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De Branges–Rovnyak Realizations of Operator-Valued Schur Functions on the Complex Right Half-Plane

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Abstract We give a controllable energy-preserving and an observable co-energy-preserving de Branges–Rovnyak functional model realization of an arbitrary given operator Schur function defined on the complex right-half plane. We work the theory out fully in the right-half plane, without using results for the disk case, in order to expose the technical details of continuous-time systems theory. At the end of the article, we make explicit the connection to the corresponding classical de Branges–Rovnyak realizations for Schur functions on the complex unit disk.

Keywords Schur function · Right half-plane · Continuous time · Functional model · De Branges–Rovnyak space · Reproducing kernel

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Contents

1 Introduction

It essentially goes back to Kalman (with earlier roots in circuit theory from the middle of the twentieth century) that any rational function ϕ holomorphic in a neighborhood of the origin with values in the space $\mathcal{B}(\mathcal{U}, \mathcal{Y})$ of bounded linear operators between two Hilbert spaces $\mathcal{U}$ (the input space) and $\mathcal{Y}$ (the output space) can be realized as the transfer function of an input/state/output linear system, i.e., there is a Hilbert space $\mathcal{X}$ (the state space) and a bounded operator system matrix $\mathbf{U} := \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] : \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$ so that $\phi(z)$ has the representation

$$\phi(z) = D + zC(1 - zA)^{-1}B. \tag{1.1}$$

If we associate with $\mathbf{U}$ the discrete-time input/state/output system

$$\Sigma_{\mathbf{U}} : \quad \begin{cases} x(t+1) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}, \tag{1.2}$$

the meaning of (1.1) is that ϕ is the *transfer function* of the i/s/o system $\Sigma_{\mathbf{U}}$ in the following sense: whenever the input string $\{u_n\}_{n\in\mathbb{Z}_+}$ is fed into the system (1.2) with the initial condition $x(0) = 0$ on the state vector, the output string $\{y(n)\}_{n\in\mathbb{Z}_+}$ is produced, such that $\widehat{y}(z) = \phi(z)\widehat{u}(z)$, where $\widehat{u}$ and $\widehat{y}$ denote the Z-transforms of $\{u(n)\}_{n\in\mathbb{Z}_+}$ and $\{y(n)\}_{n\in\mathbb{Z}_+}$:

$$\widehat{u}(z) = \sum_{n=0}^{\infty} u(n)z^n, \quad \widehat{y}(z) = \sum_{n=0}^{\infty} y_n z^n. \tag{1.3}$$

In the infinite-dimensional setting, the fact that any contractive holomorphic operator-valued function can be represented in the form (1.1) with **U** unitary comes out of the Sz.-Nagy–Foiaş model theory for completely non-unitary contraction operators; see [48]. There is a closely related but somewhat different theory of canonical functional models due to de Branges and Rovnyak [23,24] which relies on reproducing kernel Hilbert spaces. This is the direction we pursue in the present paper, assuming throughout that $\mathcal{U}$ and $\mathcal{Y}$ are separable.

Let $\mathcal{G}$ be a Hilbert space and let $\mathcal{B}(\mathcal{G})$ denote the space of bounded linear operators on $\mathcal{G}$. In general we say that a function $K : \Omega \times \Omega \to \mathcal{B}(\mathcal{G})$ is a *positive kernel* on Ω if

$$\sum_{i,j=1}^{N} \langle K(\omega_i, \omega_j)g_j, g_i\rangle_{\mathcal{G}} \geq 0 \tag{1.4}$$

for all choices of points $\omega_1, \ldots, \omega_N$ in Ω and vectors $g_1, \ldots, g_N \in \mathcal{G}$. The following theorem summarizes some useful equivalent characterizations of a positive $\mathcal{B}(\mathcal{G})$-valued kernel on Ω.

Theorem 1.1 *Given a Hilbert space $\mathcal{G}$ and a function $K : \Omega \times \Omega \to \mathcal{B}(\mathcal{G})$, the following are equivalent:*

1. *The function K is a positive kernel, i.e., condition (1.4) holds for all $\omega_1, \ldots, \omega_N$ in Ω and $g_1, \ldots, g_N \in \mathcal{G}$ for $N = 1, 2, \ldots$.*
2. *The function K is the* reproducing kernel *of a reproducing kernel Hilbert space $\mathcal{H}(K)$, i.e., there is a unique Hilbert space $\mathcal{H}(K)$ whose elements are functions $f : \Omega \to \mathcal{G}$ such that:*
   (a) For each $\omega \in \Omega$ and $g \in \mathcal{G}$, the function $\zeta \mapsto K(\zeta, \omega)g$, $\zeta \in \Omega$, belongs to $\mathcal{H}(K)$, and
   (b) the reproducing property

$$\langle f, K(\cdot, \omega)g\rangle_{\mathcal{H}(K)} = \langle f(\omega), g\rangle_{\mathcal{G}} \tag{1.5}$$

   holds for all $f \in \mathcal{H}(K)$, $\omega \in \Omega$, and $g \in \mathcal{G}$.
3. *The function K has a* Kolmogorov decomposition, *i.e., there is a Hilbert space $\mathcal{F}$ and a function $H : \Omega \to \mathcal{B}(\mathcal{F}, \mathcal{G})$ such that K has the factorization*

$$K(\zeta, \omega) = H(\zeta)H(\omega)^*, \quad \zeta, \omega \in \Omega. \tag{1.6}$$

   When the conditions 1-3 hold, one Kolmogorov decomposition (often called canonical) is produced by taking $\mathcal{F} = \mathcal{H}(K)$ as defined in item 2 and $H(\zeta)$ equal to the point-evaluation map

$$H(\zeta) = e(\zeta) : f \mapsto f(\zeta), \quad f \in \mathcal{H}(K),\ \zeta \in \Omega.$$

We will make frequent use of the following observation which is an immediate consequence of the reproducing property (1.5):

Remark 1.2 In the notation of Theorem 1.1, assume that K is a reproducing kernel for the Hilbert space $\mathcal{H}(K)$. Then the linear span

$$\operatorname{span}\{\zeta \mapsto K(\zeta, \omega)g | \omega \in \Omega,\ g \in \mathcal{G}\}$$

is dense in $\mathcal{H}(K)$.

Given two separable Hilbert spaces $\mathcal{U}$ and $\mathcal{Y}$, we let $\mathcal{S}(\mathbb{D}; \mathcal{U}, \mathcal{Y})$ denote the Schur class over the unit disk $\mathbb{D}$ consisting of functions $\phi : \mathbb{D} \to \mathcal{B}(\mathcal{U}, \mathcal{Y})$ which are holomorphic on $\mathbb{D}$ with values $\phi(z)$ equal to contraction operators from $\mathcal{U}$ into $\mathcal{Y}$. Given the Schur-class function ϕ on $\mathbb{D}$, we associate the kernel

$$\mathrm{K}_o(z, w) = \frac{1 - \phi(z)\phi(w)^*}{1 - z\overline{w}} \tag{1.7}$$

for z, w in the unit disk $\mathbb{D}$. It is well known that K_o is a positive kernel; the proof is similar to Sect. 2 below. By the Moore–Aronszajn Theorem [6, §2] (part of the proof of Theorem 1.1) one can associate the reproducing-kernel Hilbert space $\mathrm{H}_o := \mathcal{H}(\mathrm{K}_o)$ to the kernel function K_o. This space plays the role of the state space in the observable co-isometric (co-energy-preserving) de Branges–Rovnyak canonical functional model for a Schur class function ϕ. We note that this functional model is of interest not only as an alternative to the Sz.-Nagy–Foiaş model [48] for contraction operators (see [14,22,23]), but also has found applications in the context of Lax-Phillips scattering theory [36] and inverse scattering theory [3,4] as well as boundary Nevanlinna-Pick interpolation [19,41]. The following result can be found at least implicitly in the work of de Branges–Rovnyak and is given explicitly in this form in [2] and in [12].

Theorem 1.3 *Suppose that the function ϕ is in the Schur class $\mathcal{S}(\mathbb{D}; \mathcal{U}, \mathcal{Y})$ and let $\mathrm{H}_o = \mathcal{H}(\mathrm{K}_o)$ be the associated de Branges–Rovnyak space with reproducing kernel (1.7). Define operators A_o, B_o, C_o, and D_o by*

$$\begin{aligned} &\mathrm{A}_o f := z \mapsto \frac{f(z) - f(0)}{z}, \qquad &&\mathrm{B}_o u := z \mapsto \frac{\phi(z) - \phi(0)}{z} u, \\ &\mathrm{C}_o f := f(0), &&\mathrm{D}_o u := \phi(0) u, \\ & && f \in \mathrm{H}_o,\ u \in \mathcal{U},\ z \in \mathbb{D}. \end{aligned} \tag{1.8}$$

Then the operator matrix $\mathbf{U}_o := \begin{bmatrix} \mathrm{A}_o & \mathrm{B}_o \\ \mathrm{C}_o & \mathrm{D}_o \end{bmatrix}$ has the following properties:

1. *The operator $\mathbf{U}_o$ defines a co-isometry from $\begin{bmatrix} \mathrm{H}_o \\ \mathcal{U} \end{bmatrix}$ to $\begin{bmatrix} \mathrm{H}_o \\ \mathcal{Y} \end{bmatrix}$.*
2. *The pair $(\mathrm{C}_o, \mathrm{A}_o)$ is an* observable pair, *i.e.,*

$$\mathrm{C}_o \mathrm{A}_o^n f = 0 \quad \textit{for all } n = 0, 1, 2, \ldots \quad \Longrightarrow \quad f = 0 \textit{ as an element of } \mathrm{H}_o.$$

3. *We recover $\phi(z)$ from $\begin{bmatrix} \mathrm{A}_o & \mathrm{B}_o \\ \mathrm{C}_o & \mathrm{D}_o \end{bmatrix}$ as $\phi(z) = \mathrm{D}_o + z\mathrm{C}_o(1 - z\mathrm{A}_o)^{-1}\mathrm{B}_o,\ z \in \mathbb{D}$.*

4. *If* $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] : \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$ *is another operator matrix with properties 1–3 above (with* $\mathcal{X}$ *in place of* H_o*), then there is a unitary operator* $\Delta : \mathrm{H}_o \to \mathcal{X}$ *so that*

$$\begin{bmatrix} \Delta & 0 \\ 0 & 1_{\mathcal{Y}} \end{bmatrix} \begin{bmatrix} \mathrm{A}_o & \mathrm{B}_o \\ \mathrm{C}_o & \mathrm{D}_o \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \Delta & 0 \\ 0 & 1_{\mathcal{U}} \end{bmatrix}.$$

If ϕ is in the Schur class $\mathcal{S}(\mathbb{D}; \mathcal{U}, \mathcal{Y})$, then the function $\widetilde{\phi}$ defined by $\widetilde{\phi}(z) := \phi(\overline{z})^*, \ z \in \mathbb{D}$, lies in $\mathcal{S}(\mathbb{D}; \mathcal{Y}, \mathcal{U})$. Replacing ϕ by $\widetilde{\phi}$ in (1.7) leads to the dual de Branges–Rovnyak kernel given by

$$\mathrm{K}_c(z, w) := \frac{1 - \phi(\overline{z})^* \phi(\overline{w})}{1 - z\overline{w}}. \tag{1.9}$$

The Hilbert space associated to this kernel plays the role of the state-space in the following controllable, isometric (energy-preserving) de Branges–Rovnyak canonical functional model:

Theorem 1.4 *Suppose that the function* ϕ *is in the Schur class* $\mathcal{S}(\mathbb{D}; \mathcal{U}, \mathcal{Y})$ *and let* $\mathrm{H}_c = \mathcal{H}(\mathrm{K}_c)$ *be the associated dual de Branges–Rovnyak space. Define operators* $\mathrm{A}_c, \ \mathrm{B}_c, \ \mathrm{C}_c$*, and* D_c *by*

$$\begin{aligned} &\mathrm{A}_c g := z \mapsto z g(z) - \phi(\overline{z})^* \widetilde{g}(0), &&\mathrm{B}_c u := z \mapsto \big(1 - \phi(\overline{z})^* \phi(0)\big) u, \\ &\mathrm{C}_c g := \widetilde{g}(0), &&\mathrm{D}_c u := \phi(0) u, \\ & &&g \in \mathrm{H}_c, \ u \in \mathcal{U}, \ z \in \mathbb{D}, \end{aligned} \tag{1.10}$$

where $\widetilde{g}(0)$ *is the unique vector in* $\mathcal{Y}$ *such that*

$$\langle \widetilde{g}(0), y \rangle_{\mathcal{Y}} = \left\langle g, z \mapsto \frac{\phi(\overline{z})^* - \phi(0)^*}{z} y \right\rangle_{\mathrm{H}_c} \quad \text{for all } \ y \in \mathcal{Y}. \tag{1.11}$$

Then the operator matrix $\mathbf{U}_c := \left[\begin{smallmatrix} \mathrm{A}_c & \mathrm{B}_c \\ \mathrm{C}_c & \mathrm{D}_c \end{smallmatrix}\right]$ *has the following properties:*

1. *The operator* $\mathbf{U}_c$ *defines an isometry from* $\left[\begin{smallmatrix} \mathrm{H}_c \\ \mathcal{U} \end{smallmatrix}\right]$ *to* $\left[\begin{smallmatrix} \mathrm{H}_c \\ \mathcal{Y} \end{smallmatrix}\right]$.
2. *The pair* $(\mathrm{A}_c, \mathrm{B}_c)$ *is a* controllable pair*, i.e.,*

$$\overline{\operatorname{span}} \left\{ \mathrm{A}_c^n \mathrm{B}_c u | u \in \mathcal{U}, \ n \geq 0 \right\} = \mathrm{H}_c.$$

3. *We recover* $\phi(z)$ *as* $\phi(z) = \mathrm{D}_c + z\mathrm{C}_c(1 - z\mathrm{A}_c)^{-1}\mathrm{B}_c, \ z \in \mathbb{D}$.
4. *If* $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] : \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$ *is another operator matrix with properties 1–3 above (with* $\mathcal{X}$ *in place of* H_c*), then there is a unitary operator* $\Delta : \mathrm{H}_c \to \mathcal{X}$ *so that*

$$\begin{bmatrix} \Delta & 0 \\ 0 & 1_{\mathcal{Y}} \end{bmatrix} \begin{bmatrix} \mathrm{A}_c & \mathrm{B}_c \\ \mathrm{C}_c & \mathrm{D}_c \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \Delta & 0 \\ 0 & 1_{\mathcal{U}} \end{bmatrix}.$$

The cases where the canonical model $\mathbf{U}_o$ and/or $\mathbf{U}_c$ is unitary can be characterized as follows:

Theorem 1.5 *The following assertions are equivalent:*

1. *The co-isometric observable canonical model* $\mathbf{U}_o$ *is unitary.*
2. *The following two conditions both hold:*

$$\mathrm{H}_o \cap \{\phi(\cdot)u | u \in \mathcal{U}\} = \{0\} \quad \textit{and} \tag{1.12}$$

$$\phi(z)u = 0 \quad \textit{for all } z \in \mathbb{D} \quad \Longrightarrow \quad u = 0. \tag{1.13}$$

3. *The maximal factorable minorant of* $1 - \phi(z)^*\phi(z)$ *is* 0*, i.e., the only holomorphic* $a \colon \mathbb{D} \to \mathcal{B}(\mathcal{U}, \mathcal{U}')$ *with the property*

$$a(z)^*a(z) \leq 1 - \phi(z)^*\phi(z), \quad z \in \mathbb{C}, \ |z| = 1$$

is $a = 0$.

The following assertions are also equivalent:

1. *The isometry* $\mathbf{U}_c$ *is unitary.*
2. *The following two conditions both hold:*

$$\mathrm{H}_c \cap \{z \mapsto \phi(\overline{z})^*y | y \in \mathcal{Y}\} = \{0\} \quad \textit{and}$$

$$\phi(\overline{z})^*y = 0 \quad \textit{for all } z \in \mathbb{D} \quad \Longrightarrow \quad y = 0.$$

3. *The maximal factorable minorant of* $z \mapsto 1 - \phi(\overline{z})\phi(\overline{z})^*$ *is* 0.

The equivalences of the conditions one and two can be found in [2, Thms 3.2.3 and 3.3.3]. For instance, one easily sees that the conditions (1.12) and (1.13) both hold if and only if ker $(\mathbf{U}_o) = \{0\}$. In order to prove that the third assertion is equivalent to unitarity in the case of $\mathbf{U}_o$, as a first step combine Lemma 8.2, Theorem 8.7, Corollary 8.8, and Theorem 9.1 in [35] to see that the zero-maximal-factorable-minorant condition on $1-\phi(\cdot)^*\phi(\cdot)$ is equivalent to each column $\left[\begin{smallmatrix} \mathrm{A}_o \\ \mathrm{C}_o \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} \mathrm{B}_o \\ \mathrm{D}_o \end{smallmatrix}\right]$ of $\mathbf{U}_o$ being isometric. It is then an elementary exercise to argue that the whole matrix $\mathbf{U}_o = \left[\begin{smallmatrix} \mathrm{A}_o & \mathrm{B}_o \\ \mathrm{C}_o & \mathrm{D}_o \end{smallmatrix}\right]$ is isometric if it is known to be contractive with each column isometric. The proof for the case of $\mathbf{U}_c$ is the same, but with $\widetilde{\phi}$ in place of ϕ and with $\mathbf{U}_c^*$ in place of $\mathbf{U}_o$.

In addition to the functional models in Theorems 1.3 and 1.4, there is also a unitary functional model which combines $\mathbf{U}_o$ and $\mathbf{U}_c$; see e.g. Brodskiĭ [20].

There is a parallel but less well developed theory for the Schur class $\mathcal{S}(\mathbb{C}^+; \mathcal{U}, \mathcal{Y})$ consisting of holomorphic functions on the right half plane $\mathbb{C}^+$ with values equal to contraction operators between the coefficient Hilbert spaces $\mathcal{U}$ and $\mathcal{Y}$. See however [28,30] as well as [16,31] for a more general algebraic curve setting. In general, if the $\mathcal{B}(\mathcal{U}, \mathcal{Y})$-valued function φ has the property that φ extends to be holomorphic in a neighborhood of infinity rather than in a neighborhood of the origin, it is natural to work with realizations of the form

$$\varphi(\mu) = D + C(\mu - A)^{-1}B. \tag{1.14}$$

It is well known that, given any $\mathcal{B}(\mathcal{U}, \mathcal{Y})$-valued function holomorphic on a neighborhood of ∞ in the complex plane, there is a Hilbert space $\mathcal{X}$ (the state space) and a system matrix

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$$

so that φ has a representation as in (1.14). If we introduce the continuous-time input/state/output linear system

$$\Sigma_{\mathbf{U}} : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t), \end{cases} \tag{1.15}$$

then application of the Laplace transform

$$\widehat{x}(\mu) = \int_0^\infty e^{-\mu t} x(t)\, \mathrm{d}t \tag{1.16}$$

leads to the relation

$$\widehat{y}(\mu) = \varphi(\mu)\widehat{u}(\mu)$$

whenever $\big(u(\cdot), x(\cdot), y(\cdot)\big)$ is a trajectory of the system (1.15) with state-vector x satisfying the zero initial condition $x(0) = 0$.

The generalized form for the operator matrix $\mathbf{U}$ appropriate for the Schur class over $\mathbb{C}^+$ was first worked out by independently by Šmuljan [43] and Salamon [38,39]. Salamon gave a well-posed realization of an holomorphic function on $\mathbb{C}^+$ which is bounded on some complex right-half plane. Later, in [8], Arov-Nudelman specialized to the case of a Schur function, giving a passive realization. The generalized form for $\mathbf{U}$ has since been refined into the notion of scattering-conservative/energy-preserving/co-energy-preserving system node; see [45] for a comprehensive treatment, and also [15,44]. The analogue for the continuous-time setting of *co-isometric system matrix* occurring in the discrete-time setting is a *co-energy-preserving system node* while the analogue for the continuous-time setting of *isometric system matrix* occurring in the discrete-time setting is an *energy-preserving system node* (precise definitions to come in Sect. 3 below).

However, what has not been done to this point for the realization theory is the analogues of Theorems 1.3 and 1.4 for φ in the Schur class over $\mathbb{C}^+$. By using the right-half plane versions of the de Branges–Rovnyak kernels K_o and K_c, namely,

$$K_o(\mu, \lambda) = \frac{1 - \varphi(\mu)\varphi(\lambda)^*}{\mu + \overline{\lambda}}, \quad K_c(\mu, \lambda) = \frac{1 - \varphi(\overline{\mu})^*\varphi(\overline{\lambda})}{\mu + \overline{\lambda}}, \tag{1.17}$$

combined with the precise formalism of scattering energy-preserving and scattering co-energy-peserving system nodes, in this paper we obtain complete analogues of Theorems 1.3 and 1.4 for the continuous-time setting. Due to complications with unbounded operators and rigged Hilbert spaces, the formulas and analysis have a quite different flavor from that in the discrete-time/unit-disk setting.

The positivity of the kernels (1.17) is proved in Sect. 2, and in Sect. 5 we establish the following continuous-time analogue of Theorem 1.3:

Theorem 1.6 *Suppose that the function φ is in the Schur class $\mathcal{S}(\mathbb{C}^+; \mathcal{U}, \mathcal{Y})$ and let $\mathcal{H}_o = \mathcal{H}(K_o)$ be the associated de Branges–Rovnyak space with reproducing kernel K_o in (1.17). Define the following unbounded operator, which maps a dense subspace of $\left[\begin{smallmatrix} \mathcal{H}_o \\ \mathcal{U} \end{smallmatrix}\right]$ into $\left[\begin{smallmatrix} \mathcal{H}_o \\ \mathcal{Y} \end{smallmatrix}\right]$:*

$$\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}_o : \begin{bmatrix} x \\ u \end{bmatrix} \mapsto \begin{bmatrix} z \\ y \end{bmatrix}, \quad \textit{where} \tag{1.18}$$

$$z(\mu) := \mu x(\mu) + \varphi(\mu)u - y, \quad \mu \in \mathbb{C}^+, \quad \textit{and} \tag{1.19}$$

$$y := \lim_{\operatorname{Re}\eta \to \infty} \eta x(\eta) + \varphi(\eta)u, \quad \textit{defined on} \tag{1.20}$$

$$\operatorname{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_o\right) := \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in \begin{bmatrix} \mathcal{H}_o \\ \mathcal{U} \end{bmatrix} \middle| \exists y \in \mathcal{Y} : \ z \ \textit{defined in (1.19) lies in } \mathcal{H}_o \right\}.$$

Then for every $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \operatorname{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_o\right)$, the $y \in \mathcal{Y}$ such that z given in (1.19) lies in $\mathcal{H}_o$ is unique and it is given by (1.20). Moreover, the operator $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_o$ has the following properties:

1. *The operator $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_o$ is an observable co-energy-preserving system node.*
2. *The operator $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_o$ is a realization of φ, i.e., we recover $\varphi(\mu)$ through an appropriate generalization of (1.14).*
3. *If $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right] : \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \supset \operatorname{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right) \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$ is another operator with properties 1–2 above (with $\mathcal{X}$ in place of $\mathcal{H}_o$), then there is a unitary operator $\Delta : \mathcal{H}_o \to \mathcal{X}$ so that $\left[\begin{smallmatrix} \Delta & 0 \\ 0 & 1_{\mathcal{U}} \end{smallmatrix}\right]$ maps $\operatorname{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_o\right)$ one-to-one onto $\operatorname{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right)$ and*

$$\begin{bmatrix} \Delta & 0 \\ 0 & 1_{\mathcal{Y}} \end{bmatrix} \begin{bmatrix} A\&B \\ C\&D \end{bmatrix}_o = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \begin{bmatrix} \Delta & 0 \\ 0 & 1_{\mathcal{U}} \end{bmatrix}.$$

Hence the system nodes $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_o$ are unitarily similar.

It is also possible to decompose $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_o$ into unbounded operators A_o, B_o, and C_o which together with φ determine $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_o$ uniquely, similar to Theorem 1.3; see Sect. 3.1 below. This involves a rigging of the state space and hence it is too technically involved to be presented in the introduction. We have the following analogue of Theorem 1.4; the proofs and more details can be found in Sect. 4:

Theorem 1.7 *Suppose that the function φ is in the Schur class $\mathcal{S}(\mathbb{C}^+; \mathcal{U}, \mathcal{Y})$ and let $\mathcal{H}_c = \mathcal{H}(K_c)$ be the associated de Branges–Rovnyak space with reproducing kernel K_c in (1.17). There exists a system node $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c : \left[\begin{smallmatrix} \mathcal{H}_c \\ \mathcal{U} \end{smallmatrix}\right] \supset \operatorname{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c\right) \to \left[\begin{smallmatrix} \mathcal{H}_c \\ \mathcal{Y} \end{smallmatrix}\right]$*

that for arbitrary $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$ *in its domain and* $\lambda \in \mathbb{C}^+$ *satisfies*

$$\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}_c \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} \mu \mapsto -\mu x(\mu) - \varphi(\overline{\mu})^* \gamma_\lambda + \big(1 - \varphi(\overline{\mu})^* \varphi(\overline{\lambda})\big) u \\ \gamma_\lambda + \varphi(\overline{\lambda}) u \end{bmatrix}, \qquad (1.21)$$

$\mu \in \mathbb{C}^+$, *where* $\gamma_\lambda \in \mathcal{Y}$ *is uniquely determined by* λ *and* $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$.

Moreover, the operator $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ *has the following properties:*

1. *The operator* $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ *is a controllable energy-preserving system node.*
2. *The operator* $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ *is a realization of* φ, *i.e., we recover* $\varphi(\mu)$ *through the appropriate generalization of* (1.14) *mentioned earlier.*
3. *If* $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right] : \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \supset \operatorname{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right) \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$ *is another operator matrix with properties 1–2 above (with* $\mathcal{X}$ *in place of* $\mathcal{H}_c$*), then there is a unitary operator* $\Delta : \mathcal{H}_c \to \mathcal{X}$ *so that* $\left[\begin{smallmatrix} \Delta & 0 \\ 0 & 1_{\mathcal{U}} \end{smallmatrix}\right]$ *maps* $\operatorname{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c\right)$ *one-to-one onto* $\operatorname{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right)$ *and*

$$\begin{bmatrix} \Delta & 0 \\ 0 & 1_{\mathcal{Y}} \end{bmatrix} \begin{bmatrix} A\&B \\ C\&D \end{bmatrix}_c = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \begin{bmatrix} \Delta & 0 \\ 0 & 1_{\mathcal{U}} \end{bmatrix}.$$

While the papers [8] and [44] worked with linear-fractional change of variables to derive the continuous-time result from the discrete-time result, a more direct geometric approach based on the "lurking isometry" technique was used in [15]. The approach in the present paper is similar to the single-variable specialization of the work of Ball-Bolotnikov [12] for the discrete-time setting, to some extent using intuition from [29]. The main difference compared to [15] is that the canonical form of the Kolmogorov factorization of the kernel K_c (as given in part 3 of Theorem 1.1) leads to *explicit functional formulas* for the system nodes $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_o$ and $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ above. It should also be pointed out that *conservative* realizations are presented in [15] (and many of the other references below), but in the present paper we study energy-preserving and co-energy-preserving realizations, which are in a certain sense only semi-conservative.

We mention that other work of de Branges–Rovnyak (the first part of [23]) and of de Branges [22] uses reproducing kernel Hilbert spaces consisting of entire functions based on positive kernels associated with Nevanlinna-class rather than Schur-class functions. (The Nevanlinna class consists of holomorphic, even entire, functions mapping the upper half plane into an operator with positive imaginary part.) This leads to models for symmetric operators with equal deficiency indices. See [17, 18] for recent developments in this direction, which is separate from what we pursue here.

Also in [17, 18] a linear-fractional transformation is used to transfer knowledge of Schur functions on $\mathbb{D}$ to Nevanlinna families on $\mathbb{C}\backslash\mathbb{R}$. In the present article we avoid the use of such transformations in the development of the realization theory in order to expose the intricacies of the continuous-time case; only in Sect. 6 we describe how to recover the original de Branges–Rovnyak models from the models we present in Sects. 4 and 5 using a linear-fractional transformation. A functional model (as a self-adjoint linear relation) for arbitrary normalized generalized Nevanlinna pairs has been worked out directly in $\mathbb{C}\backslash\mathbb{R}$ in [34].

A general unifying formulation of the de Branges–Rovnyak models has recently been worked out by Arov-Kurula-Staffans (see [7]) for the continuous-time setting as an extension to continuous time of the earlier discrete-time realization results in [10,11]. It is possible to derive Theorems 1.6 and 1.7 from [7] and the method, outlined in Sect. 7 below, is in principle straightforward. However, filling in the details is a rather lengthy process, and for this reason we have chosen to give direct proofs of Theorems 1.6 and 1.7 here that do not rely on [7].

There have also been a number of extensions of Theorems 1.3 and 1.4 to multivariable settings; see [12] for ball and polydisk versions and [1,13] for polyhalfplane versions.

Notation

$\mathbb{C}^+$:	The complex right-half plane $\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > 0\}$
$(\cdot,\cdot)_{\mathcal{X}}, \|\cdot\|_{\mathcal{X}}$:	The inner product and norm of $\mathcal{X}$, respectively
span Ξ:	The linear span of the set Ξ; a bar on the word span denotes the closed linear span
$\mathcal{B}(\mathcal{U},\mathcal{Y}), \mathcal{B}(\mathcal{U})$:	The space of bounded linear operators from $\mathcal{U}$ to $\mathcal{Y}$ and on $\mathcal{U}$, respectively
dom (A), im(A):	The domain and range of the operator A
ker (A), res (A):	The null-space and the resolvent set of the operator A
$\mathcal{X}_1 \subset \mathcal{X} \subset \mathcal{X}_{-1}$:	Rigged Hilbert spaces associated to $A : \mathcal{X} \supset \operatorname{dom}(A) \to \mathcal{X}$, with norms constructed using some $\beta \in \mathbb{C}^+$
$\mathcal{X}_1^d \subset \mathcal{X} \subset \mathcal{X}_{-1}^d$:	The rigged Hilbert spaces associated to A^*, with norms constructed using $\overline{\beta} \in \mathbb{C}^+$, where β is used in the rigging corresponding to A. $\mathcal{X}_{\pm 1}^d$ is identified with the dual of $\mathcal{X}_{\mp 1}$ using $\mathcal{X}$ as pivot space
$A\|_{\mathcal{X}}$:	The unique extension of the operator $A \in \mathcal{B}(\mathcal{X}_1,\mathcal{X})$ to an operator in $\mathcal{B}(\mathcal{X},\mathcal{X}_{-1})$
$1_{\mathcal{X}}, 1$:	The identity operator on $\mathcal{X}$
$\mathcal{U},\mathcal{Y}$:	Separable Hilbert spaces, the input and output space, respectively
$\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right]$:	The orthogonal direct sum of the Hilbert spaces $\mathcal{X}$ and $\mathcal{U}$
$e(\mu)$:	The (bounded) point-evaluation operator in $H^2(\mathbb{C}^+;\mathcal{U})$ and $H^2(\mathbb{C}^+;\mathcal{Y})$
$e(\lambda)^*$:	The (bounded) adjoint of $e(\lambda)$. Premultiplies an element of $\mathbb{C}$ or a vector space by the (scalar) kernel $k(\mu,\lambda) = \frac{1}{\mu+\overline{\lambda}}$ of $H^2(\mathbb{C})$, so that $e(\lambda)^*u$ is the function $\mu \mapsto \frac{u}{\mu+\overline{\lambda}},\ \mu,\lambda \in \mathbb{C}^+,\ u \in \mathcal{U}$
$\mathcal{S}(\mathbb{C}^+;\mathcal{U},\mathcal{Y})$:	The Schur class on the right-half plane which consists of $\mathcal{B}(\mathcal{U},\mathcal{Y})$-valued holomorphic functions whose values are contractions
M_φ:	The multiplication operator on $H^2(\mathbb{C}^+;\mathcal{U})$ with symbol $\varphi \in \mathcal{S}(\mathbb{C}^+;\mathcal{U},\mathcal{Y})$, i.e., $(M_\varphi f)(\lambda) = \varphi(\lambda)f(\lambda),\ \lambda \in \mathbb{C}^+$
$\left[\begin{smallmatrix}A\&B\\ C\&D\end{smallmatrix}\right]_o$:	The observable co-energy-preserving functional model for $\varphi \in \mathcal{S}(\mathbb{C}^+;\mathcal{U},\mathcal{Y})$
K_o:	The reproducing kernel $K_o(\mu,\lambda) = \frac{1_{\mathcal{Y}}-\varphi(\mu)\varphi(\lambda)^*}{\mu+\overline{\lambda}}$; takes values in $\mathcal{B}(\mathcal{Y})$

$\mathcal{H}_o$:	The de Branges space with reproducing kernel K_o. This is the state space for $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_o$ and it is contractively contained in $H^2(\mathbb{C}^+;\mathcal{Y})$
$e_o(\mu)$:	The point-evaluation operator in $\mathcal{H}_o$
$e_o(\lambda)^*$:	The adjoint of $e_o(\lambda)$, maps $y \in \mathcal{Y}$ into $K_o(\cdot,\lambda)y,\ \lambda \in \mathbb{C}^+$
$\widetilde{\varphi}$:	The function $\widetilde{\varphi}(\mu) = \varphi(\overline{\mu})^*,\ \mu \in \mathbb{C}^+$, which is an element of $\mathcal{S}(\mathbb{C}^+;\mathcal{Y},\mathcal{U})$ if $\varphi \in \mathcal{S}(\mathbb{C}^+;\mathcal{U},\mathcal{Y})$
ι:	Embedding operator
$\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$:	The controllable energy-preserving functional model for $\varphi \in \mathcal{S}(\mathbb{C}^+;\mathcal{U},\mathcal{Y})$
K_c:	The reproducing kernel $K_c(\mu,\lambda) = \frac{1_{\mathcal{U}} - \widetilde{\varphi}(\mu)\widetilde{\varphi}(\lambda)^*}{\mu+\overline{\lambda}}$; takes values in $\mathcal{B}(\mathcal{U})$
$\mathcal{H}_c$:	The de Branges space with reproducing kernel K_c. This is the state space for the $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$, contractively contained in $H^2(\mathbb{C}^+;\mathcal{U})$
$e_c(\mu)$:	The point-evaluation operator in $\mathcal{H}_c$
$e_c(\lambda)^*$:	The adjoint of $e_c(\lambda)$, maps $u \in \mathcal{U}$ into $K_c(\cdot,\lambda)u,\ \lambda \in \mathbb{C}^+$
Δ:	Unitary intertwinement operator from $\mathcal{H}_o$ or $\mathcal{H}_c$ to some Hilbert space $\mathcal{X}$
Ξ_α:	Unitary intertwinement operator from $\mathrm{H}_{o,\alpha}$ to $\mathcal{H}_o$ or from $\mathrm{H}_{c,\overline{\alpha}}$ to $\mathcal{H}_c$

2 The de Branges–Rovnyak Spaces $\mathcal{H}_o$ and $\mathcal{H}_c$ Over $\mathbb{C}^+$

The topic of this section is the development of the state spaces of the functional models presented in the introduction. We begin by proving that the kernels (1.17) are positive kernels, and therefore reproducing kernels of $\mathcal{H}_o$ and $\mathcal{H}_c$. The reader is assumed to be familiar with Hardy spaces over $\mathbb{C}^+$; otherwise see e.g. [21, Sect. A.6]. It is important that $\mathcal{U}$ and $\mathcal{Y}$ are separable.

Every $\varphi \in \mathcal{S}(\mathbb{C}^+;\mathcal{U},\mathcal{Y})$ lies in $H^\infty(\mathbb{C}^+;\mathcal{B}(\mathcal{U},\mathcal{Y}))$ and therefore the multiplication operator M_φ with symbol φ maps $H^2(\mathbb{C}^+;\mathcal{U})$ into $H^2(\mathbb{C}^+;\mathcal{Y})$, and $\|M_\varphi\| = \|\varphi\|_{H^\infty}$; see [21, Theorem A.6.26]. We need the following lemma in order to show that the kernel $K_o(\mu,\lambda)$ is positive:

Lemma 2.1 *Let $\varphi \in \mathcal{S}(\mathbb{C}^+;\mathcal{U},\mathcal{Y})$ and denote the point-evaluation operator in $H^2(\mathbb{C}^+;\mathcal{Y})$ by $e_{H^2(\mathbb{C}^+;\mathcal{Y})}(\cdot)$. The following claims are true:*

1. *The adjoint of $e_{H^2(\mathbb{C}^+;\mathcal{Y})}(\lambda)$ is the operator of premultiplication with the reproducing kernel $k_\mathcal{Y}$ of $H^2(\mathbb{C}^+;\mathcal{Y})$:*

$$e(\lambda)^* y = \mu \mapsto k_\mathcal{Y}(\mu,\lambda)y, \quad y \in \mathcal{Y},\ \mu,\lambda \in \mathbb{C}^+, \quad k_\mathcal{Y}(\mu,\lambda) = \frac{1_\mathcal{Y}}{\mu+\overline{\lambda}}.$$

2. *The operator M_φ^* has the following action on the kernel functions in $H^2(\mathbb{C}^+;\mathcal{Y})$:*

$$M_\varphi^*\, e_{H^2(\mathbb{C}^+;\mathcal{Y})}(\lambda)^* y = e_{H^2(\mathbb{C}^+;\mathcal{U})}(\lambda)^*\, \varphi(\lambda)^* y, \quad \lambda \in \mathbb{C}^+,\ y \in \mathcal{Y}.$$

3. *The function K_o defined in (1.17) can be factored as*

$$K_o(\mu, \lambda) = e_{H^2(\mathbb{C}^+;\mathcal{Y})}(\mu)\,(1_{H^2(\mathbb{C}^+;\mathcal{Y})} - M_\varphi M_\varphi^*)\, e_{H^2(\mathbb{C}^+;\mathcal{Y})}(\lambda)^*, \quad \mu, \lambda \in \mathbb{C}^+. \tag{2.1}$$

In the sequel we simplify the notation, so that $k(\cdot, \lambda)$ denotes a kernel function in $H^2(\mathbb{C}^+; \mathcal{U})$, $H^2(\mathbb{C}^+; \mathcal{Y})$, or $H^2(\mathbb{C}^+; \mathbb{C})$, where it is clear from the context which one to choose. Similarly, the point-evaluation operator at μ on a possibly vector-valued H^2 space is simply denoted by $e(\mu)$.

Proof We have the following short arguments:

1. It follows from residue calculus that k is the reproducing kernel of $H^2(\mathbb{C}^+)$; see [25]. That $e(\lambda)^* y = k(\cdot, \lambda) y$ then follows from the reproducing kernel property (1.5).
2. As probably first observed in [42], by the reproducing kernel property (1.5), we have for all $u \in H^2(\mathbb{C}^+; \mathcal{U})$, $y \in \mathcal{Y}$, and $\lambda \in \mathbb{C}^+$:

$$\begin{aligned}\left(u, M_\varphi^* e(\lambda)^* y\right)_{H^2(\mathbb{C}^+;\mathcal{U})} &= \left(M_\varphi u, e(\lambda)^* y\right)_{H^2(\mathbb{C}^+;\mathcal{Y})} = \left((M_\varphi u)(\lambda), y\right)_{\mathcal{Y}} \\ &= (\varphi(\lambda) u(\lambda), y)_{\mathcal{Y}} = \left(u, e(\lambda)^* \varphi(\lambda)^* y\right)_{\mathcal{U}}.\end{aligned}$$

3. For all $\mu, \lambda \in \mathbb{C}^+$ and $y, \gamma \in \mathcal{Y}$, by using assertion 2 (in the fourth equality) we have:

$$\begin{aligned}(K_o(\mu, \lambda) y, \gamma)_{\mathcal{Y}} &= \left(\frac{1}{\mu + \overline{\lambda}} y, \gamma\right)_{\mathcal{Y}} - \left(\frac{\varphi(\mu)\varphi(\lambda)^*}{\mu + \overline{\lambda}} y, \gamma\right)_{\mathcal{Y}} \\ &= (k(\mu, \lambda) y, \gamma)_{\mathcal{Y}} - \left(k(\mu, \lambda)\varphi(\lambda)^* y, \varphi(\mu)^* \gamma\right)_{\mathcal{U}} \\ &= \left(e(\lambda)^* y, e(\mu)^* \gamma\right)_{H^2(\mathbb{C}^+;\mathcal{Y})} \\ &\quad - \left(e(\lambda)^* \varphi(\lambda)^* y, e(\mu)^* \varphi(\mu)^* \gamma\right)_{H^2(\mathbb{C}^+;\mathcal{U})} \\ &= \left(e(\lambda)^* y, e(\mu)^* \gamma\right)_{H^2(\mathbb{C}^+;\mathcal{Y})} \\ &\quad - \left(M_\varphi^* e(\lambda)^* y, M_\varphi^* e(\mu)^* \gamma\right)_{H^2(\mathbb{C}^+;\mathcal{U})} \\ &= \left((1 - M_\varphi M_\varphi^*) e(\lambda)^* y, e(\mu)^* \gamma\right)_{H^2(\mathbb{C}^+;\mathcal{Y})} \\ &= \left(e(\mu)(1 - M_\varphi M_\varphi^*) e(\lambda)^* y, \gamma\right)_{\mathcal{Y}},\end{aligned} \tag{2.2}$$

and this completes the proof. □

Using this lemma it is easy to show that K_o is a positive kernel.

Theorem 2.2 *If $\varphi \in \mathcal{S}(\mathbb{C}^+; \mathcal{U}, \mathcal{Y})$, then the function $K_o(\mu, \lambda)$ defined in (1.17) is a positive kernel.*

Proof For $\varphi \in \mathcal{S}(\mathbb{C}^+; \mathcal{U}, \mathcal{Y})$, the multiplication operator $M_\varphi : H^2(\mathbb{C}^+; \mathcal{U}) \to H^2(\mathbb{C}^+; \mathcal{Y})$ is contractive, $\|M_\varphi\| \le 1$, since $\|\varphi\|_{H^\infty(\mathbb{C}^+)} \le 1$. Hence $1 - M_\varphi M_\varphi^* \ge 0$

as an operator on $H^2(\mathbb{C}^+;\mathcal{Y})$ and thus it has a bounded positive square root $(1 - M_\varphi M_\varphi^*)^{1/2}$ on $H^2(\mathbb{C}^+;\mathcal{Y})$. From the identity (2.1) we see that $K_o(\mu, \lambda)$ has a Kolmogorov decomposition (1.6) with

$$H(\mu) = e(\mu)(1 - M_\varphi M_\varphi^*)^{1/2} : \mathcal{H}(K_o) \to \mathcal{Y}.$$

We conclude from Theorem 1.1 that K_o is a positive kernel. □

We denote the Hilbert space with reproducing kernel K_o by $\mathcal{H}_o := \mathcal{H}(K_o)$. Replacing φ by $\widetilde{\varphi}(\mu) := \varphi(\overline{\mu})^*$, $\mu \in \mathbb{C}^+$, and swapping the roles of $\mathcal{U}$ and $\mathcal{Y}$, we turn the kernel K_o into the kernel K_c in (1.17). Applying Lemma 2.1 and Theorem 2.2 to $\widetilde{\varphi}$, we obtain the following result:

Corollary 2.3 *If $\varphi \in \mathcal{S}(\mathbb{C}^+;\mathcal{U},\mathcal{Y})$ then the $\mathcal{B}(\mathcal{U})$-valued function $K_c(\mu, \lambda)$ is a positive kernel on $\mathbb{C}^+ \times \mathbb{C}^+$. Denoting $\mathcal{H}_c := \mathcal{H}(K_c)$, we have that the kernel functions of $\mathcal{H}_c$ and $H^2(\mathbb{C}^+;\mathcal{U})$ are related by $K_c(\cdot, \lambda)u = (1 - M_{\widetilde{\varphi}} M_{\widetilde{\varphi}}^*)\, k(\cdot, \lambda)u$ for all $\lambda \in \mathbb{C}^+$ and $u \in \mathcal{U}$.*

An equivalent way of defining $\mathcal{H}_o$ is to set

$$\mathcal{H}_o := \left\{ f : \mathbb{C}^+ \underset{\text{holomorphic}}{\to} \mathcal{Y} \big| \|f\|_{\mathcal{H}_o} < \infty \right\},$$

and to define the norm in $\mathcal{H}_o$ by

$$\|f\|_{\mathcal{H}_o}^2 := \sup \left\{ \|f + M_\varphi g\|_{H^2(\mathbb{C}^+;\mathcal{Y})}^2 - \|g\|_{H^2(\mathbb{C}^+;\mathcal{U})}^2 \big| g \in H^2(\mathbb{C}^+;\mathcal{U}) \right\}.$$

It can be shown that this norm equals the norm induced by the reproducing kernel K_o. This corresponds to the original definition of H_o by de Branges and Rovnyak. To give the uninitiated reader better perspective on de Branges–Rovnyak spaces, we further mention the following well-known operator-range characterization of $\mathcal{H}_o$ and $\mathcal{H}_c$. For further development of this point of view in the unit disk setting see e.g. [41].

Theorem 2.4 *Let φ be a function in the Schur class $\mathcal{S}(\mathbb{C}^+;\mathcal{U},\mathcal{Y})$. Then:*

1. *The space $\mathcal{H}_o$ can be identified as a set with the operator range*

$$\mathcal{H}_o = \operatorname{im}\big((1 - M_\varphi M_\varphi^*)^{1/2}\big) \subset H^2(\mathbb{C}^+;\mathcal{Y}) \tag{2.3}$$

with norm given by

$$\big\|(1 - M_\varphi M_\varphi^*)^{1/2} g\big\|_{\mathcal{H}_o} = \|Qg\|_{H^2(\mathbb{C}^+;\mathcal{Y})}, \quad g \in H^2(\mathbb{C}^+;\mathcal{Y}), \tag{2.4}$$

where Q is the orthogonal projection of $H^2(\mathbb{C}^+;\mathcal{Y})$ onto $\big(\ker\big(1 - M_\varphi M_\varphi^*\big)\big)^\perp$.

2. *The inclusion map*

$$\iota : f \in \mathcal{H}_o \mapsto f \in H^2(\mathbb{C}^+; , \mathcal{Y})$$

is contractive, i.e.,

$$\|f\|_{H^2(\mathbb{C}^+;\mathcal{Y})} \le \|f\|_{\mathcal{H}_o} \text{ for all } f \in \mathcal{H}_o,$$

with adjoint $\iota^* : H^2(\mathbb{C}^+;\mathcal{Y}) \to \mathcal{H}_o$ *given by*

$$\iota^* = 1 - M_\varphi M_\varphi^*.$$

Analogous results with $\mathcal{H}_c$ *in place of* $\mathcal{H}_o$ *are obtained by replacing* φ *by* $\widetilde{\varphi}$*.*

Proof The result is well known among experts but we provide a proof for the sake of completeness. The first step is to prove Assertion 1.

Define the space $\widetilde{\mathcal{H}}_o$ by

$$\widetilde{\mathcal{H}}_o := \operatorname{im}\big(1 - M_\varphi M_\varphi^*)^{1/2}\big) \subset H^2(\mathbb{C}^+;\mathcal{Y})$$

with norm given by (2.4) and let $f \in \widetilde{\mathcal{H}}_o$. Set $W = 1 - M_\varphi M_\varphi^*$ on $H^2(\mathbb{C}^+;\mathcal{Y})$, so that $\widetilde{\mathcal{H}}_o = \operatorname{im}\big(W^{1/2}\big)$. From (2.1) we see that $e_o(\lambda)^* = We(\lambda)^*$, so in particular $e_o(\lambda)^* y \in \widetilde{\mathcal{H}}_o$ for each $\lambda \in \mathbb{C}^+$ and $y \in \mathcal{Y}$. Furthermore, for $f = W^{1/2} g \in \widetilde{\mathcal{H}}_o$, we compute using (2.4):

$$\begin{aligned}\langle f, e_o(\lambda)^* y\rangle_{\widetilde{\mathcal{H}}_o} &= \langle W^{1/2} g, We(\lambda)^* y\rangle_{\widetilde{\mathcal{H}}_o} = \langle Qg, QW^{1/2} e(\lambda)^* y\rangle_{H^2(\mathbb{C}^+;\mathcal{Y})}\\ &= \langle W^{1/2} g, e(\lambda)^* y\rangle_{H^2(\mathbb{C}^+;\mathcal{Y})} = \langle f(\lambda), y\rangle_{\mathcal{Y}}.\end{aligned}$$

This shows that $e_o(\lambda)^* = K_o(\cdot,\lambda)$ works as the reproducing kernel for the space $\widetilde{\mathcal{H}}_o$, and since the positive kernel $e_o(\lambda)^*$ determines its reproducing kernel Hilbert space uniquely, we conclude that $\mathcal{H}_o = \widetilde{\mathcal{H}}_o$.

Contractive containment of $\mathcal{H}_o$ in $H^2(\mathbb{C}^+;\mathcal{Y})$ follows from the following observation:

$$\|f\|_{\mathcal{H}_0} = \|g\|_{H^2(\mathbb{C}^+;\mathcal{Y})} \ge \|(1 - M_\varphi M_\varphi^*)^{1/2} g\|_{H^2(\mathbb{C}^+;\mathcal{Y})} = \|f\|_{H^2(\mathbb{C}^+;\mathcal{Y})},$$

where we used that $1 - M_\varphi M_\varphi^*$ is contractive on $H^2(\mathbb{C}^+;\mathcal{Y})$.

Since $e_o(\lambda)$ is the restriction of $e_{H^2(\mathbb{C}^+;\mathcal{Y})}$ to $\mathcal{H}_o$, the identity (2.1) amounts to the operator identity

$$e_o(\lambda)^* = (1 - M_\varphi M_\varphi^*) e(\lambda)^*, \quad \lambda \in \mathbb{C}^+. \tag{2.5}$$

Using (2.5), we obtain that $\iota^* e(\lambda)^* y = (1 - M_\varphi M_\varphi^*)\, e(\lambda)^* y$ for all $\lambda \in \mathbb{C}^+$ and $y \in \mathcal{Y}$. Indeed, it holds for all $x \in \mathcal{H}_o$ that

$$\big((1 - M_\varphi M_\varphi^*)\, e(\lambda)^* y, x\big)_{\mathcal{H}_o} = \langle y, x(\lambda)\rangle_{\mathcal{Y}} = \big(e(\lambda)^* y, \iota_o x\big)_{H^2(\mathbb{C}^+;\mathcal{Y})},$$

and taking limits of finite linear combinations of $e(\lambda_k)^* y_k$, we obtain that $\iota^* = 1 - M_\varphi M_\varphi^*$. □

Recall that $\varphi \in \mathcal{S}(\mathbb{C}^+; \mathcal{U}, \mathcal{Y})$ is called *inner* if φ has isometric boundary values a.e. on the imaginary line.

Corollary 2.5 *If φ is inner, then M_φ is isometric from $H^2(\mathbb{C}^+; \mathcal{U})$ into $H^2(\mathbb{C}^+; \mathcal{Y})$, and $(1 - M_\varphi M_\varphi^*)^{1/2} = 1 - M_\varphi M_\varphi^*$. The operator $1 - M_\varphi M_\varphi^*$ is the orthogonal projection of $H^2(\mathbb{C}^+; \mathcal{Y})$ onto $H^2(\mathbb{C}^+; \mathcal{Y}) \ominus \big(M_\varphi H^2(\mathbb{C}^+; \mathcal{U})\big)$ and this orthogonal complement equals $\mathcal{H}_o$ isometrically.*

Proof That M_φ is isometric follows from

$$\begin{aligned}\big(M_\varphi f, M_\varphi f\big)_{H^2(\mathbb{C}^+;\mathcal{Y})} &= \frac{1}{2\pi} \int\limits_{\mathbb{R}} (\varphi(i\omega) f(i\omega), \varphi(i\omega) f(i\omega))_{\mathcal{Y}} \,\mathrm{d}\omega \\ &= (f, f)_{H^2(\mathbb{C}^+;\mathcal{U})}.\end{aligned}$$

From the isometricity of M_φ it follows that $(1 - M_\varphi M_\varphi^*)^2 = 1 - M_\varphi M_\varphi^* \geq 0$, so that $(1 - M_\varphi M_\varphi^*)^{1/2} = 1 - M_\varphi M_\varphi^*$. This is the orthogonal projection onto $\big(M_\varphi H^2(\mathbb{C}^+; \mathcal{U})\big)^\perp$, since $M_\varphi M_\varphi^*$ is the orthogonal projection onto $M_\varphi H^2(\mathbb{C}^+; \mathcal{U})$. By (2.3), $\mathcal{H}_o = \mathrm{im}\big(1 - M_\varphi M_\varphi^*\big) = \ker\big(1 - M_\varphi M_\varphi^*\big)^\perp$ and hence Q in (2.4) coincides with $1 - M_\varphi M_\varphi^*$. Then (2.4) precisely says that $\mathcal{H}_o$ is isometrically contained in $H^2(\mathbb{C}^+; \mathcal{Y})$. □

When φ is not inner, $M_\varphi H^2(\mathbb{C}^+; \mathcal{U})$ and $\mathcal{H}_o$ are not orthogonal in $H^2(\mathbb{C}^+; \mathcal{Y})$, but more general *complements in the sense of de Branges*, cf. [5] or [2, §1.5].

The following limits will be encountered frequently in the sequel.

Proposition 2.6 *Every x in $H^2(\mathbb{C}^+; \mathcal{Y})$ satisfies $x(\mu) \to 0$ in $\mathcal{Y}$ as* $\mathrm{Re}\,\mu \to +\infty$. *More precisely,*

$$\|x(\mu)\|_{\mathcal{Y}} \leq \frac{\|x\|_{H^2(\mathbb{C}^+;\mathcal{Y})}}{\sqrt{2\mathrm{Re}\,\mu}}, \quad \mu \in \mathbb{C}^+. \tag{2.6}$$

It also holds that

$$\|x(\mu)\|_{\mathcal{Y}} \leq \frac{\|x\|_{\mathcal{H}_o}}{\sqrt{2\mathrm{Re}\,\mu}}, \quad \mu \in \mathbb{C}^+, \tag{2.7}$$

and in particular the only constant function in $\mathcal{H}_o$ is the zero function. The corresponding claims hold for $H^2(\mathbb{C}^+; \mathcal{U})$ and $\mathcal{H}_c$.

Proof We verify the assertion only for $H^2(\mathbb{C}^+; \mathcal{Y})$ and $\mathcal{H}_o$. By the Cauchy-Schwarz inequality, we have for all $x \in H^2(\mathbb{C}^+; \mathcal{Y})$ that

$$\begin{aligned}|\,(x(\mu), y)_{\mathcal{Y}}\,| &= \big|\,\big(x, e(\mu)^* y\big)_{H^2(\mathbb{C}^+;\mathcal{Y})}\,\big| \\ &\leq \|x\|_{H^2(\mathbb{C}^+;\mathcal{Y})} \|e(\mu)^* y\|_{H^2(\mathbb{C}^+;\mathcal{Y})}\end{aligned}$$

$$
\begin{aligned}
&= \|x\|_{H^2(\mathbb{C}^+;\mathcal{Y})} \left(e(\mu)^* y, e(\mu)^* y\right)^{1/2}_{H^2(\mathbb{C}^+;\mathcal{Y})} \\
&= \|x\|_{H^2(\mathbb{C}^+;\mathcal{Y})} \left(\frac{1}{\mu+\overline{\mu}} y, y\right)^{1/2}_{\mathcal{Y}} \\
&\le \frac{\|x\|_{H^2(\mathbb{C}^+;\mathcal{Y})} \|y\|_{\mathcal{Y}}}{\sqrt{2\mathrm{Re}\,\mu}}, \quad \mu \in \mathbb{C}^+.
\end{aligned}
$$

From here we obtain (2.6):

$$
\|x(\mu)\|_{\mathcal{Y}} = \sup_{0\neq y\in\mathcal{Y}} \frac{|\,(x(\mu), y)_{\mathcal{Y}}\,|}{\|y\|} \le \frac{\|x\|_{H^2(\mathbb{C}^+;\mathcal{Y})}}{\sqrt{2\mathrm{Re}\,\mu}}, \quad \mu \in \mathbb{C}^+.
$$

Now (2.7) follows from (2.6) combined with the facts $\mathcal{H}_o \subset H^2(\mathbb{C}^+;\mathcal{Y})$ and $\|x\|_{H^2(\mathbb{C}^+;\mathcal{Y})} \le \|x\|_{\mathcal{H}_o}$ for all $x \in \mathcal{H}_o$; see Theorem 2.4. □

3 Background on System Nodes

In this section we recall the needed concepts from the theory of infinite-dimensional linear systems in continuous time. A comprehensive exposition of this theory can be found e.g. in [45] and coordinate-free versions of some of the results are in [29]. For more details on the following few paragraphs, see Definition 3.2.7 and Section 3.6 of [45].

3.1 Definition of a System Node and its Transfer Function

The *resolvent set* res (A) of a closed operator A on the Hilbert space $\mathcal{X}$ is the set of all $\mu \in \mathbb{C}$ such that $\mu - A$ maps dom (A) one-to-one onto $\mathcal{X}$. The generator A of a C_0 semigroup is closed and dom (A) dense in $\mathcal{X}$; see e.g. [37, Theorem 1.2.7]. Moreover, the resolvent set of a C_0 semigroup generator contains some complex right-half plane. For such a generator, dom (A) is a Hilbert space with the inner product

$$
(x, z)_{\mathrm{dom}(A)} = ((\beta - A)x, (\beta - A)z)_{\mathcal{X}}, \tag{3.1}
$$

where β is some fixed but arbitrary complex number in res (A).

Thus $\mathcal{X}_1 :=$ dom (A) with the norm $\|x\|_1 := \|(\beta - A)x\|_{\mathcal{X}}$ is a dense subspace of $\mathcal{X}$. It follows immediately from (3.1) that A maps dom $(A) = \mathcal{X}_1$ with this norm continuously into $\mathcal{X}$. Denote by $\mathcal{X}_{-1}$ the completion of $\mathcal{X}$ with respect to the norm $\|x\|_{-1} = \|(\beta - A)^{-1}x\|_{\mathcal{X}}$. The operator A can then also be considered as a continuous operator which maps the dense subspace $\mathcal{X}_1$ of $\mathcal{X}$ into $\mathcal{X}_{-1}$, and we denote the unique continuous extension of A to an operator $\mathcal{X} \to \mathcal{X}_{-1}$ by $A|_{\mathcal{X}}$. Note that res $(A) =$ res $(A|_{\mathcal{X}})$ and that $(\beta - A|_{\mathcal{X}})^{-1}$ maps $\mathcal{X}_{-1}$ unitarily onto $\mathcal{X}$.

The triple $\mathcal{X}_1 \subset \mathcal{X} \subset \mathcal{X}_{-1}$ is called a *Gelfand triple*, and the three spaces are also said to be *rigged*. The spaces $\mathcal{X}_{-1}$ corresponding to two different choices of $\beta \in$ res (A) can be identified with each other as topological vector spaces, and although the norms will be different they are equivalent to each other. The norms of $\mathcal{X}_1$ corresponding to two different choices of $\beta \in \operatorname{res}(A)$ will also be equivalent. Hence $(\alpha - A)^{-1}$ is an isomorphism from $\mathcal{X}$ to $\mathcal{X}_1$ and $(\alpha - A|_{\mathcal{X}})$ is an isomorphism from $\mathcal{X}$ to $\mathcal{X}_{-1}$ for all $\alpha \in \operatorname{res}(A)$, and these operators are unitary for $\alpha = \beta$.

Definition 3.1 A linear operator

$$\begin{bmatrix} A\&B \\ C\&D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \supset \operatorname{dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}\right) \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$$

(which is in general unbounded) is called a *system node* on the triple $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$ of Hilbert spaces if it has all of the following properties:

1. The operator $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is closed.
2. The operator

$$\begin{aligned} Ax &:= \begin{bmatrix} A\&B \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} \quad \text{defined on} \\ \operatorname{dom}(A) &:= \left\{ x \in \mathcal{X} \,\middle|\, \begin{bmatrix} x \\ 0 \end{bmatrix} \in \operatorname{dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}\right) \right\}, \end{aligned} \tag{3.2}$$

   is the generator of a C_0-semigroup on $\mathcal{X}$.
3. The operator $\begin{bmatrix} A\&B \end{bmatrix}$ can be extended to an operator $\begin{bmatrix} A|_{\mathcal{X}} & B \end{bmatrix}$ that maps $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$ continuously into $\mathcal{X}_{-1}$.
4. The domain of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ satisfies the condition

$$\operatorname{dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}\right) = \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \,\middle|\, A|_{\mathcal{X}} x + Bu \in \mathcal{X} \right\}.$$

When these conditions are satisfied, $\mathcal{U}$, $\mathcal{X}$, and $\mathcal{Y}$ are called the *input space*, *state space*, and *output space*, respectively, of the system node.

It was mentioned in the introduction that the definition of the operator-valued function $\mu \mapsto C(\mu - A)^{-1}B + D$ can be extended to arbitrary system nodes. This is often done as follows. By [45, Lemma 4.7.3], $\left[\begin{smallmatrix} 1 & (\alpha - A|_{\mathcal{X}})^{-1}B \\ 0 & 1 \end{smallmatrix}\right]$ maps $\left[\begin{smallmatrix} \operatorname{dom}(A) \\ \mathcal{U} \end{smallmatrix}\right]$ one-to-one onto $\operatorname{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right)$ for every system node $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ and $\alpha \in \operatorname{res}(A)$, and this allows us to express the domain of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ as

$$\operatorname{dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}\right) = \begin{bmatrix} \operatorname{dom}(A) \\ \{0\} \end{bmatrix} \dotplus \begin{bmatrix} (\alpha - A|_{\mathcal{X}})^{-1}B \\ 1 \end{bmatrix} \mathcal{U} \tag{3.3}$$

Note in particular that $\left[\begin{smallmatrix} (\alpha - A|_{\mathcal{X}})^{-1}B \\ 1 \end{smallmatrix}\right]$ maps $\mathcal{U}$ into the domain of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$. The following is [45, Definition 4.7.4]:

Definition 3.2 The operators A and B in Definition 3.1 are the *main operator* and *control operator* of the system node $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$, respectively. The *observation operator* $C : \operatorname{dom}(A) \to \mathcal{Y}$ of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is the operator

$$Cx := \begin{bmatrix} C\&D \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix}, \quad x \in \operatorname{dom}(A), \tag{3.4}$$

and the *transfer function* $\widehat{\mathfrak{D}} : \operatorname{res}(A) \to \mathcal{B}(\mathcal{U}, \mathcal{Y})$ of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is the operator-valued holomorphic function

$$\widehat{\mathfrak{D}}(\mu) := \begin{bmatrix} C\&D \end{bmatrix} \begin{bmatrix} (\mu - A|_{\mathcal{X}})^{-1}B \\ 1 \end{bmatrix}, \quad \mu \in \operatorname{res}(A). \tag{3.5}$$

By a *realization* of a given analytic function φ, we mean a system node $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ whose transfer function $\widehat{\mathfrak{D}}$ coincides with φ on some right-half plane

$$\mathbb{C}^+_\omega := \{\mu \in \mathbb{C} | \operatorname{Re}\mu > \omega\} \subset \operatorname{res}(A) \cap \operatorname{dom}(\varphi), \quad \omega \in \mathbb{R}.$$

Regarding the last sentence of Definition 3.2, we consider two analytic functions f and g with $\operatorname{dom}(f), \operatorname{dom}(g) \subset \mathbb{C}$ to be *identical* if there exists some complex right-half plane $\mathbb{C}^+_\omega \subset \operatorname{dom}(f) \cap \operatorname{dom}(g)$, such that f and g coincide on $\mathbb{C}^+_\omega$. In this paper we can usually take $\omega = 0$, so that $\mathbb{C}^+_\omega = \mathbb{C}^+$.

Since $(\alpha - A)^{-1}$ maps $\mathcal{X}$ one-to-one onto $\operatorname{dom}(A)$, we have that the operator $\left[\begin{smallmatrix} (\alpha-A)^{-1} & (\alpha-A|_{\mathcal{X}})^{-1}B \\ 0 & 1 \end{smallmatrix}\right]$ maps $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$ one-to-one onto $\operatorname{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right)$ for every $\alpha \in \operatorname{res}(A)$, cf. (3.3). The system node satisfies

$$\begin{aligned} &\begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \begin{bmatrix} (\alpha - A)^{-1}x & (\alpha - A|_{\mathcal{X}})^{-1}Bu \\ 0 & u \end{bmatrix} \\ &\quad = \begin{bmatrix} A(\alpha - A)^{-1}x & \alpha(\alpha - A|_{\mathcal{X}})^{-1}Bu \\ C(\alpha - A)^{-1}x & \widehat{\mathfrak{D}}(\alpha)u \end{bmatrix} \end{aligned} \tag{3.6}$$

for all $\alpha \in \operatorname{res}(A)$ and $x \in \mathcal{X}$, $u \in \mathcal{U}$. By the closed graph theorem, $C(\alpha - A)^{-1}$ is bounded from $\mathcal{X}$ into $\mathcal{Y}$, and therefore C maps $\operatorname{dom}(A)$ boundedly into $\mathcal{Y}$. Similarly, $\widehat{\mathfrak{D}}(\alpha)$ is bounded from $\mathcal{U}$ into $\mathcal{Y}$ for all $\alpha \in \operatorname{res}(A)$. It is part of condition 3 in Definition 3.1 that B maps $\mathcal{U}$ boundedly into $\mathcal{X}_{-1}$.

The Eq. (3.6) can equivalently be written, still for arbitrary $\alpha \in \operatorname{res}(A)$:

$$\begin{aligned} \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} &= \begin{bmatrix} A(\alpha - A)^{-1} & \alpha(\alpha - A|_{\mathcal{X}})^{-1}B \\ C(\alpha - A)^{-1} & \widehat{\mathfrak{D}}(\alpha) \end{bmatrix} \\ &\quad \times \begin{bmatrix} (\alpha - A)^{-1} & (\alpha - A|_{\mathcal{X}})^{-1}B \\ 0 & 1 \end{bmatrix}^{-1} \Bigg|_{\operatorname{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right)} \\ &= \begin{bmatrix} A & \alpha(\alpha - A|_{\mathcal{X}})^{-1}B \\ C & \widehat{\mathfrak{D}}(\alpha) \end{bmatrix} \\ &\quad \times \begin{bmatrix} 1 & -(\alpha - A|_{\mathcal{X}})^{-1}B \\ 0 & 1 \end{bmatrix} \Bigg|_{\operatorname{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right)}, \end{aligned} \tag{3.7}$$

where $\operatorname{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right)$ is given in (3.3). In particular,

$$\begin{aligned} \left[C\&D\right]\begin{bmatrix} x \\ u \end{bmatrix} &= C\big(x - (\alpha - A|_{\mathcal{X}})^{-1}Bu\big) + \widehat{\mathfrak{D}}(\alpha)u, \\ \begin{bmatrix} x \\ u \end{bmatrix} &\in \operatorname{dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}\right), \end{aligned} \tag{3.8}$$

for an arbitrary $\alpha \in \operatorname{res}(A)$.

Remark 3.3 By [45, Lem. 4.7.6], we can reconstruct a system node $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ from its operators $A,\ B,\ C$, and $\widehat{\mathfrak{D}}(\alpha)$, for one arbitrary $\alpha \in \operatorname{res}(A)$, in the following way: The space $\mathcal{X}_{-1}$ is obtained as the co-domain of B, and we can then extend $A : \operatorname{dom}(A) \to \mathcal{X}$ continuously into $A|_{\mathcal{X}} : \mathcal{X} \to \mathcal{X}_{-1}$. Then we define $A\&B$ via:

$$\begin{aligned} \operatorname{dom}\left(\left[A\&B\right]\right) &:= \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \middle| A|_{\mathcal{X}}x + Bu \in \mathcal{X} \right\}, \\ \left[A\&B\right] &:= \left[A|_{\mathcal{X}}\ \ B\right]\Big|_{\operatorname{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right)}, \end{aligned}$$

and finally we define $\left[C\&D\right]$ on $\operatorname{dom}\left(\left[A\&B\right]\right) = \operatorname{dom}\left(\left[C\&D\right]\right)$ by (3.8).

3.2 Controllability and Observability

We will use the following variants of controllability and observability:

Definition 3.4 Let $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be a system node and denote the component of $\operatorname{res}(A)$ that contains some right-half plane by $\rho_\infty(A)$.

We say that $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is *controllable* if

$$\operatorname{span}\left\{(\mu - A|_{\mathcal{X}})^{-1}Bu \middle| \mu \in \rho_\infty(A),\ u \in \mathcal{U}\right\}$$

is dense in the state space $\mathcal{X}$. The system node $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is *observable* if

$$\bigcap_{\mu \in \rho_\infty(A)} \ker\big(C(\mu - A)^{-1}\big) = \{0\}\,.$$

As a consequence of [45, Cor. 9.6.2 and 9.6.5], it suffices to take the linear span or intersection only over a subset $\Omega \subset \rho_\infty(A)$ with a cluster point in $\rho_\infty(A)$ instead of over the whole set $\rho_\infty(A)$; we obtain the following:

Lemma 3.5 *Let* $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ *be a controllable system node on* $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$ *and fix* $\alpha \in \operatorname{res}(A)$ *arbitrarily. Assume that* $\Omega \subset \rho_\infty(A)$ *has a cluster point in* $\rho_\infty(A)$*. Then the linear span*

$$\operatorname{span}\left\{(\mu - A|_{\mathcal{X}})^{-1}Bu - (\alpha - A|_{\mathcal{X}})^{-1}Bu \middle| \mu \in \Omega,\ u \in \mathcal{U}\right\} \tag{3.9}$$

is a dense subspace of both $\operatorname{dom}(A)$ *(with respect to the graph norm of* A*) and of* $\mathcal{X}$*, and the linear span*

$$\operatorname{span}\left\{\begin{bmatrix}(\mu - A|_{\mathcal{X}})^{-1}Bu \\ u\end{bmatrix} \middle| \mu \in \Omega,\ u \in \mathcal{U}\right\} \tag{3.10}$$

is a dense subspace of dom $\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right)$ *with respect to the graph norm of* $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$.

Proof Let $\mathcal{E}$ denote the linear span in (3.9). For $\mu \in \Omega$ and $u \in \mathcal{U}$, the resolvent identity gives

$$\begin{aligned}&\left((\mu - A|_{\mathcal{X}})^{-1}Bu - (\alpha - A|_{\mathcal{X}})^{-1}\right)Bu \\ &\quad = (\alpha - \mu)(\alpha - A)^{-1}(\mu - A|_{\mathcal{X}})^{-1}Bu \in \operatorname{dom}(A),\end{aligned}$$

Here $(\alpha - A)^{-1}$ is an isomorphism from $\mathcal{X}$ to dom (A), and so $\mathcal{E}$ is dense in dom (A) if and only if

$$\operatorname{span}\left\{(\alpha - \mu)(\mu - A|_{\mathcal{X}})^{-1}Bu | \mu \in \Omega,\ u \in \mathcal{U}\right\} \tag{3.11}$$

is dense in $\mathcal{X}$. It is easy to see that this linear span is the same as

$$\operatorname{span}\left\{(\mu - A|_{\mathcal{X}})^{-1}Bu | \mu \in \Omega \backslash \{\alpha\},\ u \in \mathcal{U}\right\},$$

and this space is dense in $\mathcal{X}$, since $\Omega\backslash\{\alpha\}$ has a cluster point in $\rho_\infty(A)$ and $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is assumed controllable. We have proved that (3.9) is dense in dom (A). Since dom (A) is dense in $\mathcal{X}$, it now follows automatically that (3.9) is dense in $\mathcal{X}$.

According to [45, Lemma 4.7.3(ix)], the following norm is equivalent to the norm on dom $\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right)$ induced by the graph of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$:

$$\left\|\begin{bmatrix} x \\ u \end{bmatrix}\right\|_\alpha := \left\|\begin{bmatrix} 1 & -(\alpha - A|_{\mathcal{X}})^{-1}B \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix}\right\|_{\left[\begin{smallmatrix}\operatorname{dom}(A) \\ \mathcal{U}\end{smallmatrix}\right]},$$

where dom (A) is equipped with the graph norm of A. Therefore the denseness of (3.10) follows if we can show that

$$\begin{bmatrix} 1 & -(\alpha - A|_{\mathcal{X}})^{-1}B \\ 0 & 1 \end{bmatrix} \operatorname{span}\left\{\begin{bmatrix}(\mu - A|_{\mathcal{X}})^{-1}Bu \\ u\end{bmatrix} \middle| \mu \in \Omega,\ u \in \mathcal{U}\right\} \tag{3.12}$$

is dense in $\left[\begin{smallmatrix}\operatorname{dom}(A) \\ \mathcal{U}\end{smallmatrix}\right]$.

Fix $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \left[\begin{smallmatrix}\operatorname{dom}(A) \\ \mathcal{U}\end{smallmatrix}\right]$ arbitrarily. We will show that $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$ can be approximated arbitrarily well by an element of the linear span in (3.12), in the norm of $\left[\begin{smallmatrix}\operatorname{dom}(A) \\ \mathcal{U}\end{smallmatrix}\right]$. By the above, we can approximate x by an element in $\mathcal{E}$, say

$$\|x - x_N\|_{\operatorname{dom}(A)} < \varepsilon, \quad \text{with} \quad x_N = \sum_{k=1}^{N}(\mu_k - A|_{\mathcal{X}})^{-1}Bu_k - (\alpha - A|_{\mathcal{X}})^{-1}Bu_k.$$

Setting $v_N := u - \sum_{k=1}^N u_k$, we obtain

$$\begin{aligned}
&\left\| \begin{bmatrix} x \\ u \end{bmatrix} - \begin{bmatrix} 1 & -(\alpha - A|_{\mathcal{X}})^{-1}B \\ 0 & 1 \end{bmatrix} \sum_{k=1}^{N} \begin{bmatrix} (\mu_k - A|_{\mathcal{X}})^{-1}Bu_k \\ u_k \end{bmatrix} \right. \\
&\qquad \left. - \begin{bmatrix} 1 & -(\alpha - A|_{\mathcal{X}})^{-1}B \\ 0 & 1 \end{bmatrix} \begin{bmatrix} (\alpha - A|_{\mathcal{X}})^{-1}Bv_N \\ v_N \end{bmatrix} \right\|_{\left[\begin{smallmatrix} \mathrm{dom}(A) \\ \mathcal{U} \end{smallmatrix}\right]} \\
&\quad = \left\| \begin{bmatrix} x - x_N \\ 0 \end{bmatrix} \right\|_{\left[\begin{smallmatrix} \mathrm{dom}(A) \\ \mathcal{U} \end{smallmatrix}\right]} < \varepsilon,
\end{aligned}$$

and hence the linear span in (3.10) is dense in dom $\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right)$. □

We next recall some properties of (scattering) passive systems, including some very recent developments.

3.3 Scattering Dissipative Operators and Passive System Nodes

The following is a recent idea from [46, Def. 2.1]; see also [47,50]:

Definition 3.6 An operator $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right] : \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \supset \mathrm{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right) \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$ is called *scattering dissipative* if it satisfies for all $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \mathrm{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right)$:

$$(z, x)_{\mathcal{X}} + (x, z)_{\mathcal{X}} \le (u, u)_{\mathcal{U}} - (y, y)_{\mathcal{Y}}, \quad \begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}. \tag{3.13}$$

If such an operator $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ has no proper extension which still satisfies (3.13), then $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is said to be *maximal scattering dissipative*. If (3.13) holds with equality then $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is called *scattering isometric*.

Note that $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is scattering isometric if and only if for all $\left[\begin{smallmatrix} x_1 \\ u_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} x_2 \\ u_2 \end{smallmatrix}\right] \in \mathrm{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right)$:

$$\begin{gathered}
(z_1, x_2)_{\mathcal{X}} + (x_1, z_2)_{\mathcal{X}} = (u_1, u_2)_{\mathcal{U}} - (y_1, y_2)_{\mathcal{Y}}, \\
\begin{bmatrix} z_k \\ y_k \end{bmatrix} = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix},
\end{gathered} \tag{3.14}$$

as can be seen by polarizing (3.13), i.e., by considering $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] = \left[\begin{smallmatrix} x_1 \\ u_1 \end{smallmatrix}\right] + \lambda \left[\begin{smallmatrix} x_2 \\ u_2 \end{smallmatrix}\right]$ and letting λ vary over $\mathbb{C}$.

The following definition differs from the standard definition of a passive system node, but combining the fact that res (A) contains some right-half plane with [45, Theorem 11.1.5], see in particular assertion (iii), we obtain that the two definitions are equivalent:

Definition 3.7 A system node is said to be *passive* if it is a scattering dissipative operator. The system node is *energy preserving* if it is scattering isometric.

The type of passivity in Definition 3.7 is commonly called *scattering passivity*, where the word "scattering" refers to the fact that we use the expression $\|u(t)\|^2 - \|y(t)\|^2$ to measure the power absorbed by the system from its surroundings at time $t \geq 0$. See the introduction to [44] for more details on this.

Lemma 3.8 *Let* $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ *be a scattering dissipative operator mapping its domain* $\operatorname{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right) \subset \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$ *into* $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$. *Then* $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ *is a system node if and only if it is closed and* $\left[\begin{smallmatrix} [1\;0]-[A\&B] \\ [0\;\sqrt{2}] \end{smallmatrix}\right]$ *maps* $\operatorname{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right)$ *onto a dense subspace of* $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$. *When this is the case,* $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ *is passive.*

Proof We begin with the *if* direction. Assume therefore that $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is a closed scattering dissipative operator and that $\left[\begin{smallmatrix} [1\;0]-[A\&B] \\ [0\;\sqrt{2}] \end{smallmatrix}\right] \operatorname{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right)$ is dense in $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$. Then the so-called internal Cayley transform

$$\mathbf{T} := \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} + \left(\begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ [C\&D] \end{bmatrix} \right) E^{-1},$$

defined on $\operatorname{dom}(\mathbf{T}) := \operatorname{im}(E)$, where

$$E = \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} [A\&B]/\sqrt{2} \\ 0 \end{bmatrix}, \quad \operatorname{dom}(E) = \operatorname{dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}\right),$$

is contractive (on its domain) by Lemma 2.2, Theorem 2.3(i), and the text in between, in [46]. Moreover, $\operatorname{dom}(\mathbf{T}) = \left[\begin{smallmatrix} [1\;0]-[A\&B] \\ [0\;\sqrt{2}] \end{smallmatrix}\right] \operatorname{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right)$, dense in $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$ by assumption. By [46, Thm 2.3](iv), it follows from the closedness of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ that $\operatorname{dom}(\mathbf{T})$ is closed, and hence $\operatorname{dom}(\mathbf{T}) = \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$. This in turn implies that $\mathbf{T}$ has no proper extensions to a contraction on $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$, and therefore $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ has no scattering dissipative extension by [46, Thm 2.3(iii)]. Hence, $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is maximal scattering dissipative. Theorem 2.5 of [46] now gives that $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is a passive system node.

Conversely, for the *only-if* direction, assume that $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is a scattering-dissipative system node, i.e., a passive system node according to Definition 3.7. Then [46, Thm 2.5] gives that $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is closed and maximal scattering dissipative, and now [46, Thm 2.4] finally yields that $\operatorname{dom}(\mathbf{T}) = \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$. □

Lemma 3.9 *For a passive system node with state space* $\mathcal{X}$ *and main operator* A, *we have* $\mathbb{C}^+ \subset \operatorname{res}(A) = \operatorname{res}(A|_{\mathcal{X}})$.

This lemma follows from [45, Theorem 11.1.5(viii)] and the rigging procedure described at the beginning of Sect. 3. Hence, when discussing controllability and observability of passive systems, we always take $\rho_\infty(A) = \mathbb{C}^+$.

3.4 Dual System Nodes

If A generates a C_0-semigroup $\mathfrak{A}$ on the Hilbert space $\mathcal{X}$, then A^* generates the C_0-semigroup $t \mapsto (\mathfrak{A}^t)^*$, according to [45, Theorem 3.5.6]. Clearly $\operatorname{res}(A^*) =$

$\{\overline{\mu} \in \mathbb{C} | \mu \in \mathrm{res}\,(A)\}$, and we denote the Gelfand triple corresponding to A^* and $\overline{\beta} \in \mathrm{res}\,(A^*)$ by $\mathcal{X}_1^d \subset \mathcal{X} \subset \mathcal{X}_{-1}^d$, where $\beta \in \mathrm{res}\,(A)$ is used in the rigging $\mathcal{X}_1 \subset \mathcal{X} \subset \mathcal{X}_{-1}$. In particular, $\mathcal{X}_1^d = \mathrm{dom}\,(A^*)$.

This makes it possible to identify the dual of $\mathcal{X}_1 = \mathrm{dom}\,(A)$ with $\mathcal{X}_{-1}^d$ using $\mathcal{X}$ as pivot space:

$$\langle x, z\rangle_{\langle \mathcal{X}_1, \mathcal{X}_{-1}^d\rangle} := (x, z)_{\mathcal{X}}, \quad x \in \mathcal{X}_1,\ z \in \mathcal{X}.$$

Similarly, the dual of $\mathrm{dom}\,(A^*)$ is identified with $\mathcal{X}_{-1}$ using $\mathcal{X}$ as pivot space.

Proposition 3.10 *Every system node* $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ *on the triple* $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$ *of Hilbert-spaces has the following properties:*

1. *The adjoint* $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]^*$ *is a system node on* $(\mathcal{Y}, \mathcal{X}, \mathcal{U})$. *The main operator of* $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]^*$ *is* $A^d = A^*$, *the control operator is* $B^d = C^* \in \mathcal{B}(\mathcal{Y}, \mathcal{X}_{-1}^d)$, *the observation operator is* $C^d = B^* \in \mathcal{B}(\mathcal{X}_1^d, \mathcal{U})$, *and the transfer function satisfies* $\widehat{\mathfrak{D}}^d(\lambda) = \widehat{\mathfrak{D}}(\overline{\lambda})^*$ *for all* $\lambda \in \mathrm{res}\,(A^*)$, *where* $\widehat{\mathfrak{D}}$ *is the transfer function of* $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$.
2. *The system node* $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ *is passive if and only if* $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]^*$ *is passive.*
3. *The system node* $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ *is controllable if and only if* $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]^*$ *is observable and vice versa.*

For a proof of the first statement see [45, Lemma 6.2.14]. The second statement follows from [45, Lemma 11.1.4]; note that passivity implies well-posedness. The third claim follows immediately on combining the first statement with Definition 3.4.

Definition 3.11 The (possibly unbounded) adjoint $\left[\begin{smallmatrix} A^d\&B^d \\ C^d\&D^d \end{smallmatrix}\right] := \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]^*$ of a system node $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is called the *causal dual system node*, or shortly just the *dual*, of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$.

We say that a system node is *co-energy preserving* if its dual system node is energy preserving. A system node that is both energy preserving and co-energy preserving is called *conservative*.

We see that a system node $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is conservative if and only if the dual system node $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]^*$ is conservative. Energy preservation is also clearly a necessary condition for conservativity, and the following important result provides a converse:

Theorem 3.12 *For every energy-preserving system node* $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$, *the following hold:*

1. *The operator* $\left[\begin{smallmatrix} [1\ 0] \\ C\&D \end{smallmatrix}\right]$ *maps* $\mathrm{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right)$ *into* $\mathrm{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]^*\right)$ *and*

$$\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}^* \begin{bmatrix} [1\ 0] \\ C\&D \end{bmatrix} = \begin{bmatrix} -A\&B \\ [0\ 1] \end{bmatrix} \quad \text{on} \quad \mathrm{dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}\right). \tag{3.15}$$

2. *The following conditions are equivalent:*
   (a) The system node $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ *is conservative.*
   (b) The operator $\left[\begin{smallmatrix} [1\ 0] \\ C\&D \end{smallmatrix}\right]$ *maps* $\mathrm{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right)$ onto $\mathrm{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]^*\right)$.

(c) The range of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right] + \left[\begin{smallmatrix} \overline{\alpha} & 0 \\ 0 & 0 \end{smallmatrix}\right]$ *is dense in* $\left[\begin{smallmatrix} \mathcal{H}_c \\ \mathcal{Y} \end{smallmatrix}\right]$ *for some, or equivalently for all,* $\alpha \in \mathbb{C}^+$.

This follows by taking $R = 1_U$, $P = 1_{\mathcal{X}}$, and $J = 1_{\mathcal{Y}}$ in [33, Thms 3.2 and 4.2]. We now finally arrive at the main part of the article: a study of the continuous-time analogue of the controllable energy-preserving model in Theorem 1.4.

4 The Controllable Energy-Preserving Functional Model

In this section we present the controllable energy-preserving model realization, which uses $\mathcal{H}_c$ as state space. Later, in Sect. 5, we show how the properties of the observable co-energy-preserving functional-model system node can be concluded from the results of this section.

4.1 Definition and Immediate Properties

Let $\varphi \in \mathcal{S}(\mathbb{C}^+; \mathcal{U}, \mathcal{Y})$ where $\mathcal{U}$ and $\mathcal{Y}$ are separable Hilbert spaces. As before, let $\mathcal{H}_c$ denote the Hilbert space whose reproducing kernel is

$$K_c(\mu, \lambda) = \frac{1 - \varphi(\overline{\mu})^* \varphi(\overline{\lambda})}{\mu + \overline{\lambda}} \tag{4.1}$$

and let $e_c(\cdot)$ be the point-evaluation mapping on $\mathcal{H}_c$, so that $e_c(\lambda)^* u = K_c(\cdot, \lambda) u$ for all $\lambda \in \mathbb{C}^+$ and $u \in \mathcal{U}$. Introduce the mapping

$$\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}_c : \begin{bmatrix} e_c(\overline{\lambda})^* u \\ u \end{bmatrix} \mapsto \begin{bmatrix} \lambda e_c(\overline{\lambda})^* u \\ \varphi(\lambda) u \end{bmatrix}, \quad u \in \mathcal{U},\ \lambda \in \mathbb{C}^+. \tag{4.2}$$

In the following lemma we show that $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ in (4.2) can be extended to a closable linear operator

$$\begin{aligned} &\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}_c : \begin{bmatrix} \mathcal{H}_c \\ \mathcal{U} \end{bmatrix} \supset \mathcal{D}_0 \to \begin{bmatrix} \mathcal{H}_c \\ \mathcal{Y} \end{bmatrix}, \quad \text{where} \\ &\mathcal{D}_0 := \operatorname{span} \left\{ \begin{bmatrix} e_c(\overline{\lambda})^* u \\ u \end{bmatrix} \middle| \lambda \in \mathbb{C}^+,\ u \in \mathcal{U} \right\}. \end{aligned} \tag{4.3}$$

Lemma 4.1 *The formula* (4.2) *extends via linearity and limit-closure to define a scattering-isometric closed linear operator* $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$.

Proof By (4.1) and the equality $K_c(\overline{\lambda}_2, \overline{\lambda}_1) = e_c(\overline{\lambda}_2) e_c(\overline{\lambda}_1)^*$, we have for all $\lambda_k \in \mathbb{C}^+$ and $u_k \in \mathcal{U}$, $k = 1, 2$, that

$$\begin{aligned} &(u_1, u_2)_{\mathcal{U}} - (\varphi(\lambda_1) u_1, \varphi(\lambda_2) u_2)_{\mathcal{Y}} \\ &\quad = (\overline{\lambda}_2 + \lambda_1) \left(e_c(\overline{\lambda}_1)^* u_1, e_c(\overline{\lambda}_2)^* u_2 \right)_{\mathcal{H}_c} \\ &\quad = \left(\lambda_1 e_c(\overline{\lambda}_1)^* u_1, e_c(\overline{\lambda}_2)^* u_2 \right)_{\mathcal{H}_c} + \left(e_c(\overline{\lambda}_1)^* u_1, \lambda_2 e_c(\overline{\lambda}_2)^* u_2 \right)_{\mathcal{H}_c}. \end{aligned} \tag{4.4}$$

If we for $k = 1, 2$ set

$$\begin{bmatrix} x_k \\ u_k \end{bmatrix} := \begin{bmatrix} e_c(\overline{\lambda}_k)^* u_k \\ u_k \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} z_k \\ y_k \end{bmatrix} := \begin{bmatrix} A\&B \\ C\&D \end{bmatrix}_c \begin{bmatrix} x_k \\ u_k \end{bmatrix} = \begin{bmatrix} \lambda_k e_c(\overline{\lambda}_k)^* u_k \\ \varphi(\lambda_k) u_k \end{bmatrix},$$

then (4.4) can be expressed as

$$\begin{aligned} (u_1, u_2)_{\mathcal{U}} - (y_1, y_2)_{\mathcal{Y}} &= (z_1, x_2)_{\mathcal{H}_c} + (x_1, z_2)_{\mathcal{H}_c}, \\ \begin{bmatrix} z_k \\ y_k \end{bmatrix} &= \begin{bmatrix} A\&B \\ C\&D \end{bmatrix}_c \begin{bmatrix} x_k \\ u_k \end{bmatrix}, \end{aligned} \tag{4.5}$$

for all $\left[\begin{smallmatrix} x_k \\ u_k \end{smallmatrix}\right] = \left[\begin{smallmatrix} e_c(\overline{\lambda}_k)^* u_k \\ u_k \end{smallmatrix}\right]$, $k = 1, 2$. If we formally extend the definition (4.2) of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ to all of $\mathcal{D}_0$ by taking linear combinations (where at this stage $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ may a priori be ill-defined, so that $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$ depends on the choice of linear combination $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] = \sum_{k=1}^N \left[\begin{smallmatrix} e_c(\overline{\lambda}_k)^* u_k \\ u_k \end{smallmatrix}\right]$ chosen to represent $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$), then the identity (4.5) continues to hold for all $\left[\begin{smallmatrix} x_1 \\ u_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} x_2 \\ u_2 \end{smallmatrix}\right]$ in the span $\mathcal{D}_0$.

We now show that this implies that $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ in (4.3) is well-defined and closable. Suppose that $x_n,\ u_n,\ z_n$, and y_n are sequences such that

$$\begin{aligned} \begin{bmatrix} z_n \\ y_n \end{bmatrix} = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix}_c \begin{bmatrix} x_n \\ u_n \end{bmatrix} \to \begin{bmatrix} z \\ y \end{bmatrix} \text{ in } \begin{bmatrix} \mathcal{H}_c \\ \mathcal{Y} \end{bmatrix} \quad &\text{and} \\ \begin{bmatrix} x_n \\ u_n \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ in } \begin{bmatrix} \mathcal{H}_c \\ \mathcal{Y} \end{bmatrix}. \end{aligned} \tag{4.6}$$

To establish that $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ is closable, we need to show that $\left[\begin{smallmatrix} z \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$. The special case where $\left[\begin{smallmatrix} x_n \\ u_n \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ for all n is exactly what is needed to see that $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ is well-defined; in this way well-definedness and closability are simultaneously handled in a single argument.

Using (4.5) and the continuity of the inner product, the hypothesis (4.6) implies that

$$-\|y\|^2_{\mathcal{Y}} = (0, 0)_{\mathcal{U}} - (y, y)_{\mathcal{Y}} = (z, 0)_{\mathcal{H}_c} + (0, z)_{\mathcal{H}_c} = 0,$$

and so $y = 0$. Applying (4.5) again, we now obtain that for all $\left[\begin{smallmatrix} x_2 \\ u_2 \end{smallmatrix}\right] \in \operatorname{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c\right)$:

$$0 = (0, y_2)_{\mathcal{U}} - (0, u_2)_{\mathcal{Y}} = (z, x_2)_{\mathcal{H}_c} + (0, z_2)_{\mathcal{H}_c}, \quad \begin{bmatrix} z_2 \\ y_2 \end{bmatrix} = \left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \right)_c \begin{bmatrix} x_2 \\ u_2 \end{bmatrix},$$

so that $z \perp x_2$ for all $\left[\begin{smallmatrix} x_2 \\ y_2 \end{smallmatrix}\right] \in \operatorname{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c\right)$. In particular, for every $\lambda \in \mathbb{C}^+$ and $u \in \mathcal{U}$ we have that $\left[\begin{smallmatrix} x_2 \\ u_2 \end{smallmatrix}\right] := \left[\begin{smallmatrix} e_c(\overline{\lambda})^* u \\ u \end{smallmatrix}\right] \in \operatorname{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c\right)$ and

$$0 = \left(z, e_c(\overline{\lambda})^* u\right)_{\mathcal{H}_c} = \left(z(\overline{\lambda}), u\right)_{\mathcal{U}}, \quad \lambda \in \mathbb{C}^+,\ u \in \mathcal{U},$$

and therefore $z(\overline{\lambda}) = 0$ for all $\lambda \in \mathbb{C}^+$. We conclude that both z and y are zero as needed to complete the proof. □

From now on, we let $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ denote the *closure* of the linear operator determined by (4.3).

Theorem 4.2 *The operator $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ is an energy-preserving system node with input space $\mathcal{U}$, state space $\mathcal{H}_c$, and output space $\mathcal{Y}$. Denoting the main and control operators of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ by A_c and B_c, respectively, we obtain that*

$$(\alpha - A_c|_{\mathcal{H}_c})^{-1} B_c = e_c(\overline{\alpha})^*, \quad \alpha \in \mathbb{C}^+. \tag{4.7}$$

In addition, $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ is controllable:

$$\overline{\operatorname{span}} \left\{ (\alpha - A_c|_{\mathcal{H}_c})^{-1} B_c u | u \in \mathcal{U},\, \alpha \in \mathbb{C}^+ \right\} = \mathcal{H}_c,$$

and $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ realizes φ:

$$\begin{bmatrix} C_c\&D_c \end{bmatrix} \begin{bmatrix} (\alpha - A_c|_{\mathcal{H}_c})^{-1} B_c \\ 1 \end{bmatrix} = \varphi(\alpha), \quad \alpha \in \mathbb{C}^+.$$

Proof We use Lemma 3.8 to prove that $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ is a passive system node. Since $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ is closed it suffices to show that the following subspace of the range of $\left[\begin{smallmatrix} [1\ 0] - [A_c\&B_c] \\ [0\ \sqrt{2}] \end{smallmatrix}\right]$ is dense in $\left[\begin{smallmatrix} \mathcal{H}_c \\ \mathcal{U} \end{smallmatrix}\right]$:

$$\begin{aligned} \mathcal{R} &:= \operatorname{span} \left\{ \begin{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} - \begin{bmatrix} A_c\&B_c \end{bmatrix} \\ \begin{bmatrix} 0 & \sqrt{2} \end{bmatrix} \end{bmatrix} \begin{bmatrix} e_c(\overline{\lambda})^* u \\ u \end{bmatrix} \,\middle|\, \lambda \in \mathbb{C}^+,\, u \in \mathcal{U} \right\} \\ &= \operatorname{span} \left\{ \begin{bmatrix} (1-\lambda) e_c(\overline{\lambda})^* u \\ \sqrt{2} u \end{bmatrix} \,\middle|\, \lambda \in \mathbb{C}^+,\, u \in \mathcal{U} \right\}. \end{aligned}$$

This space is indeed dense, since

$$\begin{aligned} \begin{bmatrix} x_2 \\ u_2 \end{bmatrix} \in \begin{bmatrix} \mathcal{H}_c \\ \mathcal{U} \end{bmatrix} \ominus \mathcal{R} \iff & \forall \lambda \in \mathbb{C}^+,\, u \in \mathcal{U} : \\ & \left(x_2, (1-\lambda) e_c(\overline{\lambda})^* u \right)_{\mathcal{H}_c} + \left(u_2, \sqrt{2} u \right)_{\mathcal{U}} = 0 \\ \iff & \forall \lambda \in \mathbb{C}^+ : \ (\overline{\lambda} - 1) x_2(\overline{\lambda}) = \sqrt{2} u_2. \end{aligned}$$

Choosing $\lambda = 1$ yields that $u_2 = 0$ and hence $x_2(\lambda) = 0$ for all $\lambda \in \mathbb{C}^+ \backslash \{1\}$. Since x_2 is holomorphic and thus continuous, also $x_2(1) = 0$ as well, and hence x_2 is the zero function in $\mathcal{H}_o$. We have established that $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ is a passive system node, which is moreover energy preserving due to Definition 3.7 and Lemma 4.1.

By Lemma 3.9, we have $\mathbb{C}^+ \subset \mathrm{res}\left(A|_{\mathcal{H}_c}\right)$. Then (4.7) follows from (4.3) and Definition 3.1.3, since for every $\alpha \in \mathbb{C}^+$:

$$\begin{aligned} \begin{bmatrix} A_c\&B_c \end{bmatrix} \begin{bmatrix} e_c(\overline{\alpha})^* u \\ u \end{bmatrix} &= \alpha e_c(\overline{\alpha})^* u = A_c|_{\mathcal{H}_c} e_c(\overline{\alpha})^* u + B_c u \\ &\iff (\alpha - A_c|_{\mathcal{H}_c}) e_c(\overline{\alpha})^* u = B_c u. \end{aligned} \tag{4.8}$$

In particular, $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ is controllable because

$$\overline{\mathrm{span}}\left\{ e_c(\overline{\alpha})^* u | u \in \mathcal{U},\ \alpha \in \mathbb{C}^+ \right\} = \mathcal{H}_c$$

by Remark 1.2. Finally, the transfer function of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ evaluated at $\alpha \in \mathbb{C}^+$ is

$$\begin{bmatrix} C_c\&D_c \end{bmatrix} \begin{bmatrix} (\alpha - A_c|_{\mathcal{H}_c})^{-1} B_c u \\ u \end{bmatrix} = \begin{bmatrix} C_c\&D_c \end{bmatrix} \begin{bmatrix} e_c(\overline{\alpha})^* u \\ u \end{bmatrix} = \varphi(\alpha) u, \tag{4.9}$$

for all $u \in \mathcal{U}$. □

The domain of the main operator A_c of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ is defined abstractly in (3.2), but we do not know how to characterize $\mathrm{dom}\,(A_c)$ explicitly. The observation operator C_c is defined in (3.4), but we have no explicit formula for the action of C_c on generic elements of $\mathrm{dom}\,(A_c)$ either. These two shortcomings will cause us significant difficulties later.

4.2 Uniqueness up to Unitary Similarity

We now prove that every controllable energy-preserving realization of φ is unitarily similar to $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$; this justifies the terminology *canonical functional-model* system node for $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$.

Theorem 4.3 *Let $\varphi \in \mathcal{S}(\mathbb{C}^+; \mathcal{U}, \mathcal{Y})$ and let $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be a controllable and energy preserving realization of φ with state space $\mathcal{X}$. Then the mapping $\Delta : \mathcal{H}_c \to \mathcal{X}$ defined by*

$$\Delta e_c(\overline{\lambda})^* u := (\lambda - A|_{\mathcal{X}})^{-1} B u, \quad \lambda \in \mathbb{C}^+,\ u \in \mathcal{U}, \tag{4.10}$$

extends by linearity and limit-closure to a unitary operator $\mathcal{H}_c \to \mathcal{X}$. Moreover, Δ intertwines $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ *with* $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$*:*

$$\begin{aligned} \mathrm{dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}\right) &= \begin{bmatrix} \Delta & 0 \\ 0 & 1_{\mathcal{U}} \end{bmatrix} \mathrm{dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}\right)_c \quad \text{and} \\ \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \begin{bmatrix} \Delta & 0 \\ 0 & 1_{\mathcal{U}} \end{bmatrix} &= \begin{bmatrix} \Delta & 0 \\ 0 & 1_{\mathcal{Y}} \end{bmatrix} \begin{bmatrix} A\&B \\ C\&D \end{bmatrix}_c, \end{aligned} \tag{4.11}$$

so that $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ are unitarily similar.

Proof The key to the proof is the following consequence of (3.3) and (3.6): for all $\lambda \in \operatorname{res}(A)$ we have that $\left[\begin{smallmatrix} (\lambda - A|_{\mathcal{X}})^{-1}Bu \\ u \end{smallmatrix}\right] \in \operatorname{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right)$ and

$$\begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \begin{bmatrix} (\lambda - A|_{\mathcal{X}})^{-1}Bu \\ u \end{bmatrix} = \begin{bmatrix} \lambda(\lambda - A|_{\mathcal{X}})^{-1}Bu \\ \varphi(\lambda)u \end{bmatrix}. \tag{4.12}$$

Since $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is assumed to be energy preserving, it follows from Lemma 3.9 that $\mathbb{C}^+ \subset \operatorname{res}(A)$ and that (3.14) is satisfied. According to (4.10), we have for all $\lambda, \mu \in \mathbb{C}^+$ and all $u, v \in \mathcal{U}$ that

$$\begin{aligned} &(\mu + \lambda)\left(\Delta e_c(\overline{\lambda})^* u, \Delta e_c(\mu)^* v\right)_{\mathcal{X}} \\ &\quad = (\mu + \lambda)\left((\lambda - A|_{\mathcal{X}})^{-1}Bu, (\overline{\mu} - A|_{\mathcal{X}})^{-1}Bv\right)_{\mathcal{X}} \\ &\quad = \left(\lambda(\lambda - A|_{\mathcal{X}})^{-1}Bu, (\overline{\mu} - A|_{\mathcal{X}})^{-1}Bv\right)_{\mathcal{X}} \\ &\qquad + \left((\lambda - A|_{\mathcal{X}})^{-1}Bu, \overline{\mu}(\overline{\mu} - A|_{\mathcal{X}})^{-1}Bv\right)_{\mathcal{X}}. \end{aligned}$$

This is by (4.12) and (3.14) equal to

$$(u, v)_{\mathcal{U}} - (\varphi(\lambda)u, \varphi(\overline{\mu})v)_{\mathcal{Y}} = (\mu + \lambda)\left(e_c(\overline{\lambda})^* u, e_c(\mu)^* v\right)_{\mathcal{H}_c},$$

where we used (4.4) in the last step. We can conclude that

$$\left(\Delta e_c(\overline{\lambda})^* u, \Delta e_c(\mu)^* v\right)_{\mathcal{X}} = \left(e_c(\overline{\lambda})^* u, e_c(\mu)^* v\right)_{\mathcal{H}_c}, \quad \overline{\lambda}, \mu \in \mathbb{C}^+,\ u, v \in \mathcal{U}. \tag{4.13}$$

Taking linear combinations, we obtain from (4.13) that for all $\overline{\lambda}_k \in \mathbb{C}^+$ and $u_k \in \mathcal{U}$:

$$\left\| \Delta \sum_{k=1}^{n} e_c(\overline{\lambda_k})^* u_k \right\|_{\mathcal{X}}^2 = \left\| \sum_{k=1}^{n} e_c(\overline{\lambda_k})^* u_k \right\|_{\mathcal{H}_c}^2. \tag{4.14}$$

Denote $\mathcal{E}_0 := \operatorname{span}\left\{e_c(\overline{\lambda})^* u | \overline{\lambda} \in \mathbb{C}^+,\ u \in \mathcal{U}\right\}$, equipped with the norm of $\mathcal{H}_c$. Then each $x \in \mathcal{E}_0$ can be written as a sum $x = \sum_{k=1}^{n} e_c(\overline{\lambda_k})^* u_k$, and (4.14) shows that the value of $\Delta \sum_{k=1}^{n} e_c(\overline{\lambda_k})^* u_k$ is independent of the particular linear combination $\sum_{k=1}^{n} e_c(\overline{\lambda_k})^* u_k$ that is used to represent x. Thus, Δ, which was originally defined only for kernel functions $e(\overline{\lambda})^* u$ with $\overline{\lambda} \in \mathbb{C}^+$ and $u \in \mathcal{U}$, has a unique extension to a linear operator $\mathcal{E}_0 \to \mathcal{X}$, which we still denote by Δ. Due to (4.14), this operator is isometric, and by (4.10) the image of $\mathcal{E}_0$ under this operator is dense in $\mathcal{X}$, since $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is assumed to be controllable. As $\mathcal{E}_0$ is dense in $\mathcal{H}_c$, we may further extend Δ to a unitary operator $\mathcal{H}_c \to \mathcal{X}$, which we still denote by Δ.

Now we prove that Δ intertwines $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ with $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$. It follows from (3.3) that $\left[\begin{smallmatrix} \Delta & 0 \\ 0 & 1 \end{smallmatrix}\right]$ maps $\mathcal{D}_0$, defined in (4.3), into $\operatorname{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right)$. By (4.10), (4.12), and (4.2), the following equality holds for all $\lambda \in \mathbb{C}^+$ and $u \in \mathcal{U}$:

$$\begin{aligned}\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}\begin{bmatrix} \Delta e_c(\overline{\lambda})^* u \\ u \end{bmatrix} &= \begin{bmatrix} A\&B \\ C\&D \end{bmatrix}\begin{bmatrix} (\lambda - A|_{\mathcal{X}})^{-1} Bu \\ u \end{bmatrix} \\ &= \begin{bmatrix} \lambda(\lambda - A|_{\mathcal{X}})^{-1} Bu \\ \varphi(\lambda)u \end{bmatrix} \\ &= \begin{bmatrix} \lambda \Delta e_c(\overline{\lambda})^* u \\ \varphi(\lambda)u \end{bmatrix} = \begin{bmatrix} \Delta \left[A_c\&B_c \right] \\ C_c\&D_c \end{bmatrix}\begin{bmatrix} e_c(\overline{\lambda})^* u \\ u \end{bmatrix}, \end{aligned} \tag{4.15}$$

which shows that $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\left[\begin{smallmatrix} \Delta & 0 \\ 0 & 1_{\mathcal{U}} \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} \Delta & 0 \\ 0 & 1_{\mathcal{Y}} \end{smallmatrix}\right]\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ coincide on $\mathcal{D}_0$. Furthermore, $\mathcal{D}_0$ is dense in $\mathrm{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c\right)$, equipped with the graph norm, by Lemma 3.5.

We next show how it follows from the above that

$$\begin{gathered}\begin{bmatrix} \Delta x \\ u \end{bmatrix} \in \mathrm{dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}\right) \text{ for all } \begin{bmatrix} x \\ u \end{bmatrix} \in \mathrm{dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}\right)_c, \quad \text{and} \\ \begin{bmatrix} A\&B \\ C\&D \end{bmatrix}\begin{bmatrix} \Delta x \\ u \end{bmatrix} = \begin{bmatrix} \Delta \left[A_c\&B_c \right] \\ C_c\&D_c \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix}, \quad \begin{bmatrix} x \\ u \end{bmatrix} \in \mathrm{dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}\right)_c. \end{gathered} \tag{4.16}$$

For every $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \mathrm{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c\right)$ there by the definition of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ exists a sequence $\left[\begin{smallmatrix} x_n \\ u_n \end{smallmatrix}\right] \in \mathcal{D}_0$, such that $\left[\begin{smallmatrix} x_n \\ u_n \end{smallmatrix}\right] \to \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$ in $\left[\begin{smallmatrix} \mathcal{H}_c \\ \mathcal{U} \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c\left[\begin{smallmatrix} x_n \\ u_n \end{smallmatrix}\right] \to \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$ in $\left[\begin{smallmatrix} \mathcal{H}_c \\ \mathcal{Y} \end{smallmatrix}\right]$. By the continuity of Δ and the fact that $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\left[\begin{smallmatrix} \Delta & 0 \\ 0 & 1_{\mathcal{U}} \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} \Delta & 0 \\ 0 & 1_{\mathcal{Y}} \end{smallmatrix}\right]\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ coincide on $\mathcal{D}_0$, this implies that

$$\begin{bmatrix} \Delta \left[A_c\&B_c \right] \\ C_c\&D_c \end{bmatrix}\begin{bmatrix} x_n \\ u_n \end{bmatrix} = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix}\begin{bmatrix} \Delta x_n \\ u_n \end{bmatrix} \to \begin{bmatrix} \Delta \left[A_c\&B_c \right] \\ C_c\&D_c \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix}.$$

Using the continuity of Δ again, we obtain that $\left[\begin{smallmatrix} \Delta x_n \\ u_n \end{smallmatrix}\right] \in \mathrm{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right)$ converges to $\left[\begin{smallmatrix} \Delta x \\ u \end{smallmatrix}\right]$ in $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$, and so by the closedness of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$, we have (4.16).

It remains to prove that $\left[\begin{smallmatrix} \Delta & 0 \\ 0 & 1 \end{smallmatrix}\right]$ maps $\mathrm{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c\right)$ *onto* $\mathrm{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right)$. As a consequence of (4.10) and Lemma 3.5, $\left[\begin{smallmatrix} \Delta & 0 \\ 0 & 1 \end{smallmatrix}\right]\mathcal{D}_0$ is dense in $\mathrm{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right)$ with the graph norm. Hence, for every $\left[\begin{smallmatrix} w \\ u \end{smallmatrix}\right] \in \mathrm{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right)$, we can find a sequence $\left[\begin{smallmatrix} w_n \\ u_n \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \Delta & 0 \\ 0 & 1 \end{smallmatrix}\right]\mathcal{D}_0$ that converges to $\left[\begin{smallmatrix} w \\ u \end{smallmatrix}\right]$ in the graph norm of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$. Writing $x_n := \Delta^{-1} w_n$, we obtain from (4.16) and the closedness of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ that $\left[\begin{smallmatrix} \Delta^{-1} w_n \\ u_n \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \Delta^{-1} w \\ u \end{smallmatrix}\right]$ in the graph norm of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$. (The details are very similar to the preceding paragraph.) Thus, for every $\left[\begin{smallmatrix} w \\ u \end{smallmatrix}\right] \in \mathrm{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right)$, we have $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] := \left[\begin{smallmatrix} \Delta^{-1} w \\ u \end{smallmatrix}\right] \in \mathrm{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c\right)$ and $\left[\begin{smallmatrix} w \\ u \end{smallmatrix}\right] = \left[\begin{smallmatrix} \Delta x \\ u \end{smallmatrix}\right]$. □

We would like to obtain explicit formulas for the main, control, and observation operators of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ acting on generic elements of $\mathcal{H}_c$, and similarly for the adjoint. It turns out that this task is much easier for $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c^*$ than for $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$, so we start with the adjoint.

4.3 Explicit Formulas for the System-Node Operators of the Dual

In reproducing-kernel Hilbert spaces, the existence of an explicit formula for the action of a given operator on kernel functions usually means that there is an equally explicit formula for the action of the adjoint on a generic functional element of the reproducing kernel Hilbert space. This phenomenon continues to hold in the present unbounded setting, as illustrated in the following proposition. We refer the reader back to Sect. 3.4 for the definition of dual system node.

The reader will observe that many of the formulas in this section have apparent singularities at some points of the form $0/0$. Since the functions are holomorphic (or conjugate holomorphic), the singularities are in fact removable and the formulas continue to hold when one applies holomorphic continuation to evaluate at such exceptional points.

By the general principles explained in Sect. 3.4, we know that $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c^*$ is a system node on $(\mathcal{Y}, \mathcal{X}, \mathcal{U})$. We now compute this dual system node.

Theorem 4.4 *The dual system node of* $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ *is the operator*

$$\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}_c^* : \begin{bmatrix} \mathcal{H}_o \\ \mathcal{Y} \end{bmatrix} \supset \operatorname{dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}\right)_c^* \to \begin{bmatrix} \mathcal{H}_o \\ \mathcal{U} \end{bmatrix}$$

given by

$$\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}_c^* : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} z \\ u \end{bmatrix}, \quad \textit{where} \tag{4.17}$$

$$z(\mu) := \mu x(\mu) + \widetilde{\varphi}(\mu) y - u, \quad \mu \in \mathbb{C}^+, \quad \textit{and} \tag{4.18}$$

$$u := \lim_{\operatorname{Re}\eta \to \infty} \eta x(\eta) + \widetilde{\varphi}(\eta) y, \quad \textit{with domain} \tag{4.19}$$

$$\operatorname{dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}_c^*\right) := \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \begin{bmatrix} \mathcal{H}_c \\ \mathcal{Y} \end{bmatrix} \middle| \exists u \in \mathcal{U} : \ z \in \mathcal{H}_c \text{ in } (4.18) \right\}. \tag{4.20}$$

For every $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \operatorname{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c^*\right)$*, the* $u \in \mathcal{U}$ *such that* z *defined in* (4.18) *lies in* $\mathcal{H}_c$ *is unique, and it is given by* (4.19).

Proof We combine the graph characterization

$$\begin{bmatrix} \mathcal{H}_c \\ \mathcal{U} \\ \mathcal{H}_c \\ \mathcal{Y} \end{bmatrix} \ominus \left(\begin{bmatrix} -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} A\&B \\ C\&D \end{bmatrix}_c \end{bmatrix} \operatorname{dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}_c\right) \right) = \begin{bmatrix} \begin{bmatrix} A\&B \\ C\&D \end{bmatrix}_c^* \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix} \operatorname{dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}_c^*\right)$$

of the adjoint of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ with the construction of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ and thus obtain that $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \operatorname{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c^*\right)$ and $\left[\begin{smallmatrix} z \\ u \end{smallmatrix}\right] = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c^* \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ if and only if

$$\begin{bmatrix} z \\ u \\ x \\ y \end{bmatrix} \in \begin{bmatrix} \mathcal{H}_c \\ \mathcal{U} \\ \mathcal{H}_c \\ \mathcal{Y} \end{bmatrix} \ominus \overline{\operatorname{span}} \left\{ \begin{bmatrix} -e_c(\mu)^* v \\ -v \\ \overline{\mu} e_c(\mu)^* v \\ \varphi(\overline{\mu}) v \end{bmatrix}, \quad v \in \mathcal{U},\ \mu \in \mathbb{C}^+ \right\}$$
$$= \bigcap_{\mu \in \mathbb{C}^+} \ker \left(\begin{bmatrix} -e_c(\mu) & -1 & \mu e_c(\mu) & \varphi(\overline{\mu})^* \end{bmatrix} \right).$$

Thus a pair $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{H}_c \\ \mathcal{Y} \end{smallmatrix}\right]$ lies in dom $\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c^*\right)$ if and only if there exist $z \in \mathcal{H}_c$ and $u \in \mathcal{U}$ such that $-z(\mu) - u + \mu x(\mu) + \widetilde{\varphi}(\mu) y = 0$ for all $\mu \in \mathbb{C}^+$, i.e., z is given by (4.18). When such a u exists, we have

$$\lim_{\operatorname{Re}\eta \to \infty} z(\eta) = \lim_{\operatorname{Re}\eta \to \infty} \eta x(\eta) + \varphi(\overline{\eta})^* y - u = 0$$

by Proposition 2.6, and hence u is given by (4.19). □

With the formulas in Theorem 4.4 as a starting point, it is possible to compute the main operator $A_c^d = A_c^*$, control operator $B_c^d = C_c^* \in \mathcal{B}(\mathcal{Y}, \mathcal{H}_{c,-1}^d)$, and observation operator $C_c^d = B_c^* \in \mathcal{B}(\mathcal{H}_{c,1}^d, \mathcal{U})$, of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c^*$ explicitly.

Proposition 4.5 *The domain* dom $\left(A_c^*\right) = \mathcal{H}_{c,1}^d$ *of* $A_c^* = A_c^d$ *is given by*

$$\operatorname{dom}\left(A_c^*\right) = \{x \in \mathcal{H}_c | \exists u \in \mathcal{U} :\ \mu \mapsto \mu x(\mu) - u \in \mathcal{H}_c\}. \tag{4.21}$$

Moreover, when $x \in$ dom $\left(A_c^*\right)$*, the associated vector* u *can be recovered from* x *using the formula* $u = \lim_{\operatorname{Re}\eta \to \infty} \eta x(\eta)$*, and*

$$(A_c^* x)(\mu) = \mu x(\mu) - \lim_{\operatorname{Re}\eta \to \infty} \eta x(\eta), \quad C_c^d x = B_c^* x = \lim_{\operatorname{Re}\eta \to \infty} \eta x(\eta). \tag{4.22}$$

Proof By Definition 3.11, (3.2), and (3.4), a function $x \in \mathcal{H}_c$ lies in dom $\left(A_c^d\right)$ if and only if $\left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in$ dom $\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c^*\right)$, and in this case

$$\begin{bmatrix} A_c^d x \\ C_c^d x \end{bmatrix} = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix}_c^* \begin{bmatrix} x \\ 0 \end{bmatrix}.$$

Comparing this to Theorem 4.4 with $y = 0$ gives the result. □

To get an explicit description of the (-1)-scaled rigged space (also called “extrapolation space”) $\mathcal{H}_{c,-1}^d$ we first need a formula for the resolvent of A_c^*.

Proposition 4.6 *Let $A_c^d = A_c^*$ be the main operator and $B_c^d = C_c^*$ be the control operator for the dual system node $\begin{bmatrix} A_c^d \& B_c^d \\ C_c^d \& D_c^d \end{bmatrix} = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix}_c^*$. Then the resolvent of A_c^* is given by*

$$\left((\overline{\alpha} - A_c^*)^{-1} x\right)(\mu) = \frac{x(\mu) - x(\overline{\alpha})}{\overline{\alpha} - \mu}, \quad \alpha, \mu \in \mathbb{C}^+, \ x \in \mathcal{H}_c. \tag{4.23}$$

Moreover, the following formulas hold:

$$\left(A_c^*(\overline{\alpha} - A_c^*)^{-1} x\right)(\mu) = \frac{\mu x(\mu) - \overline{\alpha} x(\overline{\alpha})}{\overline{\alpha} - \mu}, \quad \alpha, \mu \in \mathbb{C}^+, \ x \in \mathcal{H}_c, \tag{4.24}$$

$$\left((\overline{\alpha} - A_c^*|_{\mathcal{H}_c})^{-1} C_c^* y\right)(\mu) = \frac{\widetilde{\varphi}(\mu) - \widetilde{\varphi}(\overline{\alpha})}{\overline{\alpha} - \mu} y, \quad \alpha, \mu \in \mathbb{C}^+, \ y \in \mathcal{Y}, \tag{4.25}$$

$$B_c^*(\overline{\alpha} - A_c^*)^{-1} = e_c(\overline{\alpha}), \quad \alpha \in \mathbb{C}^+. \tag{4.26}$$

Proof For $\xi \in \operatorname{dom}\left(A_c^*\right)$, set $x = (\overline{\alpha} - A_c^*)\xi$. From the formulas (4.22) we see that

$$x(\mu) = (\overline{\alpha} - \mu)\xi(\mu) + B_c^* \xi.$$

We conclude that $B_c^* \xi = x(\overline{\alpha})$ and $x(\mu) = (\overline{\alpha} - \mu)\xi(\mu) + x(\overline{\alpha})$. Solving for ξ gives $\xi(\mu) = \frac{x(\mu) - x(\overline{\alpha})}{\overline{\alpha} - \mu}$ and formula (4.23) follows.

From $A_c^*(\overline{\alpha} - A_c^*)^{-1} = \overline{\alpha}(\overline{\alpha} - A_c^*)^{-1} - 1$, we get

$$\left(A_c^*(\overline{\alpha} - A_c^*)^{-1} x\right)(\mu) = \frac{\overline{\alpha} x(\mu) - \overline{\alpha} x(\overline{\alpha})}{\overline{\alpha} - \mu} - x(\mu) = \frac{\mu x(\mu) - \overline{\alpha} x(\overline{\alpha})}{\overline{\alpha} - \mu}$$

and formula (4.24) follows.

By (3.3), for an arbitrary $y \in \mathcal{Y}$ we can set $x := \left(\overline{\alpha} - A_c^*|_{\mathcal{H}_c}\right)^{-1} C_c^* y$ in order to get $\begin{bmatrix} x \\ y \end{bmatrix} \in \operatorname{dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}_c^*\right)$. If we further set $\begin{bmatrix} z \\ u \end{bmatrix} := \begin{bmatrix} A\&B \\ C\&D \end{bmatrix}_c^* \begin{bmatrix} x \\ y \end{bmatrix}$, then we obtain from (3.6) that

$$z = \overline{\alpha} x, \quad u = \widetilde{\varphi}(\overline{\alpha}) y.$$

(Here we use the fact that the transfer function of $\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}_c^*$ is $\widetilde{\varphi}$ since the transfer function of $\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}_c$ is φ, as observed in Proposition 3.10.) On the other hand, from Theorem 4.4 we know that $z(\mu) = \mu x(\mu) + \widetilde{\varphi}(\mu) y - u$. Combining these, we have for all $y \in \mathcal{Y}$ and $\alpha \in \mathbb{C}^+$ that

$$\overline{\alpha} x(\mu) = \mu x(\mu) + \widetilde{\varphi}(\mu) y - \widetilde{\varphi}(\overline{\alpha}) y \quad \Longrightarrow \quad x(\mu) = \frac{\widetilde{\varphi}(\mu) - \widetilde{\varphi}(\overline{\alpha})}{\overline{\alpha} - \mu} y$$

and formula (4.25) is established. Formula (4.26) follows directly from (4.7). □

We recall that the (-1)-scaled rigged space is defined as the completion of the space $\mathcal{H}_c$ in the norm

$$\|x\| = \left\| (\overline{\beta} - A_c^*)^{-1} x \right\|_{\mathcal{H}_c} \tag{4.27}$$

and that $(\overline{\beta} - A_c^*)^{-1}$ has an extension to a unitary operator from this rigged space onto $\mathcal{H}_c$. For $x \in \mathcal{H}_c$, we now have the formula (4.23) for the action of the resolvent $(\overline{\beta} - A_c^*)^{-1}$ on x. This suggests that we identify the (-1)-rigged space concretely as follows:

$$\mathcal{H}^d_{c,-1} := \left\{ x : \mathbb{C}^+ \to \mathcal{U} \middle| \mu \mapsto \frac{x(\mu) - x(\overline{\beta})}{\overline{\beta} - \mu} \in \mathcal{H}_c \right\}, \tag{4.28}$$

with norm given by

$$\|x\|_{\mathcal{H}^d_{c,-1}} := \left\| \mu \mapsto \frac{x(\mu) - x(\overline{\beta})}{\overline{\beta} - \mu} \right\|_{\mathcal{H}_c}, \tag{4.29}$$

as was also done in [26,27] in a closely related setting.

Once again the norm depends on the choice of $\overline{\beta}$ but all norms arising in this way are equivalent. Note that constant functions $x(\mu) = u$ are in $\mathcal{H}^d_{c,-1}$ with norm zero; therefore we view the space as consisting of equivalence classes, where two representatives x and ξ of the same equivalence class differ by a constant: $x(\mu) - \xi(\mu) = v$ for all $\mu \in \mathbb{C}^+$ for some $v \in \mathcal{U}$. We denote the equivalence class of x in $\mathcal{H}^d_{c,-1}$ by $[x]$; and if $[x] = [\xi]$ then we write $x \cong \xi$. Next some properties of this space $\mathcal{H}^d_{c,-1}$ are summarized:

Theorem 4.7 *The space $\mathcal{H}^d_{c,-1}$ defined in (4.28) and (4.29) is complete.*

1. *The map $\iota : x \mapsto [x]$ embeds $\mathcal{H}_c$ into $\mathcal{H}^d_{c,-1}$ as a dense subspace. A given element $[x] \in \mathcal{H}^d_{c,-1}$ is of the form $\iota(z)$ for some $z \in \mathcal{H}_c$ if and only if the function $\mu \mapsto \dfrac{x(\mu) - x(\overline{\alpha})}{\overline{\alpha} - \mu}$, $\mu \in \mathbb{C}^+$, is not only in $\mathcal{H}_c$ but in fact is in $\operatorname{dom}\left(A_c^*\right) = \mathcal{H}^d_{c,1} \subset \mathcal{H}_c$ for some, or equivalently for all, $\alpha \in \mathbb{C}^+$. When this is the case, the equivalence class representative z for $[x]$ that lies in $\mathcal{H}_c$, is uniquely determined by the decay condition at infinity:*

$$\lim_{\operatorname{Re}\eta \to \infty} z(\eta) = 0. \tag{4.30}$$

2. *Define an operator $A_c^*|_{\mathcal{H}_c} : \mathcal{H}_c \to \mathcal{H}^d_{c,-1}$ by*

$$A_c^*|_{\mathcal{H}_c} x := [\mu \mapsto \mu x(\mu)], \quad x \in \mathcal{H}_c,\ \mu \in \mathbb{C}^+. \tag{4.31}$$

When $\mathcal{H}_c$ is identified as a linear sub-manifold of $\mathcal{H}^d_{c,-1}$, then $A_c^*|_{\mathcal{H}_c}$ is the continuous extension of $A_c^* : \operatorname{dom}\left(A_c^*\right) \to \mathcal{H}_c$ to an operator $\mathcal{H}_c \to \mathcal{H}^d_{c,-1}$. Its resolvent is given by

$$\left((\overline{\alpha} - A_c^*|_{\mathcal{H}_c})^{-1}[x]\right)(\mu) = \frac{x(\mu) - x(\overline{\alpha})}{\overline{\alpha} - \mu}, \quad \alpha, \mu \in \mathbb{C}^+,\ [x] \in \mathcal{H}^d_{c,-1}, \tag{4.32}$$

and for $\overline{\alpha} = \overline{\beta}$ this is a unitary map from $\mathcal{H}^d_{c,-1}$ to $\mathcal{H}_c$.

3. *With* $\mathcal{H}^d_{c,-1}$ *identified concretely as in* (4.28)*, the action of* $C_c^* : \mathcal{Y} \to \mathcal{H}^d_{c,-1}$ *is given by*

$$C_c^* y = [\mu \mapsto \widetilde{\varphi}(\mu) y], \quad y \in \mathcal{Y},\ \mu \in \mathbb{C}^+. \tag{4.33}$$

Proof We first check that $\mathcal{H}^d_{c,-1}$ defined as in (4.28) and (4.29) is complete. Suppose that $[x_n]$ is a Cauchy sequence in $\mathcal{H}^d_{c,-1}$. Then the sequence $z_n := \mu \mapsto \frac{x_n(\mu)-x_n(\overline{\beta})}{\overline{\beta}-\mu}$, $\mu \in \mathbb{C}^+$, is Cauchy in $\mathcal{H}_c$ and it converges to some z in $\mathcal{H}_c$. We solve the equation $\frac{x(\mu)-x(\overline{\beta})}{\overline{\beta}-\mu} = z(\mu)$ to come up with

$$\begin{aligned}\frac{x(\mu)-x(\overline{\beta})}{\overline{\beta}-\mu} = z(\mu) &\Longleftrightarrow x(\mu) = x(\overline{\beta}) + (\overline{\beta}-\mu) z(\mu) \\ &\Longleftrightarrow x \cong \mu \mapsto (\overline{\beta}-\mu) z(\mu), \quad \mu \in \mathbb{C}^+;\end{aligned} \tag{4.34}$$

note that the solution is determined only up to an additive constant. By (4.34), $[\mu \mapsto (\overline{\beta}-\mu) z(\mu)] \in \mathcal{H}^d_{c,-1}$ since $z \in \mathcal{H}_c$, and $[x_n] \to [x]$ in $\mathcal{H}^d_{c,-1}$ by (4.29) and the fact that $z_n \to z$ in $\mathcal{H}_c$.

1. If $z \in \mathcal{H}_c$ then by (4.28), $\mu \to z(\mu) + v \in \mathcal{H}^d_{c,-1}$ for all $v \in \mathcal{U}$, since

$$\mu \mapsto \frac{z(\mu) + v - z(\overline{\beta}) - v}{\overline{\beta}-\mu} = (\overline{\beta} - A_c^*)^{-1} z \in \operatorname{dom}\left(A_c^*\right),\ \mu \in \mathbb{C}^+, \tag{4.35}$$

see (4.23), and hence $\iota(\mathcal{H}_c) \subset \mathcal{H}^d_{c,-1}$. From (4.35) it also follows that if $[x] = \iota(z)$ for some $z \in \mathcal{H}$ then $\mu \mapsto \frac{x(\mu)-x(\overline{\alpha})}{\overline{\alpha}-\mu} \in \operatorname{dom}\left(A_c^*\right)$ for all $\alpha \in \mathbb{C}^+$. Conversely, if $w := \mu \mapsto \frac{x(\mu)-x(\overline{\alpha})}{\overline{\alpha}-\mu} \in \operatorname{dom}\left(A_c^*\right)$ for some $\alpha \in \mathbb{C}^+$, then by (4.22):

$$\begin{aligned}\big((\overline{\alpha} - A_c^*) w\big)(\mu) - x(\mu) &= (\overline{\alpha} - \mu) w(\mu) + \lim_{\operatorname{Re}\eta\to\infty} \eta w(\eta) - x(\mu) \\ &= -x(\overline{\alpha}) + \lim_{\operatorname{Re}\eta\to\infty} \eta w(\eta),\end{aligned}$$

which is constant, so that $[x] = \iota\big((\overline{\alpha} - A_c^*) w\big)$.
It is a consequence of the estimate (2.7) that functions in $\mathcal{H}_c$ satisfy the decay condition (4.30). As two representatives of the same equivalence class differ by a constant, it is clear that there can be at most one representative of a given equivalence class which satisfies (4.30). Thus the decay condition (4.30) picks out the unique representative which is in $\mathcal{H}_c$ (assuming that the equivalence class is in the image of ι). Apart from the claim that $\iota(\mathcal{H}_c)$ is dense in $\mathcal{H}^d_{c,-1}$, Assertion 1 is proved.

2. We next suppose that $x \in \mathcal{H}_c$ and we wish to verify that $[\mu x(\mu)]$ is in $\mathcal{H}^d_{c,-1}$. Thus we must check that the function $z : \mu \mapsto \frac{\mu x(\mu) - \overline{\beta} x(\overline{\beta})}{\overline{\beta}-\mu}$ is in $\mathcal{H}_c$. But we have already verified that this expression is just the formula for $A_c^* (\overline{\beta} - A_c^*)^{-1} x$, see formula (4.24), and hence z is in $\mathcal{H}_c$ as wanted. Thus $A_c^*|_{\mathcal{H}_c}$ maps $\mathcal{H}_c$ into

$\mathcal{H}_{c,-1}$ and it follows from (4.22) that $A_c^*|_{\mathcal{H}_c}$ is an extension of A_c^* interpreted as a densely defined operator from $\mathcal{H}_c$ into $\mathcal{H}_{c,-1}$. It is straightforward to verify that the formula (4.23) for the resolvent of A_c^* extends to (4.32); use (4.31) and read (4.34) backwards. Combining (4.32) with (4.29), we obtain that $(\overline{\beta} - A_c^*|_{\mathcal{H}_c})^{-1}$ is isometric from all of $\mathcal{H}_{c,-1}^d$ into $\mathcal{H}_c$. On the other hand, $(\overline{\beta} - A_c^*|_{\mathcal{H}_c})^{-1}$ is onto $\mathcal{H}_c$ by (4.34): for every $z \in \mathcal{H}_c$, $[\mu \mapsto (\overline{\beta} - \mu)z(\mu)] \in \mathcal{H}_{c,-1}$ is mapped into z by $(\overline{\beta} - A_c^*|_{\mathcal{H}_c})^{-1}$.
By construction, once we fix our choice of $\overline{\beta} \in \mathbb{C}^+$ to define the norm on the (-1)-rigged space, the operator $(\overline{\beta} - A_c^*|_{\mathcal{H}_c})^{-1}$ is unitary from this rigged space onto $\mathcal{H}_c$. This makes precise the identification of the completion of $\mathcal{H}_c$ in the norm (4.27) with the concrete version of $\mathcal{H}_{c,-1}^d$ given by (4.28)–(4.29). Moreover, $A_c^*|_{\mathcal{H}_c}$ maps $\mathcal{H}_c$ continuously into $\mathcal{H}_{c,-1}^d$, since for all $x \in \mathcal{H}_c$:

$$\begin{aligned}\|A_c^*|_{\mathcal{H}_c}x\|_{\mathcal{H}_{c,-1}^d} &= \|(\overline{\beta} - A_c^*|_{\mathcal{H}_c})^{-1}A_c^*|_{\mathcal{H}_c}x\|_{\mathcal{H}_c} = \|\overline{\beta}(\overline{\beta} - A_c^*)^{-1}x - x\|_{\mathcal{H}_c} \\ &\le \Big| |\overline{\beta}|\,\|(\overline{\beta} - A_c^*)^{-1}\| + 1 \Big|\,\|x\|_{\mathcal{H}_c}.\end{aligned}$$

Because $\mathcal{H}_{c,-1}^d$ is a completion of $\iota(\mathcal{H}_c)$, it is clear that $\iota(\mathcal{H}_c)$ is dense in $\mathcal{H}_{c,-1}^d$. Now Assertions 1 and 2 are proved completely.

3. By Definition 3.1.3, $C_c^* y = \left[A_c^* \& C_c^*\right]\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] - A_c^*|_{\mathcal{H}_c}x$, where x is any choice of function in $\mathcal{H}_c$ for which $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ is in dom $\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c^*\right)$; here we use the fact that such an x exists for every $y \in \mathcal{Y}$ by (3.3). Using (4.18) together with (4.31), we see that

$$C_c^* y = [\mu \mapsto \mu x(\mu) + \widetilde{\varphi}(\mu)y - u - \mu x(\mu)] = [\mu \mapsto \widetilde{\varphi}(\mu)y], \quad \mu \in \mathbb{C}^+,$$

and (4.33) follows. □

We point out that $\mu \mapsto (\overline{\beta} - \mu)z(\mu)$ in (4.34) is the unique representative of $(\overline{\beta} - A_c^*|_{\mathcal{H}_c})z$ with value zero at $\overline{\beta}$; this will be useful in Sect. 4.7 below. Furthermore, it follows from (4.22) that for all $x \in \text{dom}\left(A_c^*\right)$:

$$\big((\overline{\alpha} - A_c^*)x\big)(\mu) = (\overline{\alpha} - \mu)x(\mu) + \lim_{\operatorname{Re}\eta \to \infty} \eta x(\eta),$$

so that for arbitrary $\alpha \in \mathbb{C}^+$:

$$\lim_{\operatorname{Re}\eta \to \infty} \eta x(\eta) = \big((\overline{\alpha} - A_c^*)x\big)(\overline{\alpha}), \quad x \in \text{dom}\left(A_c^*\right).$$

We next give another interpretation of this limit.

Remark 4.8 Assume that f and its distribution derivative f' both lie in $L^2(\mathbb{R}^+; \mathcal{Y})$. Then their Laplace transforms $\widehat{f}$ and $\widehat{f'}$ both lie in $H^2(\mathbb{C}^+; \mathcal{Y})$, and upon combining the general Laplace-transform formula $\widehat{f'}(\mu) = \mu\widehat{f}(\mu) - f(0)$ with (2.6), it follows that

$$\lim_{\operatorname{Re}\mu \to \infty} \mu\widehat{f}(\mu) = f(0). \tag{4.36}$$

Hence, the limit $\lim_{\mathrm{Re}\,\eta\to\infty} \eta x(\eta)$ equals $\check{x}(0)$, where $\check{x}$ is the inverse Laplace transform of x. Comparing this to (4.22), we see that A_c^d is the frequency-domain analogue of spatial derivative, the generator of an incoming left shift.

In fact, $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c^*$ is a frequency-domain analogue of the standard output-normalized shift realization of $\widetilde{\varphi}$, but with $\mathcal{H}_c$ as state space rather than an isometrically contained subspace of $H^2(\mathbb{C}^+;\mathcal{U})$; see e.g. [45, Def. 9.5.1]. The reason for choosing the state space $\mathcal{H}_c$ is that it makes $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ energy preserving, provided that we choose the appropriate norm in $\mathcal{H}_c$. The realization $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ is in general not energy preserving if we use the norm of $H^2(\mathbb{C}^+;\mathcal{U})$ on the state space; see [9] for more details on this. These statements are included only in order to provide the reader with some intuition; we make no use of them here and we give no proofs.

As a corollary of (4.33) and (4.28), we see that

$$\mu \mapsto \frac{\widetilde{\varphi}(\mu) - \widetilde{\varphi}(\overline{\alpha})}{\overline{\alpha} - \mu} y \in \mathcal{H}_c \quad \text{for all } \alpha \in \mathbb{C}^+ \text{ and } y \in \mathcal{Y}. \tag{4.37}$$

In fact the formula (4.25) identifying this expression with $(\overline{\alpha} - A_c^*|_{\mathcal{H}_c})^{-1} C_c^* y$ can now be seen as a consequence of the formula (4.33) combined with (4.32).

Finally, note that $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c^*$ can be recovered from A_c^*, C_c^*, B_c^*, and $\widetilde{\varphi}$ evaluated at one arbitrary point in $\mathbb{C}^+$, as described in Remark 3.3.

4.4 More Explicit Formulas for the Controllable Model

In this subsection we obtain more explicit formulas (to the extent possible) for the action of the operators A_c, B_c, and C_c in $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$.

Let us say that an expression of the form $(\alpha - A_c)^{-1} e_c(\lambda)^* u$, where $\alpha, \lambda \in \mathbb{C}^+$ and $u \in \mathcal{U}$, is a *regularized kernel function*. While a kernel function $e_c(\lambda)^* u$ itself may not be in $\operatorname{dom}(A_c)$, a regularized kernel function is always in $\operatorname{dom}(A_c)$. More precisely, by the first assertion in the following proposition

$$\begin{aligned} \mathcal{F}_0 &:= \operatorname{span}\left\{ \frac{e_c(\lambda)^* u - e_c(\overline{\alpha})^* u}{\alpha - \overline{\lambda}} \middle| \lambda \in \mathbb{C}^+,\ u \in \mathcal{U} \right\} \\ &= (\alpha - A_c)^{-1} \operatorname{span}\left\{ e_c(\lambda)^* u | \lambda \in \mathbb{C}^+,\ u \in \mathcal{U} \right\}, \end{aligned} \tag{4.38}$$

and by Lemma 3.5 and (4.7) this linear span is a dense subspace of $\operatorname{dom}(A_c)$ in the graph norm for all fixed $\alpha \in \mathbb{C}^+$. In particular, the difference of two kernel functions with the same $u \in \mathcal{U}$ is in $\operatorname{dom}(A_c)$. (The linear span $\mathcal{F}_0$ can also be viewed as a dense subspace of $\mathcal{H}_c$.)

The first result gives more explicit formulas for the actions of A_c and C_c on regularized kernel functions:

Proposition 4.9 *The following statements hold for the system node* $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ *in Theorem* 4.2*:*

1. *The action of the resolvent of* A_c *on kernel functions is given by*

$$(\alpha - A_c)^{-1}\big(e_c(\lambda)^* u\big) = \frac{e_c(\lambda)^* - e_c(\overline{\alpha})^*}{\alpha - \overline{\lambda}} u, \quad \alpha, \lambda \in \mathbb{C}^+,\ u \in \mathcal{U}. \tag{4.39}$$

   The formula (4.39) *uniquely determines the action of* $(\alpha - A_c)^{-1}$ *on the whole space* $\mathcal{H}_c$ *by linearity and continuity.*

2. *The main operator* A_c *and observation operator* C_c *of* $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ *have the following actions on regularized kernel functions:*

$$A_c\big((\alpha - A_c)^{-1} e_c(\lambda)^* u\big) = \frac{\overline{\lambda} e_c(\lambda)^* - \alpha e_c(\overline{\alpha})^*}{\alpha - \overline{\lambda}} u, \quad \text{and} \tag{4.40}$$

$$C_c\big((\alpha - A_c)^{-1} e_c(\lambda)^* u\big) = \frac{\varphi(\overline{\lambda}) - \varphi(\alpha)}{\alpha - \overline{\lambda}} u, \quad \alpha, \lambda \in \mathbb{C}^+,\ u \in \mathcal{U}. \tag{4.41}$$

   Moreover these formulas uniquely determine A_c *and* C_c *on the whole space* $\mathcal{H}_{c,1} =$ dom (A_c).

Proof 1. We first prove that the resolvent of A_c satisfies (4.39). For arbitrary $\alpha, \lambda \in \mathbb{C}^+$ and $u \in \mathcal{U}$ we by the definition (4.3) of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ have that $\left[\begin{smallmatrix} e_c(\lambda)^* u \\ u \end{smallmatrix}\right], \left[\begin{smallmatrix} e_c(\overline{\alpha})^* u \\ u \end{smallmatrix}\right] \in$ dom $\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c\right)$ and that

$$\begin{aligned}
\begin{bmatrix} e_c(\lambda)^* u \\ u \end{bmatrix} - \begin{bmatrix} e_c(\overline{\alpha})^* u \\ u \end{bmatrix} &= \begin{bmatrix} e_c(\lambda)^* u - e_c(\overline{\alpha})^* u \\ 0 \end{bmatrix} \in \operatorname{dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}\right)_c, \\
\begin{bmatrix} A_c \\ C_c \end{bmatrix} \big(e_c(\lambda)^* - e_c(\overline{\alpha})^*\big) u &= \begin{bmatrix} A\&B \\ C\&D \end{bmatrix}_c \left(\begin{bmatrix} e_c(\lambda)^* u \\ u \end{bmatrix} - \begin{bmatrix} e_c(\overline{\alpha})^* u \\ u \end{bmatrix}\right) \\
&= \begin{bmatrix} \overline{\lambda} e_c(\lambda)^* - \alpha e_c(\overline{\alpha})^* \\ \varphi(\overline{\lambda}) - \varphi(\alpha) \end{bmatrix} u,
\end{aligned} \tag{4.42}$$

so that in particular $e_c(\lambda)^* u - e_c(\overline{\alpha})^* u \in$ dom (A_c) and

$$A_c\big(e_c(\lambda)^* u - e_c(\overline{\alpha})^* u\big) = \overline{\lambda} e_c(\lambda)^* u - \alpha e_c(\overline{\alpha})^* u.$$

Now clearly

$$\begin{aligned}
&(\alpha - A_c)\big(e_c(\lambda)^* u - e_c(\overline{\alpha})^* u\big) \\
&\quad = \alpha e_c(\lambda)^* u - \alpha e_c(\overline{\alpha})^* u - \overline{\lambda} e_c(\lambda)^* u + \alpha e_c(\overline{\alpha})^* u \\
&\quad = (\alpha - \overline{\lambda}) e_c(\lambda)^* u,
\end{aligned}$$

which shows that the resolvent satisfies (4.39). As the span of kernel functions is dense in $\mathcal{H}_c$ and $(\alpha - A_c)^{-1}$ is a bounded linear operator on $\mathcal{H}_o$, we see that (4.39) uniquely determines $(\alpha - A_c)^{-1}$ on the whole space $\mathcal{H}_c$.

2. It follows from (4.42) and (4.39) that A_c and C_c satisfy (4.40) and (4.41), and therefore the maps A_c and C_c are determined by these formulas on the dense subspace $\mathcal{F}_0$ of their domain $\operatorname{dom}(A_c)$. Since A_c and C_c are bounded from $\operatorname{dom}(A_c)$ to $\mathcal{H}_c$ and $\mathcal{Y}$, respectively, the claim follows. $\square$

Proposition 4.9 is of a preliminary nature, and now we proceed to search for the actions of operators on generic elements of their domains. The following important result is a consequence of Theorem 3.12.

Theorem 4.10 *The following claims are true:*

1. *For arbitrary* $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \operatorname{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c\right)$*, we can set* $y := \left[C_c\&D_c\right]\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$ *and obtain*

$$\left[A_c\&B_c\right]\begin{bmatrix} x \\ u \end{bmatrix} = \mu \mapsto -\mu x(\mu) - \widetilde{\varphi}(\mu)y + u, \quad \mu \in \mathbb{C}^+. \tag{4.43}$$

Here u can be recovered from $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ *using (4.19). Moreover,*

$$(A_c x)(\mu) = -\mu x(\mu) - \widetilde{\varphi}(\mu)C_c x, \quad x \in \operatorname{dom}(A_c), \ \mu \in \mathbb{C}^+. \tag{4.44}$$

2. *We have*

$$\operatorname{dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}_c\right) \subset \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in \begin{bmatrix} \mathcal{H}_c \\ \mathcal{U} \end{bmatrix} \middle| \ \exists y \in \mathcal{Y}: \ \mu \mapsto -\mu x(\mu) - \widetilde{\varphi}(\mu)y + u \in \mathcal{H}_c \right\}, \tag{4.45}$$

and in particular

$$\operatorname{dom}(A_c) \subset \left\{ x \in \mathcal{H}_c \middle| \exists y \in \mathcal{Y}: \ \mu \mapsto \mu x(\mu) + \widetilde{\varphi}(\mu)y \in \mathcal{H}_c \right\}. \tag{4.46}$$

Proof The Eq. (4.43) follows from the energy-preserving property of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$, Theorem 3.12.1, and Theorem 4.4. Taking $u = 0$ and using the Definitions (3.2) and (3.4) of A_c and C_c, we obtain (4.44). By Definition 3.1, $\left[A_c\&B_c\right]$ maps $\operatorname{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c\right)$ into $\mathcal{H}_c$, and combining this with (4.43), we get (4.45). Taking $u = 0$ in (4.45) together with (3.2), we arrive at (4.46). $\square$

Corollary 4.11 *For every $u \in \mathcal{U}$, there exist $x \in \mathcal{H}_c$ and $y \in \mathcal{Y}$, such that*

$$\mu \mapsto -\mu x(\mu) - \widetilde{\varphi}(\mu)y + u \in \mathcal{H}_c. \tag{4.47}$$

Also, for every $y \in \mathcal{Y}$, there exist $x \in \mathcal{H}_c$ and $u \in \mathcal{U}$, such that (4.47) holds. The set of $x \in \mathcal{H}_c$, for which there exists a $u \in \mathcal{U}$, such that $\mu \mapsto -\mu x(\mu) + u \in \mathcal{H}_c$ is dense in $\mathcal{H}_c$.

Moreover, (4.47) holds if and only if $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \operatorname{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c^*\right)$ *and* $u = \left[B_c^*\&D_c^*\right]\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$.

Proof By (4.2), for all $u \in \mathcal{U}$, we have $\left[\begin{smallmatrix} e_c(\overline{\lambda})^* u \\ u \end{smallmatrix}\right] \in \mathrm{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c\right)$ for every $\lambda \in \mathbb{C}^+$, and by Theorem 4.10.2, this pair $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$ satisfies (4.47) with $y := \left[C_c\&D_c\right]\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] = \varphi(\lambda)u$. The condition (4.47) is equivalent to $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \mathrm{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c^*\right)$ and $u = \left[B_c^*\&D_c^*\right]\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ by Theorem 4.4. Now the proof is completed by combining (3.3) with the fact that $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c^*$ is a system node with input space $\mathcal{Y}$. (Recall that $\mathrm{dom}(A^*)$ is dense in $\mathcal{X}$ for a system node with main operator A^* and state space $\mathcal{X}$.) □

The inclusion (4.45) is in general not an equality and hence (4.47) does not imply $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \mathrm{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c\right)$; this brings us some difficulties later. If we know A_c, including its domain, then it will soon turn out that we can calculate $C_c x$ for generic $x \in \mathrm{dom}(A_c)$. Then the following continuous-time version of Theorem 1.4 gives a description of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$, including its domain:

Theorem 4.12 *A pair* $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{H}_c \\ \mathcal{U} \end{smallmatrix}\right]$ *lies in* $\mathrm{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c\right)$ *if and only if for some, or equivalently for all,* $\lambda \in \mathbb{C}^+$, *the function* $x - e_c(\lambda)^* u$ *lies in* $\mathrm{dom}(A_c)$. *When this is the case, for an arbitrary fixed* $\lambda \in \mathbb{C}^+$, *the action of* $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ *is*

$$\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}_c \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} \mu \mapsto -\mu x(\mu) - \varphi(\overline{\mu})^* \gamma_\lambda + \left(1 - \varphi(\overline{\mu})^* \varphi(\overline{\lambda})\right) u \\ \gamma_\lambda + \varphi(\overline{\lambda}) u \end{bmatrix}, \\ \gamma_\lambda = C_c\left(x - e_c(\lambda)^* u\right). \tag{4.48}$$

Proof By (3.3) we have that $x - (\overline{\lambda} - A_c|_{\mathcal{H}_c})^{-1} B_c u \in \mathrm{dom}(A_c)$ if and only if $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \mathrm{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c\right)$ for arbitrary $\lambda \in \mathbb{C}^+$, and this should be combined with (4.7) to verify the claim on the domains. It follows from (3.8) and (4.7) that $\left[C_c\&D_c\right]\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] = \gamma_\lambda + \varphi(\overline{\lambda})u$, and together with Theorem 4.10.1, this implies (4.48). □

In the sequel, the following bounded operator from $\mathcal{H}_c$ into $\mathcal{Y}$ plays such an important role that we give it a special notation:

Definition 4.13 We denote

$$\tau_{c,\alpha} := C_c(\alpha - A_c)^{-1}, \quad \alpha \in \mathbb{C}^+. \tag{4.49}$$

From (4.25) it then follows that for all $y \in \mathcal{Y}$:

$$(\tau_{c,\alpha})^* y = (\overline{\alpha} - A_c^*|_{\mathcal{H}_c})^{-1} C_c^* y = \mu \mapsto \frac{\widetilde{\varphi}(\mu) - \widetilde{\varphi}(\overline{\alpha})}{\overline{\alpha} - \mu}, \quad \mu \in \mathbb{C}^+. \tag{4.50}$$

Hence, if x happens to lie in $\mathrm{dom}(A)$, then taking $u = 0$ in (4.48) yields (note the absence of λ from the right-hand side)

$$\gamma_\lambda = C_c x = \tau_{c,\alpha}(\alpha - A_c)x$$

for $\alpha \in \mathbb{C}^+$, and moreover, γ_λ is the unique element in $\mathcal{Y}$, such that

$$\langle \gamma_\lambda, y \rangle_{\mathcal{Y}} = \left\langle (\alpha - A_c)x, \mu \mapsto \frac{\widetilde{\varphi}(\mu) - \widetilde{\varphi}(\overline{\alpha})}{\overline{\alpha} - \mu} y \right\rangle_{\mathcal{H}_c} \quad \text{for all } y \in \mathcal{Y}.$$

This should be compared to the definition of $\widetilde{g}(0)$ in (1.11).

In fact, the arbitrary parameter λ in Theorem 4.12 is only used for calculating $y = \left[C_c\&D_c\right]\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$, which is obviously independent of λ, and the formula for $\left[A_c\&B_c\right]\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$ in (4.48) only depends on y, not on λ; compare (4.48) to (4.43). In Theorem 1.4 there is no need for any arbitrary parameter w, because there $\left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in \mathrm{dom}\left(\left[\begin{smallmatrix} \mathrm{A}_c & \mathrm{B}_c \\ \mathrm{C}_c & \mathrm{D}_c \end{smallmatrix}\right]\right)$ for all $x \in \mathrm{H}_c$, since $\left[\begin{smallmatrix} \mathrm{A}_c & \mathrm{B}_c \\ \mathrm{C}_c & \mathrm{D}_c \end{smallmatrix}\right]$ is bounded.

Remark 4.14 The use of $\tau_{c,\alpha}$ thus allows us to calculate C_c on generic elements of $\mathrm{dom}\,(A_c)$ using $C_c = \tau_{c,\alpha}(\alpha - A_c)$ and (4.50), assuming that we have an explicit formula for A_c, but note that the formula (4.44) gives the action of A_c in terms of C_c. This circle definition can be corrected if the condition $\mathcal{H}_c \cap \{\widetilde{\varphi}(\cdot)y | y \in \mathcal{Y}\} = \{0\}$ holds. Indeed, under this assumption, the function $\widetilde{\varphi}(\cdot)y$, such that $\mu \mapsto \mu x(\mu) + \widetilde{\varphi}(\mu)y \in \mathcal{H}_c$, is uniquely determined by x. (Such a y exists for every $x \in \mathrm{dom}\,(A_c)$ by (4.46).) Note that y will in general not be uniquely determined, only the function $\widetilde{\varphi}(\cdot)y$; see Lemma 4.16 and Theorem 4.17 below for more details on this.

With the help of $\tau_{c,\alpha}$, we have an explicit formula for the resolvent operator $(\alpha - A_c)^{-1}$, $\alpha \in \mathbb{C}^+$, on generic elements of $\mathcal{H}_c$:

Corollary 4.15 *The resolvent operator $(\alpha - A_c)^{-1}$ acting on arbitrary functions in $\mathcal{H}_c$ is given explicitly by*

$$\left((\alpha - A_c)^{-1}x\right)(\mu) = \frac{x(\mu) - \widetilde{\varphi}(\mu)\tau_{c,\alpha}x}{\alpha + \mu}, \quad x \in \mathcal{H}_c,\ \alpha, \mu \in \mathbb{C}^+. \tag{4.51}$$

Proof Setting $z := (\alpha - A_c)^{-1}x$ in (4.51) and using (4.49), we obtain the equivalent condition

$$(\alpha + \mu)z(\mu) = \big((\alpha - A_c)z\big)(\mu) - \widetilde{\varphi}(\mu)C_c z, \quad z \in \mathrm{dom}\,(A_c),\ \mu \in \mathbb{C}^+,$$

which is true by (4.44). □

Using (4.51) involves calculating $\tau_{c,\alpha}x$ for generic $x \in \mathcal{H}_c$, but we have no explicit formula for this except for in the case when x is a kernel function; see (4.41). One way to calculate $\tau_{c,\alpha}x$ is to use (4.50) and calculate

$$\left(\tau_{c,\alpha}x, \gamma\right)_{\mathcal{Y}} = \left(x, \mu \mapsto \frac{\widetilde{\varphi}(\mu) - \widetilde{\varphi}(\overline{\alpha})}{\overline{\alpha} - \mu}\gamma\right)_{\mathcal{H}_c}, \quad \gamma \in \mathcal{Y}.$$

From the domain of a system node $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$, the domain of A_c is constructed using (3.2). Conversely, if we know $\mathrm{dom}\,(A_c)$, then we can recover $\mathrm{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right)$ using (3.3) as in the proof of Theorem 4.12. In particular, the following result shows that we have equality in (4.45) if and only if we have equality in (4.46).

Lemma 4.16 *The following claims are true:*

1. *The condition*

$$\operatorname{dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}_c\right) = \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in \begin{bmatrix} \mathcal{H}_c \\ \mathcal{U} \end{bmatrix} \middle| \exists y \in \mathcal{Y} : \mu \mapsto -\mu x(\mu) - \widetilde{\varphi}(\mu) y + u \in \mathcal{H}_c \right\}, \tag{4.52}$$

holds if and only if

$$\operatorname{dom}(A_c) = \{x \in \mathcal{H}_c | \exists y \in \mathcal{Y} : \ \mu \mapsto \mu x(\mu) + \widetilde{\varphi}(\mu) y \in \mathcal{H}_c\}. \tag{4.53}$$

2. *It holds that*

$$\mathcal{H}_c \cap \{\widetilde{\varphi}(\cdot) y | y \in \mathcal{Y}\} = \{0\} \tag{4.54}$$

if and only if for some (or equivalently for all) $\alpha \in \mathbb{C}^+$:

$$\operatorname{dom}(A_c) \cap \left\{ \mu \mapsto \frac{\widetilde{\varphi}(\mu) y}{\alpha + \mu} \middle| y \in \mathcal{Y} \right\} = \{0\}. \tag{4.55}$$

3. *The conditions* (4.52)–(4.55) *hold if and only if for some (or equivalently for all) $\alpha \in \mathbb{C}^+$:*

$$\mathcal{H}_c \cap \left\{ \mu \mapsto \frac{\widetilde{\varphi}(\mu) y}{\alpha + \mu} \middle| y \in \mathcal{Y} \right\} = \{0\}. \tag{4.56}$$

Proof 1. By Theorem 4.10, the domains are included in the sets on the right-hand sides of (4.52) and (4.53), so we only need to show that the converse inclusions are equivalent. First assume (4.52) and let $x \in \mathcal{H}_c$ and $y \in \mathcal{Y}$ be such that

$$\mu \mapsto \mu x(\mu) + \widetilde{\varphi}(\mu) y \in \mathcal{H}_c. \tag{4.57}$$

The working assumption (4.52) then implies that $\left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in \operatorname{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c\right)$, and according to (3.2), this precisely means that $x \in \operatorname{dom}(A_c)$.
Now assume (4.53) and (4.47). Since $\alpha K_c(\cdot, \overline{\alpha}) u \in \mathcal{H}_c$, we also have

$$\begin{aligned} &- \mu x(\mu) - \widetilde{\varphi}(\mu) y + u - \alpha K_c(\mu, \overline{\alpha}) u \\ &\quad = -\mu x(\mu) + \mu \frac{1 - \widetilde{\varphi}(\mu)\varphi(\alpha)}{\mu + \alpha} u - \widetilde{\varphi}(\mu)\big(y - \varphi(\alpha) u\big) \end{aligned}$$

in $\mathcal{H}_c$ as a function of $\mu \in \mathbb{C}^+$. Hence,

$$\exists \gamma \in \mathcal{Y} : \ \mu \mapsto -\mu x(\mu) + \mu \frac{1 - \widetilde{\varphi}(\mu)\varphi(\alpha)}{\mu + \alpha} u - \widetilde{\varphi}(\mu) \gamma \in \mathcal{H}_c, \tag{4.58}$$

since one can simply take $\gamma := y - \varphi(\alpha)u$. The statement (4.58) is by (4.53) equivalent to $x - e_c(\overline{\alpha})^* u \in \operatorname{dom}(A_c)$, and according to the first assertion in Theorem 4.12, this is equivalent to $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \operatorname{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c\right)$.

2. First assume (4.54) and suppose that $z := \mu \mapsto \frac{\widetilde{\varphi}(\mu)y}{\alpha+\mu} \in \operatorname{dom}(A_c)$ for some choice of $y \in \mathcal{Y}$ and $\alpha \in \mathbb{C}^+$. Use (4.44) and (4.49) to calculate

$$
\begin{aligned}
(\alpha - A_c)z &= \mu \mapsto (\alpha+\mu)z(\mu) + \widetilde{\varphi}(\mu)C_c z \\
&= \mu \mapsto \widetilde{\varphi}(\mu)y + \widetilde{\varphi}(\mu)C_c z \in \mathcal{H}_c \cap \widetilde{\varphi}(\cdot)\mathcal{Y}.
\end{aligned}
$$

By (4.54) this quantity is 0, and since $\alpha - A_c$ is injective, we have $z = 0$, and it follows that (4.55) holds for all $\alpha \in \mathbb{C}^+$.
Conversely, assume that (4.55) holds for some $\alpha \in \mathbb{C}^+$ and suppose that $x := \widetilde{\varphi}(\cdot)y$ is in $\mathcal{H}_c$ for some $y \in \mathcal{Y}$. Use (4.51) to calculate

$$
\begin{aligned}
(\alpha - A_c)^{-1}x &= \mu \mapsto \frac{x(\mu) - \widetilde{\varphi}(\mu)\tau_{c,\alpha}\, x}{\alpha+\mu} \\
&= \mu \mapsto \frac{\widetilde{\varphi}(\mu)}{\alpha+\mu}\left(y - \tau_{c,\alpha}\, x\right) \in \operatorname{dom}(A_c) \cap \frac{\widetilde{\varphi}(\cdot)}{\alpha+\cdot}\mathcal{Y}.
\end{aligned}
$$

By (4.55) this quantity is 0, and hence also $x = 0$, which proves (4.54).

3. First assume that (4.56) is satisfied. Then trivially (4.55) holds, since $\operatorname{dom}(A_c) \subset \mathcal{H}_c$, and we next prove that (4.53) is satisfied too. Suppose that $x \in \mathcal{H}_c$ and $y \in \mathcal{Y}$ are such that (4.57) holds. Then for every $\alpha \in \mathbb{C}^+$ it also holds that

$$
\begin{aligned}
z &:= \mu \mapsto (\alpha+\mu)x(\mu) + \widetilde{\varphi}(\mu)y \in \mathcal{H}_c \quad \text{and thus} \\
x &= \mu \mapsto \frac{z(\mu) - \widetilde{\varphi}(\mu)y}{\alpha+\mu} \in \mathcal{H}_c.
\end{aligned}
$$

On the other hand we have

$$
(\alpha - A_c)^{-1}z = \mu \mapsto \frac{z(\mu) - \widetilde{\varphi}(\mu)\tau_{c,\alpha}z}{\alpha+\mu} \in \operatorname{dom}(A_c) \subset \mathcal{H}_c,
$$

and these two together imply that $\mu \mapsto \frac{\widetilde{\varphi}(\mu)(y-\tau_{c,\alpha}z)}{\alpha+\mu} \in \mathcal{H}_c$. The working assumption (4.56) then gives that $\frac{\widetilde{\varphi}(\mu)(y-\tau_{c,\alpha}z)}{\alpha+\mu} = 0$ for all $\mu \in \mathbb{C}^+$, i.e., $\widetilde{\varphi}(\cdot)y = \widetilde{\varphi}(\cdot)\tau_{c,\alpha}z$, and this implies that $x = (\alpha - A_c)^{-1}z \in \operatorname{dom}(A_c)$.
Finally, with the objective of showing (4.56), we assume that $\alpha \in \mathbb{C}^+$ is such that (4.53) and (4.55) hold. Then we pick an $x := \mu \mapsto \frac{\widetilde{\varphi}(\mu)\eta}{\alpha+\mu} \in \mathcal{H}_c$, so that also $\alpha x \in \mathcal{H}_c$, and thus by (4.53):

$$
\begin{aligned}
x \in \operatorname{dom}(A_c) &\iff \exists \gamma \in \mathcal{Y}:\ \mu \mapsto \mu\frac{\widetilde{\varphi}(\mu)\eta}{\alpha+\mu} + \widetilde{\varphi}(\mu)\gamma \in \mathcal{H}_c \\
&\iff \exists \gamma \in \mathcal{Y}:\ \mu \mapsto \mu\frac{\widetilde{\varphi}(\mu)\eta}{\alpha+\mu} + \alpha\frac{\widetilde{\varphi}(\mu)\eta}{\alpha+\mu} + \widetilde{\varphi}(\mu)\gamma \in \mathcal{H}_c.
\end{aligned}
$$

This is seen to be true by simply choosing $\gamma := -\eta$. Thus every $x \in \mathcal{H}_c$ of the form $x(\mu) = \frac{\widetilde{\varphi}(\mu)y}{\alpha+\mu}$ also lies in dom (A_c), and (4.55) finally gives the desired result. □

4.5 Conservativity and the Extrapolation Space

Theorem 1.5 includes a characterization of the case where controllable isometric realization is in fact unitary. The corresponding situation in the present paper is that $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ is not only energy preserving, but even conservative, so that also $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c^*$ is also energy preserving. We have the following characterizations of this case:

Theorem 4.17 *The following conditions are equivalent:*

1. *The system node* $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ *is conservative.*
2. *The condition* (4.52) *holds together with the following implication:*

$$\widetilde{\varphi}(\cdot)y \in \mathcal{H}_c \Longrightarrow y = 0. \tag{4.59}$$

3. *The condition* (4.56) *holds for some (or equivalently for all)* $\alpha \in \mathbb{C}^+$ *and*

$$\widetilde{\varphi}(\mu)y = 0 \textit{ for all } \mu \in \mathbb{C}^+ \Longrightarrow y = 0. \tag{4.60}$$

4. *The function* $1-\widetilde{\varphi}(\cdot)^*\widetilde{\varphi}(\cdot)$ *has maximal factorable minorant in the right half-plane sense equal to 0, i.e., if* $a : \mathbb{C}^+ \to \mathcal{B}(\mathcal{Y}, \mathcal{Y}')$ *is holomorphic with* $a(\mu)^*a(\mu) \le 1 - \widetilde{\varphi}(\mu)^*\widetilde{\varphi}(\mu)$ *for almost all* μ *on the imaginary line* $i\mathbb{R}$*, then* $a = 0$.

Proof This proof is heavily based on Theorem 3.12. We first prove that *3. implies 2.*: Assume (4.56) and (4.60). Then (4.52) and (4.54) hold by Lemma 4.16.3. Hence, if $\widetilde{\varphi}(\cdot)y \in \mathcal{H}_c$, then $\widetilde{\varphi}(\mu)y = 0$, for all $\mu \in \mathbb{C}^+$, which by (4.60) implies $y = 0$.

We next show that *2. implies 1.* Assume that (4.52) and (4.59) hold. By Theorem 3.12.2, we need to show that $\left[\begin{smallmatrix} [1\,0] \\ [C_c\&D_c] \end{smallmatrix}\right]$ maps dom $\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c\right)$ onto dom $\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c^*\right)$. Hence fix $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in$ dom $\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c^*\right)$ arbitrarily and let u be the unique element in $\mathcal{U}$ for which (4.47) holds; see (4.20). Then $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in$ dom $\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c\right)$ by (4.52) and we may define $\eta := \left[C_c\&D_c\right]\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$. It follows from Theorem 4.10.1 that (4.47) holds also with y replaced by η, and hence $\widetilde{\varphi}(\cdot)(y-\eta) \in \mathcal{H}_c$. Now the implication (4.59) gives that $y = \eta = \left[C_c\&D_c\right]\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$.

Next we establish that *1. implies 3.*: By Theorem 3.12.2, conservativity of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ implies that $\mathcal{R} := \operatorname{im}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c + \left[\begin{smallmatrix} \overline{\alpha} & 0 \\ 0 & 0 \end{smallmatrix}\right]\right)$ is dense in $\left[\begin{smallmatrix} \mathcal{H}_c \\ \mathcal{Y} \end{smallmatrix}\right]$ for some, or equivalently for all, $\alpha \in \mathbb{C}^+$. From the construction of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ it follows that

$$\mathcal{G} := \operatorname{span}\left\{\begin{bmatrix} (\overline{\alpha}+\mu)e_c(\overline{\mu})^*u \\ \varphi(\mu)u \end{bmatrix} \middle| \mu \in \mathbb{C}^+,\ u \in \mathcal{U}\right\}$$

is dense in $\mathcal{R}$, because by the construction of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$, the graph of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c + \left[\begin{smallmatrix} \overline{\alpha} & 0 \\ 0 & 0 \end{smallmatrix}\right]$ is the closure of

$$\operatorname{span}\left\{\begin{bmatrix}(\overline{\alpha}+\mu)e_c(\overline{\mu})^*u\\ \varphi(\mu)u\\ e_c(\overline{\mu})^*u\\ u\end{bmatrix}\middle|\mu\in\mathbb{C}^+,\ u\in\mathcal{U}\right\}\quad\text{in}\quad\begin{bmatrix}\mathcal{H}_c\\ \mathcal{Y}\\ \mathcal{H}_c\\ \mathcal{U}\end{bmatrix}.$$

Therefore $\mathcal{G}$ is also dense in $\begin{bmatrix}\mathcal{H}_c\\ \mathcal{Y}\end{bmatrix}$.

Now note that we have

$$\begin{aligned}&\begin{bmatrix}z\\ y\end{bmatrix}\in\begin{bmatrix}\mathcal{H}_c\\ \mathcal{Y}\end{bmatrix}\ominus\mathcal{G}\\ &\iff\left(\begin{bmatrix}(\overline{\alpha}+\mu)e_c(\overline{\mu})^*u\\ \varphi(\mu)u\end{bmatrix},\begin{bmatrix}z\\ y\end{bmatrix}\right)_{\left[\begin{smallmatrix}\mathcal{H}_c\\ \mathcal{Y}\end{smallmatrix}\right]}=0\quad\text{for all }\mu\in\mathbb{C}^+,\ u\in\mathcal{U}\\ &\iff z\in\mathcal{H}_c\quad\text{and}\quad z(\overline{\mu})=-\frac{\widetilde{\varphi}(\overline{\mu})y}{\alpha+\overline{\mu}},\quad\mu\in\mathbb{C}^+.\end{aligned}$$

Thus, if $\mu\mapsto\frac{\widetilde{\varphi}(\mu)y}{\alpha+\mu}\in\mathcal{H}_c$ then we can denote this function by $-z$ and get that $\left[\begin{smallmatrix}z\\ y\end{smallmatrix}\right]\in\begin{bmatrix}\mathcal{H}_c\\ \mathcal{Y}\end{bmatrix}\ominus\mathcal{G}$, which then by the above implies that $z=0$ and $y=0$.

The equivalence of assertions one and four is reduced to the corresponding discrete result in Theorem 1.5 using the Cayley transform described in Sect. 6. □

Remark 4.18 We can make the following interesting observations:

1. The condition (4.59) implies (4.54). Together with (4.60), (4.54) also implies (4.59). Note that (4.60) can also be written $\cap_{\mu\in\mathbb{C}^+}\ker(\widetilde{\varphi}(\mu))=\{0\}$. Implication (4.59) holds, e.g., if $\widetilde{\varphi}$ is bounded away from zero on $i\mathbb{R}$. Indeed, in this case $\widetilde{\varphi}(\cdot)y$ is not even in $L^2(i\mathbb{R},\mathcal{U})$ for any nonzero $y\in\mathcal{Y}$, and *a fortiori*, $\widetilde{\varphi}(\cdot)y\notin\mathcal{H}_c$. In the example in Sect. 4.6 below, the implication (4.59) can be false but (4.54) is true.
2. Note that (4.56) is true if $\widetilde{\varphi}$ is inner: In this case $\mathcal{H}_c=H^2(\mathbb{C}^+;\mathcal{U})\ominus M_{\widetilde{\varphi}}H^2(\mathbb{C}^+;\mathcal{Y})$ isometrically by Corollary 2.5, and the function $\mu\mapsto\frac{\widetilde{\varphi}(\mu)y}{\alpha+\mu}$, $\mu\in\mathbb{C}^+$, is $M_{\widetilde{\varphi}}$ applied to the kernel function $e(\overline{\alpha})^*y$ in $H^2(\mathbb{C}^+;\mathcal{Y})$, cf. Lemma 2.1.2. Hence, for every co-inner φ, the model $\left[\begin{smallmatrix}A\&B\\ C\&D\end{smallmatrix}\right]_c$ is conservative.
3. On the other extreme of the situation in 2., if $\|\varphi\|_{H^\infty}=\delta<1$, then $\mathcal{H}_c$ is simply a re-normed version of $H^2(\mathbb{C}^+;\mathcal{U})$ by (2.4), since $\sqrt{1-\delta^2}\le(1-M_{\widetilde{\varphi}}M_{\widetilde{\varphi}}^*)^{1/2}\le 1$ and $Qx=x$, because $1-M_{\widetilde{\varphi}}M_{\widetilde{\varphi}}^*$ is injective. Then the intersection in (4.56) is all of $\left\{\mu\mapsto\frac{\widetilde{\varphi}(\mu)y}{\alpha+\mu}\middle|y\in\mathcal{Y},\ \mu\in\mathbb{C}^+\right\}$ and $\mathcal{H}_c$ is not conservative unless $\varphi(\mu)=0$ for all $\mu\in\mathbb{C}^+$. If φ is identically zero, then condition (4.60) is violated and hence $\left[\begin{smallmatrix}A\&B\\ C\&D\end{smallmatrix}\right]_c$ is not conservative in this case either. Thus, $\left[\begin{smallmatrix}A\&B\\ C\&D\end{smallmatrix}\right]_c$ can be conservative only if $\|\varphi\|_{H^\infty(\mathbb{C}^+;\mathcal{U},\mathcal{Y})}=1$.

In the rest of this subsection, we assume that (4.56) holds, which is true e.g. if $\left[\begin{smallmatrix}A\&B\\ C\&D\end{smallmatrix}\right]_c$ is conservative. In this case we can proceed to identify $\mathcal{H}_{c,-1}$ concretely and calculate $A_c|_{\mathcal{H}_c}$ and B_c explicitly. In addition to (4.56), we make vital use of the characterization (4.53) of dom (A_c).

Since dom (A_c) is given by (4.53) and $\mathcal{H}_{c,-1} = (\beta - A_c|_{\mathcal{H}_c})\mathcal{H}_c$, the formula (4.51) for the resolvent of A_c suggests the following concrete identification of the extrapolation space:

$$\mathcal{H}_{c,-1} = \left\{ x : \mathbb{C}^+ \to \mathcal{Y} \middle| \exists y \in \mathcal{Y} : \ \mu \mapsto \frac{x(\mu) + \widetilde{\varphi}(\mu)y}{\beta + \mu} \in \mathcal{H}_c \right\}. \tag{4.61}$$

The property (4.54) guarantees us that the function $\widetilde{\varphi}(\cdot)y$ in (4.61) is uniquely determined by x (whenever at least one such function exists). Note that the choice of y is is in general not unique. Now we set

$$\|x\|_{\mathcal{H}_{c,-1}} = \left\| \mu \mapsto \frac{x(\mu) + \widetilde{\varphi}(\mu)y}{\beta + \mu} \right\|_{\mathcal{H}_c}, \quad x \in \mathcal{H}_{c,-1}. \tag{4.62}$$

We have $\widetilde{\varphi}(\cdot)\gamma \in \mathcal{H}_{c,-1}$ with zero norm for all $\gamma \in \mathcal{Y}$; simply choose $y := -\gamma$ in (4.61) and (4.62). Conversely, if $\|x\|_{\mathcal{H}_{c,-1}} = 0$, then $\mu \mapsto \frac{x(\mu)+\widetilde{\varphi}(\mu)y}{\beta+\mu} = 0$ for all $\mu \in \mathbb{C}^+$, i.e., $x(\mu) = -\widetilde{\varphi}(\mu)y$ for all $\mu \in \mathbb{C}^+$. Hence, the elements of $\mathcal{H}_{c,-1}$ are equivalence classes of functions modulo the subspace $\widetilde{\varphi}(\cdot)\mathcal{Y}$. These equivalence classes are denoted as $[x]$, where x is any particular representative of the equivalence class. We summarize the properties of the space $\mathcal{H}_{c,-1}$ as follows:

Theorem 4.19 *Assume that (4.56) holds. Then the space $\mathcal{H}_{c,-1}$ given by (4.61) and (4.62) is complete and the following claims are true:*

1. *The map $\iota : x \mapsto [x]$ embeds $\mathcal{H}_c$ into $\mathcal{H}_{c,-1}$ as a dense subspace. A given element $[z] \in \mathcal{H}_{c,-1}$ is of the form $\iota(x)$ for some $x \in \mathcal{H}_c$ if and only if there is a choice of y in $\mathcal{Y}$ so that the function $\mu \mapsto \dfrac{z(\mu) + \widetilde{\varphi}(\mu)y}{\alpha + \mu}$ is not only in $\mathcal{H}_c$ but also in $\mathcal{H}_{c,1} = \mathrm{dom}\,(A_c)$ for some (or equivalently for every) $\alpha \in \mathbb{C}^+$.*
2. *Define an operator $A_c|_{\mathcal{H}_c} : \mathcal{H}_c \to \mathcal{H}_{c,-1}$ by*

$$A_c|_{\mathcal{H}_c} x := [\mu \mapsto -\mu x(\mu)], \quad x \in \mathcal{H}_c, \ \mu \in \mathbb{C}^+. \tag{4.63}$$

When $\mathcal{H}_c$ is identified as a linear sub-manifold of $\mathcal{H}_{c,-1}$ via the embedding map ι above, then $A_c|_{\mathcal{H}_c}$ is the unique extension of $A_c : \mathrm{dom}\,(A_c) \to \mathcal{H}_c$ to a continuous operator $\mathcal{H}_c \to \mathcal{H}_{c,-1}$. Moreover, the following operator is unitary from $\mathcal{H}_{c,-1}$ to $\mathcal{H}_c$

$$\big((\beta - A_c|_{\mathcal{H}_c})^{-1}[x]\big)(\mu) = \frac{x(\mu) + \widetilde{\varphi}(\mu)y}{\beta + \mu}, \quad [x] \in \mathcal{H}_{c,-1}, \ \mu \in \mathbb{C}^+, \tag{4.64}$$

where the condition $\mu \mapsto \frac{x(\mu)+\widetilde{\varphi}(\mu)y}{\beta+\mu} \in \mathcal{H}_c$ uniquely determines $\widetilde{\varphi}(\cdot)y$.

3. *The action of $B_c : \mathcal{U} \to \mathcal{H}_{c,-1}$ is given by*

$$B_c u := [\mu \mapsto u], \quad u \in \mathcal{U}, \ \mu \in \mathbb{C}^+.$$

Proof Completeness of $\mathcal{H}_{c,-1}$ is proved precisely the same way as completeness of $\mathcal{H}^d_{c,-1}$ was proved in Theorem 4.7: For a Cauchy sequence $[x_n]$ in $\mathcal{H}_{c,-1}$, denote the limit of the Cauchy sequence $z_n := \mu \mapsto \frac{x_n(\mu)+\widetilde{\varphi}(\mu)y_n}{\beta+\mu}$ in $\mathcal{H}_c$ by z. Then $[x_n] \to [x]$ in $\mathcal{H}_{c,-1}$, where $x(\mu) = (\beta+\mu)z(\mu),\ \mu \in \mathbb{C}^+$.

1. It is clear that $\iota(\mathcal{H}_c) \subset \mathcal{H}_{c,-1}$, since by (4.51),

$$\mu \mapsto \frac{x(\mu)+\widetilde{\varphi}(\mu)y}{\beta+\mu} = (\beta - A_c)^{-1}x \in \mathcal{H}_{c,1} \subset \mathcal{H}_c$$

with $y = -\tau_{c,\alpha}x$. We prove denseness of $\mathcal{H}_c$ in $\mathcal{H}_{c,-1}$ in the proof of assertion two.
Next assume that the function $g(\mu) := \frac{z(\mu)+\widetilde{\varphi}(\mu)y}{\alpha+\mu}$ lies in $\operatorname{dom}(A_c)$ for some $\alpha \in \mathbb{C}^+$. We need to prove that $x := (\alpha - A_c)g \in \mathcal{H}_c$ has the property $[z] = \iota(x)$. We may use formula (4.44) to compute

$$\begin{aligned}\big((\alpha - A_c)g\big)(\mu) &= (\alpha+\mu)g(\mu) + \widetilde{\varphi}(\mu)C_c g\\ &= z(\mu) + \widetilde{\varphi}(\mu)y + \widetilde{\varphi}(\mu)C_c g\\ &= z(\mu) + \widetilde{\varphi}(\mu)\big(y + C_c g\big) = x(\mu),\end{aligned}$$

and we can conclude that $[z] = [x]$, where $x \in \mathcal{H}_c$.
The converse implication is seen as follows. Assume that $[z] = [x]$ with $x \in \mathcal{H}_c$ and let $\alpha \in \mathbb{C}^+$ be arbitrary. Then $z(\mu) = x(\mu) + \widetilde{\varphi}(\mu)\gamma$ for some $\gamma \in \mathcal{Y}$, and by (4.51) it holds that

$$\big((\alpha - A_c)^{-1}x\big)(\mu) = \frac{x(\mu) - \widetilde{\varphi}(\mu)\tau_{c,\alpha}x}{\alpha+\mu} = \frac{z(\mu) - \widetilde{\varphi}(\mu)(\tau_{c,\alpha}x + \gamma)}{\alpha+\mu}.$$

Choosing $y := -\tau_{c,\alpha}x - \gamma$, we thus have that $\mu \mapsto \frac{z(\mu)+\widetilde{\varphi}(\mu)y}{\alpha+\mu} \in \operatorname{dom}(A_c)$ for every $\alpha \in \mathbb{C}^+$.

2. If $x \in \mathcal{H}_c$ and $z(\mu) = -\mu x(\mu)$, then $[\beta x - z] \in \mathcal{H}_{c,-1}$ since

$$\mu \mapsto \frac{\beta x(\mu) - z(\mu)}{\beta+\mu} = x \in \mathcal{H}_c; \tag{4.65}$$

take $y = 0$ in (4.61). Moreover, $\beta x \in \mathcal{H}_c \subset \mathcal{X}_{c,-1}$, and it follows that $[z] = [\mu \mapsto -\mu x(\mu)] \in \mathcal{H}_{c,-1}$. This shows that (4.63) defines an operator from all of $\mathcal{H}_c$ into $\mathcal{H}_{c,-1}$. If it happens that $x \in \operatorname{dom}(A_c)$ then $A_c|_{\mathcal{H}_c}x = [A_c x]$ by (4.44), and hence $A_c|_{\mathcal{H}_c}$ is an extension of A_c.
The operator in (4.64) maps $\mathcal{H}_{c,-1}$ into $\mathcal{H}_c$ and it equals $(\beta - A_c|_{\mathcal{H}_c})^{-1}$, because (4.63) gives

$$\big[(\beta - A_c|_{\mathcal{H}_c})^{-1}[x]\big] = \left[\mu \mapsto \frac{x(\mu)}{\beta+\mu}\right], \quad \mu \in \mathbb{C}^+,$$

and as $[x] \in \mathcal{H}_{c,-1}$, there by (4.61) exists a $y \in \mathcal{Y}$ such that $\mu \mapsto \frac{x(\mu)+\widetilde{\varphi}(\mu)y}{\beta+\mu} \in \mathcal{H}_c$. By (4.61), (4.62), and (4.64), $(\beta - A_c|_{\mathcal{H}_c})^{-1}$ maps $\mathcal{H}_{c,-1}$ isometrically into $\mathcal{H}_c$. On the other hand, for an arbitrary $z \in \mathcal{H}_c$, $x := \mu \mapsto (\beta + \mu)z(\mu)$ satisfies $[x] \in \mathcal{H}_{c,-1}$ and $(\beta - A_c|_{\mathcal{H}_c})^{-1}[x] = z$. Thus $(\beta - A_c|_{\mathcal{H}_c})^{-1}$ is onto $\mathcal{H}_c$, and from here it follows by a standard argument that $A_c|_{\mathcal{H}_c} \in \mathcal{B}(\mathcal{H}_c, \mathcal{H}_{c,-1})$.

3. Combining (4.7) with the formula for $A_c|_{\mathcal{H}_c}$, we see that for arbitrary $\alpha \in \mathbb{C}^+$:

$$\begin{aligned} B_c u &= (\alpha - A_c|_{\mathcal{H}_c}) e_c(\overline{\alpha})^* u \\ &= \left[\mu \mapsto (\alpha + \mu)\frac{1 - \widetilde{\varphi}(\mu)\varphi(\alpha)}{\mu + \alpha} u\right] = [\mu \mapsto u], \quad u \in \mathcal{U},\ \mu \in \mathbb{C}^+. \end{aligned}$$

□

It follows from Assertion 3 in Theorem 4.19 that all constant functions $[\mu \mapsto u]$ are in $\mathcal{H}_{c,-1}$, but here they have non-zero norm in general. This can actually be seen directly in (4.61), by choosing $y := -\varphi(\beta)u$; then the function in (4.62) is $e_c(\overline{\beta})^* u$, and $\|[u]\|_{\mathcal{H}_{c,-1}} = \left\|e_c(\overline{\beta})^* u\right\|_{\mathcal{H}_c} \neq 0$.

So far we only know the resolvent of $A_c|_{\mathcal{H}_c}$ at the single point β corresponding to the rigging. Based on (4.51) and (4.64), it seems plausible to guess that for other $\alpha \in \mathbb{C}^+$ the resolvent at α would be

$$\big((\alpha - A_c|_{\mathcal{H}_c})^{-1}[x]\big)(\mu) = \frac{x(\mu) + \widetilde{\varphi}(\mu)\gamma}{\alpha + \mu}, \quad [x] \in \mathcal{H}_{c,-1},\ \alpha, \mu \in \mathbb{C}^+, \tag{4.66}$$

for some $\gamma \in \mathcal{Y}$, which depends on $[x]$ and α.

Proposition 4.20 *Assume that (4.56) holds. For $[x] \in \mathcal{H}_{c,-1}$, let $y \in \mathcal{Y}$ be such that $\mu \mapsto \frac{x(\mu)+\widetilde{\varphi}(\mu)y}{\beta+\mu} \in \mathcal{H}_c$. Then (4.66) holds with the choice*

$$\gamma := y + (\beta - \alpha)\tau_{c,\alpha}\frac{x(\cdot) + \widetilde{\varphi}(\cdot)y}{\beta + \cdot}.$$

Proof We can use the resolvent formula

$$(\alpha - A_c|_{\mathcal{H}_c})^{-1} = (\beta - A_c|_{\mathcal{H}_c})^{-1} + (\beta - \alpha)(\alpha - A_c)^{-1}(\beta - A_c|_{\mathcal{H}_c})^{-1}$$

together with (4.51) and (4.64) to calculate (for $\mu \in \mathbb{C}^+$):

$$\begin{aligned} \big((\alpha - A_c|_{\mathcal{H}_c})^{-1}[x]\big)(\mu) &= \frac{x(\mu) + \widetilde{\varphi}(\mu)y}{\beta + \mu} \\ &\quad + (\beta - \alpha)\frac{\frac{x(\mu)+\widetilde{\varphi}(\mu)y}{\beta+\mu} - \widetilde{\varphi}(\mu)\tau_{c,\alpha}\frac{x(\cdot)+\widetilde{\varphi}(\cdot)y}{\beta+\cdot}}{\alpha + \mu}. \end{aligned} \tag{4.67}$$

Straightforward simplifications show that (4.67) minus the right-hand side of (4.66) equals

$$\frac{\widetilde{\varphi}(\mu)}{\alpha+\mu}\left(y-\gamma+(\beta-\alpha)\tau_{c,\alpha}\frac{x(\cdot)+\widetilde{\varphi}(\cdot)y}{\beta+\cdot}\right), \tag{4.68}$$

which proves the claim. □

4.6 An Example: Constant Schur Functions

We illustrate some of the results so far using the case of a constant φ. Assume that $\varphi(\mu) = D_c$ for all $\mu \in \mathbb{C}^+$, so that $\varphi \in \mathcal{S}(\mathbb{C}^+;\mathcal{U},\mathcal{Y})$ if and only if $\|D_c\| \le 1$. Then it follows from (4.50) that $\tau_{c,\alpha} = 0$ and $C_c = 0$, and Corollary 4.15 then yields that $(\alpha - A_c)^{-1}x = \mu \mapsto x(\mu)/(\alpha+\mu),\ \mu \in \mathbb{C}^+$, and by Theorem 4.10, $(A_c x)(\mu) = -\mu x(\mu),\ \mu \in \mathbb{C}^+$, for $x \in \operatorname{dom}(A_c)$, where

$$\operatorname{dom}(A_c) = (\alpha - A_c)^{-1}\mathcal{H}_c = \left\{\mu \mapsto \frac{z(\mu)}{\alpha+\mu}\,\middle|\, z \in \mathcal{H}_c,\ \mu \in \mathbb{C}^+\right\}. \tag{4.69}$$

Here $\alpha \in \mathbb{C}^+$ is arbitrary, and by Definition 3.1, we also have

$$\operatorname{dom}(A_c) = \{x \in \mathcal{H} | A_c x \in \mathcal{H}_c\} = \{x \in \mathcal{H}_c | \mu \mapsto \mu x(\mu) \in \mathcal{H}_c\},$$

so that (4.53) holds with the additional simplification that we only need to consider $y = 0$.

Now Theorem 4.12 gives (for some arbitrary $\alpha \in \mathbb{C}^+$)

$$\begin{aligned}
&\operatorname{dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}_c\right) \\
&= \left\{\begin{bmatrix} x \\ u \end{bmatrix} \in \begin{bmatrix} \mathcal{H}_c \\ \mathcal{U} \end{bmatrix} \,\middle|\, \frac{(\alpha+\mu)x(\mu) - (1 - D_c^* D_c)u}{\alpha+\mu} \in \operatorname{dom}(A_c)\right\} \\
&= \left\{\begin{bmatrix} x \\ u \end{bmatrix} \in \begin{bmatrix} \mathcal{H}_c \\ \mathcal{U} \end{bmatrix} \,\middle|\, \mu \mapsto -\mu x(\mu) + (1 - D_c^* D_c)u \in \mathcal{H}_c\right\}, \quad \text{and} \\
&\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}_c \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} \mu \mapsto -\mu x(\mu) + (1 - D_c^* D_c)u \\ D_c u \end{bmatrix},
\end{aligned} \tag{4.70}$$

where we used (4.69) and that $\alpha x \in \mathcal{H}_c$ in the second equality. Note that the arbitrary parameter $\lambda \in \mathbb{C}^+$ in (4.48) does not appear here.

In (4.70) the state part is purely for energy accounting, since the output is independent of the current state. If it happens that D_c is isometric, then the energy is preserved without any state needing to absorb energy, and indeed $\mathcal{H}_c = \{0\}$ in this case, as can easily be seen from the reproducing kernel K_c of $\mathcal{H}_c$. Then the realization consists only of the static operator D_c. If D_c is not isometric, then the function $B_c u = \mu \mapsto (1 - D_c^* D_c)u,\ \mu \in \mathbb{C}^+$, never lies in $\mathcal{H}_c$ unless it is zero. Thus B_c is *strictly unbounded* (in the terminology of [32]), and it is interesting that both A_c and B_c are unbounded even though the transfer function φ is rational, even constant.

We make the following observations on the dual system node $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c^*$: Due to Theorem 4.4, the first equality holds in

$$\begin{aligned} \operatorname{dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}_c^*\right) &= \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \begin{bmatrix} \mathcal{H}_c \\ \mathcal{Y} \end{bmatrix} \,\middle|\, \right. \\ &\qquad \left. \exists u \in \mathcal{U} : \ \mu \mapsto \mu x(\mu) + D_c^* y - u \in \mathcal{H}_c \right\} \\ &= \begin{bmatrix} \operatorname{dom}\left(A_c^*\right) \\ \mathcal{Y} \end{bmatrix}, \end{aligned} \tag{4.71}$$

where as usual

$$\operatorname{dom}\left(A_c^*\right) = \{x \in \mathcal{H}_c | \exists u \in \mathcal{U} : \ \mu \mapsto \mu x(\mu) - u \in \mathcal{H}_c\},$$

because for all $x \in \mathcal{H}_c$ and $y \in \mathcal{Y}$, it holds that

$$\exists u \in \mathcal{U} : \ \mu \mapsto \mu x(\mu) + D_c^* y - u \in \mathcal{H}_c \iff \exists v \in \mathcal{U} : \ \mu \mapsto \mu x(\mu) - v \in \mathcal{H}_c.$$

It is a rare convenience that the domain of a system node decomposes into a product space in this way; it is for instance not the case for $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ itself.

The fact that $C_c^* = 0$ can also be seen in (4.33), since

$$\|C_c^* y\|_{\mathcal{H}^d_{c,-1}} = \left\|[\mu \mapsto D_c^* y]\right\|_{\mathcal{H}^d_{c,-1}} = 0, \quad y \in \operatorname{dom} A_c^*.$$

A consequence of $C_c^* = 0$ is that $[A_c^* \& C_c^*][xy] = A_c^* x = \mu \mapsto \mu x(\mu) - \lim_{\eta \to \infty} \eta x(\eta)$. This agrees with (4.18)–(4.19), since $\lim_{\mathrm{Re}\eta \to \infty} \widetilde{\varphi}(\eta) y = \widetilde{\varphi}(\mu) y, \ \ \mu \in \mathbb{C}^+$, and these terms cancel in (4.18).

Due to Proposition 2.6, we have that $\widetilde{\varphi}(\cdot) y = \mu \mapsto D_c^* y \in \mathcal{H}_c$ only if $D_c^* y = 0$, and thus (4.54) holds. The implication (4.59), on the other hand, holds if and only if D_c^* is injective, i.e., D_c has range dense in $\mathcal{Y}$. By Theorem 4.17, D_c has dense range and (4.56) holds if and only if $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ in (4.70) is conservative.

4.7 Reproducing Kernels of the Rigged Spaces

We finish our study of the controllable realization by calculating the reproducing kernels associated to the rigged spaces. In this subsection we do not make any additional assumptions, such as (4.56), unless otherwise indicated.

Recall that the state space $\mathcal{H}_c$ for the system node $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ is a reproducing kernel Hilbert space with reproducing kernel K_c as in (4.1). By construction the 1-scaled rigged space $\mathcal{H}^d_{c,1}$ of the dual also consists of $\mathcal{U}$-valued functions and it is not difficult to see that the point-evaluation map $e^d_{c,1}(\mu)$ is bounded in $\mathcal{H}^d_{c,1}$-norm and hence $\mathcal{H}^d_{c,1}$ is also a reproducing kernel Hilbert space. The same is true for $\mathcal{H}_{c,1}$.

Technically the (-1)-scaled rigged spaces $\mathcal{H}^d_{c,-1}$ and $\mathcal{H}_{c,-1}$ are not reproducing kernel Hilbert spaces since they consist of equivalence classes of functions rather than of functions. However, while point-evaluation is not well-defined on $\mathcal{H}^d_{c,-1}$, the map $[z] \mapsto z(\mu) - z(\overline{\alpha})$ is well-defined for all fixed $\alpha \in \mathbb{C}^+$; this amounts to evaluating the unique member z of the equivalence class $[z]$ normalized to satisfy $z(\overline{\alpha}) = 0$. Here it is most convenient to choose $\overline{\alpha} = \overline{\beta}$, the parameter used to define the $\mathcal{H}^d_{c,1}$ and $\mathcal{H}^d_{c,-1}$ norms.

If we define $e^d_{c,-1}(\mu) : \mathcal{H}^d_{c,-1} \to \mathcal{U}$ by

$$e^d_{c,-1}(\mu)[z] := z(\mu) - z(\overline{\beta}), \quad [z] \in \mathcal{H}^d_{c,-1},\ \mu \in \mathbb{C}^+,$$

then $e^d_{c,-1}(\mu)$ is also bounded for every $\mu \in \mathbb{C}^+$. More precisely, $\|e^d_{c,-1}\| \le |\overline{\beta} - \mu|\, \|e_c(\mu)\|$, since we by (4.32) have

$$\begin{aligned}\left\| e^d_{c,-1}(\mu)[z] \right\|_{\mathcal{U}} = \|z(\mu) - z(\overline{\beta})\|_{\mathcal{U}} &= \left\| e_c(\mu)(\overline{\beta} - \mu)\big((\overline{\beta} - A^*_c|_{\mathcal{H}_c})^{-1}[z]\big) \right\|_{\mathcal{U}} \\ &\le \|e_c(\mu)\|\, |\overline{\beta} - \mu|\, \|[z]\|_{\mathcal{H}^d_{c,-1}}, \quad [z] \in \mathcal{H}^d_{c,-1}.\end{aligned}$$

We may then consider the function

$$K^d_{c,-1}(\mu, \lambda) = e^d_{c,-1}(\mu) e^d_{c,-1}(\lambda)^*, \quad \mu, \lambda \in \mathbb{C}^+,$$

to be the reproducing kernel for $\mathcal{H}^d_{c,-1}$.

Suppose that $x \in \mathcal{H}_{c,-1}$. With assumption (4.56) in force, there is a unique choice of function $\widetilde{\varphi}(\cdot) y_x$ so that $\mu \mapsto \frac{x(\mu) + \widetilde{\varphi}(\mu) y_x}{\beta + \mu} \in \mathcal{H}_c$. The space $\mathcal{H}_{c,-1}$ is defined as equivalence classes whereby $[x] = [x']$ means that $x - x' = \mu \mapsto \widetilde{\varphi}(\mu) y$, for some choice of $y \in \mathcal{Y}$. To define a point evaluation $e_{c,-1}(\lambda)$ on $\mathcal{H}_{c,-1}$, we have to choose a canonical representative of each equivalence class. We choose as our canonical representative the function $\mu \mapsto x(\mu) + \widetilde{\varphi}(\mu) y_x$ above. Thus we define $e_{c,-1}(\lambda) \colon \mathcal{H}_{c,-1} \to \mathcal{U}$ by

$$e_{c,-1}(\lambda) \colon [x] \mapsto x(\lambda) + \widetilde{\varphi}(\lambda) y_x, \quad \frac{x + \widetilde{\varphi}(\cdot) y_x}{\beta + \cdot} \in \mathcal{H}_c,\ \lambda \in \mathbb{C}^+, \tag{4.72}$$

and consider $K_{c,-1}(\mu, \lambda) := e_{c,-1}(\mu) e_{c,-1}(\lambda)^*$, $\mu, \lambda \in \mathbb{C}^+$, to be the reproducing kernel of $\mathcal{H}_{c,-1}$. The operator $e_{c,-1}(\lambda)$ is bounded for all $\lambda \in \mathbb{C}^+$, since

$$\left\| e_{c,-1}(\lambda)[x] \right\|_{\mathcal{U}} = \left\| (\beta + \lambda)\, e_c(\lambda)\, \frac{x + \widetilde{\varphi}(\cdot) y_x}{\beta + \cdot} \right\|_{\mathcal{U}} \le |\beta + \lambda|\, \|e_c(\lambda)\|\ \|[x]\|_{\mathcal{H}_{c,-1}}.$$

Proposition 4.21 *We have the following formulas for the kernel functions associated with the Hilbert spaces $\mathcal{H}^d_{c,\pm 1}$ and $\mathcal{H}_{c,\pm 1}$ (for $\mu, \lambda \in \mathbb{C}^+$):*

$$K_c(\mu, \lambda) = B^*_c (\mu - A^*_c)^{-1} (\overline{\lambda} - A_c|_{\mathcal{H}_c})^{-1} B_c, \tag{4.73}$$

$$K_{c,1}^d(\mu,\lambda) = B_c^*(\mu - A_c^*)^{-1}(\overline{\beta} - A_c^*)^{-1} \times (\beta - A_c)^{-1}(\overline{\lambda} - A_c|_{\mathcal{H}_c})^{-1}B_c, \tag{4.74}$$

$$K_{c,-1}^d(\mu,\lambda) = (\overline{\beta} - \mu)(\beta - \overline{\lambda})B_c^*(\mu - A_c^*)^{-1}(\overline{\lambda} - A_c|_{\mathcal{H}_c})^{-1}B_c, \tag{4.75}$$

$$K_{c,1}(\mu,\lambda) = B_c^*(\mu - A_c^*)^{-1}(\beta - A_c)^{-1} \times (\overline{\beta} - A_c^*)^{-1}(\overline{\lambda} - A_c|_{\mathcal{H}_c})^{-1}B_c, \tag{4.76}$$

$$K_{c,-1}(\mu,\lambda) = (\beta + \mu)(\overline{\beta} + \overline{\lambda})B_c^*(\mu - A_c^*)^{-1}(\overline{\lambda} - A_c|_{\mathcal{H}_c})^{-1}B_c. \tag{4.77}$$

The above state-space formulas correspond to the following purely function-theoretic formulas (for $\mu, \lambda \in \mathbb{C}^+$*):*

$$K_c(\mu,\lambda) = \frac{1 - \varphi(\overline{\mu})^*\varphi(\overline{\lambda})}{\mu + \overline{\lambda}},$$

$$K_{c,1}^d(\mu,\lambda) = \frac{K_c(\mu,\lambda) - K_c(\mu,\overline{\beta})}{(\overline{\beta} - \mu)(\beta - \overline{\lambda})} - \frac{K_c(\overline{\beta},\lambda) - K_c(\overline{\beta},\overline{\beta})}{(\overline{\beta} - \mu)(\beta - \overline{\lambda})}, \tag{4.78}$$

$$K_{c,-1}^d(\mu,\lambda) = (\overline{\beta} - \mu)(\beta - \overline{\lambda})K_c(\mu,\lambda), \tag{4.79}$$

$$K_{c,1}(\mu,\lambda) = \frac{\kappa_c(\mu,\lambda) - \varphi(\overline{\mu})^*\tau_{c,\beta}\left(\kappa_c(\cdot,\lambda)\right)}{\beta + \mu}, \quad \textit{where} \quad \kappa_c(\mu,\lambda) := \frac{K_c(\mu,\lambda) - K_c(\overline{\beta},\lambda)}{\overline{\beta} - \mu}, \tag{4.80}$$

$$K_{c,-1}(\mu,\lambda) = (\beta + \mu)(\overline{\beta} + \overline{\lambda})K_c(\mu,\lambda). \tag{4.81}$$

Here the point $\beta \in \mathbb{C}^+$ *appearing in the formulas must be chosen to be the same* β *which was used in the rigging* $\mathcal{H}_{c,1} \subset \mathcal{H}_c \subset \mathcal{H}_{c,-1}$*, so that* $\overline{\beta}$ *corresponds to the rigging* $\mathcal{H}_{c,1}^d \subset \mathcal{H}_c \subset \mathcal{H}_{c,-1}^d$*. The formulas* (4.77) *and* (4.81) *depend on Theorem 4.19 and hence they are established only for the case when* (4.56) *holds.*

Proof To see (4.73), combine part 3 of Theorem 1.1 with formula (4.7):

$$K_c(\mu,\lambda) = e_c(\mu)e_c(\lambda)^* = B_c^*(\mu - A_c^*|_{\mathcal{H}_c})^{-1}(\overline{\lambda} - A_c|_{\mathcal{H}_c})^{-1}B_c.$$

As for (4.74), we use that $(\overline{\beta} - A_c^*)^{-1}$ is a unitary transformation from $\mathcal{H}_c$ to $\mathcal{H}_{c,1}^d$. Hence any $f \in \mathcal{H}_{c,1}^d$ has the form $f = (\overline{\beta} - A_c^*)^{-1}g$ for a unique $g \in \mathcal{H}_c$, and for $\lambda \in \mathbb{C}_+$ we can compute

$$\begin{aligned}
\langle f(\lambda), u\rangle_{\mathcal{U}} &= \langle((\overline{\beta} - A_c^*)^{-1}g)(\lambda), u\rangle_{\mathcal{U}} \\
&= \langle(\overline{\beta} - A_c^*)^{-1}g, e_c(\lambda)^*u\rangle_{\mathcal{H}_c} \\
&= \langle g, (\beta - A_c)^{-1}e_c(\lambda)^*u\rangle_{\mathcal{H}_c} \\
&= \langle f, (\overline{\beta} - A_c^*)^{-1}(\beta - A_c)^{-1}e_c(\lambda)^*u\rangle_{\mathcal{H}_{c,1}^d}.
\end{aligned}$$

Combining this with the fact that the point-evaluation operator in $\mathcal{H}_{c,1}^d$ is the restriction of $e_c(\cdot)$ to $\mathcal{H}_{c,1}^d$, we obtain (4.74). In order to get (4.78), we continue calculating

$$e_c(\mu)(\overline{\beta}-A_c^*)^{-1}(\beta-A_c)^{-1}e_c(\lambda)^*u = e_c(\mu)(\overline{\beta}-A_c^*)^{-1}\frac{e_c(\lambda)^*-e_c(\overline{\beta})^*}{\beta-\overline{\lambda}}u$$
$$= \frac{\frac{K_c(\mu,\lambda)-K_c(\mu,\overline{\beta})}{\beta-\overline{\lambda}} - \frac{K_c(\overline{\beta},\lambda)-K_c(\overline{\beta},\overline{\beta})}{\beta-\overline{\lambda}}}{\overline{\beta}-\mu}u,$$

where we used (4.39) and (4.23) in the first and second equalities, respectively.

The formula (4.76) follows immediately on replacing $\overline{\beta}-A_c^*$ by $\beta-A_c$ in (4.74), since $(\beta-A_c)^{-1}$ is the appropriate unitary operator from $\mathcal{H}_c$ to $\mathcal{H}_{c,1}$ instead of $(\overline{\beta}-A_c^*)^{-1}$. Using (4.23) and (4.51), we obtain (4.80).

In order to establish (4.75), we use that $(\overline{\beta}-A_c^*|_{\mathcal{H}_c})$ is a unitary transformation from $\mathcal{H}_c$ to $\mathcal{H}_{c,-1}^d$. Thus any $[f]\in\mathcal{H}_{c,-1}^d$ has the form $[f]=(\overline{\beta}-A_c^*|_{\mathcal{H}_c})g$ with $g\in\mathcal{H}_c$. Furthermore, from (4.34) we have that the unique representative f for the equivalence class $[f]$ satisfying $f(\overline{\beta})=0$ is given by $f(\mu)=(\overline{\beta}-\mu)g(\mu)$. Hence, we have, for $\lambda\in\mathbb{C}^+$,

$$\begin{aligned}\langle f(\lambda),u\rangle_{\mathcal{U}} &= \langle(\overline{\beta}-\lambda)g(\lambda),u\rangle_{\mathcal{U}}\\ &= \langle g,(\beta-\overline{\lambda})e_c(\lambda)^*u\rangle_{\mathcal{H}_c}\\ &= \langle f,(\beta-\overline{\lambda})(\overline{\beta}-A_c^*|_{\mathcal{H}_c})e_c(\lambda)^*u\rangle_{\mathcal{H}_{c,-1}^d}.\end{aligned} \tag{4.82}$$

This proves that $\left(e_{c,-1}^d(\lambda)\right)^*u = (\beta-\overline{\lambda})(\overline{\beta}-A_c^*|_{\mathcal{H}_c})e_c(\lambda)^*u$, and combining this with (4.31) we obtain (4.79), again choosing the representative with value zero at $\overline{\beta}$. Now (4.75) follows from (4.79) and (4.73).

In order to obtain (4.81) and (4.77), we use (4.72) to compute

$$\begin{aligned}\langle[x],e_{c,-1}(\lambda)^*u\rangle_{\mathcal{H}_{c,-1}} &= \langle e_{c,-1}(\lambda)[x],u\rangle_{\mathcal{U}} = \langle x(\lambda)+\widetilde{\varphi}(\lambda)y_x,u\rangle_{\mathcal{U}}\\ &= \left\langle\frac{x(\lambda)+\widetilde{\varphi}(\lambda)y_x}{\beta+\lambda},(\overline{\beta}+\overline{\lambda})u\right\rangle_{\mathcal{U}},\end{aligned}$$

and using (4.63) with the unitarity of $\beta-A_c|_{\mathcal{H}_c}$, we further obtain that this equals

$$\begin{aligned}&\left\langle\mu\mapsto\frac{x(\mu)+\widetilde{\varphi}(\mu)y_x}{\beta+\mu},\mu\mapsto K_c(\mu,\lambda)(\overline{\beta}+\overline{\lambda})u\right\rangle_{\mathcal{H}_c}\\ &= \left\langle[x],[\mu\mapsto(\beta+\mu)(\overline{\beta}+\overline{\lambda})K_c(\mu,\lambda)u]\right\rangle_{\mathcal{H}_{c,-1}}.\end{aligned}$$

We conclude that

$$e_{c,-1}(\lambda)^* = [\mu\mapsto(\beta+\mu)(\overline{\beta}+\overline{\lambda})K_c(\mu,\lambda)],$$

and since $\mu\mapsto(\overline{\beta}+\overline{\lambda})K_c(\mu,\lambda)\in\mathcal{H}_c$, we have $e_{c,-1}(\mu)e_{c,-1}(\lambda)^* = (\beta+\mu)(\overline{\beta}+\overline{\lambda})K_c(\mu,\lambda)$. □

The following result follows from (4.74), (4.76), and the unitarity of $(\beta-A_c|_{\mathcal{H}_c})^{-1}$ and $(\overline{\beta}-A_c^*|_{\mathcal{H}_c})^{-1}$.

Corollary 4.22 *For all $\mu, \lambda \in \mathbb{C}^+$ and $u, v \in \mathcal{U}$, we have*

$$\left(K^d_{c,1}(\mu,\lambda)u, v\right)_{\mathcal{U}} = (K_c(\cdot,\lambda)u, K_c(\cdot,\mu)v)_{\mathcal{H}_{c,-1}} \quad \text{and}$$
$$\left(K_{c,1}(\mu,\lambda)u, v\right)_{\mathcal{U}} = (K_c(\cdot,\lambda)u, K_c(\cdot,\mu)v)_{\mathcal{H}^d_{c,-1}}.$$

We now turn our attention to the observable functional model.

5 The Observable Co-Energy-Preserving Model

In this section we present the observable co-energy-preserving model realization of a given $\varphi \in \mathcal{S}(\mathbb{C}^+;\mathcal{U},\mathcal{Y})$ which uses the reproducing kernel Hilbert space $\mathcal{H}_o$ as state space. We have already seen in Sect. 2 that $K_o(\mu,\lambda) = \frac{1-\varphi(\mu)\varphi(\lambda)^*}{\mu+\overline{\lambda}}$ is a positive kernel with associated reproducing kernel Hilbert space denoted as $\mathcal{H}_o$. Rather than defining the system node $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_o$ directly, it is more tractable to first define its adjoint $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_o^*$. The adjoint is first defined via the mapping

$$\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}_o^* : \begin{bmatrix} e_o(\overline{\lambda})^* y \\ y \end{bmatrix} \mapsto \begin{bmatrix} \lambda e_o(\overline{\lambda})^* y \\ \widetilde{\varphi}(\lambda) y \end{bmatrix}, \quad y \in \mathcal{Y},\ \lambda \in \mathbb{C}^+, \tag{5.1}$$

cf. (4.2). One can then mimic the proof of Lemma 4.1 to see that this mapping can be extended uniquely to a well-defined closed linear operator $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_o^*$. One can then mimic the whole development of Sect. 4 to arrive at the sought-after results for the observable co-energy-preserving case here.

A logically more efficient (if not as intuitively appealing) procedure is to reduce the results for the observable co-energy-preserving case to those of Sect. 4 for the controllable energy-preserving case by the following duality trick. As was noted in Proposition 3.10, if $\varphi(\mu)$ is the transfer function of the system node $\left[\begin{smallmatrix} A\&B \\ C\&C \end{smallmatrix}\right]$, then $\left[\begin{smallmatrix} A^d\&B^d \\ C^d\&D^d \end{smallmatrix}\right] := \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]^*$ is a system node with transfer function equal to $\widetilde{\varphi}(\mu) = \varphi(\overline{\mu})^*$. Observe that the transformation $\varphi \mapsto \widetilde{\varphi}$ maps the Schur class $\mathcal{S}(\mathbb{C}^+;\mathcal{U},\mathcal{Y})$ bijectively to the Schur class $\mathcal{S}(\mathbb{C}^+;\mathcal{Y},\mathcal{U})$. Given a Schur-class function $\varphi \in \mathcal{S}(\mathbb{C}^+;\mathcal{U},\mathcal{Y})$, let $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c^\sim$ be the controllable energy-preserving canonical functional-model system node constructed as in Sect. 4 but associated with $\widetilde{\varphi}$ rather than with φ. Then its adjoint

$$\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}_c^\sim\right)^* =: \begin{bmatrix} \widetilde{A}^d_c\&\widetilde{B}^d_c \\ \widetilde{C}^d_c\&\widetilde{D}^d_c \end{bmatrix}$$

is also a system node which has transfer function $(\widetilde{\varphi})^\sim = \varphi$. Since $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c^\sim$ is controllable and energy-preserving by construction, as was observed in Theorem 4.2, it follows that $\left[\begin{smallmatrix} \widetilde{A}^d\&\widetilde{B}^d_c \\ \widetilde{C}^d_d\&\widetilde{D}^d_c \end{smallmatrix}\right]$ is observable and co-energy-preserving. One can then define the associated *observable, co-energy-preserving canonical functional-model system* *node associated with* φ by

$$\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}_o = \begin{bmatrix} \widetilde{A}_c^d\&\widetilde{B}_c^d \\ \widetilde{C}_c^d\&\widetilde{D}_c^d \end{bmatrix}.$$

Note that every result concerning the controllable energy-preserving canonical functional-model system node $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ obtained in Sect. 4 carries over to a corresponding result for $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_o$: Simply apply the result from Sect. 4 but with $\widetilde{\varphi}$ in place of φ and then express the result in terms of operators associated with the adjoint system node $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_o = \left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c^{\sim}\right)^*$ rather than with $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c^{\sim}$ itself. In this section we state the most interesting results but leave all of the proofs to the reader. The reader is invited to supply the proofs by either of the two routes sketched above.

The following result is the observable, co-energy-preserving analogue of Lemma 4.1, and Theorems 4.2 and 4.4 combined.

Theorem 5.1 *Suppose that we are given a function* $\varphi \in \mathcal{S}(\mathbb{C}_+; \mathcal{U}, \mathcal{Y})$ *and define* $\mathcal{H}_o = \mathcal{H}(K_o)$ *as above.*

1. *The mapping* (5.1) *which was defined initially only on elements of the form* $\left[\begin{smallmatrix} e_o(\lambda)^* y \\ y \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{H}_o \\ \mathcal{Y} \end{smallmatrix}\right]$ *extends by linearity and limit-closure uniquely to a closed linear operator*

$$\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}_o^* : \begin{bmatrix} \mathcal{H}_o \\ \mathcal{Y} \end{bmatrix} \supset \operatorname{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right)_o^* \to \begin{bmatrix} \mathcal{H}_o \\ \mathcal{U} \end{bmatrix} \tag{5.2}$$

which is a controllable, energy-preserving system node having transfer function equal to $\widetilde{\varphi}(\mu) = \varphi(\overline{\mu})^*, \ \ \mu \in \mathbb{C}^+$.

2. *The adjoint of the system node* $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_o^*$ *given by* (5.2)*, namely*

$$\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}_o : \begin{bmatrix} \mathcal{H}_o \\ \mathcal{U} \end{bmatrix} \supset \operatorname{dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}_o\right) \to \begin{bmatrix} \mathcal{H}_o \\ \mathcal{Y} \end{bmatrix}, \tag{5.3}$$

is an observable, co-energy-preserving system node with transfer function equal to φ.

3. *The system node* (5.3) *can be characterized more directly as follows:*

$$\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}_o : \begin{bmatrix} x \\ u \end{bmatrix} \mapsto \begin{bmatrix} z \\ y \end{bmatrix}, \quad \textit{where} \tag{5.4}$$

$$z(\mu) := \mu x(\mu) + \varphi(\mu)u - y, \quad \mu \in \mathbb{C}^+, \quad \textit{and} \tag{5.5}$$

$$y := \lim_{\operatorname{Re}\eta \to \infty} \eta x(\eta) + \varphi(\eta)u, \quad \textit{defined on}$$

$$\operatorname{dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}_o\right) := \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in \begin{bmatrix} \mathcal{H}_o \\ \mathcal{U} \end{bmatrix} \middle| \exists y \in \mathcal{Y} : \ z \in \mathcal{H}_o \textit{ in (5.5)} \right\}. \tag{5.6}$$

For every $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \operatorname{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_o\right)$*, the* $y \in \mathcal{Y}$ *such that* z *given in* (5.5) *lies in* $\mathcal{H}_o$ *is unique and it is given by* (5.6)*.*

4. *The kernel functions* $K_o(\cdot,\lambda) = e_o(\lambda)^*,\ \lambda \in \mathbb{C}^+$, *for the space* $\mathcal{H}_o$ *are given by*

$$e_o(\lambda)^* = (\overline{\lambda} - A_o^*|_{\mathcal{H}_o})^{-1} C_o^*, \quad \lambda \in \mathbb{C}^+.$$

Remark 4.8 applies with minor modifications to $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_o$. The following result (the analogue of Theorem 4.3) explains the canonical property for the functional-model system node $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_o$.

Theorem 5.2 *Let* $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ *be an observable and co-energy-preserving realization of* φ *with state space* $\mathcal{X}$. *Define the operator* $\Delta : \mathcal{H}_o \to \mathcal{X}$ *as the unique continuous linear extension of the mapping*

$$\Delta : e_o(\lambda)^* y \mapsto \left(\overline{\lambda} - A^*|_{\mathcal{X}}\right)^{-1} C^* y, \quad \lambda \in \mathbb{C}^+,\ y \in \mathcal{Y}.$$

Then Δ *is unitary from* $\mathcal{H}_o$ *to* $\mathcal{X}$, *the operator* $\left[\begin{smallmatrix} \Delta & 0 \\ 0 & 1_{\mathcal{U}} \end{smallmatrix}\right]$ *maps* dom $\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_o\right)$ *one-to-one onto* dom $\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right)$, *and*

$$\begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \begin{bmatrix} \Delta & 0 \\ 0 & 1_{\mathcal{U}} \end{bmatrix} = \begin{bmatrix} \Delta & 0 \\ 0 & 1_{\mathcal{Y}} \end{bmatrix} \begin{bmatrix} A\&B \\ C\&D \end{bmatrix}_o.$$

The proof is simply an application of Theorem 4.3 to $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]^*$ and $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_o^*$. The following result provides formulas involving the resolvent of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_o$.

Theorem 5.3 *The main operator* A_o *of* $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_o$ *is given explicitly by*

$$(A_o x)(\mu) = \mu x(\mu) - \lim_{\operatorname{Re}\eta\to\infty} \eta x(\eta), \quad \mu \in \mathbb{C}^+,$$

for x *in* dom $(A_o) = \{x \in \mathcal{H}_o | \exists y \in \mathcal{Y} :\ \mu \mapsto \mu x(\mu) - y \in \mathcal{H}_o\}$, *and the observation operator is*

$$C_o x = \lim_{\operatorname{Re}\eta\to\infty} \eta x(\eta), \quad x \in \operatorname{dom}(A_o).$$

The resolvent of A_o *is given by*

$$\left((\alpha - A_o)^{-1} x\right)(\mu) = \frac{x(\mu) - x(\alpha)}{\alpha - \mu}, \quad \alpha, \mu \in \mathbb{C}^+,\ x \in \mathcal{H}_o. \tag{5.7}$$

Denoting the control operator of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_o$ *by* B_o, *we have the following formulas:*

$$\begin{aligned} \left(A_o(\alpha - A_o)^{-1} x\right)(\mu) &= \frac{\mu x(\mu) - \alpha x(\alpha)}{\alpha - \mu}, \quad \alpha, \mu \in \mathbb{C}^+,\ x \in \mathcal{H}_o, \\ \left((\alpha - A_o|_{\mathcal{H}_o})^{-1} B_o u\right)(\mu) &= \frac{\varphi(\mu) - \varphi(\alpha)}{\alpha - \mu} u, \quad \alpha, \mu \in \mathbb{C}^+,\ u \in \mathcal{U}, \\ C_o(\alpha - A_o)^{-1} x &= x(\alpha), \quad \alpha \in \mathbb{C}^+,\ x \in \mathcal{H}_o. \end{aligned} \tag{5.8}$$

5.1 The Dual System Node and Extrapolation Spaces

Just as in Sect. 4.3 for the controllable, energy-preserving case, the formula (5.7) for the resolvent of A_o suggests a way to concretely identify the (-1)-scaled rigged space $\mathcal{H}_{o,-1}$ defined abstractly as the completion of the space $\mathcal{H}_o$ in the norm

$$\|x\| = \|(\beta - A_o)^{-1}\|_{\mathcal{H}_o}.$$

Namely we define

$$\mathcal{H}_{o,-1} = \left\{ x : \mathbb{C}^+ \to \mathcal{Y} \middle| \mu \mapsto \frac{x(\mu) - x(\beta)}{\beta - \mu} \in \mathcal{H}_o \right\} \tag{5.9}$$

with norm given by

$$\|x\|_{\mathcal{H}_{o,-1}} = \left\| \mu \mapsto \frac{x(\mu) - x(\beta)}{\beta - \mu} \right\|_{\mathcal{H}_o}. \tag{5.10}$$

We emphasize again that the $\mathcal{H}_{o,-1}$ norm (and inner product) depends on the choice of $\beta \in \mathbb{C}_+$; different choices of β give different norms although all such norms are equivalent. The elements of $\mathcal{H}_{o,-1}$ are equivalence classes of functions modulo constant terms. We have the following analogue of Theorem 4.7:

Theorem 5.4 *The space $\mathcal{H}_{o,-1}$ given by (5.9) and (5.10) is complete.*

1. *The map $\iota : x \mapsto [x]$ embeds $\mathcal{H}_o$ into $\mathcal{H}_{o,-1}$ as a dense subspace. A given element $[z] \in \mathcal{H}_{o,-1}$ is of the form $\iota(x)$ for some $x \in \mathcal{H}_o$ if and only if the function $\mu \mapsto \dfrac{z(\mu) - z(\beta)}{\beta - \mu}$, $\mu \in \mathbb{C}^+$, is not only in $\mathcal{H}_o$ but in fact is in $\operatorname{dom}(A_o) = \mathcal{H}_{o,1} \subset \mathcal{H}_o$. When this is the case, the equivalence class representative x for $[z]$, for which $x \in \mathcal{H}_o$, is uniquely determined by the decay condition at infinity:*

$$\lim_{\operatorname{Re}\eta \to \infty} x(\eta) = 0.$$

2. *Define an operator $A_o|_{\mathcal{H}_o} : \mathcal{H}_o \to \mathcal{H}_{o,-1}$ by*

$$A_o|_{\mathcal{H}_o} x := [\mu \mapsto \mu x(\mu)], \quad x \in \mathcal{H}_o, \ \mu \in \mathbb{C}^+.$$

   When $\mathcal{H}_o$ is identified as a linear sub-manifold of $\mathcal{H}_{o,-1}$, then $A_o|_{\mathcal{H}_o}$ is the unique extension of $A_o : \operatorname{dom}(A_o) \to \mathcal{H}_o$ to an operator in $\mathcal{B}(\mathcal{H}_o; \mathcal{H}_{o,-1})$. Moreover, $(\beta - A_o|_{\mathcal{H}_o})^{-1}$ is a unitary map from $\mathcal{H}_{o,-1}$ to $\mathcal{H}_o$.
3. *With $\mathcal{H}_{o,-1}$ identified concretely as in (5.9), the action of $B_o : \mathcal{U} \to \mathcal{H}_{o,-1}$ is given by*

$$B_o u := [\mu \mapsto \varphi(\mu) u], \quad u \in \mathcal{U}, \ \mu \in \mathbb{C}^+.$$

We shall now present formulas for the action of the operators of *the dual of* $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_o$ on generic functions in their domains. For $\alpha \in \mathbb{C}^+$ we define $\tau_{o,\alpha} : \mathcal{H}_o \to \mathcal{U}$ by

$$\tau_{o,\alpha} := B_o^*(\alpha - A_o^*)^{-1}, \quad \alpha \in \mathbb{C}^+,$$

and it follows from (5.8) that

$$(\tau_{o,\alpha})^* u = \mu \mapsto \frac{\varphi(\mu) - \varphi(\overline{\alpha})}{\overline{\alpha} - \mu} u, \quad u \in \mathcal{U},\ \mu \in \mathbb{C}^+.$$

The map $\tau_{o,\alpha}$ enters into the explicit formula for the resolvent of A_o^* acting on generic elements of $\mathcal{H}_o$, as described in the following analogue of Corollary 4.15 and Theorem 4.10:

Theorem 5.5 *The following claims are true:*

1. *For arbitrary* $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \operatorname{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_o^*\right)$, *if we set* $u := \left[B_o^*\&D_o^*\right]\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$, *then we get*

$$\left[A_o^*\&C_o^*\right]\begin{bmatrix} x \\ y \end{bmatrix} = \mu \mapsto -\mu x(\mu) - \varphi(\mu)u + y, \quad \mu \in \mathbb{C}^+.$$

It follows that

$$\operatorname{dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}\right)_o^* \subset \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \begin{bmatrix} \mathcal{H}_c \\ \mathcal{Y} \end{bmatrix} \middle| \; \exists u \in \mathcal{U} : \ \mu \mapsto -\mu x(\mu) - \varphi(\mu)u + y \in \mathcal{H}_o \right\}.$$

2. *For* $x \in \operatorname{dom}\left(A_o^*\right)$ *the function* $A_o^* x \in \mathcal{H}_o$ *satisfies the identity*

$$\left(A_o^* x\right)(\mu) = -\mu x(\mu) - \varphi(\mu) B_o^* x, \quad \mu \in \mathbb{C}^+,$$

and in particular,

$$\operatorname{dom}\left(A_o^*\right) \subset \{x \in \mathcal{H}_o | \exists u \in \mathcal{U} : \ \mu \mapsto \mu x(\mu) + \varphi(\mu)u \in \mathcal{H}_o\}. \tag{5.11}$$

3. *If we know* A_o^*, *then we can characterize* $\operatorname{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_o^*\right)$ *and* $\left[B_c^*\&D_c^*\right]$ *as follows, for an arbitrary* $\lambda \in \mathbb{C}^+$:

$$\operatorname{dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}_o^*\right) = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \begin{bmatrix} \mathcal{H}_o \\ \mathcal{Y} \end{bmatrix} \middle| x - e_o(\overline{\lambda})^* y \in \operatorname{dom}\left(A_o^*\right) \right\},$$

$$\left[B_c^*\&D_c^*\right]\begin{bmatrix} x \\ y \end{bmatrix} = \tau_{o,\lambda}(\lambda - A_o^*)\big(x - e_o(\lambda)^* y\big) + \widetilde{\varphi}(\overline{\lambda});$$

neither of these two depends on the choice of $\lambda \in \mathbb{C}^+$.

4. *We have the following formula for the resolvent of A_o^*:*

$$\big((\overline{\alpha} - A_o^*)^{-1}x\big)(\mu) = \frac{x(\mu) - \varphi(\mu)\,\tau_{o,\overline{\alpha}}\,x}{\overline{\alpha} + \mu}, \quad \alpha, \mu \in \mathbb{C}^+,\ x \in \mathcal{H}_o, \tag{5.12}$$

and the action of this resolvent on kernel functions of $\mathcal{H}_o$ is

$$(\overline{\alpha} - A_o^*)^{-1} e_o(\lambda)^* y = \frac{e_o(\lambda)^* - e_o(\alpha)^*}{\overline{\alpha} - \overline{\lambda}} y, \quad \alpha, \lambda \in \mathbb{C}^+,\ y \in \mathcal{Y}. \tag{5.13}$$

The formula (5.13) is useful when calculating the reproducing kernel of $\mathcal{H}_{o,1}$.

Similar to Lemma 4.16, with an added assumption, it is possible to strengthen the containment in (5.11) to an equality. Moreover, we obtain a characterization of when the observable energy-preserving realization is in fact conservative, cf. Theorem 1.5.

Theorem 5.6 *The following two conditions are equivalent:*

1. *For some (or equivalently for every) $\alpha \in \mathbb{C}^+$, the state space $\mathcal{H}_o$ has the property*

$$\mathcal{H}_o \cap \left\{ \mu \mapsto \frac{\varphi(\mu)u}{\alpha + \mu} \,\middle|\, u \in \mathcal{U} \right\} = \{0\}. \tag{5.14}$$

2. *We have both*

$$\mathcal{H}_o \cap \{\varphi(\cdot)u | u \in \mathcal{U}\} = \{0\} \quad \textit{and} \tag{5.15}$$

$$\mathrm{dom}\left(A_o^*\right) = \{x \in \mathcal{H}_o | \exists u \in \mathcal{U} :\ \mu \mapsto \mu x(\mu) + \varphi(\mu)u \in \mathcal{H}_o\}. \tag{5.16}$$

The conditions (5.14)–(5.16) hold together with the implication $\varphi(\cdot)u = 0 \Rightarrow u = 0$ if and only if $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_o$ is conservative. This is in turn true if and only if $1 - \varphi(\cdot)^*\varphi(\cdot)$ has maximal factorable minorant in the right half-plane sense equal to 0.

When (5.15) holds, a given $x \in \mathcal{H}_o$ in $\mathrm{dom}\left(A_o^*\right)$ determines the function $\varphi(\cdot)u$ appearing in (5.11) uniquely through the formula

$$\varphi(\mu)u = \varphi(\mu)B_o^*x, \quad \mu \in \mathbb{C}^+. \tag{5.17}$$

If $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_o$ is conservative, then $x \in \mathrm{dom}\left(A_o^*\right)$ further determines the vector $u \in \mathcal{U}$ uniquely in (5.17).

Recall that $\mathcal{H}^d_{o,-1}$ is defined to be the completion of $\mathcal{H}_o$ in the norm $\|x\| = \|(\overline{\beta} - A_o^*)^{-1}x\|$. With assumption (5.14) in force, Theorem 5.6 assures us that $\mathrm{dom}\left(A_o^*\right)$ is given by (5.16). Then formula (5.12) suggests the following concrete identification of the (-1)-rigged space:

$$\mathcal{H}^d_{o,-1} := \left\{ x : \mathbb{C}^+ \to \mathcal{Y} \,\middle|\, \exists u \in \mathcal{U} :\ \mu \mapsto \frac{x(\mu) + \varphi(\mu)u}{\overline{\beta} + \mu} \in \mathcal{H}_o \right\}. \tag{5.18}$$

Now the property (5.15) guarantees us that the choice of $\varphi(\cdot)u$ in (5.18) is uniquely determined by x (whenever at least one suitable $u \in \mathcal{U}$ exists). For $x \in \mathcal{H}^d_{o,-1}$ we set

$$\|x\|_{\mathcal{H}^d_{o,-1}} := \left\| \mu \mapsto \frac{x(\mu) + \varphi(\mu)u}{\overline{\beta} + \mu} \right\|_{\mathcal{H}_o}. \tag{5.19}$$

Thus elements of $\mathcal{H}^d_{o,-1}$ are equivalence classes of functions modulo the subspace $\varphi(\cdot)\mathcal{U}$. These equivalence classes are denoted as $[x]$ where x is any particular representative of the equivalence class. We summarize the properties of the space $\mathcal{H}^d_{o,-1}$ as follows; see also Theorem 4.19:

Proposition 5.7 *Assume that (5.14) holds. Then the space $\mathcal{H}^d_{o,-1}$ given by (5.18)–(5.19) above is complete, and moreover:*

1. *The map $\iota : x \mapsto [x]$ embeds $\mathcal{H}_o$ into $\mathcal{H}^d_{o,-1}$ as a dense subspace. A given element $[z] \in \mathcal{H}^d_{o,-1}$ is of the form $\iota(x)$ for some $x \in \mathcal{H}_o$ if and only if there is a choice of u so that the function $\mu \mapsto \dfrac{z(\mu) + \varphi(\mu)u}{\overline{\beta} + \mu}$, $\mu \in \mathbb{C}^+$, is not only in $\mathcal{H}_o$ but also in $\mathcal{H}^d_{o,1} = \operatorname{dom}(A^*_o)$. This choice of $\varphi(\cdot)u$ is then unique.*
2. *Define an operator $A^*_o|_{\mathcal{H}_o} : \mathcal{H}_o \to \mathcal{H}^d_{o,-1}$ by*

$$A^*_o|_{\mathcal{H}_o} x := [\mu \mapsto -\mu x(\mu)], \quad x \in \mathcal{H}_o,\ \mu \in \mathbb{C}^+.$$

When $\mathcal{H}_o$ is identified as a linear sub-manifold of $\mathcal{H}^d_{o,-1}$ as in statement 1, then $A^*_o|_{\mathcal{H}_o}$ is the unique extension of $A^*_o : \operatorname{dom}(A^*_o) \to \mathcal{H}_o$ to an operator in $\mathcal{B}(\mathcal{H}_o; \mathcal{H}^d_{o,-1})$. The resolvent of $A^*_o|_{\mathcal{H}_o}$ is given by

$$\big((\overline{\alpha} - A^*_o|_{\mathcal{H}_c})^{-1}[x]\big)(\mu) = \frac{x(\mu) + \varphi(\mu)u}{\overline{\alpha} + \mu}, \quad [x] \in \mathcal{H}^d_{o,-1},\ \alpha, \mu \in \mathbb{C}^+,$$

where the condition $\mu \mapsto \frac{x(\mu)+\varphi(\mu)u}{\overline{\alpha}+\mu} \in \mathcal{H}_o$ uniquely determines $\varphi(\cdot)u$. Moreover, $(\overline{\beta} - A^*_o|_{\mathcal{H}_c})^{-1}$ is unitary from $\mathcal{H}_{c,-1}$ to $\mathcal{H}_c$.
3. *The action of $C^*_o : \mathcal{Y} \to \mathcal{H}^d_{o,-1}$ is given by*

$$C^*_o y := [\mu \mapsto y], \quad y \in \mathcal{Y},\ \mu \in \mathbb{C}^+.$$

We next present a collection of reproducing-kernel formulas. This is the dual version of Proposition 4.21 and Corollary 4.22. Again, $\mathcal{H}_{o,-1}$ and $\mathcal{H}^d_{o,-1}$ are not spaces of functions, but we can identify them with the Hilbert spaces with reproducing kernels

$$K_{o,-1}(\mu, \lambda) = e_{o,-1}(\mu)\big(e_{o,-1}(\lambda)\big)^* \quad \text{and}$$
$$K^d_{o,-1}(\mu, \lambda) = e^d_{o,-1}(\mu)\big(e^d_{o,-1}(\lambda)\big)^*,$$

respectively, where

$$e_{o,-1}(\mu)[z] := z(\mu) - z(\beta), \quad [z] \in \mathcal{H}_{o,-1},\ \mu \in \mathbb{C}^+, \quad \text{and}$$
$$e^d_{o,-1}(\mu) : (\overline{\beta} - A_o^*|_{\mathcal{H}_o})x \mapsto (\overline{\beta} + \mu)x(\mu), \quad x \in \mathcal{H}_o,\ \mu \in \mathbb{C}^+,$$

are bounded operators that point-evaluate well-chosen representatives of the equivalence classes in $\mathcal{H}_{o,-1}$ and $\mathcal{H}^d_{o,-1}$.

Proposition 5.8 *We have the following formulas for the kernel functions associated with the reproducing kernel Hilbert spaces $\mathcal{H}_o$, $\mathcal{H}_{o,\pm 1}$, and $\mathcal{H}^d_{o,\pm 1}$ (with $\mu, \lambda \in \mathbb{C}^+$):*

$$\begin{aligned}
K_o(\mu,\lambda) &= C_o(\mu - A_o)^{-1}(\overline{\lambda} - A_o^*|_{\mathcal{H}_o})^{-1}C_o^* = \frac{1 - \varphi(\mu)\varphi(\lambda)^*}{\mu + \overline{\lambda}},\\
K_{o,1}(\mu,\lambda) &= C_o(\mu - A_c)^{-1}(\beta - A_o)^{-1}(\overline{\beta} - A_o^*)^{-1}(\overline{\lambda} - A_o^*|_{\mathcal{H}_o})^{-1}C_o^*\\
&= \frac{K_o(\mu,\lambda) - K_o(\mu,\beta)}{(\beta - \mu)(\overline{\beta} - \overline{\lambda})} - \frac{K_o(\beta,\lambda) - K_o(\beta,\beta)}{(\beta - \mu)(\overline{\beta} - \overline{\lambda})}\\
K_{o,-1}(\mu,\lambda) &= (\beta - \mu)(\overline{\beta} - \overline{\lambda})C_o(\mu - A_o)^{-1}(\overline{\lambda} - A_o^*|_{\mathcal{H}_o})^{-1}C_o^*\\
&= (\beta - \mu)(\overline{\beta} - \overline{\lambda})K_o(\mu,\lambda),\\
K^d_{o,1}(\mu,\lambda) &= C_o(\mu - A_c)^{-1}(\overline{\beta} - A_o^*)^{-1}(\beta - A_o)^{-1}(\overline{\lambda} - A_o^*|_{\mathcal{H}_o})^{-1}C_o^*\\
&= \frac{\kappa_o(\mu,\lambda) - \varphi(\mu)\tau_{o,\overline{\beta}}\left(\kappa_o(\cdot,\lambda)\right)}{\overline{\beta} + \mu}, \quad \textit{where}\\
&\qquad \kappa_o(\mu,\lambda) := \frac{K_o(\mu,\lambda) - K_o(\beta,\lambda)}{\beta - \mu},\\
K^d_{o,-1}(\mu,\lambda) &= (\overline{\beta} + \mu)(\beta + \overline{\lambda})C_o(\mu - A_o)^{-1}(\overline{\lambda} - A_o^*|_{\mathcal{H}_o})^{-1}C_o^* &(5.20)\\
&= (\overline{\beta} + \mu)(\beta + \overline{\lambda})K_o(\mu,\lambda). &(5.21)
\end{aligned}$$

Here β is the parameter used in the construction of the rigging dom $(A_o) \subset \mathcal{H}_o \subset \mathcal{H}_{o,-1}$ *as usual. The formulas* (5.20) *and* (5.21) *have only been established under the assumption* (5.14).

Moreover, for all $\mu, \lambda \in \mathbb{C}^+$ and $y, \gamma \in \mathcal{Y}$, we have

$$\left(K_{o,1}(\mu,\lambda)y, \gamma\right)_{\mathcal{Y}} = (K_o(\cdot,\lambda)y, K_o(\cdot,\mu)\gamma)_{\mathcal{H}^d_{c,-1}} \quad \textit{and}$$
$$\left(K^d_{o,1}(\mu,\lambda)y, \gamma\right)_{\mathcal{Y}} = (K_o(\cdot,\lambda)y, K_o(\cdot,\mu)\gamma)_{\mathcal{H}_{c,-1}}.$$

This completes our study of the observable functional model.

6 Recovering the Classical de Branges–Rovnyak Models

In this section we use the so-called internal Cayley transformation to recover the original de Branges–Rovnyak models (1.8) and (1.10). This Cayley transformation is described in detail in [45, §12.3]; here we only include the small fragments of the theory that we need.

Following [45, Thm 12.3.6], the *Cayley transform with parameter* $\alpha \in \mathrm{res}\,(A)$ of the system node $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is the bounded operator $\left[\begin{smallmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{smallmatrix}\right]$ from $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$ into $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$ defined by

$$\begin{aligned} &\mathbf{A} := (\overline{\alpha} + A)(\alpha - A)^{-1}, \quad \mathbf{B} := \sqrt{2\mathrm{Re}\,\alpha}\,(\alpha - A|_{\mathcal{X}})^{-1}B, \\ &\mathbf{C} := \sqrt{2\mathrm{Re}\,\alpha}\,C(\alpha - A)^{-1}, \quad \text{and} \quad \mathbf{D} := \widehat{\mathfrak{D}}(\alpha), \end{aligned} \tag{6.1}$$

where A, $A|_{\mathcal{X}}$, and B are as in Definition 3.1, and C and $\widehat{\mathfrak{D}}$ are given by (3.4) and (3.5), respectively.

We interpret the bounded operator $\left[\begin{smallmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{smallmatrix}\right]$ described by (6.1) as the connecting operator of a discrete-time system with the same input space $\mathcal{U}$, state space $\mathcal{X}$, and output space $\mathcal{Y}$ as $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$:

$$\begin{bmatrix} x(k+1) \\ y(k) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}, \quad k \in \mathbb{Z}^+. \tag{6.2}$$

As in the introduction, the transfer function of the system (6.2) is

$$\widehat{\mathbf{D}}(z) = z\mathbf{C}(1 - z\mathbf{A})^{-1}\mathbf{B} + \mathbf{D}. \tag{6.3}$$

The reader should be warned that the transfer function of a discrete-time system is defined as $\mathbf{C}(z - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$ in [45], but the results can be translated from one setting to the other by interchanging z and $1/z$.

Recall that a discrete-time system with input space $\mathcal{U}$, state space $\mathcal{X}$, and output space $\mathcal{Y}$ is (scattering) passive, energy preserving, or co-energy preserving, if and only if the connecting operator is contractive, isometric, or co-isometric, respectively, from $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$; see e.g. [44, §5].

We use the linear fractional transformation

$$z_\alpha(\mu) := \frac{\alpha - \mu}{\overline{\alpha} + \mu}, \quad \mu \in \mathbb{C}^+ \Longleftrightarrow \mu_\alpha(z) = \frac{\alpha - \overline{\alpha}z}{1 + z}, \quad z \in \mathbb{D}, \tag{6.4}$$

also referred to as a Cayley transformation, to map $\mathbb{C}^+$ one-to-one onto $\mathbb{D}$. In the sequel, we often abbreviate $z_\alpha(\cdot) = z(\cdot)$ and $\mu_\alpha(\cdot) = \mu(\cdot)$. By combining the resolvent identity

$$(\mu - A|_{\mathcal{X}})^{-1} - (\alpha - A|_{\mathcal{X}})^{-1} = (\alpha - \mu)(\alpha - A)^{-1}(\mu - A|_{\mathcal{X}})^{-1}, \quad \mu, \alpha \in \mathrm{res}\,(A),$$

with the definition (3.5) of the transfer function $\widehat{\mathfrak{D}}$, one can verify that the transfer function in (6.3) satisfies

$$\widehat{\mathbf{D}}(z) = \widehat{\mathfrak{D}}\big(\mu_\alpha(z)\big), \quad \frac{1}{z} \in \mathrm{res}\,(\mathbf{A}), \tag{6.5}$$

if $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{smallmatrix}\right]$ are related by (6.1).

Remark 6.1 According to [44, Thms 3.1 and 3.2], the Cayley transform $\left[\begin{smallmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{smallmatrix}\right]$ is a contraction (isometric) for some/for all $\alpha \in \mathbb{C}^+$ if and only if the original system $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is passive (energy preserving). Moreover, $\left[\begin{smallmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{smallmatrix}\right]$ is controllable (observable) if and only if $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is controllable (observable). The convention here that the transfer function has the form $z\mathbf{C}(1-z\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}$ rather than $\mathbf{C}(z-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}$ has no influence on these general facts; see also [45, Sect. 12.3].

It is easy to show that the adjoint of $\left[\begin{smallmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{smallmatrix}\right]$ in (6.1) is the Cayley transform of the dual of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ with parameter $\overline{\alpha} \in \operatorname{res}(A^*)$ along lines similar to the proof of [45, Lem. 6.2.14]. Hence $\left[\begin{smallmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{smallmatrix}\right]$ is a co-isometry for some/for all $\alpha \in \mathbb{C}^+$ if and only if $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is a co-energy-preserving system node.

6.1 The Observable Co-Energy-Preserving Models

It follows immediately from Theorem 5.3 and (6.1) that the internal Cayley transform with parameter $\alpha \in \mathbb{C}^+$ of the observable co-energy-preserving model $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_o$ for the Schur function φ on $\mathbb{C}^+$ is the operator $\begin{bmatrix} \mathbf{A}_o & \mathbf{B}_o \\ \mathbf{C}_o & \mathbf{D}_o \end{bmatrix}$, where

$$\begin{aligned} (\mathbf{A}_o x)(\mu) &= \frac{\overline{\alpha}+\mu}{\alpha-\mu}\, x(\mu) - \frac{2\operatorname{Re}\alpha}{\alpha-\mu}\, x(\alpha), \quad x \in \mathcal{H}_o,\ \mu \in \mathbb{C}^+, \\ (\mathbf{B}_o u)(\mu) &= \sqrt{2\operatorname{Re}\alpha}\, \frac{\varphi(\mu)-\varphi(\alpha)}{\alpha-\mu} u, \quad u \in \mathcal{U},\ \mu \in \mathbb{C}^+, \\ \mathbf{C}_o x &= \sqrt{2\operatorname{Re}\alpha}\, x(\alpha), \quad x \in \mathcal{H}_o, \quad \text{and} \\ \mathbf{D}_o u &= \varphi(\alpha) u, \quad u \in \mathcal{U}. \end{aligned} \tag{6.6}$$

The system (6.6) is observable and isometric, and by (6.5) the transfer function ϕ_α of (6.6) satisfies

$$\phi_\alpha(z) = \varphi\big(\mu_\alpha(z)\big), \quad z \in \mathbb{D}. \tag{6.7}$$

We denote the Hilbert space with reproducing kernel

$$\mathrm{K}_{o,\alpha}(z,w) := \frac{1-\phi_\alpha(z)\phi_\alpha(w)^*}{1-z\overline{w}}, \quad z, w \in \mathbb{D}, \tag{6.8}$$

by $\mathrm{H}_{o,\alpha}$. By assertion 4 of Theorem 1.3, the operator $\left[\begin{smallmatrix} \mathbf{A}_o & \mathbf{B}_o \\ \mathbf{C}_o & \mathbf{D}_o \end{smallmatrix}\right]$ must be unitarily similar to the corresponding de Branges–Rovnyak discrete-time model realization $\left[\begin{smallmatrix} \mathrm{A}_o & \mathrm{B}_o \\ \mathrm{C}_o & \mathrm{D}_o \end{smallmatrix}\right]$ in (1.8), constructed using the transfer function $\phi_\alpha \in \mathcal{S}(\mathbb{D}; \mathcal{U}, \mathcal{Y})$ in (6.7). The following result describes this unitary similarity:

Proposition 6.2 *For arbitrary $\varphi \in \mathcal{S}(\mathbb{C}^+; \mathcal{U}, \mathcal{Y})$ and $\alpha \in \mathbb{C}^+$, the following claims are true:*

1. *Let $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_o$ be the canonical observable co-energy-preserving model for $\varphi \in \mathcal{S}(\mathbb{C}^+; \mathcal{U}, \mathcal{Y})$. Then $\left[\begin{smallmatrix} \mathbf{A}_o & \mathbf{B}_o \\ \mathbf{C}_o & \mathbf{D}_o \end{smallmatrix}\right]$ in (6.6) is the Cayley transform with parameter α of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_o$.*

2. *The following linear operator maps* $\mathrm{H}_{o,\alpha}$ *unitarily onto* $\mathcal{H}_o$*:*

$$(\Xi_\alpha \xi)(\mu) := \frac{\sqrt{2\mathrm{Re}\,\alpha}}{\overline{\alpha}+\mu}\,\xi\big(z_\alpha(\mu)\big), \quad \xi \in \mathrm{H}_{o,\alpha}, \mu \in \mathbb{C}^+, \tag{6.9}$$

where α *is the same in* (6.4) *and* (6.9)*. The inverse is*

$$\big((\Xi_\alpha)^{-1}\zeta\big)(z) = \frac{\sqrt{2\mathrm{Re}\,\alpha}}{1+z}\,\zeta\big(\mu_\alpha(z)\big), \quad \alpha \in \mathbb{C}^+,\ \zeta \in \mathcal{H}_o,\ z \in \mathbb{D}. \tag{6.10}$$

3. *Let* $\left[\begin{smallmatrix} \mathrm{A}_o & \mathrm{B}_o \\ \mathrm{C}_o & \mathrm{D}_o \end{smallmatrix}\right]$ *be the de Branges–Rovnyak model realization in* (1.8) *of* ϕ_α*. The operator* Ξ_α *intertwines* $\left[\begin{smallmatrix} \mathbf{A}_o & \mathbf{B}_o \\ \mathbf{C}_o & \mathbf{D}_o \end{smallmatrix}\right]$ *and* $\left[\begin{smallmatrix} \mathrm{A}_o & \mathrm{B}_o \\ \mathrm{C}_o & \mathrm{D}_o \end{smallmatrix}\right]$*:*

$$\begin{bmatrix} \mathbf{A}_o\Xi_\alpha & \mathbf{B}_o \\ \mathbf{C}_o\Xi_\alpha & \mathbf{D}_o \end{bmatrix} = \begin{bmatrix} \Xi_\alpha\mathrm{A}_o & \Xi_\alpha\mathrm{B}_o \\ \mathrm{C}_o & \mathrm{D}_o \end{bmatrix}. \tag{6.11}$$

Proof We leave it to the reader to verify assertion 1 as described above (6.6). In order to prove assertion 2, we for notational reasons first relate w to μ as z is related to λ in (6.4):

$$w_\alpha(\lambda) := \frac{\alpha-\lambda}{\overline{\alpha}+\lambda}, \quad \lambda \in \mathbb{C}^+ \Longleftrightarrow \lambda_\alpha(w) = \frac{\alpha-\overline{\alpha}w}{1+w}, \quad w \in \mathbb{D}. \tag{6.12}$$

The key to the unitarity of Ξ_α is the following relationship between the reproducing kernels of $\mathcal{H}_o$ and $\mathrm{H}_{o,\alpha}$:

$$\mathrm{K}_o\big(z(\mu), w(\lambda)\big) = \frac{(\overline{\alpha}+\mu)(\alpha+\overline{\lambda})}{2\mathrm{Re}\,\alpha} K_o(\mu,\lambda), \quad \mu,\lambda \in \mathbb{C}^+, \tag{6.13}$$

which follows from the fact that

$$\frac{1-\phi_\alpha\big(z(\mu)\big)\phi_\alpha\big(w(\lambda)\big)^*}{1-z(\mu)\overline{w(\lambda)}} = \frac{1-\varphi(\mu)\varphi(\lambda)^*}{1-\frac{\alpha-\mu}{\overline{\alpha}+\mu}\frac{\overline{\alpha}-\overline{\lambda}}{\alpha+\overline{\lambda}}} = \frac{(\overline{\alpha}+\mu)(\alpha+\overline{\lambda})}{2\mathrm{Re}\,\alpha}\,\frac{1-\varphi(\mu)\varphi(\lambda)^*}{\mu+\overline{\lambda}}.$$

Combining (6.13) with (6.9) gives that the action of Ξ_α on kernel functions e_o in $\mathrm{H}_{o,\alpha}$ is

$$\Big(\Xi_\alpha \mathrm{e}_o\big(w(\lambda)\big)^* y\Big)(\mu) = \frac{\alpha+\overline{\lambda}}{\sqrt{2\mathrm{Re}\,\alpha}} e_o(\lambda)^* y, \quad \lambda \in \mathbb{C}^+,\ y \in \mathcal{Y}. \tag{6.14}$$

It now follows that Ξ_α is isometric, since (using (6.13) in the second equality)

$$\begin{aligned}\left(\Xi_\alpha \mathrm{e}_o\big(w(\lambda)\big)^* y, \Xi_\alpha \mathrm{e}_o\big(z(\mu)\big)^*\gamma\right)_{\mathcal{H}_o} &= \left(\frac{\alpha+\overline{\lambda}}{\sqrt{2\mathrm{Re}\,\alpha}} e_o(\lambda)^* y, \frac{\alpha+\overline{\mu}}{\sqrt{2\mathrm{Re}\,\alpha}} e_o(\mu)^*\gamma\right)_{\mathcal{H}_o} \\ &= \left(\mathrm{K}_o\big(z(\mu), w(\lambda)\big)y, \gamma\right)_{\mathcal{Y}} \\ &= \left(\mathrm{e}_o\big(w(\lambda)\big)^* y, \mathrm{e}_o\big(z(\mu)\big)^*\gamma\right)_{\mathrm{H}_o}.\end{aligned}$$

The Eq. (6.14) moreover implies that the range of Ξ_α contains the dense subspace span $\left\{e_o(\lambda)^* y | \lambda \in \mathbb{C}^+,\ y \in \mathcal{Y}\right\}$ of $\mathcal{H}_o$. We conclude that Ξ_α is unitary as claimed. Formula (6.10) follows from (6.9) by denoting the right-hand side of (6.9) by $\zeta(\mu)$, changing the variable from $\mu \in \mathbb{C}^+$ to $z \in \mathbb{D}$ using (6.4), and solving for $\xi(z)$.

The following calculations use (1.8) and prove (6.11):

$$\begin{aligned}(\Xi_\alpha \mathrm{B}_o u)(\mu) &= \frac{\sqrt{2\mathrm{Re}\,\alpha}}{\overline{\alpha}+\mu}\,\frac{\phi\big(z(\mu)\big)-\phi(0)}{z(\mu)} u = \sqrt{2\mathrm{Re}\,\alpha}\,\frac{\varphi(\mu)-\varphi(\alpha)}{\alpha-\mu} u \\ &= (\mathbf{B}_o u)(\mu), \\ (\Xi_\alpha \mathrm{A}_o \xi)(\mu) &= \frac{\sqrt{2\mathrm{Re}\,\alpha}}{\overline{\alpha}+\mu}\,\frac{\xi\big(z(\mu)\big)-\xi(0)}{z(\mu)} = \sqrt{2\mathrm{Re}\,\alpha}\,\frac{\xi\big(z(\mu)\big)-\xi(0)}{\alpha-\mu}, \\ (\mathbf{A}_o \Xi_\alpha \xi)(\mu) &= \frac{\overline{\alpha}+\mu}{\alpha-\mu}\,\frac{\sqrt{2\mathrm{Re}\,\alpha}}{\overline{\alpha}+\mu}\,\xi\big(z(\mu)\big) - \frac{2\mathrm{Re}\,\alpha}{\alpha-\mu}\,\frac{\sqrt{2\mathrm{Re}\,\alpha}}{\overline{\alpha}+\alpha}\,\xi\big(z(\alpha)\big) \\ &= \sqrt{2\mathrm{Re}\,\alpha}\,\frac{\xi\big(z(\mu)\big)-\xi(0)}{\alpha-\mu} = (\Xi_\alpha \mathrm{A}_o \xi)(\mu), \quad \text{and} \\ \mathbf{C}_o \Xi_\alpha \xi = \sqrt{2\mathrm{Re}\,\alpha}\,(\Xi_\alpha \xi)(\alpha) &= \sqrt{2\mathrm{Re}\,\alpha}\,\frac{\sqrt{2\mathrm{Re}\,\alpha}}{\overline{\alpha}+\alpha}\,\xi\big(z(\alpha)\big) = \xi(0) = \mathrm{C}_o \xi,\end{aligned}$$

valid for all $\xi \in \mathrm{H}_o$, $\mu \in \mathbb{C}^+$, and $u \in \mathcal{U}$. □

Note the interesting fact that

$$K^d_{o,-1}(\mu, \lambda) = (2\mathrm{Re}\,\beta)\,\mathrm{K}_o\big(z_\beta(\mu), w_\beta(\lambda)\big), \quad \mu, \lambda \in \mathbb{C}^+,$$

cf. (5.20) and (6.13). We have no explanation for this coincidence.

The operator Ξ_α is called the inverse Cayley transform in [45]. It is the frequency-domain analogue of the inverse Laguerre transform; see [45, Thm 12.3.1 and Def. 12.3.2]. This unitary mapping can be used for transferring knowledge from the very well-known disk setting to the half-plane setting. For instance, by (6.9), the condition (5.14) holds if and only if the only function in $\mathrm{H}_{o,\alpha}$ of the form $\phi_\alpha(\cdot)u$ is the zero function; also note that by (6.7) we have $\varphi(\mu)u = 0$ for all $\mu \in \mathbb{C}^+$ if and only if $\phi_\alpha(z)u = 0$ for all $z \in \mathbb{D}$. Thus, the conditions (5.14) and $\varphi(\cdot)u = 0 \Rightarrow u = 0$ hold if and only if the conditions (1.12) and (1.13) hold with $\phi = \phi_\alpha$. By the last assertion of Theorem 1.3, this is the case if and only if the corresponding observable co-isometric realization $\mathbf{U}_o$ is unitary, which by Remark 6.1 is equivalent to $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_o$ being conservative. This provides an alternative proof of the statement that (5.14) and $\varphi(\cdot)u = 0 \Rightarrow u = 0$ hold together if and only if $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_o$ is conservative; see Theorem 5.6. It is moreover easy to see that the internal Cayley transformation can be used to convert the statements 3 in Theorem 1.5 to statement 4 in Theorem 4.17 and the corresponding statement in Theorem 5.6.

6.2 The Controllable Energy-Preserving Models

We have seen in (1.7) and (1.9) that the model space $\mathrm{H}_c = \mathcal{H}(\mathrm{K}_c)$ over $\mathbb{D}$ arises in the same way as $\mathrm{H}_o = \mathcal{H}(\mathrm{K}_o)$ but with $\widetilde{\phi}(z) = \phi(\overline{z})^*$ in place of ϕ. Similarly for the models over $\mathbb{C}^+$, the model space $\mathcal{H}_c = \mathcal{H}(K_c)$ arises in the same way as $\mathcal{H}_o = \mathcal{H}(K_o)$ but with $\widetilde{\varphi}(\mu) = \varphi(\overline{\mu})^*$ in place of $\varphi(\mu)$; see (1.17). If we assume that ϕ_α and φ are related according to (6.7), then we see that

$$\widetilde{\varphi}(\mu) = \varphi(\overline{\mu})^* = \phi\big(z_\alpha(\overline{\mu})\big)^* = \phi\left(\overline{z_{\overline{\alpha}}(\mu)}\right)^* = \widetilde{\phi}\big(z_{\overline{\alpha}}(\mu)\big). \tag{6.15}$$

This suggests that the appropriate mapping of $\mathbb{C}^+$ onto $\mathbb{D}$ for the energy-preserving setting should be $\mu \mapsto z_{\overline{\alpha}}(\mu)$ rather than $\mu \mapsto z_\alpha(\mu)$. Indeed, defining

$$\mathrm{K}_{c,\overline{\alpha}}(z,w) := \frac{1-\phi_{\overline{\alpha}}(\overline{z})^*\phi_{\overline{\alpha}}(\overline{w})}{1-z\overline{w}}, \quad z, w \in \mathbb{D}, \tag{6.16}$$

we obtain

$$\mathrm{K}_{c,\overline{\alpha}}\big(z_{\overline{\alpha}}(\mu), w_{\overline{\alpha}}(\lambda)\big) = \frac{(\alpha+\mu)(\overline{\alpha}+\overline{\lambda})}{2\mathrm{Re}\,\alpha} K_c(\mu,\lambda), \quad \mu, \lambda \in \mathbb{C}^+, \tag{6.17}$$

and this leads to the following unitary similarity result for the discrete-time controllable realizations:

Proposition 6.3 *Let $\varphi \in \mathcal{S}(\mathbb{C}^+;\mathcal{U},\mathcal{Y})$ and $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ be the controllable energy-preserving realization of φ given by the operator closure of (4.3). For arbitrary $\alpha \in \mathbb{C}^+$, the following claims are true:*

1. *The Cayley transform with parameter α of the canonical controllable energy-preserving model $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ is the isometry $\left[\begin{smallmatrix} \mathbf{A}_c & \mathbf{B}_c \\ \mathbf{C}_c & \mathbf{D}_c \end{smallmatrix}\right] : \left[\begin{smallmatrix} \mathcal{H}_c \\ \mathcal{U} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{H}_c \\ \mathcal{Y} \end{smallmatrix}\right]$ given by*

$$\begin{aligned}
(\mathbf{A}_c x)(\mu) &= \frac{\overline{\alpha}-\mu}{\alpha+\mu}\,x(\mu) - \frac{2\mathrm{Re}\,\alpha}{\alpha+\mu}\,\widetilde{\varphi}(\mu)\tau_{c,\alpha}x, \quad x \in \mathcal{H}_c,\ \mu \in \mathbb{C}^+,\\
\mathbf{B}_c u &= \sqrt{2\mathrm{Re}\,\alpha}\,e_c(\overline{\alpha})^* u, \quad u \in \mathcal{U},\\
\mathbf{C}_c x &= \sqrt{2\mathrm{Re}\,\alpha}\,\tau_{c,\alpha}x, \quad x \in \mathcal{H}_c, \quad \textit{and}\\
\mathbf{D}_c u &= \varphi(\alpha)u, \quad u \in \mathcal{U}.
\end{aligned} \tag{6.18}$$

2. *Let $\mathrm{H}_{c,\alpha}$ be the Hilbert space with reproducing kernel (6.16). Then $\Xi_{\overline{\alpha}}$ in (6.9) is unitary from $\mathrm{H}_{c,\alpha}$ to $\mathcal{H}_c$.*

3. *The operator* $\Xi_{\overline{\alpha}}$ *intertwines* $\left[\begin{smallmatrix} \mathbf{A}_c & \mathbf{B}_c \\ \mathbf{C}_c & \mathbf{D}_c \end{smallmatrix}\right]$ *in* (6.18) *and* $\left[\begin{smallmatrix} \mathrm{A}_c & \mathrm{B}_c \\ \mathrm{C}_c & \mathrm{D}_c \end{smallmatrix}\right]$ *in* (1.10)*:*

$$\begin{bmatrix} \mathbf{A}_c \Xi_{\overline{\alpha}} & \mathbf{B}_c \\ \mathbf{C}_c \Xi_{\overline{\alpha}} & \mathbf{D}_c \end{bmatrix} = \begin{bmatrix} \Xi_{\overline{\alpha}} \mathrm{A}_c & \Xi_{\overline{\alpha}} \mathrm{B}_c \\ \mathrm{C}_c & \mathrm{D}_c \end{bmatrix}. \tag{6.19}$$

Proof The formula for $\mathbf{B}_c$ follows from (6.1) and (4.7), and the formula for $\mathbf{C}_c$ from (6.1) combined with (4.49). By (4.44) and (4.49), we have

$$\big(A_c(\alpha - A_c)^{-1}x\big)(\mu) = -\mu\big((\alpha - A_c)^{-1}x\big)(\mu) - \widetilde{\varphi}(\mu)\tau_{c,\alpha}x, \quad x \in \mathcal{H}_c,\ \mu \in \mathbb{C}^+,$$

and combining this with (4.51) and (6.1) gives the formula for $\mathbf{A}_c$. Due to (6.17) we have

$$\Xi_{\overline{\alpha}}\, \mathrm{e}_c\big(w_{\overline{\alpha}}(\lambda)\big)^* u = \frac{\overline{\alpha} + \overline{\lambda}}{\sqrt{2\mathrm{Re}\,\alpha}} e_c(\lambda)^* u, \quad \lambda \in \mathbb{C}^+,\ u \in \mathcal{U}, \tag{6.20}$$

and the unitarity of $\Xi_{\overline{\alpha}}$ follows from the argument in the proof of Proposition 6.2.2.

We have that $\mathbf{B}_c = \Xi_{\overline{\alpha}} \mathrm{B}_c$, since by Theorem 1.4 and (6.20), it holds for all $u \in \mathcal{U}$ that

$$\Xi_{\overline{\alpha}} \mathrm{B}_c u = \Xi_{\overline{\alpha}} \mathrm{e}_c(0)^* u = \Xi_{\overline{\alpha}} \mathrm{e}_c\big(w_{\overline{\alpha}}(\overline{\alpha})\big)^* u = \frac{\overline{\alpha} + \alpha}{\sqrt{2\mathrm{Re}\,\alpha}}\, e_c(\overline{\alpha})^* u = \mathbf{B}_c u.$$

Moreover $\mathrm{C}_c = \mathbf{C}_c \Xi_{\overline{\alpha}}$, because by Theorem 1.4, (6.15), the unitarity of $\Xi_{\overline{\alpha}}$, and (4.50), we have the following equalities, valid for all $x \in \mathrm{H}_{c,\alpha}$ and $y \in \mathcal{Y}$:

$$\begin{aligned}
(\mathrm{C}_c x, y)_{\mathcal{Y}} &= \left(x, z \mapsto \frac{\widetilde{\phi}(z) - \widetilde{\phi}(0)}{z} y\right)_{\mathrm{H}_c} \\
&= \left(\Xi_{\overline{\alpha}} x,\ \Xi_{\overline{\alpha}}\left(z \mapsto \frac{\widetilde{\phi}(z) - \widetilde{\phi}(0)}{z}\right) y\right)_{\mathcal{H}_c} \\
&= \left(\Xi_{\overline{\alpha}} x, \left(\mu \mapsto \frac{\sqrt{2\mathrm{Re}\,\alpha}}{\alpha + \mu}\, \frac{\widetilde{\phi}\big(z_{\overline{\alpha}}(\mu)\big) - \widetilde{\phi}\big(z_{\overline{\alpha}}(\overline{\alpha})\big)}{z_{\overline{\alpha}}(\mu)}\right) y\right)_{\mathcal{H}_c} \\
&= \left(\Xi_{\overline{\alpha}} x, \left(\mu \mapsto \sqrt{2\mathrm{Re}\,\alpha}\, \frac{\widetilde{\varphi}(\mu) - \widetilde{\varphi}(\overline{\alpha})}{\overline{\alpha} - \mu}\right) y\right)_{\mathcal{H}_c} \\
&= \left(\sqrt{2\mathrm{Re}\,\alpha}\, \tau_{c,\alpha}\, \Xi_{\overline{\alpha}}\, x, y\right)_{\mathcal{Y}}.
\end{aligned}$$

Finally, by Theorem 1.4, (6.18), and (6.15), it holds for $x \in \mathcal{H}_c$ and $\mu \in \mathbb{C}^+$ that

$$\begin{aligned}
(\mathbf{A}_c \Xi_{\overline{\alpha}} x)(\mu) &= \frac{\overline{\alpha} - \mu}{\alpha + \mu} \frac{\sqrt{2\mathrm{Re}\,\alpha}}{\alpha + \mu}\, x\big(z_{\overline{\alpha}}(\mu)\big) - \frac{2\mathrm{Re}\,\alpha}{\alpha + \mu}\, \widetilde{\varphi}(\mu)\, \tau_{c,\alpha}\, \Xi_{\overline{\alpha}}\, x \\
&= \frac{\sqrt{2\mathrm{Re}\,\alpha}}{\alpha + \mu}\Big(z_{\overline{\alpha}}(\mu)\, x\big(z_{\overline{\alpha}}(\mu)\big) - \widetilde{\phi}\big(z_{\overline{\alpha}}(\mu)\big)\, \mathrm{C}_c\, x\Big) \\
&= (\Xi_{\overline{\alpha}} \mathrm{A}_c x)(\mu).
\end{aligned}$$

□

7 Final Remarks

We have developed a realization theory for arbitrary $\varphi \in \mathcal{S}(\mathbb{C}^+;\mathcal{U},\mathcal{Y})$ that is completely analogous to the classical case worked out by de Branges and Rovnyak on the complex unit disk $\mathbb{D}$. The same general principles carry over from the discrete case, but unboundedness of most of the involved operators makes it more complicated to work out the details. By avoiding linear fractional transformations, we obtain more insight into intricacies specific to continuous-time systems, such as the Hilbert space riggings $\mathcal{H}_{c,1} \subset \mathcal{H}_c \subset \mathcal{H}_{c,-1}$, $\mathcal{H}_{o,1} \subset \mathcal{H}_o \subset \mathcal{H}_{o,-1}$, $\mathcal{H}^d_{c,1} \subset \mathcal{H}_c \subset \mathcal{H}^d_{c,-1}$, and $\mathcal{H}^d_{o,1} \subset \mathcal{H}_o \subset \mathcal{H}^d_{o,-1}$.

Formulas for the canonical models $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$ and $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_o$, as well as their component operators, are summarized in the following tables:

Formulas related to $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_o$.

$\mathcal{H}_o$:	$\mathcal{H}(K_o)$, the state space of the observable model
$\mathcal{H}_{o,1}$:	$\{x \in \mathcal{H}_o \mid \exists y \in \mathcal{Y}:\ \mu \mapsto \mu x(\mu) - y \in \mathcal{H}_o\}$, the domain of A_o
$\mathcal{H}_{o,-1}$:	$\left\{x : \mathbb{C}^+ \to \mathcal{Y} \mid \mu \mapsto \dfrac{x(\mu)-x(\alpha)}{\alpha-\mu} \in \mathcal{H}_o\right\}$, equivalence classes modulo constants
A_o:	$x \mapsto \big(\mu \mapsto \mu x(\mu) - \lim_{\mathrm{Re}\,\eta\to\infty} \eta x(\eta)\big)$, maps $\mathcal{H}_{o,1}$ boundedly into $\mathcal{H}_o$
$A_o\vert_{\mathcal{H}_o}$:	$x \mapsto [\mu \mapsto \mu x(\mu)]$, element of $\mathcal{B}(\mathcal{H}_o,\mathcal{H}_{o,-1})$
B_o:	$u \mapsto [\mu \mapsto \varphi(\mu)u]$, operator in $\mathcal{B}(\mathcal{U},\mathcal{H}_{o,-1})$
C_o:	$x \mapsto \lim_{\mathrm{Re}\,\eta\to\infty} \eta x(\eta)$, element of $\mathcal{B}(\mathcal{H}_{o,1},\mathcal{Y})$
$(\alpha - A_o\vert_{\mathcal{H}_o})^{-1}$:	$x \mapsto \left(\mu \mapsto \dfrac{x(\mu)-x(\alpha)}{\alpha-\mu}\right)$, operator in $\mathcal{B}(\mathcal{H}_{o,-1},\mathcal{H}_o)$
$\mathrm{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_o\right)$:	$\left\{\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{H}_o \\ \mathcal{U} \end{smallmatrix}\right] \middle\vert\, \exists y \in \mathcal{Y}:\ \mu \mapsto \mu x(\mu) + \varphi(\mu)u - y \in \mathcal{H}_o\right\}$
$\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_o$:	$\begin{bmatrix} x \\ u \end{bmatrix} \mapsto \begin{bmatrix} \mu \mapsto \mu x(\mu) + \varphi(\mu)u - y \\ y \end{bmatrix}$, where x and u determine y via $y = \lim_{\mathrm{Re}\,\eta\to\infty} \eta x(\eta) + \varphi(\eta)u$

Formulas related to $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$.

$\mathcal{H}_c$:	$\mathcal{H}(K_c)$, the state space of the controllable model
$(\alpha - A_c)^{-1}$:	$x \mapsto \dfrac{x(\mu) - \widetilde{\varphi}(\mu)\tau_{c,\alpha}\, x}{\alpha+\mu}$, element of $\mathcal{B}(\mathcal{H}_c;\mathcal{H}_{c,1})$
A_c:	$x \mapsto \big(\mu \mapsto -\mu x(\mu) - \widetilde{\varphi}(\mu) C_c x\big)$, $x \in \mathrm{dom}\,(A_c)$
C_c:	$\tau_{c,\alpha}(\alpha - A_c) \in \mathcal{B}(\mathcal{H}_{c,1},\mathcal{Y})$
$\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c$:	$\begin{bmatrix} x \\ u \end{bmatrix} \mapsto \begin{bmatrix} \mu \mapsto -\mu x(\mu) - \varphi(\overline{\mu})^*\gamma_\lambda + \big(1 - \varphi(\overline{\mu})^*\varphi(\overline{\lambda})\big)u \\ \gamma_\lambda + \varphi(\overline{\lambda})u \end{bmatrix}$, where $\gamma_\lambda = C_c\big(x - e_c(\lambda)^* u\big)$, for arbitrary $\lambda \in \mathbb{C}^+$

The following formulas are valid only under the assumption (4.56):

$\mathcal{H}_{c,1}$:	$\{x \in \mathcal{H}_c \mid \exists y \in \mathcal{Y}:\ \mu \mapsto \mu x(\mu) + \widetilde{\varphi}(\mu) y \in \mathcal{H}_c\}$; this is the domain of A_c

$\mathcal{H}_{c,-1}$: $\left\{x : \mathbb{C}^+ \to \mathcal{U} \,\middle|\, \exists y \in \mathcal{Y} : \ \mu \mapsto \dfrac{x(\mu) + \widetilde{\varphi}(\mu)y}{\beta + \mu} \in \mathcal{H}_c\right\}$, consists of equivalence classes modulo the subspace $\widetilde{\varphi}(\cdot)\mathcal{Y} \subset \mathcal{H}_{c,-1}$

dom $\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c\right)$: $\left\{\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{H}_c \\ \mathcal{U} \end{smallmatrix}\right] \,\middle|\, \exists y \in \mathcal{Y} : \mu \mapsto -\mu x(\mu) - \widetilde{\varphi}(\mu)y + u \in \mathcal{H}_c\right\}$

$A_c|_{\mathcal{H}_c}$: $[\mu \mapsto -\mu x(\mu)]$, lies in $\mathcal{B}(\mathcal{H}_c, \mathcal{H}_{c,-1})$

B_c: $u \mapsto [\mu \mapsto u]$, lies in $\mathcal{B}(\mathcal{U}, \mathcal{H}_{c,-1})$

Note that the reason for making the assumption (4.56) is that it allows us to characterize the spaces $\mathcal{H}_{c,\pm1}$ and dom $\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_c\right)$. The formulas for A_c and C_c a circle definition. This can be avoided in case $\widetilde{\varphi}(\cdot) \cap \mathcal{H}_c = \{0\}$ which holds if (4.56) holds, since A_c can then be defined without using C_c; see Remark 4.14.

We next describe how to derive Theorems 1.6 and 1.7 from [7] by using the same method as was used in [10] to derive Theorems 1.2 and 1.3, replacing the unit disk by the right half-plane. The multiplication operator M_φ induced by a Schur function $\varphi \in \mathcal{S}(\mathbb{C}^+; \mathcal{U}, \mathcal{Y})$ defines a contraction from $H^2(\mathbb{C}^+; \mathcal{U})$ into $H^2(\mathbb{C}^+; \mathcal{U})$. The graph of this operator is a maximal nonnegative subspace $\widehat{\mathfrak{W}}$ of the Kreĭn space $H^2(\mathbb{C}^+; \mathcal{W})$, where $\mathcal{W} = \mathcal{U} \boxplus -\mathcal{Y}$ (i.e., the Kreĭn space $\mathcal{W}$ is the orthogonal sum of $\mathcal{U}$ and the anti-space $-\mathcal{Y}$ of $\mathcal{Y}$). This subspace is invariant under multiplication by the function $\lambda \mapsto e^{-\lambda}$. The inverse Laplace transform maps $\widehat{\mathfrak{W}}$ onto a maximal nonnegative right-shift invariant subspace $\mathfrak{W}_+$ of the Kreĭn space $L^2(\mathbb{R}^+; \mathcal{W})$, which using the terminology of [7] is called a (time domain) passive future behavior in $\mathcal{W}$.

In [7] three different canonical state/signal realizations of $\mathfrak{W}_+$ are constructed, one which is controllable and energy preserving, another which is observable and co-energy preserving, and a third which is simple and conservative. These three realizations are given in the time domain setting, but they can be mapped into frequency domain realizations by arguing as in [10, Section 9], with the unit disk $\mathbb{D}$ replaced by the right half-plane $\mathbb{C}^+$. From these frequency domain realizations one can recover the input/state/output realizations in Theorems 1.6 and 1.7 (as well as an additional simple conservative one) by using the fundamental decomposition $\mathcal{W} = \mathcal{U} \boxplus -\mathcal{Y}$ of $\mathcal{W}$ to get input/state/output representations of scattering type of the canonical state/signal representations in [10], as was done in [10, Section 10] in the discrete-time setting.

Finally, we mention that a planned project for the future is to develop a canonical-model of a conservative closely-connected (or simple) system node realization of a Schur-class function over $\mathbb{C}^+$ in the spirit of the present paper.

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