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C_0-semigroups for hyperbolic partial differential equations on a one-dimensional spatial domain

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Abstract. Hyperbolic partial differential equations on a one-dimensional spatial domain are studied. This class of systems includes models of beams and waves as well as the transport equation and networks of non-homogeneous transmission lines. The main result of this paper is a simple test for C_0-semigroup generation in terms of the boundary conditions. The result is illustrated with several examples.

1. Introduction and main result

Consider the following class of partial differential equations

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta, t) &= \left(P_1 \frac{\partial}{\partial \zeta} + P_0\right)(\mathcal{H}(\zeta)x(\zeta, t)), \qquad \zeta \in [0, 1], t \geq 0, \\ x(\zeta, 0) &= x_0(\zeta), \end{aligned} \tag{1}$$

where P_1 is an invertible $n \times n$ Hermitian matrix, P_0 is a $n \times n$ matrix and $\mathcal{H}(\zeta)$ is a positive $n \times n$ Hermitian matrix for a.e. $\zeta \in (0, 1)$ satisfying $\mathcal{H}, \mathcal{H}^{-1} \in L^\infty(0, 1; \mathbb{C}^{n\times n})$. This class of Cauchy problems covers in particular the wave equation, the transport equation and the Timoshenko beam equation, and also coupled beam and wave equations. These Cauchy problems are also known as Hamiltonian partial differential equations or port-Hamiltonian systems, see [3,6] and in particular the Ph.D thesis [7]. The boundary conditions are of the form

$$\tilde{W}_B \begin{bmatrix} (\mathcal{H}x)(1,t) \\ (\mathcal{H}x)(0,t) \end{bmatrix} = 0, \tag{2}$$

where $\tilde{W}_B$ is an $n \times 2n$-matrix. Define

$$Ax := \left(P_1 \frac{\mathrm{d}}{\mathrm{d}\zeta} + P_0\right)(x), \qquad x \in D(A), \tag{3}$$

on $X_p := L^p(0, 1; \mathbb{C}^n)$, $1 \leq p < \infty$, with the domain

$$D(A) := \left\{ x \in \mathcal{W}^{1,p}(0, 1; \mathbb{C}^n) \mid \tilde{W}_B \begin{bmatrix} x(1) \\ x(0) \end{bmatrix} = 0 \right\}. \tag{4}$$

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Then, the partial differential equation (1) with the boundary conditions (2) can be written as the abstract differential equation

$$\dot{x}(t) = A\mathcal{H}x(t), \qquad x(0) = x_0.$$

If we equip X_2 with the energy norm $\langle \cdot, \mathcal{H}\cdot \rangle$, then $A\mathcal{H}$ generates a contraction semigroup (or an unitary C_0-group) on $(X_2, \langle \cdot, \mathcal{H}\cdot \rangle)$ if and only if A is dissipative on $(X_2, \langle \cdot, \cdot \rangle)$(or A and $-A$ are dissipative on $(X_2, \langle \cdot, \cdot \rangle)$, respectively) [1,3,4]. Matrix conditions to guarantee generation of a contraction semigroup or of a unitary group have been obtained [1,3,4]. The following theorem extends these results.

THEOREM 1.1. *Let* $W_B := \tilde{W}_B \left[\begin{smallmatrix} P_1 & -P_1 \\ I & I \end{smallmatrix}\right]^{-1}$ *and* $\Sigma := \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]$.

1. *The following statements are equivalent:*
   - *(a)* $A\mathcal{H}$ *with domain* $D(A\mathcal{H}) := \{x \in X_2 \mid \mathcal{H}x \in D(A)\} = \mathcal{H}^{-1}D(A)$ *generates a contraction semigroup on* $(X_2, \langle \cdot, \mathcal{H}\cdot \rangle)$*;*
   - *(b)* *Re* $\langle Ax, x \rangle \leq 0$ *for every* $x \in D(A)$*;*
   - *(c)* *Re* $P_0 \leq 0$ *and* $u^* P_1 u - y^* P_1 y \leq 0$ *for every* $\left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \in$ *ker* $\tilde{W}_B$*;*
   - *(d)* *Re* $P_0 \leq 0$*,* $W_B \Sigma W_B^* \geq 0$ *and rank* $\tilde{W}_B = n$.
2. *The following statements are equivalent:*
   - *(a)* $A\mathcal{H}$ *with domain* $D(A\mathcal{H}) := \{x \in X_2 \mid \mathcal{H}x \in D(A)\} = \mathcal{H}^{-1}D(A)$ *generates a unitary* C_0*-group on* $(X_2, \langle \cdot, \mathcal{H}\cdot \rangle)$*;*
   - *(b)* *Re* $\langle Ax, x \rangle = 0$ *for every* $x \in D(A)$*;*
   - *(c)* *Re* $P_0 = 0$ *and* $u^* P_1 u - y^* P_1 y = 0$ *for every* $\left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \in$ *ker* $\tilde{W}_B$*;*
   - *(d)* *Re* $P_0 = 0$*,* $W_B \Sigma W_B^* = 0$ *and rank* $\tilde{W}_B = n$.

Theorem 1.1 was proved in [3, Theorem 7.2.4] with the additional assumptions that $P_0^* = -P_0$ and rank $\tilde{W}_B = n$. The extension to non-skew-adjoint matrices P_0 is in [1]. However, the equivalence with (c) is not explicitly shown in the above references, and it is assumed that rank $\tilde{W}_B = n$. A short proof of Theorem 1.1 is in the following section.

By the assumptions on $\mathcal{H}$, it is clear that the norm on $(X_2, \langle \cdot, \mathcal{H}\cdot \rangle)$ is equivalent to the standard norm on X_2. Hence, if $A\mathcal{H}$ generates a contraction (or a unitary group) with respect to the energy norm for some $\mathcal{H}$, then it will generate a C_0-semigroup (C_0-group) on X_2 equipped with the standard norm as well.

The following corollary follows immediately.

COROLLARY 1.2. *The following statements are equivalent:*

1. *A generates a contraction semigroup on* $(X_2, \langle \cdot, \cdot \rangle)$*,*
2. *$A\mathcal{H}$ generates a contraction semigroup on* $(X_2, \langle \cdot, \mathcal{H}\cdot \rangle)$.

Corollary 1.2 implies that whether or not $A\mathcal{H}$ generates a contraction semigroup on the energy space $(X_2, \langle \cdot, \mathcal{H}\cdot \rangle)$ is independent of the Hamiltonian density $\mathcal{H}$: A is the generator of a contraction semigroup on $(X_2, \langle \cdot, \cdot \rangle)$ if and only if $A\mathcal{H}$ generates a contraction semigroup on $(X_2, \langle \cdot, \mathcal{H}\cdot \rangle)$. The condition of a contraction semigroup is essential here. For a counterexample, see Example 3.2 or [8, Section 6].

DEFINITION 1.3. *An operator $\mathcal{A}$ generates a quasi-contractive semigroup if $\mathcal{A} - \omega I$ generates a contraction semigroup for some $\omega \in \mathbb{R}$.* □

COROLLARY 1.4. *If Re $P_0 \leq 0$, then $A\mathcal{H}$ generates a quasi-contractive semigroup on $(X_2, \langle\cdot, \mathcal{H}\cdot\rangle)$ if and only if $A\mathcal{H}$ generates a contraction semigroup on $(X_2, \langle\cdot, \mathcal{H}\cdot\rangle)$.*

The proof of Corollary 1.4 will be given in Sect. 2.

Theorem 1.1 characterizes boundary conditions for which $A\mathcal{H}$ generates a contraction semigroup or a unitary group. However, other boundary conditions may still lead to a C_0-semigroup. To characterize those, we diagonalize $P_1\mathcal{H}(\zeta)$. It is easy to see that the eigenvalues of $P_1\mathcal{H}(\zeta)$ are the same as the eigenvalues of $\mathcal{H}(\zeta)^{\frac{1}{2}} P_1 \mathcal{H}(\zeta)^{\frac{1}{2}}$. Hence, by Sylvester's law of inertia, the number of positive and negative eigenvalues of $P_1\mathcal{H}(\zeta)$ equal those of P_1. We denote by n_1 the number of positive and by $n_2 = n - n_1$ the number of negative eigenvalues of P_1. Hence, we can find matrices such that

$$P_1\mathcal{H}(\zeta) = S^{-1}(\zeta) \begin{bmatrix} \Lambda(\zeta) & 0 \\ 0 & \Theta(\zeta) \end{bmatrix} S(\zeta), \qquad \text{a.e. } \zeta \in (0, 1), \tag{5}$$

with $\Lambda(\zeta)$ and $\Theta(\zeta)$ diagonal matrices of size $n_1 \times n_1$ and $n_2 \times n_2$, respectively.

The main result of this paper is the following theorem that provides easily checked conditions for when the operator $A\mathcal{H}$ generates a C_0-semigroup on X_p. These cover the situation where $A\mathcal{H}$ may not generate a contraction semigroup.

THEOREM 1.5. *Assume that S, Λ and Θ in (5) are continuously differentiable on $[0, 1]$ and that rank $\tilde{W}_B = n$. Define $Z^+(\zeta)$ to be the span of eigenvectors of $P_1\mathcal{H}(\zeta)$ corresponding to its positive eigenvalues. Similarly, we define $Z^-(\zeta)$ to be the span of eigenvectors of $P_1\mathcal{H}(\zeta)$ corresponding to its negative eigenvalues. We write $\tilde{W}_B$ as*

$$\tilde{W}_B = \begin{bmatrix} W_1 & W_0 \end{bmatrix} \tag{6}$$

with $W_1, W_0 \in \mathbb{C}^{n \times n}$. Then, the following statements are equivalent:

1. *The operator $A\mathcal{H}$ defined by (3)–(4) generates a C_0-semigroup on X_p.*
2. $W_1\mathcal{H}(1)Z^+(1) \oplus W_0\mathcal{H}(0)Z^-(0) = \mathbb{C}^n$.

The proof of Theorem 1.5 will be given in the next section.

REMARK 1.6. *1. In Kato [9, Chapter II], conditions on $P_1\mathcal{H}$ are given guaranteeing that S, Λ and Θ are continuously differentiable.*

2. *In [2], a more restrictive version of Theorem 1.5 that applies when $\mathcal{H} = I$ and $p = 2$ was proven by a different approach. In [2] estimates for the growth bound are given.*
3. *Theorem 1.5 implies that if $A\mathcal{H}$ generates a C_0-semigroup on one X_p, then $A\mathcal{H}$ generates a C_0-semigroup on every X_p, $1 \leq p < \infty$. A similar statement does not hold for contraction semigroups. Example 3.3, given later in this paper, illustrates this point.*

□

2. Proof of Theorems 1.1 and 1.5 and Corollary 1.4

Proof of Theorem 1.1: Since the proof of Part 2 is similar to that of Part 1 we only present the details for Part 1.

The implication (a) $\Rightarrow$ (b) follows directly from the Lumer–Phillips theorem and Lemma 7.2.3 in [3]. Next, we show the implication (b) $\Rightarrow$ (c). It is easy to see that

$$\mathrm{Re}\langle Ax, x\rangle = x(1)^* P_1 x(1) - x(0)^* P_1 x(0) + \mathrm{Re}\int_0^1 x(\zeta)^* P_0 x(\zeta) d\zeta \tag{7}$$

holds for every $x \in D(A)$. Choosing $x \in W^{1,2}(0, 1; \mathbb{C}^n)$ with $x(0) = x(1) = 0$, we obtain $\mathrm{Re}\, P_0 \leq 0$. For every $u, y \in \mathbb{C}^n$ and every $\varepsilon > 0$, there exists a function in $x \in W^{1,2}(0, 1; \mathbb{C}^n)$ such that $x(0) = u$, $x(1) = y$ and the L^2-norm of x is less than ε. Choosing this function in Eq. (7) and letting ε go to zero implies the second assertion in (c), see also Lemma 2.4 of [1]. The implication (d) $\Rightarrow$ (a) follows from Theorem 2.3 of [1], see also [4]. Hence, it remains to show (c) $\Rightarrow$ (d).

We introduce the notation $f_1 = x(1)$ and $f_0 = x(0)$. Then, the condition in (c) can be written as

$$\begin{bmatrix} f_1^* & f_0^* \end{bmatrix} \begin{bmatrix} P_1 & 0 \\ 0 & -P_1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_0 \end{bmatrix} \leq 0, \quad \text{for } \begin{bmatrix} f_1 \\ f_0 \end{bmatrix} \in \ker \tilde{W}_B. \tag{8}$$

Since $\tilde{W}_B$ is an $n \times 2n$ matrix, its kernel has dimension $2n$ minus its rank. Hence, this dimension will be larger or equal to n. Since P_1 is an invertible Hermitian $n \times n$ matrix, the matrix $\left[\begin{smallmatrix} P_1 & 0 \\ 0 & -P_1 \end{smallmatrix}\right]$ will have n positive and n negative eigenvalues. This implies that if $v^* \left[\begin{smallmatrix} P_1 & 0 \\ 0 & -P_1 \end{smallmatrix}\right] v \leq 0$ for all v in a linear subspace, then that subspace has at most dimension n. Combining these two facts, the dimension of the kernel of $\tilde{W}_B$ equals n, and so $\tilde{W}_B$ is a matrix of rank n.

Defining $\left[\begin{smallmatrix} y_1 \\ y_0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} P_1 & -P_1 \\ I & I \end{smallmatrix}\right] \left[\begin{smallmatrix} f_1 \\ f_0 \end{smallmatrix}\right]$, and using (8), an easy calculation shows

$$y_1^* y_0 + y_0^* y_1 \leq 0, \quad \text{for } \begin{bmatrix} y_1 \\ y_0 \end{bmatrix} \in \ker W_B. \tag{9}$$

We write W_B as $W_B = [W_1\ W_2]$. Now, it is easy to see that $W_1 + W_2$ is invertible (we refer to page 87 in [3] for the details). Defining $V := (W_1 + W_2)^{-1}(W_1 - W_2)$, we obtain

$$W_B = \frac{1}{2}(W_1 + W_2)\,[I + V, I - V]\,.$$

Let $\left[\begin{smallmatrix} f \\ e \end{smallmatrix}\right] \in \ker W_B$ be arbitrary. By [3, Lemma 7.3.2], there exists a vector ℓ such that $\left[\begin{smallmatrix} f \\ e \end{smallmatrix}\right] = \left[\begin{smallmatrix} I-V \\ -I-V \end{smallmatrix}\right] \ell$. This implies

$$0 \geq f^* e + e^* f = \ell^*(-2I + 2V^* V)\ell, \tag{10}$$

This inequality holds for any $\left[\begin{smallmatrix} f \\ e \end{smallmatrix}\right] \in \ker W_B$. Since the $n \times 2n$ matrix W_B has rank n, its kernel has dimension n, and so the set of vectors ℓ satisfying $\left[\begin{smallmatrix} f \\ e \end{smallmatrix}\right] = \left[\begin{smallmatrix} I-V \\ -I-V \end{smallmatrix}\right] \ell$ for some $\left[\begin{smallmatrix} f \\ e \end{smallmatrix}\right] \in \ker W_B$ equals the whole space $\mathbb{C}^n$. Thus, (10) implies that $V^*V \leq I$, and by [3, Lemma 7.3.1] we obtain $W_B \Sigma W_B^* \geq 0$. □

Proof of Corollary 1.4: As $A\mathcal{H} - \omega I$ generates a contraction semigroup, Theorem 1.1 implies $W_B \Sigma W_B^* \leq 0$ and rank $\tilde{W}_B = n$. Thanks to Re $P_0 \leq 0$ and Theorem 1.1, finally $A\mathcal{H}$ generates a contraction semigroup. □

The following proposition is needed for the proof of Theorem 1.5.

PROPOSITION 2.1. *([8, Theorem 3.3] [3, Theorem 13.3.1] for $p = 2$ and [8, Theorem 3.3 and Section 7] for $1 \leq p < \infty$) Suppose $K, Q \in \mathbb{C}^{n\times n}$, $\Lambda \in C^1([0, 1]; \mathbb{C}^{n_1\times n_1})$ is a diagonal real matrix-valued function with (strictly) positive functions on the diagonal and $\Theta \in C^1([0, 1]; \mathbb{C}^{n_2\times n_2})$, $n_1 + n_2 = n$, is a diagonal real matrix-valued function with (strictly) negative functions on the diagonal. We split a function $g \in L^p(0, 1; \mathbb{C}^n)$ as*

$$g(\zeta) = \begin{bmatrix} g_+(\zeta) \\ g_-(\zeta) \end{bmatrix}, \tag{11}$$

where $g_+(\zeta) \in \mathbb{C}^{n_1}$ and $g_-(\zeta) \in \mathbb{C}^{n_2}$.

Then, the operator $\tilde{A} : D(\tilde{A}) \subset X_p \to X_p$ defined by

$$\tilde{A} \begin{bmatrix} g_+ \\ g_- \end{bmatrix} = \frac{d}{d\zeta} \left(\begin{bmatrix} \Lambda & 0 \\ 0 & \Theta \end{bmatrix} \begin{bmatrix} g_+ \\ g_- \end{bmatrix} \right) \tag{12}$$

$$D(\tilde{A}) = \left\{ \begin{bmatrix} g_+ \\ g_- \end{bmatrix} \in W^{1,p}(0, 1, \mathbb{C}^n) \mid K \begin{bmatrix} \Lambda(1)g_+(1) \\ \Theta(0)g_-(0) \end{bmatrix} + Q \begin{bmatrix} \Lambda(0)g_+(0) \\ \Theta(1)g_-(1) \end{bmatrix} = 0 \right\} \tag{13}$$

generates a C_0-semigroup on X_p if and only if K is invertible.

Proof of Theorem 1.5: We define the new state variable $g := Sx$. Since S defines a boundedly invertible operator on $L^p(0, 1; \mathbb{C}^n)$, the operator $A\mathcal{H}$ generates a C_0-semigroup if and only if $SA\mathcal{H}S^{-1}$ generates a C_0-semigroup. We define

$$\Delta := \begin{bmatrix} \Lambda & 0 \\ 0 & \Theta \end{bmatrix}.$$

Then, we obtain

$$\begin{aligned} (SA\mathcal{H}S^{-1}g)(\zeta) &= \frac{\mathrm{d}}{\mathrm{d}\zeta}(\Delta(\zeta)g(\zeta)) + S(\zeta)\frac{\mathrm{d}S^{-1}}{\mathrm{d}\zeta}(\zeta)\Delta(\zeta)g(\zeta) \\ &\quad + S(\zeta)P_0\mathcal{H}(\zeta)S^{-1}(\zeta)g(\zeta) \\ D(SA\mathcal{H}S^{-1}) &= \left\{ g \in W^{1,p}(0, 1; \mathbb{C}^n) \mid \tilde{W}_B \begin{bmatrix} (\mathcal{H}S^{-1}g)(1) \\ (\mathcal{H}S^{-1}g)(0) \end{bmatrix} = 0 \right\}. \end{aligned} \tag{14}$$

Since the last two operators in (14) are bounded, $SA\mathcal{H}S^{-1}$ generates a C_0-semigroup if and only if the operator

$$A_S g = \frac{\mathrm{d}}{\mathrm{d}\zeta}(\Delta g) \tag{15}$$

$$D(A_S) = \left\{ g \in W^{1,p}(0,1;\mathbb{C}^{n\times n}) \mid \tilde{W}_B \begin{bmatrix} (\mathcal{H}S^{-1}g)(1) \\ (\mathcal{H}S^{-1}g)(0) \end{bmatrix} = 0 \right\} \tag{16}$$

generates a C_0-semigroup on X_p. We split the matrices $W_1(\mathcal{H}S^{-1})(1)$ and $W_0(\mathcal{H}S^{-1})(0)$ as

$$W_1(\mathcal{H}S^{-1})(1) = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \qquad W_0(\mathcal{H}S^{-1})(0) = \begin{bmatrix} U_1 & U_2 \end{bmatrix},$$

where $U_1, V_1 \in \mathbb{C}^{n\times n_1}$ and $U_2, V_2 \in \mathbb{C}^{n\times n_2}$, and as in (11) write

$$g(\zeta) = \begin{bmatrix} g_+(\zeta) \\ g_-(\zeta) \end{bmatrix}, \tag{17}$$

where $g_+(\zeta) \in \mathbb{C}^{n_1}$ and $g_-(\zeta) \in \mathbb{C}^{n_2}$. Then,

$$\begin{aligned} 0 = \tilde{W}_B \begin{bmatrix} (\mathcal{H}S^{-1}g)(1) \\ (\mathcal{H}S^{-1}g)(0) \end{bmatrix} &= \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} g_+(1) \\ g_-(1) \end{bmatrix} + \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} g_+(0) \\ g_-(0) \end{bmatrix} \\ &= \begin{bmatrix} V_1 \; U_2 \end{bmatrix} \begin{bmatrix} g_+(1) \\ g_-(0) \end{bmatrix} + \begin{bmatrix} U_1 & V_2 \end{bmatrix} \begin{bmatrix} g_+(0) \\ g_-(1) \end{bmatrix} \\ &= \begin{bmatrix} V_1 & U_2 \end{bmatrix} \begin{bmatrix} \Lambda(1)^{-1} & 0 \\ 0 & \Theta(0)^{-1} \end{bmatrix} \begin{bmatrix} \Lambda(1)g_+(1) \\ \Theta(0)g_-(0) \end{bmatrix} \\ &\quad + \begin{bmatrix} U_1 & V_2 \end{bmatrix} \begin{bmatrix} \Lambda(0)^{-1} & 0 \\ 0 & \Theta(1)^{-1} \end{bmatrix} \begin{bmatrix} \Lambda(0)g_+(0) \\ \Theta(1)g_-(1) \end{bmatrix}. \end{aligned}$$

Thus, by Proposition 2.1, the operator A_S as defined in (15) and (16) generates a C_0-semigroup if and only if the matrix

$$K = \begin{bmatrix} V_1 & U_2 \end{bmatrix} \begin{bmatrix} \Lambda(1)^{-1} & 0 \\ 0 & \Theta(0)^{-1} \end{bmatrix}$$

is invertible. Since the matrix $\begin{bmatrix} \Lambda(1)^{-1} & 0 \\ 0 & \Theta(0)^{-1} \end{bmatrix}$ is invertible, A_S generates a C_0-semigroup if and only if $\begin{bmatrix} V_1 \; U_2 \end{bmatrix}$ is invertible. Now, $\begin{bmatrix} V_1 \; U_2 \end{bmatrix}$ is invertible if and only if for every $f \in \mathbb{C}^n$ there exists $x \in \mathbb{C}^{n_1}$ and $y \in \mathbb{C}^{n_2}$ such that

$$\begin{aligned} f = \begin{bmatrix} V_1 & U_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} V_1 & U_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} U_1 & V_2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} 0 \\ y \end{bmatrix} \\ &= W_1(\mathcal{H}S^{-1})(1) \begin{bmatrix} x \\ 0 \end{bmatrix} + W_0(\mathcal{H}S^{-1})(0) \begin{bmatrix} 0 \\ y \end{bmatrix}. \end{aligned} \tag{18}$$

Referring, to Eq. (5) the columns of $S^{-1}(\zeta)$ are the eigenvectors of $P_1\mathcal{H}(\zeta)$. The eigenvectors corresponding to the positive eigenvalues forms the first n_1 columns. Thus, $S^{-1}(1)\begin{bmatrix} x \\ 0 \end{bmatrix}$ is in $Z^+(1)$. Similarly, $S^{-1}(0)\begin{bmatrix} 0 \\ y \end{bmatrix}$ is in $Z^-(0)$. Thus, $\begin{bmatrix} V_1 & U_2 \end{bmatrix}$ is invertible if and only if

$$W_1\mathcal{H}(1)Z^+(1) \oplus W_0\mathcal{H}(0)Z^-(0) = \mathbb{C}^n,$$

which concludes the proof. $\square$

3. Examples

The following three examples are provided as illustration of Theorem 1.5.

EXAMPLE 3.1. *Consider the one-dimensional transport equation on the interval* $(0, 1)$*:*

$$\frac{\partial x}{\partial t}(\zeta, t) = \frac{\partial \mathcal{H}x}{\partial \zeta}(\zeta, t), \qquad x(\zeta, 0) = x_0(\zeta),$$
$$\begin{bmatrix} w_1 & w_0 \end{bmatrix}\begin{bmatrix} (\mathcal{H}x)(1, t) \\ (\mathcal{H}x)(0, t) \end{bmatrix} = 0,$$

where $\mathcal{H} \in C^1[0, 1]$ *with* $\mathcal{H}(\zeta) > 0$ *for every* $\zeta \in [0, 1]$.

An easy calculation shows $P_1\mathcal{H} = \mathcal{H}$ *and thus* $Z^+(1) = \mathbb{C}$ *and* $Z^-(0) = \{0\}$. *Thus, by Theorem 1.5 the corresponding operator*

$$A\mathcal{H}x = \frac{\partial}{\partial \zeta}(\mathcal{H}x),$$
$$D(A\mathcal{H}) = \left\{ x \in W^{1,p}(0, 1) \mid \begin{bmatrix} w_1 & w_0 \end{bmatrix}\begin{bmatrix} (\mathcal{H}x)(1) \\ (\mathcal{H}x)(0) \end{bmatrix} = 0 \right\},$$

generates a C_0*-semigroup on* $L^p(0, 1)$ *if and only if* $w_1 \neq 0$. *Further, by Theorem 1.1,* $A\mathcal{H}$ *generates a contraction semigroup (unitary* C_0*-group) on* $L^2(0, 1)$ *equipped with the scalar product* $\langle \cdot, \mathcal{H}\cdot \rangle$ *if and only if* $w_1^2 \geq w_0^2$ $(w_1^2 = w_0^2)$. $\square$

EXAMPLE 3.2. *An (undamped) vibrating string can be modeled by*

$$\frac{\partial^2 w}{\partial t^2}(\zeta, t) = \frac{1}{\rho(\zeta)}\frac{\partial}{\partial \zeta}\left(T(\zeta)\frac{\partial w}{\partial \zeta}(\zeta, t)\right), \qquad t \geq 0, \zeta \in (0, 1), \tag{19}$$

where $\zeta \in [0, 1]$ *is the spatial variable,* $w(\zeta, t)$ *is the vertical position of the string at place* ζ *and time* t, $T(\zeta) > 0$ *is the Young's modulus of the string and* $\rho(\zeta) > 0$ *is the mass density, which may vary along the string. We assume that* T *and* ρ *are positive and continuously differentiable functions on* $[0, 1]$. *By choosing the state variables*

$x_1 = \rho \frac{\partial w}{\partial t}$ *(momentum) and* $x_2 = \frac{\partial w}{\partial \zeta}$ *(strain), the partial differential equation* (19) *can equivalently be written as*

$$\frac{\partial}{\partial t}\begin{bmatrix} x_1(\zeta,t) \\ x_2(\zeta,t)\end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0\end{bmatrix}\frac{\partial}{\partial \zeta}\left(\begin{bmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta)\end{bmatrix}\begin{bmatrix} x_1(\zeta,t) \\ x_2(\zeta,t)\end{bmatrix}\right)$$
$$= P_1 \frac{\partial}{\partial \zeta}\left(\mathcal{H}(\zeta)\begin{bmatrix} x_1(\zeta,t) \\ x_2(\zeta,t)\end{bmatrix}\right), \tag{20}$$

where $P_1 = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0\end{smallmatrix}\right]$ *and* $\mathcal{H}(\zeta) = \left[\begin{smallmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta)\end{smallmatrix}\right]$.

The boundary conditions for (20) *are*

$$\begin{bmatrix} W_1 & W_0 \end{bmatrix}\begin{bmatrix} (\mathcal{H}x)(1,t) \\ (\mathcal{H}x)(0,t)\end{bmatrix} = 0,$$

where $\begin{bmatrix} W_1 & W_0 \end{bmatrix}$ *is a* 2×4*-matrix with rank 2, or equivalently, the partial differential equation* (19) *is equipped with the boundary conditions*

$$\begin{bmatrix} W_1 & W_0 \end{bmatrix}\begin{bmatrix} \rho\frac{\partial w}{\partial t}(1,t) \\ \frac{\partial w}{\partial \zeta}(1,t) \\ \rho\frac{\partial w}{\partial t}(0,t) \\ \frac{\partial w}{\partial \zeta}(0,t)\end{bmatrix} = 0.$$

Defining $\gamma = \sqrt{T(\zeta)/\rho(\zeta)}$, *the matrix function* $P_1\mathcal{H}$ *can be factorized as*

$$P_1\mathcal{H} = \begin{bmatrix} \gamma & -\gamma \\ \rho^{-1} & \rho^{-1}\end{bmatrix}\begin{bmatrix} \gamma & 0 \\ 0 & -\gamma\end{bmatrix}\begin{bmatrix} (2\gamma)^{-1} & \rho/2 \\ -(2\gamma)^{-1} & \rho/2\end{bmatrix}.$$

This implies $Z^+(1) = span \left[\begin{smallmatrix} T(1) \\ \gamma(1)\end{smallmatrix}\right]$ *and* $Z^-(0) = span \left[\begin{smallmatrix} -T(0) \\ \gamma(0)\end{smallmatrix}\right]$. *Thus, by Theorem* 1.5 *the corresponding operator*

$$(A\mathcal{H}x)(\zeta) = \begin{bmatrix} 0 & 1 \\ 1 & 0\end{bmatrix}\frac{\partial}{\partial \zeta}\left(\begin{bmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta)\end{bmatrix}x(\zeta)\right);$$
$$D(A\mathcal{H}) = \left\{x \in W^{1,p}(0,1;\mathbb{C}^2) \mid \begin{bmatrix} W_1 & W_0 \end{bmatrix}\begin{bmatrix} (\mathcal{H}x)(1) \\ (\mathcal{H}x)(0)\end{bmatrix} = 0\right\},$$

generates a C_0*-semigroup on* $L^p(0,1;\mathbb{C}^2)$ *if and only if*

$$W_1\begin{bmatrix} \gamma(1) \\ T(1)\end{bmatrix} \oplus W_0\begin{bmatrix} -\gamma(0) \\ T(0)\end{bmatrix} = \mathbb{C}^2,$$

or equivalently if the vectors $W_1\left[\begin{smallmatrix} \gamma(1) \\ T(1)\end{smallmatrix}\right]$ *and* $W_0\left[\begin{smallmatrix} -\gamma(0) \\ T(0)\end{smallmatrix}\right]$ *are linearly independent.*

If $W_1 := I$ *and* $W_0 := \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1\end{smallmatrix}\right]$, *then* $A\mathcal{H}$ *generates a* C_0*-semigroup if and only if the vectors* $\left[\begin{smallmatrix} \gamma(1) \\ T(1)\end{smallmatrix}\right]$ *and* $\left[\begin{smallmatrix} \gamma(0) \\ T(0)\end{smallmatrix}\right]$ *are linearly independent. Thus, not only the nature of the boundary conditions but also Young's modulus and the mass density on the interval* $[0, 1]$ *affect whether or not* $A\mathcal{H}$ *generates a* C_0*-semigroup.* □

EXAMPLE 3.3. *Consider the following network of three transport equations on the interval* (0, 1)*:*

$$\frac{\partial x_j}{\partial t}(\zeta,t) = \frac{\partial x_j}{\partial \zeta}(\zeta,t), \quad t \geq 0, \quad \zeta \in (0,1), \quad j = 1,2,3,$$

$$x_j(\zeta,0) = x_{j,0}(\zeta), \quad \zeta \in (0,1), \quad j = 1,2,3$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1(1,t) \\ x_2(1,t) \\ x_3(1,t) \\ x_1(0,t) \\ x_2(0,t) \\ x_3(0,t) \end{bmatrix} = 0, \quad t \geq 0.$$

Writing $x = \left[\begin{smallmatrix} x_1 \\ x_2 \\ x_3 \end{smallmatrix}\right]$*, the corresponding operator* $A : D(A) \subset L^p(0,1;\mathbb{C}^3) \to L^p(0,1;\mathbb{C}^3)$ *is*

$$(Ax)(\zeta) = \frac{\partial x}{\partial \zeta}(\zeta),$$

$$D(A) = \left\{ x \in W^{1,p}(0,1;\mathbb{C}^3) \mid \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x(1) \\ x(0) \end{bmatrix} = 0 \right\}.$$

In this example, $\mathcal{H} = P_1 = I$ *and* $P_0 = 0$ *and therefore the assumptions on* S, Λ *and* Θ *are satisfied. An easy calculation yields*

$$x^*(1)x(1) - x^*(0)x(0) = 2x_1(0)x_3(0)$$

for every $x \in D(A)$*. Theorem* 1.1 *implies that* A *does not generate a contraction semigroup on* $L^2(0,1;\mathbb{C}^3)$.

However, by Theorem 1.5 A *generates a* C_0*-semigroup on* $L^p(0,1;\mathbb{C}^3)$ *for* $1 \leq p < \infty$*: In this example,* $Z^+(\zeta) = \mathbb{C}^3$, $Z^-(\zeta) = \{0\}$, $W_1 = I$ *and* $W_0 = \left[\begin{smallmatrix} 0 & 0 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{smallmatrix}\right]$. *Thus,*

$$W_1 Z^+(1) \oplus W_0 Z^-(0) = \mathbb{C}^3.$$

Finally, [5, Corollary2.1.6] implies that A *generates a contraction semigroup on* $L^1(0,1;\mathbb{C}^3)$.

Summarizing, A *generates a* C_0*-semigroup on* $L^p(0,1;\mathbb{C}^3)$ *for* $1 \leq p < \infty$ *and in fact a contraction semigroup on* $L^1(0,1;\mathbb{C}^3)$ *but it does not generate a contraction semigroup on* $L^2(0,1;\mathbb{C}^3)$. □

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