a path $(i_1, \ldots, i_r)$ is a *directed path* from node i_1 to node i_r if for all $h = 1, \ldots, r-1$ it holds that $(i_h, i_{h+1}) \in \Gamma$. In a digraph Γ, $j \neq i$ is a *successor* of i and i is a *predecessor* of j if there exists a directed path from i to j, and j is a *immediate successor* of i and i is a *immediate predecessor* of j if $(i,j) \in \Gamma$. Given a digraph Γ on N and $i \in N$, the sets of all predecessors, all immediate predecessors, all immediate successors, and all successors of i in Γ we denote by $P_\Gamma(i)$, $O_\Gamma(i)$, $F_\Gamma(i)$, and $S_\Gamma(i)$ correspondingly; moreover, $\bar{P}_\Gamma(i) = P_\Gamma(i) \cup \{i\}$ and $\bar{S}_\Gamma(i) = S_\Gamma(i) \cup \{i\}$.

Given a graph Γ on N, two nodes i and j in N are *connected* if there exists a path from node i to node j. Graph Γ on N is *connected* if any two nodes in N are connected. For a graph Γ on N and a coalition $S \subseteq N$, the *subgraph of* Γ *on* S is the graph $\Gamma|_S = \{\{i,j\} \in \Gamma \mid i,j \in S\}$ on S when Γ is undirected and the digraph $\Gamma|_S = \{(i,j) \in \Gamma \mid i,j \in S\}$ on S when Γ is directed. Given a graph Γ on N, a coalition $S \subseteq N$ is *connected* if the subgraph $\Gamma|_S$ is connected. For a graph Γ on N and coalition $S \subseteq N$, $C^\Gamma(S)$ is the set of all connected subcoalitions of S, S/Γ is the set of maximally connected subcoalitions of S, called the *components of* S, and $(S/\Gamma)_i$ is the component of S containing player $i \in S$. Notice that S/Γ is a partition of S. Besides, for any coalition structure $\mathcal{P}$, the graph $\Gamma^c(\mathcal{P}) = \bigcup_{P \in \mathcal{P}} \Gamma_P^c$, splits into completely connected components $P \in \mathcal{P}$, and $N/\Gamma^c(\mathcal{P}) = \mathcal{P}$. For any $\langle v, \Gamma\rangle \in \mathcal{G}_N^\Gamma$, a payoff vector $x \in \mathbb{R}^N$ is *component efficient* if $x(C) = v(C)$, for every $C \in N/\Gamma$. Later on when for avoiding confusion it is necessary to specify the set of nodes N in a graph Γ, we write Γ_N instead of Γ.

Following Myerson [11], we assume that for $\langle v, \Gamma\rangle \in \mathcal{G}_N^\Gamma$ cooperation is possible only between connected players and consider a *restricted game* $v^\Gamma \in \mathcal{G}_N$ defined as

$$v^\Gamma(S) = \sum_{C \in S/\Gamma} v(C), \quad \text{for all } S \subseteq N. \tag{1}$$

The *core* $C(v,\Gamma)$ of $\langle v,\Gamma\rangle \in \mathcal{G}_N^\Gamma$ is defined as a set of component efficient payoff vectors that are not dominated by any connected coalition, i.e.,

$$C(v,\Gamma)=\{x\in\mathbb{R}^N \mid x(C)=v(C),\ \forall C\in N/\Gamma,\ \text{and } x(T)\geq v(T),\ \forall T\in C^\Gamma(N)\}. \tag{2}$$

It is easy to see that $C(v,\Gamma)=C(v^\Gamma)$.

Below along with communication structures given by arbitrary undirected graphs we consider also those given by cycle-free undirected graphs and by directed graphs—line-graphs with linearly ordered players, rooted and sink forests. In an undirected graph Γ a path $(i_1,\dots,i_r)$, $r\geq 3$, is a *cycle* in Γ if $\{i_r,i_1\}\in\Gamma$. An undirected graph is *cycle-free* if it contains no cycles. A directed graph Γ is a *rooted tree* if there is one node in N, called a *root*, having no predecessors in Γ and there is a unique directed path in Γ from this node to any other node in N. A directed graph Γ is a *sink tree* if the directed graph composed by the same set of links as Γ but with the opposite orientation is a rooted tree; in this case the root of a tree changes its meaning to the absorbing sink. A directed graph is a *rooted/sink forest* if it is composed by a number of disjoint rooted/sink trees. A *line-graph* is a directed graph that contains links only between subsequent nodes. Without loss of generality we may assume that in a line-graph nodes are ordered according to the natural order from 1 to n, i.e., line-graph $\Gamma\subseteq\{(i,i+1)\mid i=1,\dots,n-1\}$.

2.4. Games with two-level communication structures

We now consider situations in which the players are partitioned into a coalition structure $\mathcal{P}$ and are linked to each other by communication graphs. First, there is a communication graph Γ_M between the a priori unions M in the partition $\mathcal{P}$. Second, for each a priori union $N_k,\ k\in M$, there is a communication graph Γ_k between the players in N_k. Given a player set $N\subset\mathbb{N}$ and a coalition structure $\mathcal{P}\in\mathfrak{P}_N$, a *two-level graph (communication) structure* on N is a tuple $\Gamma_{\mathcal{P}}=\langle\Gamma_M,\{\Gamma_{N_k}\}_{k\in M}\rangle$. For every $N\subset\mathbb{N}$ and $\mathcal{P}\in\mathfrak{P}_N$ by $\mathfrak{G}_N^{\mathcal{P}}$ we denote the set of all two-level graph structures on N with fixed $\mathcal{P}$. Let $\mathfrak{G}_N^P=\bigcup_{\mathcal{P}\in\mathfrak{P}_N}\mathfrak{G}_N^{\mathcal{P}}$ be the set of all two-level graph structures on N, and let $\mathcal{G}_N^{P\Gamma}=\mathcal{G}_N\times\mathfrak{G}_N^P$. A pair $\langle v,\Gamma_{\mathcal{P}}\rangle\in\mathcal{G}_N^{P\Gamma}$ constitutes a *game with two-level graph (communication) structure*, or simply *two-level graph game* or $P\Gamma$*-game*, on N. A $P\Gamma$*-value* is a mapping that assigns for every $N\subset\mathbb{N}$ and every $\langle v,\Gamma_{\mathcal{P}}\rangle\in\mathcal{G}_N^{P\Gamma}$ a payoff vector $\xi(v,\Gamma_{\mathcal{P}})\in\mathbb{R}^N$.

Observe that $P\Gamma$-games $\langle v,\Gamma_{\langle N\rangle}\rangle$ and $\langle v,\Gamma_{\{N\}}\rangle$ with trivial coalition structures reduce to Γ-game $\langle v,\Gamma_N\rangle$. In what follows for simplicity of notation and when it causes no ambiguity we denote graphs Γ_{N_k} within a priori unions N_k, $k\in M$, by Γ_k.

Given $\langle v,\Gamma_{\mathcal{P}}\rangle\in\mathcal{G}_N^{P\Gamma}$, one can consider *$\Gamma$-games within a priori unions* $\langle v_k,\Gamma_k\rangle\in\mathcal{G}_{N_k}^\Gamma$ with $v_k=v|_{N_k},\ k\in M$, that model the bargaining within a priori unions for distribution of their total shares among their members taking also into account a limited cooperation within each union N_k given by the communication graph Γ_k. Moreover, since every two-level graph structure $\Gamma_{\mathcal{P}}$ assumes a coalition structure $\mathcal{P}$ to be given, it is natural to consider a quotient game between a priori unions that models the upper level bargaining between a priori unions for their shares in the total payoff. When at least two a priori unions are negotiating for their shares, then similar to the classical quotient game of Owen, only the cumulative interests of each entire a priori union are taken into account. In such situations the information about limited cooperation within different a priori unions is not relevant and simply might be not known between the unions. But at the same time each a priori union knowing its own interior limited cooperation ability is able to re-evaluate its real individual capacity. So, for any $\langle v,\Gamma_{\mathcal{P}}\rangle\in\mathcal{G}_N^{P\Gamma}$ we define a *quotient game* $v_{\Gamma_{\mathcal{P}}}\in\mathcal{G}_M$ as

$$v_{\Gamma_{\mathcal{P}}}(Q)=\begin{cases} v_k^{\Gamma_k}(N_k), & Q=\{k\},\ k\in M\\ v\left(\bigcup\limits_{k\in Q}N_k\right), & |Q|>1,\end{cases}\qquad \text{for all } Q\subseteq M. \tag{3}$$

Observe that for a $P\Gamma$-game for which all graphs $\Gamma_k,\ k\in M$, are connected, the quotient game $v_{\Gamma_{\mathcal{P}}}$ coincides with the Owen quotient game $v_{\mathcal{P}}$. Next recall that when unions negotiate for their shares, their cooperation possibilities are restricted by the communication graph Γ_M on the level of the unions. So, one can consider a *quotient Γ-game* $\langle v_{\Gamma_{\mathcal{P}}},\Gamma_M\rangle\in\mathcal{G}_M^\Gamma$.

Furthermore, given a Γ-value ϕ, for any $\langle v,\Gamma_{\mathcal{P}}\rangle\in\mathcal{G}_N^{P\Gamma}$ with a graph structure Γ_M on the level of a priori unions suitable for application of ϕ to the corresponding quotient Γ-game $\langle v_{\Gamma_{\mathcal{P}}},\Gamma_M\rangle$,[1] along with a subgame v_k within a priori union N_k, $k\in M$, one can also consider a *ϕ_k-game* v_k^ϕ defined as

$$v_k^\phi(S)=\begin{cases}\phi_k(v_{\Gamma_{\mathcal{P}}},\Gamma_M), & S=N_k,\\ v(S), & S\neq N_k,\end{cases}\quad \text{for all } S\subseteq N_k, \tag{4}$$

where $\phi_k(v_{\Gamma_{\mathcal{P}}},\Gamma_M)$ is the payoff to N_k given by ϕ in $\langle v_{\Gamma_{\mathcal{P}}},\Gamma_M\rangle$. In particular, for any $x\in\mathbb{R}^M$, a x_k-game v_k^x within N_k, $k\in M$, is defined by

$$v_k^x(S)=\begin{cases}x_k, & S=N_k,\\ v(S), & S\neq N_k,\end{cases}\quad \text{for all } S\subseteq N_k.$$

In this context it is natural to consider Γ-games $\langle v_k^\phi,\Gamma_k\rangle$, $k\in M$, as well.

[1] In general, Γ-values can be applied only to Γ-games determined by graphs of certain types; for more detailed discussion see Section 3.

The *core* $C(v,\Gamma_{\mathcal{P}})$ of $\langle v,\Gamma_{\mathcal{P}}\rangle \in \mathcal{G}_N^{P\Gamma}$ is the set of payoff vectors that are

(i) component efficient both in the quotient Γ-game $\langle v_{\Gamma_{\mathcal{P}}},\Gamma_M\rangle$ and in all graph games within a priori unions $\langle v_k,\Gamma_k\rangle$, $k\in M$, containing more than one player,

(ii) not dominated by any connected coalition:

$$C(v,\Gamma_{\mathcal{P}}) = \Big\{x\in\mathbb{R}^N \mid \Big[\big[x^{\mathcal{P}}(K)=v_{\Gamma_{\mathcal{P}}}(K),\ \forall K\in M/\Gamma_M\big] \text{ and } \big[x^{\mathcal{P}}(Q)\geq v_{\Gamma_{\mathcal{P}}}(Q),\ \forall Q\in C^{\Gamma_M}(M)\big]\Big] \\ \text{and } \Big[\big[x(C)=v(C),\ \forall C\in N_k/\Gamma_k, C\neq N_k\big] \text{ and } \big[x(S)\geq v(S),\ \forall S\in C^{\Gamma_k}(N_k)\big], \forall k\in M\Big]\Big\}. \tag{5}$$

Remark 1. In the above definition of the core the condition of component efficiency on components equal to the entire a priori unions at the level within a priori unions is excluded. The reason is the following. By definition of a quotient game, for any $k\in M$, $v_{\Gamma_{\mathcal{P}}}(\{k\}) = v_k^{\Gamma_k}(N_k)$. If $N_k\in N_k/\Gamma_k$, i.e., if Γ_k is connected, $v_k^{\Gamma_k}(N_k)=v(N_k)$, and therefore, $v_{\Gamma_{\mathcal{P}}}(\{k\})=v(N_k)$. Besides by definition, $x^{\mathcal{P}}(\{k\}) = x_k^{\mathcal{P}} = x(N_k)$, for all $k\in M$. Furthermore, singleton coalitions are always connected, i.e., $\{k\}\in C^{\Gamma_M}(M)$, for all $k\in M$. Thus, in case $N_k\in N_k/\Gamma_k$ and $\{k\}\notin M/\Gamma_M$, the presence of a stronger condition $x(N_k)=v(N_k)$ at the level within a priori unions may conflict with a weaker condition $x^{\mathcal{P}}(\{k\})\geq v_{\Gamma_{\mathcal{P}}}(\{k\})$ which in this case at the level of a priori unions is the same as $x(N_k)\geq v(N_k)$; as a result this can lead to the emptiness of the core. Observe also that in case $\{k\}\in M/\Gamma_M$ and $N_k\in N_k/\Gamma_k$, the component efficiency condition $x^{\mathcal{P}}(\{k\})=v_{\Gamma_{\mathcal{P}}}(\{k\})$ on the level between a priori unions is simply the same as component efficiency condition $x(N_k)=v(N_k)$ at the level within a priori unions.

The next statement easily follows from the latter definition.

Proposition 1. *For any* $\langle v,\Gamma_{\mathcal{P}}\rangle\in\mathcal{G}_N^{P\Gamma}$ *and* $x\in\mathbb{R}^N$,

$$x\in C(v,\Gamma_{\mathcal{P}}) \Longleftrightarrow \big[x^{\mathcal{P}}\in C(v_{\Gamma_{\mathcal{P}}},\Gamma_M)\big] \quad \textit{and} \quad \big[x_{N_k}\in C(v_k^{x^{\mathcal{P}}},\Gamma_k),\ \forall k\in M\colon n_k>1\big].$$

Remark 2. The claim $x_{N_k}\in C(v_k^{x^{\mathcal{P}}},\Gamma_k)$, $k\in M$, is vital only if $N_k\in N_k/\Gamma_k$, i.e., if Γ_k is connected; when Γ_k is disconnected, it can be replaced by $x_{N_k}\in C(v_k,\Gamma_k)$, as well.

3. Uniform approach to component efficient Γ-values

We show now that a number of known component efficient Γ-values for games with communication structures given by undirected and directed graphs of different types can be approached within the single framework. This unified approach will be employed later in Section 4 for the construction of $P\Gamma$-values reflecting the two-stage distribution procedure.

A Γ-value ξ is *component efficient* (CE) if, for any Γ-game $\langle v,\Gamma\rangle\in\mathcal{G}_N^{\Gamma}$, for all $C\in N/\Gamma$,

$$\sum_{i\in C}\xi_i(v,\Gamma)=v(C).$$

3.1. CE values for undirected graph games

3.1.1. The Myerson value

The *Myerson value* [11] is defined for any Γ-game $\langle v,\Gamma\rangle\in\mathcal{G}_N^{\Gamma}$ with arbitrary undirected graph Γ as the Shapley value of the restricted game v^{Γ}:

$$\mu_i(v,\Gamma)=Sh_i(v^{\Gamma}),\quad \text{for all } i\in N.$$

The Myerson value is characterized by two axioms of component efficiency and fairness.

A Γ-value ξ is *fair* (F) if, for any Γ-game $\langle v,\Gamma\rangle\in\mathcal{G}_N^{\Gamma}$, for every link $\{i,j\}\in\Gamma$, it holds that

$$\xi_i(v,\Gamma)-\xi_i(v,\Gamma\setminus\{i,j\})=\xi_j(v,\Gamma)-\xi_j(v,\Gamma\setminus\{i,j\}).$$

3.1.2. The position value

The *position value* introduced in Meessen [10] and developed in Borm et al. [3] is defined for any Γ-game $\langle v,\Gamma\rangle\in\mathcal{G}_N^{\Gamma}$ with arbitrary undirected graph Γ. The position value in $\langle v,\Gamma\rangle$ assigns to each player the sum of his individual value $v(i)$ and half of the value of each link he is involved in, where the value of a link is defined as the Shapley payoff to this link in the associated link game on links of Γ:

$$\pi_i(v,\Gamma) = v(i) + \frac{1}{2}\sum_{l\in\Gamma_i} Sh_l(\Gamma, v^0_\Gamma), \quad \text{for all } i \in N,$$

where $\Gamma_i = \{l \in \Gamma \mid l \ni i\}$, v^0 is the zero-normalization of v, i.e., for all $S \subseteq N$, $v^0(S) = v(S) - \sum_{i\in S} v(i)$, and for any zero-normalized game $v \in \mathcal{G}_N$ and a graph Γ, the associated link game $\langle \Gamma, v_\Gamma\rangle$ between links in Γ is defined as

$$v_\Gamma(\Gamma') = v^{\Gamma'}(N), \quad \text{for all } \Gamma' \in 2^\Gamma.$$

Slikker [14] characterizes the position value on the class of all graph games via component efficiency and balanced link contributions.

A Γ-value ξ meets *balanced link contributions* (BLC) if for any Γ-game $\langle v,\Gamma\rangle \in \mathcal{G}^\Gamma_N$ and $i, j \in N$, it holds that

$$\sum_{h|\{i,h\}\in\Gamma} \big[\xi_j(v,\Gamma) - \xi_j(v,\Gamma\setminus\{i,h\})\big] = \sum_{h|\{j,h\}\in\Gamma} \big[\xi_i(v,\Gamma) - \xi_i(v,\Gamma\setminus\{j,h\})\big].$$

3.1.3. The average tree solution

The *average tree solution* (*AT solution*) for undirected cycle-free Γ-games introduced in Herings et al. [7] in any Γ-game $\langle v,\Gamma\rangle \in \mathcal{G}^\Gamma_N$ with cycle-free undirected graph Γ assigns to any player $i \in N$ the average of his tree value payoffs in all rooted spanning trees[2] in the subgraph $\langle (N/\Gamma)_i, \Gamma|_{(N/\Gamma)_i}\rangle$:

$$AT_i(v,\Gamma) = \frac{1}{|(N/\Gamma)_i|}\sum_{j\in(N/\Gamma)_i} t_i(v, T(j)), \quad \text{for all } i \in N,$$

where $T(j)$, $j \in (N/\Gamma)_i$, is a rooted tree on $(N/\Gamma)_i$ with j as root and composed of all links of undirected cycle-free subgraph $\langle (N/\Gamma)_i, \Gamma|_{(N/\Gamma)_i}\rangle$ with orientation directed away from the root and t is the tree value that in any digraph game $\langle v,\Gamma\rangle$ on N with Γ being a rooted forest (in particular, in Γ-game $\langle v, T(j)\rangle$ with rooted-tree digraph $T(j)$ on the player set $(N/\Gamma)_i$ as in the formula above) assigns to each player his contribution to all his successors in Γ when he joins them, i.e.,

$$t_i(v,\Gamma) = v(\bar{S}_\Gamma(i)) - \sum_{h\in F_\Gamma(i)} v(\bar{S}_\Gamma(h)), \quad \text{for all } i \in N. \tag{6}$$

Remark that the AT solution is very attractive from the algorithmic point of view because the order of its computational complexity is equal to n while the order of computational complexity of the Myerson value is $n!$.

In Herings et al. [7] it is shown that the AT solution defined on the class of superadditive cycle-free graph games is stable and on the entire class of cycle-free graph games it is characterized via two axioms of component efficiency and component fairness.

A Γ-value ξ is *component fair* (CF) if, for any cycle-free Γ-game $\langle v,\Gamma\rangle \in \mathcal{G}^\Gamma_N$, for every link $\{i,j\} \in \Gamma$, it holds that

$$\frac{1}{|(N/\Gamma\setminus\{i,j\})_i|}\sum_{t\in(N/\Gamma\setminus\{i,j\})_i}\Big(\xi_t(v,\Gamma) - \xi_t(v,\Gamma\setminus\{i,j\})\Big)$$
$$= \frac{1}{|(N/\Gamma\setminus\{i,j\})_j|}\sum_{t\in(N/\Gamma\setminus\{i,j\})_j}\Big(\xi_t(v,\Gamma) - \xi_t(v,\Gamma\setminus\{i,j\})\Big).$$

3.2. CE values for directed graph games

3.2.1. Values for line-graph games

The following three values for line-graph Γ-games are studied in van den Brink et al. [15], namely, the *upper equivalent solution* given by

$$\xi^{UE}_i(v,\Gamma) = v^\Gamma(\{1,\ldots,i-1,i\}) - v^\Gamma(\{1,\ldots,i-1\}), \quad \text{for all } i \in N,$$

the *lower equivalent solution* given by

$$\xi^{LE}_i(v,\Gamma) = v^\Gamma(\{i,i+1,\ldots,n\}) - v^\Gamma(\{i+1,\ldots,n\}), \quad \text{for all } i \in N$$

and the *equal loss solution* given for all $i \in N$ by

$$\xi^{EL}_i(v,\Gamma) = \frac{\big(v^\Gamma(\{1,\ldots,i\}) - v^\Gamma(\{1,\ldots,i-1\})\big) + \big(v^\Gamma(\{i,\ldots,n\}) - v^\Gamma(\{i+1,\ldots,n\})\big)}{2}.$$

[2] Given an undirected graph Γ on N, a rooted tree Γ' on N is a *spanning tree* of Γ if for every $(i,j) \in \Gamma'$ it holds that $\{i,j\} \in \Gamma$.

All these three solutions for superadditive line-graph Γ-games appear to be stable. Moreover, on the entire class of line-graph games each one of them is characterized via component efficiency and one of the axioms of upper equivalence, lower equivalence, and equal loss correspondingly.

A Γ-value ξ is *upper equivalent* (UE) if, for any line-graph Γ-game $\langle v, \Gamma\rangle \in \mathcal{G}_N^{\Gamma}$, for any $i = 1, \ldots, n-1$, for all $j = 1, \ldots, i$, it holds that

$$\xi_j(v, \Gamma \setminus \{i, i+1\}) = \xi_j(v, \Gamma).$$

A Γ-value ξ is *lower equivalent* (LE) if, for any line-graph Γ-game $\langle v, \Gamma\rangle \in \mathcal{G}_N^{\Gamma}$, for any $i = 1, \ldots, n-1$, for all $j = i+1, \ldots, n$, it holds that

$$\xi_j(v, \Gamma \setminus \{i, i+1\}) = \xi_j(v, \Gamma).$$

A Γ-value ξ possesses the *equal loss property* (EL) if, for any line-graph Γ-game $\langle v, \Gamma\rangle \in \mathcal{G}_N^{\Gamma}$, for any $i = 1, \ldots, n-1$, it holds that

$$\sum_{j=1}^{i}\Big(\xi_j(v, \Gamma) - \xi_j(v, \Gamma \setminus \{i, i+1\})\Big) = \sum_{j=i+1}^{n}\Big(\xi_j(v, \Gamma) - \xi_j(v, \Gamma \setminus \{i, i+1\})\Big).$$

3.2.2. Tree-type values for forest-graph games

The tree value defined by (6) and the sink value

$$s_i(v, \Gamma) = v(\bar{P}_\Gamma(i)) - \sum_{j \in O_\Gamma(i)} v(\bar{P}_\Gamma(j)), \quad \text{for all } i \in N,$$

respectively for rooted-/sink-forest digraph games are studied in Khmelnitskaya [9]. Both tree and sink values are stable on the subclass of superadditive games. Moreover, the tree and sink values on the entire class of rooted-/sink-forest Γ-games can be characterized via component efficiency and successor/predecessor equivalence correspondingly.

A Γ-value ξ is *successor equivalent* (SE) if for any rooted forest Γ-game $\langle v, \Gamma\rangle \in \mathcal{G}_N^{\Gamma}$, for every link $\{i, j\} \in \Gamma$, for all $k \in \bar{S}_\Gamma(j)$, it holds that

$$\xi_k(v, \Gamma \setminus \{i, j\}) = \xi_k(v, \Gamma).$$

A Γ-value ξ is *predecessor equivalent* (PE) if for any sink forest Γ-game $\langle v, \Gamma\rangle \in \mathcal{G}_N^{\Gamma}$, for every link $\{i, j\} \in \Gamma$, for all $k \in \bar{P}_\Gamma(i)$, it holds that

$$\xi_k(v, \Gamma \setminus \{i, j\}) = \xi_k(v, \Gamma).$$

3.3. Uniform framework

Notice that each one of the considered above Γ-values for Γ-games with *suitable* graph structures is characterized by two axioms, CE and one or another *deletion link* (DL) property, reflecting the relevant reaction of a Γ-value on deletion of a link in the communication graph, i.e.,

CE + F for all undirected Γ-games $\Longleftrightarrow \mu(v, \Gamma)$,
CE + BLC for all undirected Γ-games $\Longleftrightarrow \pi(v, \Gamma)$,
CE + CF for undirected cycle-free Γ-games $\Longleftrightarrow AT(v, \Gamma)$,
CE + UE for line-graph Γ-games $\Longleftrightarrow UE(v, \Gamma)$,
CE + LE for line-graph Γ-games $\Longleftrightarrow LE(v, \Gamma)$,
CE + EL for line-graph Γ-games $\Longleftrightarrow EL(v, \Gamma)$,
CE + SE for rooted forest Γ-games $\Longleftrightarrow t(v, \Gamma)$,
CE + PE for sink forest Γ-games $\Longleftrightarrow s(v, \Gamma)$.

In the sequel for the unification of presentation and simplicity of notation, we identify each one of mentioned above Γ-values with the corresponding DL axiom. For a given DL, let $\mathcal{G}_N^{DL} \subseteq \mathcal{G}_N^{\Gamma}$ be a set of all $\langle v, \Gamma\rangle \in \mathcal{G}_N^{\Gamma}$ with Γ suitable for DL application. To summarize,

$$\text{CE} + \text{DL} \quad \text{on } \mathcal{G}_N^{DL} \Longleftrightarrow DL(v, \Gamma),$$

where DL is one of the axioms F, BLC, CF, LE, UE, EL, SE, or PE. Whence, $F(v, \Gamma) = \mu(v, \Gamma)$ and $BLC(v, \Gamma) = \pi(v, \Gamma)$ for all undirected Γ-games, $CF(v, \Gamma) = AT(v, \Gamma)$ for all undirected cycle-free Γ-games, $UE(v, \Gamma)$, $LE(v, \Gamma)$, and $EL(v, \Gamma)$ are UE, LE, and EL solutions correspondingly for all line-graph Γ-games, $SE(v, \Gamma) = t(v, \Gamma)$ for all rooted forest Γ-games, and $PE(v, \Gamma) = s(v, \Gamma)$ for all sink forest Γ-games.

4. Two-level graph game values

4.1. Component efficient $P\Gamma$-values

Henceforth we focus on $P\Gamma$-values that reflect a two-stage distribution procedure when at first the quotient Γ-game $\langle v_{\Gamma_\mathcal{P}}, \Gamma_M\rangle$ is played between a priori unions N_k, $k \in M$, and then the total payoffs y_k obtained by a priori unions are distributed among their members by playing Γ-games $\langle v_k^y, \Gamma_k\rangle$. As solutions on both steps the component efficient Γ-values are applied, might be different for the upper level between a priori unions and the lower level within a priori unions and also possibly different for different a priori unions.

We start with adaptation the notions of component efficiency and discussed above deletion link properties to $P\Gamma$-values and show that similar to component efficient Γ-values, the deletion link properties uniquely define component efficient $P\Gamma$-values on a class of admissible $P\Gamma$-games. The involvement of different deletion link properties, depending on the considered graph structure, allows to pick the most favorable among the others appropriate combinations of Γ-values applied on both levels between and within a priori unions. Moreover, the consideration of only one specific combination of Γ-values restricts the variability of applications since Γ-values developed for Γ-games defined by undirected graphs are not applicable for Γ-games with, for example, directed rooted forest graph structures, and vice versa.

First we introduce two new axioms of component efficiency with respect to $P\Gamma$-values that inherit the idea of component efficiency for Γ-values and also incorporate the quotient game property[3] of the Owen value [12] in a sense that the vector of total payoffs to a priori unions coincides with the payoff vector in the quotient game.

A $P\Gamma$-value ξ is *component efficient in quotient* (CEQ) if, for any $\langle v, \Gamma_\mathcal{P}\rangle \in \mathcal{G}_N^{P\Gamma}$, for each $K \in M/\Gamma_M$,

$$\sum_{k\in K}\sum_{i\in N_k}\xi_i(v,\Gamma_\mathcal{P}) = v_{\Gamma_\mathcal{P}}(K).$$

A $P\Gamma$-value ξ is *component efficient within a priori unions* (CEU) if, for any $\langle v, \Gamma_\mathcal{P}\rangle \in \mathcal{G}_N^{P\Gamma}$, for every $k \in M$ and all $C \in N_k/\Gamma_k$, $C \neq N_k$,

$$\sum_{i\in C}\xi_i(v,\Gamma_\mathcal{P}) = v(C).$$

Remark that CEU becomes redundant if considered on the subclass $P\Gamma$-games for which all graphs Γ_k, $k \in M$, are connected.

Next we reconsider the deletion link properties, now with respect to $P\Gamma$-values. Recall that every $P\Gamma$-value is defined as a mapping $\xi\colon \mathcal{G}_N^{P\Gamma} \to \mathbb{R}^N$ assigning a payoff vector to any $P\Gamma$-game on the player set N. A mapping $\xi = \{\xi_i\}_{i\in N}$ generates on the domain of $P\Gamma$-games on N a mapping $\xi^\mathcal{P}\colon \mathcal{G}_N^{P\Gamma} \to \mathbb{R}^M$, $\xi^\mathcal{P} = \{\xi_k^\mathcal{P}\}_{k\in M}$, with $\xi_k^\mathcal{P} = \sum_{i\in N_k}\xi_i$, $k \in M$, that assigns to every $P\Gamma$-game on N a vector of total payoffs to all a priori unions and m mappings $\xi_{N_k}\colon \mathcal{G}_N^{P\Gamma} \to \mathbb{R}^{N_k}$, $\xi_{N_k} = \{\xi_i\}_{i\in N_k}$, $k \in M$, assigning payoffs to players within a priori unions. Since there are many $P\Gamma$-games $\langle v, \Gamma_\mathcal{P}\rangle$ with the same quotient Γ-game $\langle v_{\Gamma_\mathcal{P}}, \Gamma_M\rangle$, there exists a variety of mappings $\psi_\mathcal{P}\colon \mathcal{G}_M^\Gamma \to \mathcal{G}_N^{P\Gamma}$ assigning to any Γ-game $\langle u, \Gamma\rangle \in \mathcal{G}_M^\Gamma$ on the player set M some $P\Gamma$-game $\langle v, \Gamma_\mathcal{P}\rangle \in \mathcal{G}_N^{P\Gamma}$ on the player set N such that $v_{\Gamma_\mathcal{P}} = u$ and $\Gamma_M = \Gamma$. Notice that in general, it is not necessarily that $\psi_\mathcal{P}(v_{\Gamma_\mathcal{P}}, \Gamma_M) = \langle v, \Gamma_\mathcal{P}\rangle$. However, for some fixed $P\Gamma$-game $\langle v^*, \Gamma^*_{\mathcal{P}*}\rangle$ one can always choose a mapping $\psi^*_\mathcal{P}$ such that $\psi^*_\mathcal{P}(v^*_{\Gamma_\mathcal{P}}, \Gamma^*_M) = \langle v^*, \Gamma^*_{\mathcal{P}*}\rangle$. Every mapping $\xi^\mathcal{P}\circ\psi_\mathcal{P}\colon \mathcal{G}_M^\Gamma \to \mathbb{R}^M$ by definition is a Γ-value on the player set M that, in particular, can be applied to the quotient Γ-game $\langle v_{\Gamma_\mathcal{P}}, \Gamma_M\rangle \in \mathcal{G}_M^\Gamma$ of some $P\Gamma$-game $\langle v, \Gamma_\mathcal{P}\rangle \in \mathcal{G}_N^{P\Gamma}$. Similarly, for a given Γ-value $\phi\colon \mathcal{G}_M^\Gamma \to \mathbb{R}^M$ assigning a payoff vector to any $P\Gamma$-game on the player set M, in particular to the quotient Γ-game $\langle v_{\Gamma_\mathcal{P}}, \Gamma_M\rangle \in \mathcal{G}_M^\Gamma$ of some $P\Gamma$-game $\langle v, \Gamma_\mathcal{P}\rangle \in \mathcal{G}_N^{P\Gamma}$, for every $k \in M$ there exists a variety of mappings $\psi_k^\phi\colon \mathcal{G}_{N_k}^\Gamma \to \mathcal{G}_N^{P\Gamma}$ assigning to any Γ-game $\langle u, \Gamma\rangle \in \mathcal{G}_{N_k}^\Gamma$ on N_k some $P\Gamma$-game $\langle v, \Gamma_\mathcal{P}\rangle \in \mathcal{G}_N^{P\Gamma}$ on N such that $v_k^\phi = u$ and $\Gamma_k = \Gamma$. Every mapping $\xi_{N_k}\circ\psi_k^\phi\colon \mathcal{G}_{N_k}^\Gamma \to \mathbb{R}^{N_k}$, $k \in M$, by definition is a Γ-value on the player set N_k that, in particular, can be applied to Γ-games $\langle v_k^\phi, \Gamma_k\rangle \in \mathcal{G}_{N_k}^\Gamma$ of some $P\Gamma$-game $\langle v, \Gamma_\mathcal{P}\rangle \in \mathcal{G}_N^{P\Gamma}$ and a Γ-value ϕ chosen to be applied on the upper level to the quotient Γ-game $\langle v_{\Gamma_\mathcal{P}}, \Gamma_M\rangle$.

Each $P\Gamma$-value under scrutiny determines a two-stage distribution procedure in which the distribution of the total payoffs to a priori unions and the following after redistribution of these payoffs among the unions' members are due to the Γ-values generated respectively on the quotient level and on the level of a priori unions. So, it makes sense to introduce axioms presenting the properties of $P\Gamma$-values not only in terms of the $P\Gamma$-values but also in terms of the generated on both levels Γ-values. While the efficiency properties combining the distribution results of both stages we formulate in terms of a

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[3] A P-value ξ satisfies *the quotient game property*, if for any $\langle v, \mathcal{P}\rangle \in \mathcal{G}_N^P$, for all $k \in M$,

$$\xi_k(v_\mathcal{P}, \{M\}) = \xi_k(v_\mathcal{P}, \langle M\rangle) = \sum_{i\in N_k}\xi_i(v,\mathcal{P}).$$

$P\Gamma$-value itself, the deletion link properties that determine the type of the distribution procedures on each level we present in terms of the corresponding Γ-values.

For a given $(m+1)$-tuple of deletion link axioms $\langle \mathrm{DL}^{\mathcal{P}}, \{\mathrm{DL}^k\}_{k\in M}\rangle$ consider a set of $P\Gamma$-games $\mathcal{G}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}} \subseteq \mathcal{G}_N^{P\Gamma}$ composed of $P\Gamma$-games $\langle v, \Gamma_{\mathcal{P}}\rangle$ with graph structures $\Gamma_{\mathcal{P}} = \langle \Gamma_M, \{\Gamma_k\}_{k\in M}\rangle$ such that $\langle v_{\Gamma_{\mathcal{P}}}, \Gamma_M\rangle \in \mathcal{G}_M^{DL^{\mathcal{P}}}$ and $\langle v_k^{DL^{\mathcal{P}}}, \Gamma_k\rangle \in \mathcal{G}_{N_k}^{DL^k},\ k\in M$.

A $P\Gamma$-value ξ defined on $\mathcal{G}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$ *satisfies $(m+1)$-tuple of deletion link axioms* $\langle \mathrm{DL}^{\mathcal{P}}, \{\mathrm{DL}^k\}_{k\in M}\rangle$ if every Γ-value $\xi^{\mathcal{P}}\circ\psi_{\mathcal{P}}$ meets $\mathrm{DL}^{\mathcal{P}}$ axiom and every Γ-value $\xi_{N_k}\circ\psi_k^{DL^{\mathcal{P}}},\ k\in M$, meets the corresponding DL^k axiom.

Remark 3. It is worth to emphasize that a $(m+1)$-tuple of deletion link axioms $\langle \mathrm{DL}^{\mathcal{P}}, \{\mathrm{DL}^k\}_{k\in M}\rangle$ imposed on a $P\Gamma$-value ξ defined on $\mathcal{G}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$ in fact does not impose the deletion link properties directly on the $P\Gamma$-value ξ but on the corresponding generated by ξ Γ-values defined on $\mathcal{G}_M^{DL^{\mathcal{P}}}$ and $\mathcal{G}_{N_k}^{DL^k},\ k\in M$.

Our goal is to show that component efficiency in quotient, component efficiency within a priori unions and a tuple of deletion link axioms $\langle \mathrm{DL}^{\mathcal{P}}, \{\mathrm{DL}^k\}_{k\in M}\rangle$ uniquely define a $P\Gamma$-value. But before stating the main result we discuss the limitations of the model. First observe that the consideration of $P\Gamma$-values satisfying both CEQ and CEU is possible only for $P\Gamma$-games $\langle v, \Gamma_{\mathcal{P}}\rangle$ meeting the condition:

(i) for all nonsingleton components on the quotient level $K\in M/\Gamma_M$, $|K|>1$, for which all $\Gamma_k,\ k\in K$, are disconnected, i.e., $N_k\notin N_k/\Gamma_k$, it holds that
$$\sum_{k\in K}\sum_{C\in N_k/\Gamma_k} v(C) = v\left(\bigcup_{k\in K} N_k\right).$$

Remark that if at least one graph $\Gamma_k,\ k\in K$, is connected, the condition (i) becomes redundant.

Next, it turns out that a two-stage distribution procedure that first applies the $DL^{\mathcal{P}}$-value as a solution for the quotient game $\langle v_{\Gamma_{\mathcal{P}}}, \Gamma_M\rangle$ and then distributes the payoffs $DL_k^{\mathcal{P}}(v_{\Gamma_{\mathcal{P}}}, \Gamma_M)$, $k\in M$, obtained by a priori unions among their members using the corresponding DL^k-values is applicable not for all $P\Gamma$-games of the class $\mathcal{G}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$. Indeed, the two-stage distribution procedure assumes the benefits of cooperation between a priori unions to be distributed fully among single players, i.e., the solutions within all Γ-games $\langle v_k^{DL^{\mathcal{P}}}, \Gamma_k\rangle,\ k\in M$, need to provide an efficient distribution of the corresponding amounts $DL_k^{\mathcal{P}}(v_{\Gamma_{\mathcal{P}}}, \Gamma_M)$. Since we concentrate on component efficient solutions, it is important to ensure that the requirement of efficiency does not conflict with component efficiency which is equivalent to the claim that for every $k\in M$,
$$\sum_{C\in N_k/\Gamma_k} v_k^{DL^{\mathcal{P}}}(C) = DL_k^{\mathcal{P}}(v_{\Gamma_{\mathcal{P}}}, \Gamma_M).$$

If Γ_k is connected, i.e. if N_k is the only element of N_k/Γ_k, then the last equality holds automatically since by definition $v_k^{DL^{\mathcal{P}}}(N_k) \overset{(4)}{=} DL_k^{\mathcal{P}}(v_{\Gamma_{\mathcal{P}}}, \Gamma_M)$. Moreover, for every $k\in M$ being a singleton component $\{k\}\in M/\Gamma_M$, this equality holds also true when Γ_k is disconnected. Indeed, if $\{k\}\in M/\Gamma_M$, then due to the component efficiency of the $DL^{\mathcal{P}}$-value it holds that $DL_k^{\mathcal{P}}(v_{\Gamma_{\mathcal{P}}}, \Gamma_M) = v_{\Gamma_{\mathcal{P}}}(\{k\})$. But by definition of the quotient game $v_{\Gamma_{\mathcal{P}}}$ and the Myerson restricted game, $v_{\Gamma_{\mathcal{P}}}(\{k\}) \overset{(3)}{=} v_k^{\Gamma_k}(N_k) \overset{(1)}{=} \sum_{C\in N_k/\Gamma_k} v_k(C) = \sum_{C\in N_k/\Gamma_k} v(C)$. However, in general we can apply the described above two-stage procedure only to $P\Gamma$-games $\langle v, \Gamma_{\mathcal{P}}\rangle$ meeting the condition:

(ii) for all $k\in M$ such that
  (a) $\{k\}$ is not a singleton component on the quotient level, i.e., $\{k\}\notin M/\Gamma_M$,
  (b) Γ_k is disconnected, i.e., $N_k\notin N_k/\Gamma_k$,
  it holds that
$$\sum_{C\in N_k/\Gamma_k} v(C) = DL_k^{\mathcal{P}}(v_{\Gamma_{\mathcal{P}}}, \Gamma_M).$$

Denote by $\bar{\mathcal{G}}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$ the set of all $P\Gamma$-games $\langle v, \Gamma_{\mathcal{P}}\rangle \in \mathcal{G}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$ meeting the conditions (i) and (ii).

Remark 4. In general, without restrictions on the characteristic function, class of $P\Gamma$-games $\bar{\mathcal{G}}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$ is not closed under the modification of a two-level graph structure. Indeed, for a nonadditive characteristic function it might happen that the deletion of a link in one of the graphs Γ_M or $\Gamma_k,\ k\in M$, composing a two-level graph structure $\Gamma_{\mathcal{P}}$, may lead the resulting $P\Gamma$-game out of the class $\bar{\mathcal{G}}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$ since for the resulting $P\Gamma$-game conditions (i) and (ii) might be violated. Also for this reason we introduce a $(m+1)$-tuple of deletion link axioms $\langle \mathrm{DL}^{\mathcal{P}}, \{\mathrm{DL}^k\}_{k\in M}\rangle$ not in terms of a $P\Gamma$-value defined on $\bar{\mathcal{G}}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$ but in terms of the generated on the upper and lower levels Γ-values. These Γ-values are defined correspondingly on $\mathcal{G}_M^{DL^{\mathcal{P}}}$ and $\mathcal{G}_{N_k}^{DL^k},\ k\in M$, and do not face such problems.

For applications involving disconnected graphs Γ_k in a priori unions forming nonsingleton components on the quotient level, i.e., for $k \in M$ such that $|(M/\Gamma_M)_k| > 1$, the requirements (i) and (ii) appear to be too demanding. But both conditions (i) and (ii) are redundant when for all nonsingleton components $C \in M/\Gamma_M$ graphs Γ_k, $k \in C$, are connected. It is worth to emphasize the following remark.

Remark 5. Every $P\Gamma$-game $\langle v, \Gamma_{\mathcal{P}}\rangle \in \mathcal{G}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$ for which for all nonsingleton components $K \in M/\Gamma_M$ graphs Γ_k, $k \in K$, are connected, in particular, when all graphs Γ_k, $k \in M$, are connected, belongs to $\bar{\mathcal{G}}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$.

Theorem 1. *There is a unique $P\Gamma$-value defined on $\bar{\mathcal{G}}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$ that meets CEQ, CEU, and $\langle DL^{\mathcal{P}}, \{DL^k\}_{k\in M}\rangle$, and for any $\langle v, \Gamma_{\mathcal{P}}\rangle \in \bar{\mathcal{G}}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$ it is given by*

$$\xi_i(v, \Gamma_{\mathcal{P}}) = \begin{cases} DL^{\mathcal{P}}_{k(i)}(v_{\Gamma_{\mathcal{P}}}, \Gamma_M), & N_{k(i)} = \{i\}, \\ DL^{k(i)}_i(v^{DL^{\mathcal{P}}}_{k(i)}, \Gamma_{k(i)}), & n_{k(i)} > 1, \end{cases} \quad \text{for all } i \in N. \tag{7}$$

From now on we refer to the $P\Gamma$-value ξ as to the *$\langle DL^{\mathcal{P}}, \{DL^k\}_{k\in M}\rangle$-value.*

Proof. I. First prove that the $P\Gamma$-value given by (7) is the unique one on $\bar{\mathcal{G}}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$ that satisfies CEQ, CEU, and $\langle DL^{\mathcal{P}}, \{DL^k\}_{k\in M}\rangle$. Take a $P\Gamma$-value ξ on $\bar{\mathcal{G}}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$ meeting CEQ, CEU, and $\langle DL^{\mathcal{P}}, \{DL^k\}_{k\in M}\rangle$. Let $\langle v^*, \Gamma^*_{\mathcal{P}}\rangle \in \bar{\mathcal{G}}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$ with $\Gamma^*_{\mathcal{P}} = \langle \Gamma^*_M, \{\Gamma^*_k\}_{k\in M}\rangle$, and let $v^*_{\Gamma_{\mathcal{P}}}$ denote its quotient game. Notice that by choice of $\langle v^*, \Gamma^*_{\mathcal{P}*}\rangle$, it holds that $\langle v^*_{\Gamma_{\mathcal{P}}}, \Gamma^*_M\rangle \in \mathcal{G}_M^{DL^{\mathcal{P}}}$ and $\langle (v^*)^{DL^{\mathcal{P}}}_k, \Gamma^*_k\rangle \in \mathcal{G}_{N_k}^{DL^k}$ for all $k \in M$.

Step 1. Level of a priori unions.

Consider the mapping $\psi^*_{\mathcal{P}}\colon \mathcal{G}_M^{DL^{\mathcal{P}}} \to \bar{\mathcal{G}}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$ that assigns to any Γ-game $\langle u, \Gamma\rangle \in \mathcal{G}_M^{DL^{\mathcal{P}}}$ the $P\Gamma$-game $\langle v, \Gamma_{\mathcal{P}}\rangle \in \bar{\mathcal{G}}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$ such that $v_{\Gamma_{\mathcal{P}}} = u$ and $\Gamma_M = \Gamma$, and satisfies the condition $\psi^*_{\mathcal{P}}(v^*_{\Gamma_{\mathcal{P}}}, \Gamma^*_M) = \langle v^*, \Gamma^*_{\mathcal{P}}\rangle$. By definition of $\xi^{\mathcal{P}}$, for any $\langle u, \Gamma\rangle \in \mathcal{G}_M^{DL^{\mathcal{P}}}$ and $\langle v, \Gamma_{\mathcal{P}}\rangle = \psi^*_{\mathcal{P}}(u, \Gamma)$ it holds that

$$(\xi^{\mathcal{P}} \circ \psi^*_{\mathcal{P}})_k(u, \Gamma) = \sum_{i\in N_k} \xi_i(v, \Gamma_{\mathcal{P}}), \quad \text{for all } k \in M. \tag{8}$$

Since ξ meets CEQ, for any $\langle v, \Gamma_{\mathcal{P}}\rangle \in \bar{\mathcal{G}}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$, for all $K \in M/\Gamma_M$,

$$\sum_{k\in K}\sum_{i\in N_k} \xi_i(v, \Gamma_{\mathcal{P}}) = v_{\Gamma_{\mathcal{P}}}(K).$$

Combining the last two equalities and taking into account that by definition of $\psi^*_{\mathcal{P}}$, $v_{\Gamma_{\mathcal{P}}} = u$ and $\Gamma_M = \Gamma$, we obtain that for any $\langle u, \Gamma\rangle \in \mathcal{G}_M^{DL^{\mathcal{P}}}$, for every $K \in M/\Gamma$,

$$\sum_{k\in K} (\xi^{\mathcal{P}} \circ \psi^*_{\mathcal{P}})_k(u, \Gamma) = u(K),$$

i.e., the Γ-value $\xi^{\mathcal{P}} \circ \psi^*_{\mathcal{P}}$ on $\mathcal{G}_M^{DL^{\mathcal{P}}}$ satisfies CE. From the characterization results for Γ-values, discussed above in Section 3, it follows that CE and $DL^{\mathcal{P}}$ together guarantee that for any $\langle u, \Gamma\rangle \in \mathcal{G}_M^{DL^{\mathcal{P}}}$,

$$(\xi^{\mathcal{P}} \circ \psi^*_{\mathcal{P}})_k(u, \Gamma) = DL^{\mathcal{P}}_k(u, \Gamma), \quad \text{for all } k \in M.$$

In particular, the last equality is valid for $\langle u, \Gamma\rangle = \langle v^*_{\Gamma_{\mathcal{P}}}, \Gamma^*_M\rangle \in \mathcal{G}_M^{DL^{\mathcal{P}}}$, i.e.,

$$(\xi^{\mathcal{P}} \circ \psi^*_{\mathcal{P}})_k(v^*_{\Gamma_{\mathcal{P}}}, \Gamma^*_M) = DL^{\mathcal{P}}_k(v^*_{\Gamma_{\mathcal{P}}}, \Gamma^*_M), \quad \text{for all } k \in M$$

wherefrom, because of (8) and by choice of $\psi^*_{\mathcal{P}}$,

$$\sum_{i\in N_k} \xi_i(v^*, \Gamma^*_{\mathcal{P}}) = DL^{\mathcal{P}}_k(v^*_{\Gamma_{\mathcal{P}}}, \Gamma^*_M), \quad \text{for all } k \in M.$$

Hence, due to arbitrary choice of the $P\Gamma$-game $\langle v^*, \Gamma^*_{\mathcal{P}}\rangle$ it follows that for any $\langle v, \Gamma_{\mathcal{P}}\rangle \in \bar{\mathcal{G}}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$,

$$\sum_{i\in N_k} \xi_i(v, \Gamma_{\mathcal{P}}) = DL^{\mathcal{P}}_k(v_{\Gamma_{\mathcal{P}}}, \Gamma_M), \quad \text{for all } k \in M. \tag{9}$$

Notice that for $k \in M$ such that $N_k = \{i\}$, equality (9) reduces to

$$\xi_i(v, \Gamma_{\mathcal{P}}) = DL^{\mathcal{P}}_{k(i)}(v_{\Gamma_{\mathcal{P}}}, \Gamma_M), \quad \text{for all } i \in N \text{ s.t. } N_{k(i)} = \{i\}. \tag{10}$$

Step 2. Level of single players within a priori unions.

Consider $k' \in M$ for which $n_{k'} > 1$. Let the mapping $\psi^*_{k'}: \mathcal{G}^{DL^{k'}}_{N_{k'}} \to \bar{\mathcal{G}}^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}_N$ assign to $\langle u, \Gamma\rangle \in \mathcal{G}^{DL^{k'}}_{N_{k'}}$ the $P\Gamma$-game $\langle v, \Gamma_{\mathcal{P}}\rangle \in \bar{\mathcal{G}}^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}_N$ such that $v^{DL^{\mathcal{P}}}_{k'} = u$ and $\Gamma_{k'} = \Gamma$, and let $\psi^*_{k'}$ meet the condition $\psi^*_{k'}((v^*)^{DL^{\mathcal{P}}}_{k'}, \Gamma^*_{k'}) = \langle v^*, \Gamma^*_{\mathcal{P}^*}\rangle$. By definition of $\xi_{N_{k'}}$, for any $\langle u, \Gamma\rangle \in \mathcal{G}^{DL^{k'}}_{N_{k'}}$ and $\langle v, \Gamma_{\mathcal{P}}\rangle = \psi^*_{k'}(u, \Gamma)$ it holds that

$$(\xi_{N_{k'}} \circ \psi^*_{k'})_i(u, \Gamma) = \xi_i(v, \Gamma_{\mathcal{P}}), \quad \text{for all } i \in N_{k'}. \tag{11}$$

Since ξ meets CEU, for any $\langle v, \Gamma_{\mathcal{P}}\rangle \in \bar{\mathcal{G}}^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}_N$, for all $C \in N_{k'}/\Gamma_{k'}$, $C \neq N_{k'}$,

$$\sum_{i\in C} \xi_i(v, \Gamma_{\mathcal{P}}) = v(C).$$

From (9) it follows, in particular, that for any $\langle v, \Gamma_{\mathcal{P}}\rangle \in \bar{\mathcal{G}}^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}_N$ such that $N_{k'} \in N_{k'}/\Gamma_{k'}$,

$$\sum_{i\in N_{k'}} \xi_i(v, \Gamma_{\mathcal{P}}) = DL^{\mathcal{P}}_{k'}(v_{\Gamma_{\mathcal{P}}}, \Gamma_M).$$

Combining the last two equalities with (11) and recalling that by choice of $\psi^*_{k'}$, $v^{DL^{\mathcal{P}}}_{k'} = u$ and $\Gamma_{k'} = \Gamma$, and therefore, for any $C \in N_{k'}/\Gamma$, $C \neq N_{k'}$, it holds that $v(C) = v|_{N_{k'}}(C) = v^{DL^{\mathcal{P}}}_{k'}(C) = u(C)$, we obtain that for any $\langle u, \Gamma\rangle \in \mathcal{G}^{DL^{k'}}_{N_{k'}}$, for every $C \in N_{k'}/\Gamma$,

$$\sum_{i\in C} (\xi_{N_{k'}} \circ \psi^*_{k'})_i(u, \Gamma) = \begin{cases} DL^{\mathcal{P}}_{k'}(v_{\Gamma_{\mathcal{P}}}, \Gamma_M), & C = N_{k'}, \\ u(C), & C \neq N_{k'}, \end{cases}$$

with $\langle v_{\Gamma_{\mathcal{P}}}, \Gamma_M\rangle$ being the quotient Γ-game for $\langle v, \Gamma_{\mathcal{P}}\rangle = \psi^*_{k'}(u, \Gamma)$. Whence, on a set of Γ-games $\mathcal{G}^{DL^{k'}}_{N_{k'}}(DL^{\mathcal{P}}_{k'})$ defined as

$$\mathcal{G}^{DL^{k'}}_{N_{k'}}(DL^{\mathcal{P}}_{k'}) = \left\{\langle u, \Gamma\rangle \in \mathcal{G}^{DL^{k'}}_{N_{k'}} \mid u(N_{k'}) = DL^{\mathcal{P}}_{k'}(v_{\Gamma_{\mathcal{P}}}, \Gamma_M) \text{ for } \langle v, \Gamma_{\mathcal{P}}\rangle = \psi^*_{k'}(u, \Gamma)\right\},$$

the Γ-value $\xi_{N_{k'}} \circ \psi^*_{k'}$ meets CE. CE together with $\mathrm{DL}^{k'}$ guarantee that for any $\langle u, \Gamma\rangle \in \mathcal{G}^{DL^{k'}}_{N_{k'}}(DL^{\mathcal{P}}_{k'})$,

$$(\xi_{N_{k'}} \circ \psi^*_{k'})_i(u, \Gamma) = DL^{k'}_i(u, \Gamma), \quad \text{for all } i \in N_{k'}.$$

Observe that by choice of $\psi^*_{k'}$, $\langle (v^*)^{DL^{\mathcal{P}}}_{k'}, \Gamma^*_{k'}\rangle \in \mathcal{G}^{DL^{k'}}_{N_{k'}}(DL^{\mathcal{P}}_{k'})$. Hence, in particular, the last equality holds on the Γ-game $\langle (v^*)^{DL^{\mathcal{P}}}_{k'}, \Gamma^*_{k'}\rangle$, i.e.,

$$(\xi_{N_{k'}} \circ \psi^*_{k'})_i((v^*)^{DL^{\mathcal{P}}}_{k'}, \Gamma^*_{k'}) = DL^{k'}_i((v^*)^{DL^{\mathcal{P}}}_{k'}, \Gamma^*_{k'}), \quad \text{for all } i \in N_{k'}$$

wherefrom, since (11) and by choice of $\psi^*_{k'}$, we obtain that

$$\xi_i(v^*, \Gamma^*_{\mathcal{P}}) = DL^{k'}_i((v^*)^{DL^{\mathcal{P}}}_{k'}, \Gamma^*_{k'}), \quad \text{for all } i \in N_{k'}.$$

Due to the arbitrary choice of both, $\langle v^*, \Gamma^*_{\mathcal{P}}\rangle$ and $k' \in M$ for which $n_{k'} > 1$, it holds that for any $\langle v, \Gamma_{\mathcal{P}}\rangle \in \bar{\mathcal{G}}^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}_N$,

$$\xi_i(v, \Gamma_{\mathcal{P}}) = DL^{k(i)}_i(v^{DL^{\mathcal{P}}}_{k(i)}, \Gamma_{k(i)}), \quad \text{for all } i \in N \text{ s.t. } n_{k(i)} > 1. \tag{12}$$

Observe that the proof of equality (12) is based on equality (9) only when $N_k \in N_k/\Gamma_k$, but (9) holds for all N_k, $k \in M$. To exclude any conflict we show now that on $\bar{\mathcal{G}}^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}_N$ (12) agrees with (9) when $N_k \notin N_k/\Gamma_k$ as well. Let $\langle v, \Gamma_{\mathcal{P}}\rangle \in \bar{\mathcal{G}}^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}_N$ be such that for some $k'' \in M$ it holds that $n_{k''} > 1$ and $N_{k''} \notin N_{k''}/\Gamma_{k''}$. Then,

$$\sum_{i\in N_{k''}} \xi_i(v, \Gamma_{\mathcal{P}}) = \sum_{C\in N_{k''}/\Gamma_{k''}} \sum_{i\in C} \xi_i(v, \Gamma_{\mathcal{P}}) \overset{(12)}{=} \sum_{C\in N_{k''}/\Gamma_{k''}} \sum_{i\in C} DL^{k''}_i(v^{DL^{\mathcal{P}}}_{k''}, \Gamma_{k''}).$$

Whence, due to component efficiency of $DL^{k''}$-value and since for every $C \in N_{k''}/\Gamma_{k''}$, $C \subsetneq N_{k''}$, it holds that $v^{DL^{\mathcal{P}}}_{k''}(C) = v_{k''}(C) = v|_{N_{k''}}(C) = v(C)$, we obtain

$$\sum_{i\in N_{k''}} \xi_i(v, \Gamma_{\mathcal{P}}) = \sum_{C\in N_{k''}/\Gamma_{k''}} v(C).$$

Since $\langle v, \Gamma_{\mathcal{P}}\rangle \in \bar{\mathcal{G}}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$, then by definition of the class $\bar{\mathcal{G}}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$ it holds that

$$\sum_{C\in N_k/\Gamma_k} v(C) = DL_k^{\mathcal{P}}(v_{\Gamma_{\mathcal{P}}}, \Gamma_M), \quad \text{for all } k \in M\colon N_k \notin N_k/\Gamma_k. \tag{13}$$

Combining the last two equalities we obtain that (9) holds for k'' as well.

Notice now that (10) and (12) together produce formula (7).

II. To complete the proof we verify that the $P\Gamma$-value ξ on $\bar{\mathcal{G}}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$ given by (7) meets all axioms CEQ, CEU, and $\langle \mathrm{DL}^{\mathcal{P}}, \{\mathrm{DL}^k\}_{k\in M}\rangle$. Consider arbitrary $\langle v, \Gamma_{\mathcal{P}}\rangle \in \bar{\mathcal{G}}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$. To simplify discussion and w.l.o.g. we assume that for all $k \in M, n_k > 1$. Consider some $k \in M$ and let $C \in N_k/\Gamma_k$. Because of component efficiency of DL^k-value, from (7) it follows that

$$\sum_{i\in C} \xi_i(v, \Gamma_{\mathcal{P}}) = v_k^{DL^{\mathcal{P}}}(C). \tag{14}$$

If $C \neq N_k$, then $v_k^{DL^{\mathcal{P}}}(C) = v_k(C) = v|_{N_k}(C) = v(C)$. Hence, due to arbitrary choice of k, ξ satisfies CEU. Moreover, from (14) and by definition of $DL_k^{\mathcal{P}}$-game $v_k^{DL^{\mathcal{P}}}$, it also follows that

$$\sum_{i\in N_k} \xi_i(v, \Gamma_{\mathcal{P}}) = DL_k^{\mathcal{P}}(v_{\Gamma_{\mathcal{P}}}, \Gamma_M), \quad \text{for all } k \in M\colon N_k \in N_k/\Gamma_k.$$

Observe that due to validity of equality (13), the just proved CEU provides that on $\bar{\mathcal{G}}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$ for all $k \in M$ for which $N_k \notin N_k/\Gamma_k$ the last equality holds as well:

$$\sum_{i\in N_k} \xi_i(v, \Gamma_{\mathcal{P}}) = \sum_{C\in N_k/\Gamma_k} \sum_{i\in C} \xi_i(v, \Gamma_{\mathcal{P}}) \overset{CEU}{=} \sum_{C\in N_k/\Gamma_k} v(C) \overset{(13)}{=} DL_k^{\mathcal{P}}(v_{\Gamma_{\mathcal{P}}}, \Gamma_M).$$

Hence,

$$\sum_{i\in N_k} \xi_i(v, \Gamma_{\mathcal{P}}) = DL_k^{\mathcal{P}}(v_{\Gamma_{\mathcal{P}}}, \Gamma_M), \quad \text{for all } k \in M. \tag{15}$$

Consider $K \in M/\Gamma_M$.

$$\sum_{k\in K} \sum_{i\in N_k} \xi_i(v, \Gamma_{\mathcal{P}}) \overset{(15)}{=} \sum_{k\in K} DL_k^{\mathcal{P}}(v_{\Gamma_{\mathcal{P}}}, \Gamma_M).$$

Whence and due to component efficiency of $DL^{\mathcal{P}}$-value we obtain that ξ meets CEQ. Next, let a mapping $\psi_{\mathcal{P}}\colon \mathcal{G}_M^{DL^{\mathcal{P}}} \to \bar{\mathcal{G}}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$ assign to any $\langle u, \Gamma\rangle \in \mathcal{G}_M^{DL^{\mathcal{P}}}$ the $P\Gamma$-game $\langle v, \Gamma_{\mathcal{P}}\rangle \in \bar{\mathcal{G}}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$ such that $v_{\Gamma_{\mathcal{P}}} = u$ and $\Gamma_M = \Gamma$. Then, for any $\langle u, \Gamma\rangle \in \mathcal{G}_M^{DL^{\mathcal{P}}}$ and $\langle v, \Gamma_{\mathcal{P}}\rangle = \psi_{\mathcal{P}}^*(u, \Gamma)$ by definition of $\xi^{\mathcal{P}}$ and due to (15) it holds

$$(\xi^{\mathcal{P}} \circ \psi_{\mathcal{P}})_k(u, \Gamma) = \xi_k^{\mathcal{P}}(v, \Gamma_{\mathcal{P}}) = \sum_{i\in N_k} \xi_i(v, \Gamma_{\mathcal{P}}) \overset{(15)}{=} DL_k^{\mathcal{P}}(v_{\Gamma_{\mathcal{P}}}, \Gamma_M), \quad \text{for all } k \in M.$$

Hence, $(\xi^{\mathcal{P}} \circ \psi_{\mathcal{P}})(u, \Gamma) = DL^{\mathcal{P}}(u, \Gamma)$, i.e., Γ-value $\xi^{\mathcal{P}} \circ \psi_{\mathcal{P}}$ meets $\mathrm{DL}^{\mathcal{P}}$. Similarly we can show that for every $k \in M$, Γ-value $\xi_{N_k} \circ \psi_k^{DL^{\mathcal{P}}}$ satisfies DL^k. ■

A simple algorithm for computing the $\langle DL^{\mathcal{P}}, \{DL^k\}_{k\in M}\rangle$-value of a $P\Gamma$-game $\langle v, \Gamma_{\mathcal{P}}\rangle \in \bar{\mathcal{G}}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$ follows from Theorem 1:

- compute the $DL^{\mathcal{P}}$-value of $\langle v_{\Gamma_{\mathcal{P}}}, \Gamma_M\rangle$;
- distribute the rewards $DL_k^{\mathcal{P}}(v_{\Gamma_{\mathcal{P}}}, \Gamma_M)$, $k \in M$, obtained by a priori unions among single players applying the DL^k-values to Γ-games $\langle v_k^{DL^{\mathcal{P}}}, \Gamma_k\rangle$ within a priori unions.

Example 1. Consider a numerical example for the $\langle LE, \underbrace{CF, \ldots, CF}_{m}\rangle$-value ξ of a $P\Gamma$-game $\langle v, \Gamma_{\mathcal{P}}\rangle$ with communication structure $\Gamma_{\mathcal{P}} = \langle \Gamma_M, \{\Gamma_k\}_{k\in M}\rangle$ given by directed line-graph Γ_M and undirected cycle-free graphs Γ_k, $k \in M$. As we will see below in Section 5, the $\langle LE, \underbrace{CF, \ldots, CF}_{m}\rangle$-value provides a reasonable solution for the river game with multiple users.

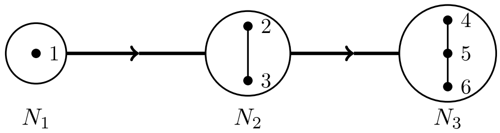

Fig. 2.

Assume that N contains 6 players, a game v is defined as follows:

$$v(\{i\}) = 0, \quad \text{for all } i \in N;$$
$$v(\{2,3\}) = 1, \qquad v(\{4,5\}) = v(\{4,6\}) = 2.8, \qquad v(\{5,6\}) = 2.9,$$
$$\text{otherwise } v(\{i,j\}) = 0, \quad \text{for all } i,j \in N;$$
$$v(\{1,2,3\}) = 2, \qquad v(\{1,2,3,i\}) = 3, \quad \text{for } i = 4,5,6; \text{ otherwise } v(S) = |S|, \text{ if } |S| \geq 3;$$

and a two-level communication structure is depicted on Fig. 2.

In this case $N = N_1 \cup N_2 \cup N_3$;

$$N_1 = \{1\}, \qquad N_2 = \{2,3\}, \qquad N_3 = \{4,5,6\}; \qquad \Gamma_1 = \emptyset, \qquad \Gamma_2 = \{\{2,3\}\}, \qquad \Gamma_3 = \{\{4,5\},\{5,6\}\};$$
$$M = \{1,2,3\}; \qquad \Gamma_M = \{(1,2),(2,3)\};$$

the quotient game $v_{\Gamma_{\mathcal{P}}}$ is given by

$$v_{\Gamma_{\mathcal{P}}}(\{1\}) = 0, \qquad v_{\Gamma_{\mathcal{P}}}(\{2\}) = 1, \qquad v_{\Gamma_{\mathcal{P}}}(\{3\}) = 3,$$
$$v_{\Gamma_{\mathcal{P}}}(\{1,2\}) = 2, \qquad v_{\Gamma_{\mathcal{P}}}(\{2,3\}) = 5, \qquad v_{\Gamma_{\mathcal{P}}}(\{1,3\}) = 4, \qquad v_{\Gamma_{\mathcal{P}}}(\{1,2,3\}) = 6;$$

the restricted quotient game $v_{\Gamma_{\mathcal{P}}}^{\Gamma_M}$ is

$$v_{\Gamma_{\mathcal{P}}}^{\Gamma_M}(\{1\}) = 0, \qquad v_{\Gamma_{\mathcal{P}}}^{\Gamma_M}(\{2\}) = 1, \qquad v_{\Gamma_{\mathcal{P}}}^{\Gamma_M}(\{3\}) = 3,$$
$$v_{\Gamma_{\mathcal{P}}}^{\Gamma_M}(\{1,2\}) = 2, \qquad v_{\Gamma_{\mathcal{P}}}^{\Gamma_M}(\{2,3\}) = 5, \qquad v_{\Gamma_{\mathcal{P}}}^{\Gamma_M}(\{1,3\}) = v_{\Gamma_{\mathcal{P}}}^{\Gamma_M}(\{1\}) + v_{\Gamma_{\mathcal{P}}}^{\Gamma_M}(\{3\}) = 3,$$
$$v_{\Gamma_{\mathcal{P}}}^{\Gamma_M}(\{1,2,3\}) = 6;$$

the games v_k, $k = 1,2,3$, within a priori unions N_k are given respectively by

$$v_1(\{1\}) = 0;$$
$$v_2(\{2\}) = v_2(\{3\}) = 0, \qquad v_2(\{2,3\}) = 1;$$
$$v_3(\{4\}) = v_3(\{5\}) = v_3(\{6\}) = 0, \qquad v_3(\{4,5\}) = v_3(\{4,6\}) = 2.8, \qquad v_3(\{5,6\}) = 2.9,$$
$$v_3(\{4,5,6\}) = 3;$$

and the restricted games $v_k^{\Gamma_k}$, $k = 1,2,3$, within a priori unions N_k are

$$v_1^{\Gamma_1}(\{1\}) = 0;$$
$$v_2^{\Gamma_2}(\{2\}) = v_2^{\Gamma_2}(\{3\}) = 0, \qquad v_2^{\Gamma_2}(\{2,3\}) = 1;$$
$$v_3^{\Gamma_3}(\{4\}) = v_3^{\Gamma_3}(\{5\}) = v_3^{\Gamma_3}(\{6\}) = 0, \qquad v_3^{\Gamma_3}(\{4,5\}) = 2.8, \qquad v_3^{\Gamma_3}(\{4,6\}) = 0,$$
$$v_3^{\Gamma_3}(\{5,6\}) = 2.9, \qquad v_3^{\Gamma_3}(\{4,5,6\}) = 3.$$

Following the algorithm above, the $P\Gamma$-value ξ can be obtained by finding of the LE solution in the line-graph quotient game $\langle v_{\Gamma_{\mathcal{P}}}, \Gamma_M \rangle$ and thereafter the total payoffs to the a priori unions $LE_k(v_{\Gamma_{\mathcal{P}}}, \Gamma_M)$, $k \in M$, should be distributed according to the AT solution applied to cycle-free graph LE-games within a priori unions, i.e., for all $i \in N$, $\xi_i(v, \Gamma_{\mathcal{P}}) = AT_i(v_{k(i)}^{LE}, \Gamma_{k(i)})$. Simple computations show that

$$LE_1(v_{\Gamma_{\mathcal{P}}}, \Gamma_M) = v_{\Gamma_{\mathcal{P}}}^{\Gamma_M}(\{1,2,3\}) - v_{\Gamma_{\mathcal{P}}}^{\Gamma_M}(\{2,3\}) = 1,$$
$$LE_2(v_{\Gamma_{\mathcal{P}}}, \Gamma_M) = v_{\Gamma_{\mathcal{P}}}^{\Gamma_M}(\{2,3\}) - v_{\Gamma_{\mathcal{P}}}^{\Gamma_M}(\{3\}) = 2,$$
$$LE_3(v_{\Gamma_{\mathcal{P}}}, \Gamma_M) = v_{\Gamma_{\mathcal{P}}}^{\Gamma_M}(\{3\}) = 3;$$
$$AT_1(v_1^{LE}, \Gamma_1) = LE_1 = 1,$$
$$AT_2(v_2^{LE}, \Gamma_2) = [[LE_2 - v_2(\{3\})] + v_2(\{2\})]/2 = (2+0)/2 = 1,$$
$$AT_3(v_2^{LE}, \Gamma_2) = [v_2(\{3\}) + [LE_2 - v_2(\{2\})]]/2 = (0+2)/2 = 1,$$
$$AT_4(v_3^{LE}, \Gamma_3) = [[LE_3 - v_3(\{5,6\})] + v_3(\{4\}) + v_3(\{4\})]/3 = [(3-2.9)+0+0]/3 = \frac{1}{30},$$

$$AT_5(v_3^{LE}, \Gamma_3) = [[v_3(\{5,6\}) - v_3(\{6\})] + [LE_3 - v_3(\{4\}) - v_3(\{6\})] + [v_3(\{4,5\}) - v_3(\{4\})]]/3 = (2.9 + 3 + 2.8)/3 = 2\frac{27}{30},$$

$$AT_6(v_3^{LE}, \Gamma_3) = [v_3(\{6\}) + v_3(\{6\}) + [LE_3 - v_3(\{4,5\})]]/3 = [0 + 0 + (3 - 2.8)]/3 = \frac{2}{30}.$$

Thus, $\xi(v, \Gamma_{\mathcal{P}}) = (1,\ 1,\ 1,\ \frac{1}{30},\ 2\frac{27}{30},\ \frac{2}{30})$.

It was already mentioned before that the $P\Gamma$-games $\langle v, \Gamma_{\langle N\rangle}\rangle$ and $\langle v, \Gamma_{\{N\}}\rangle$ reduce to the Γ-game $\langle v, \Gamma_N\rangle$. Whence, any $\langle F, \{DL^k\}_{k\in N}\rangle$-value of $\langle v, \Gamma_{\langle N\rangle}\rangle$ and any $\langle DL, F\rangle$-value of $\langle v, \Gamma_{\{N\}}\rangle$ coincide with the Myerson value of $\langle v, \Gamma_N\rangle$; moreover, if the graph Γ_N is complete, they coincide also with the Shapley value and the Owen value. Besides, note that in a $P\Gamma$-game $\langle v, \Gamma_{\mathcal{P}}\rangle$ with any coalition structure $\mathcal{P}$, empty graph Γ_M, and complete graphs $\Gamma_k,\ k \in M$, any $\langle DL^{\mathcal{P}}, \underbrace{F, \ldots, F}_{m}\rangle$-value coincides with the Aumann–Drèze value of the P-game $\langle v, \mathcal{P}\rangle$. Moreover, in a $P\Gamma$-game $\langle v, \Gamma_{\mathcal{P}}\rangle$ with any coalition structure $\mathcal{P}$ and complete graphs Γ_M and Γ_k, $k \in M$, the $\langle F, \underbrace{F, \ldots, F}_{m}\rangle$-value coincides with the two-step Shapley value introduced in Kamijo [8] of the P-game $\langle v, \mathcal{P}\rangle$. However, the $\langle DL^{\mathcal{P}}, \{DL^k\}_{k\in M}\rangle$-value of a $P\Gamma$-game $\langle v, \Gamma_{\mathcal{P}}\rangle$ with nontrivial coalition structure $\mathcal{P}$ never coincides with the Owen value (and therefore with the value of Vázquez-Brage et al. [16], as well). Indeed, in our model no cooperation is allowed between a proper subcoalition of any a priori union with members of other a priori unions. On the contrary, the Owen model assumes that every subcoalition of any chosen a priori union may represent this union in the negotiation procedure with other entire a priori unions.

4.2. Stability

Theorem 2. *If the set of DL axioms is restricted to CF, LE, UE, EL, SE, and PE, then the $\langle DL^{\mathcal{P}}, \{DL^k\}_{k\in M}\rangle$-value of any superadditive $\langle v, \Gamma_{\mathcal{P}}\rangle \in \bar{\mathcal{G}}_N^{DL^{\mathcal{P}}, \{DL^k\}_{k\in M}}$ belongs to the core $C(v, \Gamma_{\mathcal{P}})$.*

Remark 6. Under the hypothesis of Theorem 2 all $\langle DL^{\mathcal{P}}, \{DL^k\}_{k\in M}\rangle$-values are combinations of the AT solution for undirected cycle-free Γ-games, the UE, LE, and EL solutions for line-graph Γ-games, and the tree/sink value for rooted/sink forest Γ-games, that are stable on the class of superadditive Γ-games (cf. [7,15,4,9]).

Proof. For any superadditive $P\Gamma$-game $\langle v, \Gamma_{\mathcal{P}}\rangle$ the quotient game $v_{\Gamma_{\mathcal{P}}}$ and games $v_k,\ k \in M$, within a priori unions are superadditive as well. Due to Remark 6, $DL(v, \Gamma) \in C(v, \Gamma)$ for every superadditive Γ-game $\langle v, \Gamma\rangle \in \mathcal{G}_N^{DL}$. Whence,

$$DL^{\mathcal{P}}(v_{\Gamma_{\mathcal{P}}}, \Gamma_M) \in C(v_{\Gamma_{\mathcal{P}}}, \Gamma_M), \tag{16}$$

$$DL^k(v_k, \Gamma_k) \in C(v_k, \Gamma_k), \quad \text{for all } k \in M\colon n_k > 1. \tag{17}$$

From (16) and because every singleton coalition is connected it follows that

$$DL_k^{\mathcal{P}}(v_{\Gamma_{\mathcal{P}}}, \Gamma_M) \ge v_{\Gamma_{\mathcal{P}}}(\{k\}) \overset{(3)}{=} v_k^{\Gamma_k}(N_k), \quad \text{for all } k \in M\colon n_k > 1.$$

Observe that if $N_k \in N_k/\Gamma_k$, the games $v_k^{\Gamma_k}$ and v_k coincide. Therefore, because of the last inequality, the $DL_k^{\mathcal{P}}$-game $v_k^{DL^{\mathcal{P}}}$ is superadditive as well. Thus,

$$DL^k(v_k^{DL^{\mathcal{P}}}, \Gamma_k) \in C(v_k^{DL^{\mathcal{P}}}, \Gamma_k), \quad \text{for all } k \in M\colon n_k > 1 \text{ and } N_k \in N_k/\Gamma_k. \tag{18}$$

If $N_k \notin N_k/\Gamma_k$, then by definition $C(v_k^{DL^{\mathcal{P}}}, \Gamma_k) \overset{(2)}{=} C(v_k, \Gamma_k)$. Besides, by definition any of the following Γ-values: the AT solution for undirected cycle-free Γ-games, the UE, LE, and EL solutions for line-graph Γ-games, and the tree/sink values for rooted/sink forest Γ-games, is defined via the corresponding restricted game. Hence, if $N_k \notin N_k/\Gamma_k$, then $DL^k(v_k^{DL^{\mathcal{P}}}, \Gamma_k) = DL^k(v_k, \Gamma_k)$. Wherefrom together with the previous equality and because of (18) and (17) we arrive at

$$DL^k(v_k^{DL^{\mathcal{P}}}, \Gamma_k) \in C(v_k^{DL^{\mathcal{P}}}, \Gamma_k), \quad \text{for all } k \in M\colon n_k > 1. \tag{19}$$

As it is shown in part II of the proof of Theorem 1 (equality (15)), the vector

$$\langle DL^{\mathcal{P}}, \{DL^k\}_{k\in M}\rangle^{\mathcal{P}}(v, \Gamma_{\mathcal{P}}) = \left\{\sum_{i\in N_k} \langle DL^{\mathcal{P}}, \{DL^k\}_{k\in M}\rangle_i(v, \Gamma_{\mathcal{P}})\right\}_{k\in M}$$

is the $DL^{\mathcal{P}}$-value for the quotient Γ-game $\langle v_{\Gamma_{\mathcal{P}}}, \Gamma_M\rangle$. Therefore, from (16) we obtain that

$$\langle DL^{\mathcal{P}}, \{DL^k\}_{k\in M}\rangle^{\mathcal{P}}(v, \Gamma_{\mathcal{P}}) \in C(v_{\Gamma_{\mathcal{P}}}, \Gamma_M). \tag{20}$$

Further,

$$\langle DL^{\mathscr{P}}, \{DL^k\}_{k\in M}\rangle|_{N_k}(v, \Gamma_{\mathscr{P}}) \overset{(7)}{=} DL^k(v_k^{DL^{\mathscr{P}}}, \Gamma_k), \quad \text{for all } k \in M{:}\, n_k > 1.$$

Whence together with (19) it follows that

$$\langle DL^{\mathscr{P}}, \{DL^k\}_{k\in M}\rangle|_{N_k}(v, \Gamma_{\mathscr{P}}) \in C(v_k^{DL^{\mathscr{P}}}, \Gamma_k), \quad \text{for all } k \in M{:}\, n_k > 1. \tag{21}$$

Due to Proposition 1, (20) and (21) ensure that

$$\langle DL^{\mathscr{P}}, \{DL^k\}_{k\in M}\rangle(v, \Gamma_{\mathscr{P}}) \in C(v, \Gamma_{\mathscr{P}}). \quad \blacksquare$$

Return back to Example 1 and notice that it illustrates Theorem 2 as well. Observe that v is superadditive and $\xi(v, \Gamma_{\mathscr{P}}) = \langle LE, CF, CF, CF\rangle(v, \Gamma_{\mathscr{P}}) \in C(v, \Gamma_{\mathscr{P}})$. But $\phi(v, \Gamma_{\mathscr{P}}) = \langle F, F, F, F\rangle(v, \Gamma_{\mathscr{P}})$ being the combination of the Myerson values, i.e., $\phi_i(v, \Gamma_{\mathscr{P}}) = \mu_i(v^{\mu}_{k(i)}, \Gamma_{k(i)})$, $i \in N$, does not belong to $C(v, \Gamma_{\mathscr{P}})$. Indeed, $\phi(v, \Gamma_{\mathscr{P}}) = (0.5, 1, 1, \frac{2}{3}, 2\frac{7}{60}, \frac{43}{60})$. However, since $\phi_4 + \phi_5 = 2\frac{47}{60} < v_3^{\Gamma_3}(\{4, 5\}) = 2.8 = 2\frac{48}{60}$, $\phi_{N_3} \notin C(v^{\mu}_3, \Gamma_3)$. Whence, due to Proposition 1, $\phi(v, \Gamma_{\mathscr{P}}) \notin C(v, \Gamma_{\mathscr{P}})$.

Due to Proposition 1, every core selecting $P\Gamma$-value meets the weaker properties of CEQ and CEU together. Whence together with Theorem 2 the next theorem follows.

Theorem 3. *If the set of DL axioms is restricted to CF, UE, LE, EL, SE, and PE, then the $\langle DL^{\mathscr{P}}, \{DL^k\}_{k\in M}\rangle$-value of a superadditive $\langle v, \Gamma_{\mathscr{P}}\rangle \in \bar{\mathscr{G}}_N^{DL^{\mathscr{P}}, \{DL^k\}_{k\in M}}$ is the unique core selector that satisfies $(m + 1)$-tuple of axioms $\langle DL^{\mathscr{P}}, \{DL^k\}_{k\in M}\rangle$.*

Now let $\langle v, \Gamma_{\mathscr{P}}\rangle$ be a superadditive $P\Gamma$-game in which all graphs in $\Gamma_{\mathscr{P}} = \langle \Gamma_M, \{\Gamma_k\}_{k\in M}\rangle$ are either undirected cycle-free, or directed line-graphs or rooted/sink forests, and besides all Γ_k, $k \in M$, are connected. Then there exists a $(m + 1)$-tuple of $\langle \mathrm{DL}^{\mathscr{P}}, \{\mathrm{DL}^k\}_{k\in M}\rangle$ axioms of types CF, UE, LE, EL, SE, or PE, for which the communication structure $\Gamma_{\mathscr{P}} = \langle \Gamma_M, \{\Gamma_k\}_{k\in M}\rangle$ is suitable. Due to Remark 5, $\langle v, \Gamma_{\mathscr{P}}\rangle \in \bar{\mathscr{G}}_N^{DL^{\mathscr{P}}, \{DL^k\}_{k\in M}}$. Whence together with Theorem 2 we obtain the validity of the next theorem.

Theorem 4. *For every superadditive $\langle v, \Gamma_{\mathscr{P}}\rangle \in \bar{\mathscr{G}}_N^{DL^{\mathscr{P}}, \{DL^k\}_{k\in M}}$ for which all graphs in $\Gamma_{\mathscr{P}} = \langle \Gamma_M, \{\Gamma_k\}_{k\in M}\rangle$ are either undirected cycle-free, or directed line-graphs or rooted/sink forests, and all graphs Γ_k, $k \in M$, are connected, it holds that $C(v, \Gamma_{\mathscr{P}}) \neq \emptyset$.*

It is worth to remark that if among the graphs Γ_k, $k \in M$, at least one is disconnected, then it is impossible to guarantee that $\langle v, \Gamma_{\mathscr{P}}\rangle$ meets condition (ii) and therefore belongs to $\bar{\mathscr{G}}_N^{DL^{\mathscr{P}}, \{DL^k\}_{k\in M}}$.

4.3. Harsanyi dividends

Consider now $\langle DL^{\mathscr{P}}, \{DL^k\}_{k\in M}\rangle$-values with respect to the distribution of Harsanyi dividends. Since for every $v \in \mathscr{G}_N$ and $S \subseteq N$ it holds that $v(S) = \sum_{T\subseteq S} \lambda^v_T$, where λ^v_T is the dividend of T in v, the Harsanyi dividend of a coalition has a natural interpretation as an extra revenue from cooperation among its players that they could not realize staying in proper subcoalitions. How the value under scrutiny distributes the dividend of a coalition among the players provides the important information concerning the interest of different players to form the coalition. This information is especially important for games with limited cooperation when it might happen that one player (or some group of players) is responsible for forming the coalition. In this case, if such a player obtains no quota from the dividend of the coalition, he may simply block at all the coalition formation. This happens, for example, with some values for line-graph games (see discussion in Brink et al. [15]).

Because of Theorem 1, every $\langle DL^{\mathscr{P}}, \{DL^k\}_{k\in M}\rangle$-value is a combination of the $DL^{\mathscr{P}}$-value in the quotient Γ-game and DL^k-values, $k \in M$, for the corresponding Γ-games within a priori unions. Whence and by definition of a $P\Gamma$-game we obtain

Proposition 2. *In any $\langle v, \Gamma_{\mathscr{P}}\rangle \in \mathscr{G}_N^{P\Gamma}$ the only feasible coalitions are either $S = \bigcup_{k\in Q} N_k$, $Q \subseteq M$, or $S \subset N_k$, $k \in M$. Every $\langle DL^{\mathscr{P}}, \{DL^k\}_{k\in M}\rangle$-value distributes λ^v_S of $S = \bigcup_{k\in Q} N_k$ according to the $DL^{\mathscr{P}}$-value and of $S \subset N_k$ according to the DL^k-value.*

5. Sharing a river with multiple users

Ambec and Sprumont [1] approach the problem of optimal water allocation for a given river with certain capacity over the agents (countries) located along the river from the game theoretic point of view. Their model assumes that between each pair of neighboring agents there is an additional inflow of water. Each agent, in principle, can use all the inflow between itself and its upstream neighbor, however, this allocation in general is not optimal in respect to total welfare. To obtain more profitable allocation it is allowed to allocate more water to downstream agents which in turn can compensate the extra water obtained by side-payments to upstream ones. The problem of optimal water allocation is approached as the problem of optimal welfare distribution. Brink et al. [15] show that the Ambec–Sprumont river game model can be naturally embedded into the framework of a line-graph Γ-game. In Khmelnitskaya [9]the line-graph river model is extended to the rooted-tree and sink-tree digraph model of a river with a delta or with multiple sources respectively. All these models consider each

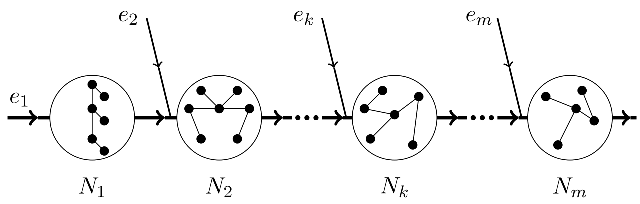

Fig. 3. A line-graph river.

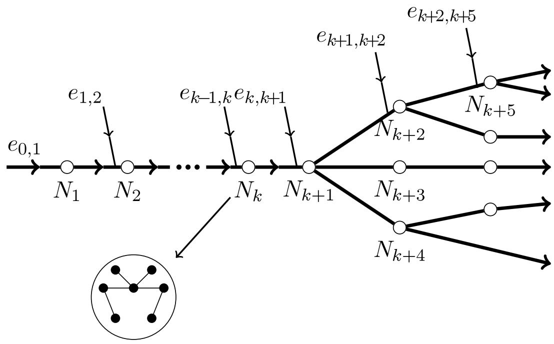

Fig. 4. A river with delta.

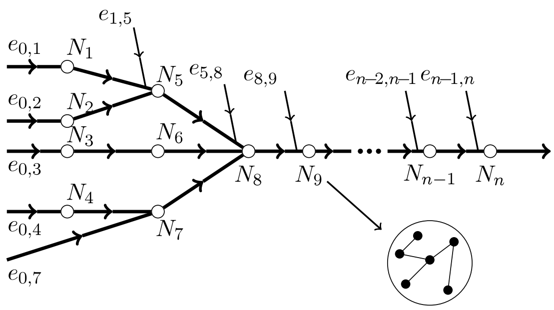

Fig. 5. A river with multiple sources.

agent as a single unit. We extend the model to multiple agents assuming that each agent represents a community of users. However, in our model no cooperation between single users or proper subgroups of users belonging to different agents is allowed, i.e., the presence of international firms having branches at different levels along the river is excluded.

Let $N = \bigcup_{k \in M} N_k$ be a set players (users of water) composed of the communities of users N_k, $k \in M$, located along the river and numbered successively from upstream to downstream. Let $e_{lk} \geq 0$, $k \in M$, l is a predecessor of k, be the inflow of water in front of the most upstream community(ies) (in this case $l = 0$) or the inflow of water entering the river between neighboring communities in front of N_k. Moreover, we assume that each N_k is equipped by a connected pipe system binding all its members. Without loss of generality we may assume that all graphs Γ_k, $k \in M$, presenting pipe systems within communities N_k are cycle free; otherwise it is always possible to close some pipes responsible for cycles. Indeed, for a graph with cycles there is a final set of cycle-free subgraphs with the same set of nodes as in the original graph. We always can choose one of them that minimizes the technological costs of water transportation within the community. Figs. 3–5 illustrate the model.

Following Ambec and Sprumont [1] it is assumed that for each community N_k there is a quasi-linear utility function given by $u^k(x_k, t_k) = b^k(x_k) + t_k$, where x_k is the total amount of water allocated to N_k, $b^k: \mathbb{R}_+ \to \mathbb{R}$ is a continuous nondecreasing function determining the benefit $b^k(x_k)$ of N_k through the consumption of the amount x_k of water, and t_k is a monetary compensation to N_k. Moreover, in case of a river with a delta it is also assumed that if a splitting of the river into branches occurs after a certain N_k, then this community takes, besides its own quota, also the responsibility to split the rest of the water flow such as to guarantee the realization of the water distribution plan for the successors. Further, we assume that if the total shares of water to all N_k, $k \in M$, are fixed, then for each N_k there exists a mechanism presented in terms of a TU game v_k that distributes optimally the obtained share of water among its members. We do not discuss how the games v_k, $k \in M$, are constructed and leave this open outside the scope of the paper.

In the model no cooperation is allowed among single users from different levels along the course of the river. Thus, the problem of optimal water allocation fits the framework of the introduced above $P\Gamma$-game and as its solution we may consider a $P\Gamma$-value that in turn is a combination of solutions for a line-graph, rooted-tree, or sink-tree Γ-game among $N_k,\ k \in M$, and cycle-free graph games within each N_k. In accordance with the results obtained in [1,15,9] the optimal water distribution among $N_k,\ k \in M$, can be modeled as a line-graph, rooted-tree, or sink-tree superadditive river game. If all games $v_k, k \in M$, determining water distribution within communities are superadditive too, then the corresponding $P\Gamma$-values appear to be selectors of the core of the river game with multiple users.

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