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Modeling values for TU-games using generalized versions of consistency, standardness and the null player property

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Abstract In the paper we discuss three general properties of values of TU-games: λ-standardness, general probabilistic consistency and some modifications of the null player property. Necessary and sufficient conditions for different families of efficient, linear and symmetric values are given in terms of these properties. It is shown that the results obtained can be used to get new axiomatizations of several classical values of TU-games.

Keywords TU-game · Null player axiom · λ-Standardness · Consistency · Probabilistic consistency · The Shapley value · The per-capita value · The solidarity value · The equal split value

1 Introduction

Classical cooperative game in the *characteristic function form* (also called a *TU-game*) is a function $v : 2^N \mapsto \mathbb{R}$ with a finite set N as the *grand coalition* of the players (agents, individuals). For each *coalition S* (a subset of N), $v(S)$ represents the *worth*

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of S (the gain possible to be achieved jointly by all the players from S when they collaborate). One of the essential problems addressed in the papers on this topic is the following: Assuming that all the players in N will collaborate and create (finally) the grand coalition N, how to divide "fairly" the whole worth $v(N)$ between them? In the literature, many various procedures (called values) have been proposed for solving this problem, and the classical *Shapley value* (Shapley 1953) is the most prominent and the best known one among them. However this value has the so-called *null player property* that does not allow for solidarity between the players, assigning zero payoff to every "unproductive player"; that is to every player i with all his *marginal contributions* $v(S \cup i) - v(S) = 0$ for $S \subset N$ (called then a *null player*). On the opposite side, there is a very classical value allowing for the total solidarity between the players, which radically rejects the null player property—the so-called *equal split value*. It awards every player in the grand coalition N, independently of his contributions, $v(S \cup i) - v(S)$ to coalitions S, the same payoff equal to $\frac{v(N)}{n}$ where n is the cardinality of N. In the literature, many other values for TU-games "standing" between the Shapley value and the equal split value were constructed and characterized by proper sets of axioms. The most interesting results here are axiomatizations of the class of convex combinations of those two values (called egalitarian Shapley values) found in the two recent papers by van den Brink et al. (2013) and Casajus and Huettner (2013).

Two of the properties used most frequently to axiomatize the values of cooperative games are the null player axiom and consistency. The consistency property for a value states that if all the players are supposed to be paid according to a payoff vector in the original game, then the players in every "reduced game" (with a smaller number of players) can achieve some payoff vector closely related with the value of the reduced game. A survey of consistency properties of several classical values for cooperative games can be found in Driessen (1991). The present paper introduces some generalized versions of both the properties, the null player axiom and consistency, and studies different axiomatizations using them. The results obtained here have allowed to get new axiomatizations for several classical values.

The first main result (Theorem 1 in Sect. 3) gives necessary and sufficient conditions for efficient, linear and symmetric values to satisfy some generalized version of the classical null player property. Next, a number of its applications to several classical values for TU-games and their convex combinations is shown. Among others, we get an axiomatization for a new value for TU-games called the *per-capita value*, as well as new axiomatizations of the equal split value and the solidarity value.

In Hart and Mas-Colell (1989) and in Sobolev (1973), one can find the main results related to the consistency of the Shapley value. Hart and Mas-Colell construct an elegant theory of potential for cooperative games and apply it to get an axiomatization of the Shapley value with the consistency axiom. Their theory is extended in Driessen and Radzik (2003) where the authors give an axiomatization based on generalized consistency properties for some wide class of values of cooperative games. The main results given there have been completed and remarkably strengthened in Sect. 4 of our paper. The group of three results (Propositions 5 and 6, and Theorem 2) establishes necessary and sufficient conditions for the class of efficient, linear and symmetric values to satisfy the so-called λ-standardness and a general consistency property. The latter generalizes the consistency property considered by Hart and Mas-Colell (1989)

and is based on some modification of Sobolev's approach to arbitrary values of TU-games.

The results of Sect. 4 have been used to get next two results (Theorems 3 and 4 in Sect. 5), that are related to a construction of a new subclass of values satisfying the λ-standardness and the so-called *probabilistic consistency*. As an application of this result, we get new axiomatizations of the Shapley value, the per-capita value, the solidarity value and the equal split value in terms of these two properties. Also a wide discussion of the results obtained here is given.

Section 2 is our preliminary one, where we recall some standard definitions and several basic axioms desired for the values of TU-games, and we quote three results from the literature, fundamental for our considerations. We also recall the formulas for five particular values, essential for illustrating the results obtained in the paper. The last Sect. 6 is devoted to the proofs of the theorems.

2 Preliminaries

Let $N = \{1, 2, \cdots, n\}$, with $n \geq 2$, be a fixed finite set of n players. Subsets of N are called *coalitions* while N is called the *grand coalition*.

The cardinality of a set X will be denoted by $|X|$. For brevity, throughout the paper, the cardinality of sets (coalitions) N, S and T will be also denoted by appropriate small letters n, s and t, respectively. All the set inclusions "⊂" are meant to be weak. Also, for notational convenience, we will write singleton $\{i\}$ as i.

A *(transferable utility) game* (on N) is any function $v : 2^N \rightarrow \mathcal{R}$ with $v(\emptyset) = 0$, where $\mathcal{R}$ denotes the set of real numbers. Then for any coalition S in N, $v(S)$ describes the *worth* of the coalition S when all the players in S collaborate. A game v is *monotonic* if $v(S) \leq v(T)$ for any $S \subset T \subset N$. The set of all games v on N is denoted by Γ_N.

For a coalition $T \subset N$, the *unanimity game* u_T is defined by $u_T(S) = 1$ for $S \supset T$ and $u_T(S) = 0$ otherwise, for $S \subset N$.

A *value* $\Phi(v) = (\Phi_1(v), \ldots, \Phi_n(v))$ on Γ_N is thought of as a vector-valued mapping $\Phi : \Gamma_N \rightarrow \mathcal{R}^n$, which uniquely determines, for each game $v \in \Gamma_N$, a distribution of the total wealth available to all the players $1, 2, \ldots, n$, through their participation in the game v. We quickly recall several basic properties a value Φ may have.

A value Φ is called *efficient* if $\sum_{i \in N} \Phi_i(v) = v(N)$ for all games v. If $\Phi(\alpha v + \beta w) = \alpha \Phi(v) + \beta \Phi(w)$ for all games v and w and for all reals α and β, a value Φ is called *linear*. If the last equality holds for $\alpha = \beta = 1$, a value is *additive*. A player i is called a *null player* (*dummy player*) in game v if $v(S \cup i) = v(S)$ ($v(S \cup i) = v(S) + v(i)$) for every coalition $S \subset N \setminus i$. If $\Phi_i(v) = 0$ in case of any null player i in game v, we say that a value Φ satisfies the null player axiom. If a value Φ satisfies the equality $\Phi_{\pi i}(N, \pi v) = \Phi_i(v)$ for all $i \in N$ and every permutation π of the player set N, then we say that Φ satisfies the *anonymity* axiom, sometimes also called the *symmetry* axiom (here πv is defined as game πv by $\pi v(\pi(S)) = v(S)$ for $S \subset N$). If a value Φ satisfies $\Phi_i(v) = \Phi_j(v)$ if $v(S \cup i) = v(S \cup j)$ for any $S \subset N \setminus \{i, j\}$, we say that it has the *equal treatment* property. This property is weaker than symmetry.

In this paper we will mainly discuss values which satisfy efficiency, symmetry and linearity. Hence, for brevity, every value satisfying those three properties will be shortly called an *ESL-value*.

Now we recall four classical ESL-values (the *equal split value*, the *Shapley value*, the *solidarity value*, the *δ-discounted Shapley value*) and define a new one essential for the illustration of the results obtained in the paper. The first two values are standard in cooperative game theory. The solidarity value was introduced in Nowak and Radzik (1994). In the recent paper of Calvo (2008), two variations of the non-cooperative model for games in coalitional form, introduced by Hart and Mas-Colell (1996), were proposed, and two new, very interesting NTU-values have been introduced: the random marginal and random removal values. It turned out that for TU-games, the random marginal value coincides with the Shapley value and that, which was completely surprising, the random removal value coincides with the solidarity value.

The fourth value, called the *δ-discounted Shapley value* is constructed as a certain modification of the Shapley value, while the fifth new one, called the *per-capita value*, is a modification of the solidarity value.

The *equal split value* Φ^{Eq}. By definition, it is the ESL-value on Γ_N defined by $\Phi^{Eq}(v) = (\Phi_i^{Eq}(v))_{i \in N}$, where

$$\Phi_i^{Eq}(v) = \frac{v(N)}{n}, \quad i \in N. \tag{1}$$

So, the equal split value divides the worth $v(N)$ of the grand coalition N equally between all the players, independently of the worth of other coalitions.

The *Shapley value* Φ^{Sh}. It is the classical ESL-value on Γ_N (introduced in Shapley 1953), of the form $\Phi^{Sh}(v) = (\Phi_i^{Sh}(v))_{i \in N}$, where

$$\Phi_i^{Sh}(v) = \sum_{S \subset N \setminus i} \frac{s!(n-s-1)!}{n!} [v(S \cup i) - v(S)], \quad i \in N. \tag{2}$$

It is known that the Shapley value Φ^{Sh} on Γ_N is the unique ESL-value which satisfies the null player axiom.

The *solidarity value* Φ^{So}. This is an ESL-value on Γ_N discussed in the paper of Nowak and Radzik (1994). It is uniquely determined by the three classical axioms, *efficiency, additivity and symmetry*, and by a modification of the null player axiom, called *A-null player* axiom. We quickly recall this axiom. To express it we need to define, for any non-empty coalition $T \subset N$ and a game v, the quantity

$$A^v(S) = \frac{1}{s} \sum_{k \in S} [v(S) - v(S \setminus k)], \tag{3}$$

where s means the cardinality of S. Clearly, $A^v(S)$ can be seen as the *average marginal contribution* of a member of a coalition S. The axiom is as follows:

A- NULL PLAYER AXIOM: If $i \in N$ is an A-null player in a game v, that is, $A^v(S) = 0$ for every coalition S containing player i, then $\Phi_i(v) = 0$.

It is shown in Nowak and Radzik (1994) that for $v \in \Gamma_N$, the solidarity value is of the form $\Phi^{So}(v) = (\Phi_i^{So}(v))_{i \in N}$, where

$$\Phi_i^{So}(v) = \sum_{S \ni i} \frac{(n-s)!(s-1)!}{n!} A^v(S), \quad i \in N. \tag{4}$$

Using (4), one can easily deduce that the solidarity value has the following equivalent form

$$\Phi_i^{So}(v) = \frac{v(N)}{n+1} + \sum_{S \subset N \setminus i} \frac{s!(n-s-1)!}{n!} \left[\frac{v(S \cup i)}{s+2} - \frac{v(S)}{s+1} \right], \quad i \in N. \tag{5}$$

The δ-*discounted Shapley value* Φ^{Sh^δ}. It is the value on Γ_N of the form $\Phi^{Sh^\delta}(v) = (\Phi_i^{Sh^\delta}(v))_{i \in N}$ with $\delta \in \mathcal{R}$, where

$$\Phi_i^{Sh^\delta}(v) = \sum_{S \subset N \setminus i} \frac{s!(n-s-1)!}{n!} [\delta^{n-s-1} v(S \cup i) - \delta^{n-s} v(S)], \quad i \in N. \tag{6}$$

(Here, by definition, $0^0 = 1$.) An equivalent form of this generalization of the Shapley value was first introduced in Joosten et al. (1994), and next axiomatized (in terms of consistency) by Joosten (1996, Chapter 5). The name "δ-discounted Shapley value" comes from Driessen and Radzik (2003).

The *per-capita value* Φ^{Pc}. It is a modification of the solidarity value of the form $\Phi^{Pc}(v) = (\Phi_i^{Pc}(v))_{i \in N}$, where

$$\Phi_i^{Pc}(v) = n \cdot \sum_{S \subset N \setminus i} \frac{s!(n-s-1)!}{n!} \left[\frac{v(S \cup i)}{s+1} - \frac{v(S)}{s} \right], \quad i \in N. \tag{7}$$

(Here, by definition, $\frac{v(S)}{s} = 0$ if $S = \emptyset$.) This new value (not studied in the literature yet) is different from the per-capita Shapley value introduced in Example 3 in Driessen and Radzik (2003).

To end with, we quote three useful facts from the literature. The first one belongs to Ruiz et al. (1998) (see Lemma 9 there).

Proposition 1 *A value Φ on Γ_N is linear, efficient and satisfies the equal treatment property if and only if there exists a unique collection of real constants $\{\lambda_s\}_{s=1,\dots,n-1}$ such that for every game $v \in \Gamma_N$ the value payoff vector $(\Phi_i(v))_{i \in N}$ is of the following form:*

$$\Phi_i(v) = \frac{v(N)}{n} + \sum_{\substack{S \subseteq N \\ S \ni i}} \frac{\lambda_s}{s} v(S) - \sum_{\substack{S \subseteq N \\ \emptyset \neq S \not\ni i}} \frac{\lambda_s}{n-s} v(S), \quad i \in N. \tag{8}$$

Remark 1 It is easy to see that any value of the form (8) satisfies the symmetry property. On the other hand, the last property is stronger than equal treatment. Hence Proposition 1 implies the following: A value Φ on Γ_N is an ESL-value if and only

if it is efficient, linear and satisfies the equal treatment property. Therefore, in all the theorems, the phrase "ESL-value on Γ_N" can be replaced with that equivalent "weaker form".

The second result (see statement (iii) of Theorem 3 in Driessen and Radzik (2003)) gives another characterization of the set of ESL-values.

Proposition 2 *A value Φ on Γ_N is an ESL-value if and only if there exists a unique collection of real constants $\{b_s\}_{s=0,1,\dots,n}$ with $b_n = 1$ and $b_0 = 0$ such that for every game $v \in \Gamma_N$ the value payoff vector $(\Phi_i(v))_{i \in N}$ is of the following form:*

$$\Phi_i(v) = \sum_{S \subset N \setminus i} \frac{s!(n-s-1)!}{n!}[b_{s+1}v(S \cup i) - b_s v(S)], \quad i \in N. \tag{9}$$

Remark 2 One can easily see that the formula (9) generalizes the classical formula (2) for the Shapley value, giving this value for the constants $b_s = 1$ for $1 \le s \le n$, and giving the equal split value for the constants $b_s = 0$ for all $1 \le s < n$. On the other hand, it follows from (5) that (9) with $b_s = \frac{1}{s+1}$ for $1 \le s < n$, describes the formula for the solidarity value. One can also check that if we rewrite formula (8) with the help of new parameters $b_1, b_2, \dots, b_{n-1}$, putting there $\lambda_s = b_s \cdot \binom{n}{s}$ for $1 \le s < n$, then (8) coincides with the formula (9).

The third result (Theorems 1 and 2 in Radzik and Driessen (2013)) gives two more detailed versions of Proposition 2. To quote it we need to introduce the next two very desirable properties of values.

FAIR TREATMENT: Let $i, j \in N$ and $v \in \Gamma_N$. If $v(S \cup i) \ge v(S \cup j)$ for all $S \subset N \setminus i \setminus j$ then $\Phi_i(v) \ge \Phi_j(v)$.

MONOTONICITY: Let v be a *monotonic game*, that is satisfying $v(S) \le v(T)$ whenever $S \subset T$. Then for each player $i \in N$, $\Phi_i(v) \ge 0$.

It is worth mentioning that the first property (*fair treatment*) appears in the literature under different names, such as *desirability* (see, e.g. Peleg and Sudhölter 2003), or *local monotonocity* (see, e.g. Levinský and Silársky 2004), while *monotonicity* is sometimes called *weak monotonicity* (see Weber 1988). Besides, four different natural types of monotonicity are studied in Malawski (2013) and an interesting discussion about their mutual relations is given.

Proposition 3 *Let Φ be an ESL-value on Γ_N with its representation of the form (9). Then (a) Φ satisfies fair treatment if and only if*

$$b_n = 1 \quad and \quad b_s \ge 0 \quad for \quad s = 1, 2, \dots, n-1. \tag{10}$$

(b) Φ satisfies fair treatment and monotonicity if and only if

$$b_n = 1 \quad and \quad 0 \le b_s \le 1 \quad for \quad s = 1, 2, \dots, n-1. \tag{11}$$

Remark 3 First note (by Proposition 2) that all the five values, Φ^{Eq}, Φ^{Sh}, Φ^{So}, Φ^{Sh^δ} and Φ^{Pc} defined above, are ESL-values on Γ_N. A straightforward comparison of the formulas (1), (2) and (5) with (9) shows that the constants b_s in the representation formula (9) for the values Φ^{Eq}, Φ^{Sh}, Φ^{So} satisfy inequalities (11). Therefore these three values satisfy the fair treatment and monotonicity properties. In a similar way we show that the value Φ^{Sh^δ} with $0 \le \delta \le 1$ also satisfies those two properties. But comparing (7) with (9), we see that the constants b_s in the representation formula for the per-capita value are of the form $b_s = \frac{n}{s}$ for $s = 1, 2, \ldots, n-1$. So, by Proposition 3, the per-capita value satisfies the fair treatment property but not the monotonicity property.

3 Theorem on ESL-values with generalized null player properties

In this section we formulate our first main result (Theorem 1) where a wide subfamily of ESL-values on Γ_N is axiomatized with the help of some theoretical generalization of the null player property. Next we show that this result immediately implies axiomatizations (some of them are new) of eight values of TU-games.

Let $N = \{1, 2, \cdots, n\}$ with $n \ge 2$ be a fixed finite set. Let $\alpha \in \mathcal{R}$ and let $\bar{\beta} = (\beta_k)_{k=0}^n$ be a sequence of real numbers. We begin with the following definition generalizing the classical notion of null player in a game.

Definition 1 Player i is called a $\bar{\beta}$-null player in a game $v \in \Gamma_N$ if $\beta_{s+1} v(S \cup i) - \beta_s v(S) = 0$ for any $S \subset N \setminus i$.

Consider now the following property of a value Φ on Γ_N.

$(\bar{\beta}, \alpha)$- NULL PLAYER PAYOFF: If player i is a $\bar{\beta}$-null player in a game $v \in \Gamma_N$, then $\Phi_i(N, v) = \alpha \frac{v(N)}{n}$.

Now we are ready to formulate the following characterization theorem, the proof of which will be given in Sect. 6.

Theorem 1 *Let $\alpha \in \mathcal{R}$ and let $\bar{\beta}$ be a sequence of real numbers with $\beta_0 = 0$ and $\beta_k \ne 0$ for $1 \le k \le n$. A value Φ on Γ_N is an ESL-value satisfying the $(\bar{\beta}, \alpha)$-null player payoff property if and only if for every game $v \in \Gamma_N$ the value payoff vector $\Phi(v) = (\Phi_i(v))_{i \in N}$ is of the following form*

$$\Phi_i(v) = \alpha \frac{v(N)}{n} + \frac{1-\alpha}{\beta_n} \sum_{S \subset N \setminus i} \frac{s!(n-s-1)!}{n!} [\beta_{s+1} v(S \cup i) - \beta_s v(S)]. \quad (12)$$

Now we will present eight immediate corollaries to Theorem 1. They give some axiomatizations of the equal split value, the per-capita value, the δ-discounted Shapley value, the solidarity value, and convex combinations of some pairs of these values.

The first result repeats the classical one of Shapley (1953). Namely, taking the sequence $\bar{\beta}$ with $\beta_k = 1$ for $k \ge 1$, and $\alpha = 0$ in Theorem 1, we immediately get the following corollary.

Corollary 1 *A value Φ on Γ_N is an ESL-value satisfying the null player axiom if and only if Φ is the Shapley value $\Phi^{Sh} = \{\Phi_i^{Sh}\}_{i \in N}$ of the form (2).*

Now consider the following two modifications of the null player axiom with a clear interpretation.

NULL PLAYER AVERAGE PAYOFF: If player i is a null player in a game $v \in \Gamma_N$, then $\Phi_i(v) = \frac{v(N)}{n}$.

NULL PLAYER α- AVERAGE PAYOFF: Let $0 \le \alpha \le 1$. If player i is a null player in a game $v \in \Gamma_N$, then $\Phi_i(v) = \alpha \frac{v(N)}{n}$.

Now taking the sequence $\bar{\beta}$ with $\beta_k = 1$ for $k \ge 1$, and $\alpha = 1$ or $\alpha \in [0, 1]$ in Theorem 1, we get the next two corollaries.

Corollary 2 *A value Φ on Γ_N is an ESL-value satisfying the null player average payoff property if and only if Φ is the equal split value $\Phi^{Eq} = \{\Phi_i^{Eq}\}_{i \in N}$ of the form (1).*

Corollary 3 *Let $0 \le \alpha \le 1$. A value Φ on Γ_N is an ESL-value satisfying the null player α-average payoff property if and only if Φ is the α-egalitarian Shapley value $\widetilde{\Phi}^\alpha$ of the form $\widetilde{\Phi}^\alpha = \alpha \Phi^{Eq} + (1 - \alpha)\Phi^{Sh}$.*

Remark 4 The null player α-average payoff property can be seen as α-degree of solidarity between the players in a game. So the main idea to consider values with such property is to "add solidarity" to the problem of distribution of the grand coalition worth. The α-egalitarian Shapley values $\widetilde{\Phi}^\alpha$ are a consequence of this approach. The first axiomatization of the value $\widetilde{\Phi}^\alpha$ (equivalent to that of Corollary 3) was found in Joosten et al. (1994). It turns out that for any $0 \le \alpha \le 1$ the value $\widetilde{\Phi}^\alpha$ has two other, very desired properties: fair treatment and monotonicity. To see this, it is enough to note that the constants $\{b_s\}_{s=0,1,\dots,n}$ for $\widetilde{\Phi}^\alpha$ in its representation (9) are of the form $b_n = 1$ and $b_s = 1 - \alpha$ for $s < n$, and next to apply Proposition 3. It is also worth mentioning that three interesting axiomatizations of the whole class of the egalitarian Shapley values $\{\widetilde{\Phi}^\alpha : 0 \le \alpha \le 1\}$ have been found recently, two in van den Brink et al. (2013) and one in Casajus and Huettner (2013). As far as the equal split value Φ^{Eq} is concerned, its three other axiomatizations (different from that of Corollary 2) are given in van den Brink (2007).

Now, let us introduce the following modification of the notion of null player in a game (with a clear interpretation).

Definition 2 Player i is called a per-capita null player in a game $v \in \Gamma_N$ if for any $S \subset N \setminus i$, $\frac{v(S \cup i)}{s+1} - \frac{v(S)}{s} = 0$. (Here, by definition, $\frac{0}{0} = 0$.)

One can think that a per-capita null player is not desirable for any coalition, because after joining him the average per player of any coalition's worth does not change, and thereby no coalition may want to join him. Consequently, such a player should get nothing. On the other hand, when we assume the "highest" degree of solidarity between the players, then a per-capita null player should be awarded $\frac{v(N)}{n}$. The third solution is to award such a player with something between 0 and $\frac{v(N)}{n}$. This leads us to the following three axioms.

PER- CAPITA NULL PLAYER: If player i is a per-capita null player in a game $v \in \Gamma_N$, then $\Phi_i(v) = 0$.

PER- CAPITA NULL PLAYER AVERAGE PAYOFF: If player i is a per-capita null player in a game $v \in \Gamma_N$, then $\Phi_i(v) = \frac{v(N)}{n}$.

PER- CAPITA NULL PLAYER α- AVERAGE PAYOFF: Let $0 \leq \alpha \leq 1$. If player i is a per-capita null player in a game $v \in \Gamma_N$, then $\Phi_i(v) = \alpha \frac{v(N)}{n}$.

Now taking the sequence $\bar{\beta}$ with $\beta_k = \frac{1}{k}$ for $k \geq 1$, and $\alpha = 0$ or $\alpha = 1$ or $0 \leq \alpha \leq 1$ in Theorem 1, we easily get the next three new axiomatizations.

Corollary 4 *A value Φ on Γ_N is an ESL-value satisfying the per-capita null player property if and only if Φ is the per-capita value $\Phi^{Pc} = \{\Phi_i^{Pc}\}_{i \in N}$ of the form (7).*

Corollary 5 *A value Φ on Γ_N is an ESL-value satisfying the per-capita null player average payoff property if and only if Φ is the equal split value Φ^{Eq}.*

By analogy to the α-egalitarian Shapley value, let us call the value $\Phi = \alpha \Phi^{Eq} + (1 - \alpha)\Phi^{Pc}$ the α-*egalitarian per-capita value*.

Corollary 6 *Let $0 \leq \alpha \leq 1$. A value Φ on Γ_N is an ESL-value satisfying the per-capita null player α-average payoff property if and only if Φ is the α-egalitarian per-capita value Ψ^α of the form $\Psi^\alpha = \alpha \Phi^{Eq} + (1 - \alpha)\Phi^{Pc}$.*

Remark 5 For $0 \leq \alpha \leq 1$, the α-egalitarian per-capita value Ψ^α satisfies the fair treatment property, but rather surprisingly it satisfies monotonicity only for $\frac{n-1}{n} \leq \alpha \leq 1$, as implied by Proposition 3. It suffices to note that $b_s = (1 - \alpha) \cdot \frac{n}{s}$ for $s = 1, \ldots, n - 1$ in the representation (9) of Ψ^α. As a consequence, the per-capita value Φ^{Pc} also satisfies the fair treatment property, but not monotonicity.

A further modification of the null player notion to that of δ-*reducing player*, and the corresponding property were introduced and discussed in van den Brink and Funaki (2010).

Definition 3 Let $0 \leq \delta \leq 1$. Player i is called a δ-reducing player in a game $v \in \Gamma_N$ if for any $S \subset N \setminus i$, $v(S \cup i) = \delta \cdot v(S)$.

δ- NULL PLAYER: If player i is a δ-reducing player in a game $v \in \Gamma_N$, then $\Phi_i(v) = 0$.

Now taking the sequence $\bar{\beta}$ with $\beta_k = \delta^{-k}$ for $k \geq 1$, and $\alpha = 0$ in Theorem 1, we immediately get the result obtained in van den Brink and Funaki (2010).

Corollary 7 *Let $0 \leq \delta \leq 1$. A value Φ on Γ_N is an ESL-value satisfying the δ-reducing player property if and only if Φ is the δ-discounted Shapley value $\Phi^{Sh^\delta} = \{\Phi_i^{Sh^\delta}\}_{i \in N}$ of the form (6).*

The last corollary gives a theoretical characterization of the solidarity value. We will use it in the discussion of its structure in Remark 6 below. To formulate it we need to modify (slightly) the definition of per-capita null player in a game and the axiom related to it.

Definition 4 Player i is called an almost per-capita null player in a game $v \in \Gamma_N$ if for any $S \subset N \setminus i$, $\frac{v(S \cup i)}{s+2} - \frac{v(S)}{s+1} = 0$.

ALMOST PER- CAPITA NULL PLAYER AVERAGE PAYOFF: If player i is an almost per-capita null player in a game $v \in \Gamma_N$, then $\Phi_i(v) = \frac{v(N)}{n+1}$.

Now taking the sequence $\bar{\beta}$ with $\beta_k = \frac{1}{k+1}$, and $\alpha = \frac{n}{n+1}$ in Theorem 1, we get the new axiomatization of the solidarity value.

Corollary 8 *A value Φ on Γ_N is an ESL-value satisfying the almost per-capita null player average payoff property if and only if Φ is the solidarity value $\Phi^{So} = \{\Phi_i^{So}\}_{i \in N}$ of the form (5).*

Remark 6 The theoretical axiomatization of the solidarity value in the last corollary does not seem to have a convincing interpretation. However, for large $n = |N|$ it is very similar to the axiomatization of the equal split value described in Corollary 5. Considering this fact, one could think that these two values are "close" for large n. In fact, it turns out that the solidarity value can be seen as "asymptotically equivalent" to the equal split value. This problem is studied in Radzik (2013). The asymptotic behavior of some other values, e.g. the discounted Shapley value, remains an open question.

4 Consistency theory

In the paper of Hart and Mas-Colell (1989) the authors introduced some natural notion of consistency of TU-values and showed that the Shapley value is uniquely determined by this consistency property and another one, called the standardness for two-person games. In Driessen and Radzik (2003) we generalized their approach, considering generalized forms for both of these properties. The main results obtained there are completed in this section (Proposition 5 and Theorem 2) to "if and only if" forms. They are formulated in terms of a generalized consistency and generalized standardness.

In the previous section, a value Φ of cooperative games was understood as a function defined on the set Γ_N of all games v with a fixed grand coalition $N = \{1, 2, \ldots, n\}$ with $n \geq 2$. Now, we extend this definition in a natural way. Namely, throughout the next two sections, a value Φ will be a function defined on the class Γ of all TU-games (M, v) with finite grand coalitions $M \subset \{1, 2, \ldots\}$ of cardinality $|M| \geq 2$, such that $\Phi(M, v) = (\Phi_i(M, v))_{i \in M} \in \mathcal{R}^{|M|}$ for each M. For any finite set M, let Γ_M be the set of all games of the form (M, v). Now the definition of an ESL-value on Γ must be slightly modified. We say that a value Φ satisfies the efficiency or the linearity axiom if for every finite set $M \subset \{1, 2, \ldots\}$, its restriction to Γ_M satisfies one of these axioms, respectively. A value Φ satisfies the symmetry axiom (or is symmetric) if for all finite subsets M_1 and M_2 of the set $\{1, 2, \ldots\}$ with $|M_1| = |M_2|$, for all games (M_1, v) and for every mapping $\pi : M_1 \overset{1-1}{\to} M_2$, $\Phi_{\pi i}(M_2, \pi v) = \Phi_i(M_1, v)$, where $i \in M_1$. (Here πv is defined by $\pi v(\pi(S)) = v(S)$). In such cases we will say that a value Φ is an ESL-value on Γ.

Now we can repeat Proposition 2 in the version for a value Φ on Γ.

Proposition 4 *A value Φ on Γ is an ESL-value if and only if for any finite $M \subset \{1, 2, \ldots\}$ there exists a unique collection of real constants $\{b_{m,s}\}_{s=0,1,\ldots,m}$ with*

$b_{m,m} = 1$ *and* $b_{m,0} = 0$ *such that for every game* $(M, v) \in \Gamma_M$, *the i-th component of the value payoff vector* $(\Phi_i(M, v))_{i \in M}$ *is of the following form:*

$$\Phi_i(M, v) = \sum_{S \subset M \setminus i} \frac{s!(m - s - 1)!}{m!} [b_{m,s+1} v(S \cup i) - b_{m,s} v(S)], \quad i \in M. \quad (13)$$

We begin with two general definitions which are basic for our paper. They come from Driessen and Radzik (2003). The first one is related to some kind of generalized standardness for two-person games, and the second one describes some general notion of consistency which extends the reduced game property introduced by Hart and Mas-Colell (1989).

Definition 5 Let $\lambda \in \mathcal{R}$. We say that a value Φ is λ-*standard* for two-person games if for every two-person game $(\{i, j\}, v)$ and every player $k \in \{i, j\}$ it holds that

$$\Phi_k(v) = \lambda v(k) + \frac{1}{2} [v(i, j) - \lambda v(i) - \lambda v(j)]. \quad (14)$$

The λ-standardness of a value for two-person games expresses the fact that the value allocates the *surplus* $[v(i, j) - \lambda v(i) - \lambda v(j)]$ equally to the players i and j after each player k concedes to get his *weighted individual worth* $\lambda v(k)$.

Remark 7 It is not difficult to check that the value Φ of the form (13) is λ-standard for two-person games if and only if $\lambda = b_{2,1}$. Therefore the solidarity value is $\frac{1}{2}$-standard, the Shapley value is 1-standard, the per-capita value is 2-standard, and the equal split value is 0-standard for two-person games. Generally, a value Φ is λ-standard for some λ if and only if Φ is an ESL-value on the set of two-person games. This fact can be easily concluded from (14). It is worth mentioning that Joosten et al. (1994) introduced some equivalent definition to λ-standardness of a value for two-person games.

Now we introduce a property generalizing the consistency property introduced by Hart and Mas-Colell (1989). It comes from Driessen and Radzik (2003) and is basic for the rest of our paper.

Definition 6 Let $\mathcal{D} = \{(A_{k,s}, B_{k,s})\}_{s=1,...,k}^{k=2,3,...}$ be a collection of pairs of real numbers. Given a game (M, v) with $|M| = m \geq 3$ and a player $i \in M$, the corresponding reduced game $(M \setminus i, v_M^{\Phi,i})$ associated with Φ and $\mathcal{D}$ is defined to be as follows: for all nonempty sets $S \subset M \setminus i$

$$v_M^{\Phi,i}(S) = A_{m-1,s} [v(S \cup i) - \Phi_i(S \cup i, v)] + B_{m-1,s} v(S). \quad (15)$$

We say that a value Φ possesses the $\mathcal{D}$-reduced game property on Γ or is consistent on Γ with respect to the reduced games $v_M^{\Phi,i}$ of the form (15) if it satisfies the following condition: for every game $(M, v) \in \Gamma$ with $m \geq 3$,

$$\Phi_j(M \setminus i, v_M^{\Phi,i}) = \Phi_j(M, v) \quad for \ all \ \ i \in M \ \ and \ \ j \in M \setminus i. \quad (16)$$

(Hart and Mas-Colell (1989)) considered the reduced game $v_M^{\Phi,i}$ with the constants $A_{k,s} \equiv 1$ and $B_{k,s} \equiv 0$.)

The reduced games $v_M^{\Phi,i}$ of the form (15) have a natural interpretation when the constants $A_{k,s}, B_{k,s}$ are nonnegative and satisfy, $A_{k,s} + B_{k,s} = 1$ for all k, s. It is presented in the first paragraph of Sect. 5.

Now we are ready to formulate three results about consistency properties of values, two auxiliary ones (Propositions 5 and 6) and Theorem 2 basic for this section. Their proofs are given in Sect. 6.

We need to introduce additional notation. Let $\mathbb{D}$ be the family of all collections $\mathcal{D} = \{(A_{k,s}, B_{k,s})\}_{s=1,\ldots,k}^{k=2,3,\ldots}$ of pairs of real constants.

The first result gives necessary and sufficient conditions for a value to be efficient. It essentially strengthens the result of Lemma 1 in Driessen and Radzik (2003), where only the sufficient condition was discussed. Notice that λ-standardness of a value Φ (considered there) is a stronger assumption than efficiency of Φ for two-person games, which follows from Remark 7.

Proposition 5 *Let $\mathcal{D} \in \mathbb{D}$ and assume that a value Φ possesses $\mathcal{D}$-reduced game property on Γ. Then Φ is efficient on Γ if and only if Φ is efficient on two-person games and the constants in collection $\mathcal{D}$ satisfy:*

$$A_{k,k} = 1 \quad and \quad B_{k,k} = 0 \quad for \quad k = 2, 3, \ldots. \tag{17}$$

The second result discusses the properties and a possible uniqueness of a consistent value. It essentially strengthens and remarkably simplifies the results of Theorems 1 and 2 in Driessen and Radzik (2003), showing that conditions (i) and (ii) are not necessary to prove the ESL property for a value Φ in Theorem 1, and that only conditions (i) and (iii) are sufficient to prove the uniqueness of Φ in Theorem 2 there.

Proposition 6 *Let $(\lambda, \mathcal{D}) \in \mathcal{R} \times \mathbb{D}$ with $\mathbb{D}$ fulfilling (17), and assume that a value Φ is λ-standard for two-person games and possesses $\mathcal{D}$-reduced game property on Γ. Then the value Φ is unique. Moreover it is an ESL-value on Γ.*

In view of Propositions 5 and 6, one could ask when an ESL-value Φ possesses $\mathcal{D}$-reduced game property for some $\mathcal{D}$. (Note that in view of Remark 7, it always is λ-standard for two-person games for some λ.) The answer to this question is given in Theorem 2 which substantially strengthens Theorem 4 in Driessen and Radzik (2003).

Theorem 2 *Let $(\lambda, \mathcal{D}) \in \mathcal{R} \times \mathbb{D}$ and let Φ be an efficient value on Γ. Then the value Φ is λ-standard for two-person games and possesses $\mathcal{D}$-reduced game property on Γ if and only Φ is the unique ESL-value on Γ such that the constants $\{b_{m,s}\}_{s=1,\ldots,m}^{m=2,3,\ldots}$ in its representation (13), the constant λ and the constants $A_{k,s}, B_{k,s}$ in collection $\mathcal{D}$ satisfy the equalities (17) and the system of equations:*

$$\begin{cases} b_{2,1} & = \lambda \\ b_{m,s} A_{m,s} & = x_{m,s} \\ b_{m,s} B_{m,s} & = b_{m+1,s} - x_{m,s-1} \quad m = 2, 3, \ldots \quad s = 1, \ldots, m-1, \end{cases} \tag{18}$$

where $\{x_{m,s}\}_{s=0,1,\ldots,m}^{m=2,3,\ldots}$ *is the set of real numbers uniquely determined by the system*

$$
\begin{cases}
x_{m,m} = m + 1 & \text{for } m \geq 2 \\
x_{m,s} = (s+1) \sum_{l=s+1}^{m} \frac{b_{l+1,s+1}}{l(l+1)} x_{m,l} & \text{for } m \geq 2 \text{ and } 0 \leq s \leq m - 1.
\end{cases}
\tag{19}
$$

Corollary 9 *Let* Φ *be an ESL-value on* Γ *with all the constants* $\{b_{m,s}\}_{s=0,1,\ldots,m}^{m=1,2,\ldots}$ *in its representation (13) different from zero. Then*

(a) there is a unique pair $(\lambda, \mathcal{D}) \in \mathcal{R} \times \mathbb{D}$ *with* $\mathcal{D}$ *fulfilling (17) such that* Φ *is* λ*-standard for two-person games and possesses* $\mathcal{D}$*-reduced game property on* Γ*;*

(b) the pair $(\lambda, \mathcal{D})$ *is the unique solution of the system (18), (19).*

Proof First note that for every $m \geq 2$ the system of equations (19) allows to calculate in turn the constants: $x_{m,m}, x_{m,m-1}, \ldots, x_{m,0}$. Hence, the constant λ and all the constants $A_{m,s}, B_{m,s}$ are uniquely determined by the system of equations (18), because $b_{m,s} \neq 0$ for all m and s. Consequently, by ($\Leftarrow$) of Theorem 2, the proof is completed. $\qquad \square$

Remark 8 If some of the constants $b_{m,s}$ in representation (13) of an ESL-value Φ on Γ happen to be zero, the system of equations (18), (19) may fail. To see this, it suffices to consider the case of Φ with $b_{m,1} = b_{m,2} = \ldots = b_{m,m-2} = 0$ and $b_{m,m-1} \neq 0$ for all $m \geq 2$. Namely, by the second equality in system (18), $x_{m,m-2} = 0$. But then (19) implies that $x_{m,m-2} = \frac{b_{m,m-1} \cdot b_{m+1,m}}{m} \neq 0$, a contradiction. Therefore (by Theorem 2) such value Φ is not consistent with respect to any reduced game of the form (15). On the other hand, it may also happen that every collection $\mathcal{D}$ of the constants $A_{k,s}, B_{k,s}$ is a solution of the system (18), (19). Namely, considering $\Phi = \Phi^{Eq}$, we have $b_{m,1} = b_{m,2} = \ldots = b_{m,m-1} = 0$ for all $m \geq 2$ in its representation (13), and it is easy to verify that, in fact, arbitrary constants $A_{k,s}, B_{k,s}$ satisfy the system (18) then.

Remark 9 It is worth mentioning that the formulas for all the constants $A_{k,s}, B_{k,s}$ of collection $\mathcal{D}$ can be obtained explicitly from the system (18), (19) when the parameters $b_{k,s}$ of Φ are of the form $b_{k,s} = \lambda_k \beta_s \neq 0$ for all k, s (see Corollary 1 in Driessen and Radzik 2003). However, in general, they have a very complex form without any natural interpretation. The problem can be highly simplified when we restrict our considerations to the natural class consisting of ESL-values Φ on Γ with the additional property that the constants $b_{k,s}$ in their representation (13) are independent of k for all $s < k$. One can easily see (considering (1), (2) and (5)) that the Shapley value, the equal split value and the solidarity value have such a property. It turns out that Theorem 2 can be used to show (in the next section) that there are very simple necessary and sufficient conditions for a value in this class to be "probabilistically consistent", that is to satisfy $A_{k,s} + B_{k,s} = 1$ for all k, s.

5 Probabilistically consistent ESL-values

In this section we widely discuss (in two theorems and several corollaries and remarks) values for TU-games possessing $\mathcal{D}$-reduced game property with some collections of

nonnegative real numbers in $\mathcal{D}$ satisfying the following additional condition: $A_{k,s} + B_{k,s} = 1$ for all k, s. In this case, the interpretation for the above consistency property seems to be very natural: Let Φ be a fixed value on a game (M, v) and suppose that one of the players, say player i, is removed. Further suppose that for every coalition $S \subset M \setminus i$ there is still a chance (with a probability equal to $A_{m-1,s}$) that i will join that coalition S or not (with a probability equal to $B_{m-1,s}$). Since a new game $(M \setminus i, v)$ arises, we can ask, how to estimate the worth of a coalition S in this new reduced game. In case when player i does not join the coalition S, its worth is the same as before, that is $v(S)$. However, when player i joins the coalition S, the worth of the coalition S after removing that player could be in a natural way estimated as the difference $v(S \cup i) - \Phi_i(S \cup i, v)$. Thus the worth of the coalition S in the reduced game with the player set equal to $M \setminus i$ is simply interpreted as the expected worth given in the definition above. Now the consistency property requires that any player j different from i should get the same in both games with respect to the considered value.

In view of the above comment, we introduce the following definition of probabilistic consistency.

Definition 7 A value Φ is probabilistically consistent on Γ if there is a collection of pairs of nonnegative real numbers $\mathcal{D} = \{(A_{k,s}, B_{k,s})\}_{s=1,\ldots,k}^{k=2,3,\ldots}$ satisfying

$$A_{k,s} + B_{k,s} = 1 \quad for \ all \ \ 2 \leq s \leq k, \ k \geq 2 \,, \tag{20}$$

such that Φ possesses the $\mathcal{D}$-reduced game property on Γ with respect to the reduced games $v_M^{\Phi,i}$ of the form (15).

Now we are ready to formulate our next two results (Theorems 3 and 4 - their proofs are shifted to Sect. 6) determining a set of probabilistically consistent values in some subfamily of the family of the ESL-values.

The first theorem characterizes the family of δ-discounted Shapley value Φ^{Sh^δ} on Γ in terms of the classical consistency property introduced in Hart and Mas-Colell (1989); that is, with respect to the reduced games $v_M^{\Phi,i}$ of the form

$$v_M^{\Phi,i}(S) = v(S \cup i) - \Phi_i(S \cup i, v) \quad \text{for} \quad S \subset M \setminus i \,. \tag{21}$$

Note that these reduced games are of the form (15) with $A_{m-1,s} \equiv 1$ and $B_{m-1,s} \equiv 0$, so any consistent value with respect to such $v_M^{\Phi,i}$ is also probabilistically consistent.

It is also worth mentioning here that a characterization equivalent to the one presented in (a) of Theorem 3, was first given (without proof) in Joosten et al. (1994), and next in Joosten (1996, Proposition 5.32) with the proof following the Hart and Mas-Collel approach. However, it turns out that this characterization is also a simple consequence of Theorem 2. Therefore, in spite of the fact that the part ($\Longleftarrow$) of the statement (a) of Theorem 3 was shown in Example 1 of Driessen and Radzik (2003), for the sake of completeness we write it in the following form:

Theorem 3 *Let $\delta \in \mathcal{R}$. Then*

(a) *an ESL-value Φ on Γ is δ-standard for two-person games, and consistent with respect to the reduced games $v_M^{\Phi,i}$ of the form (21) if and only if Φ is the δ-discounted Shapley value Φ^{Sh^δ} of the form (6).*

(b) *the value Φ^{Sh^δ} satisfies the fair treatment and monotonicity properties if and only if $0 \leq \delta \leq 1$.*

The second theorem characterizes some family of values Φ on Γ having the probabilistic consistency property with respect to generalized reduced games $v_M^{\Phi,i}$ of the form (15) with arbitrary constants $A_{m-1,s}$ and $B_{m-1,s}$ satisfying (20). In view of the comment made in Remark 9, we restrict our consideration to the family of ESL-values with the constants $b_{m,s}$ (in their representation (13)) independent of m for all $s < m$. So putting $b_{m,s} = \beta_s$ in (13), we get the family of ESL-values Φ on Γ described by sequences $\bar{\beta} = (\beta_s)_{s \geq 0}$ in the following way: For any game $(M, v) \in \Gamma$ and $i \in M$, the value payoff vector $\Phi(M, v) = (\Phi_i(M, v))_{i \in M}$ in this family is of the following form:

$$\Phi_i(M, v) = \frac{v(M)}{m}(1 - \beta_m) + \sum_{S \subset M \setminus i} \frac{s!(m - s - 1)!}{m!}[\beta_{s+1}v(S \cup i) - \beta_s v(S)].$$
(22)

[Note that after replacing $b_{m,s}$ by β_s for $s < m$ in the formula (13), it becomes equivalent to (22).]

Now we are ready to formulate our next theorem.

Theorem 4 *Let $\lambda \in \mathcal{R}$ and assume that a value $\Phi = (\Phi_i(v))_{i \in M}$ on Γ is of the form (22) for some real sequence $\{\beta_n\}_{n \geq 0}$ with $\beta_0 = 0$. Then*

(i) *Φ is a λ-standard for two-person games and probabilistically consistent value if and only if*

$$\beta_1 = \lambda, \quad 0 \leq \lambda \leq 1$$
(23)

and

$$\beta_s = \beta_s^\lambda := \frac{\lambda}{(1 - \lambda)s + \lambda} \quad for \quad s = 1, 2, \ldots .$$
(24)

(ii) *If the equalities (24) hold for some $0 \leq \lambda \leq 1$, then the value Φ is probabilistically consistent with respect to the reduced games $v_{M,\lambda}^{\Phi,i} = v_M^{\Phi,i}$ of the form (15) with the constants $A_{k,k} = 1$ and $B_{k,k} = 0$ for $k = 2, 3, \ldots,$*

$$A_{m-1,s} = A_{m-1,s}^\lambda := \frac{(1 - \lambda)s + \lambda}{(1 - \lambda)(m - 1) + \lambda} \quad and$$
$$B_{m-1,s} = B_{m-1,s}^\lambda := 1 - A_{m-1,s}^\lambda$$
(25)

for $m \geq 2$ and $1 \leq s \leq m - 1$. If additionally $\lambda \neq 0$, $v_{M,\lambda}^{\Phi,i}$ are the only reduced games with respect to which the value Φ given by (22)–(24) is probabilistically consistent.

Now we give several corollaries where, among other things, the Shapley value, the per-capita value, the solidarity value and the equal split value are discussed in terms of probabilistic consistency. Before that some notation will be introduced.

For a real λ and a sequence $\beta_1^\lambda, \beta_2^\lambda, \ldots$ determined by (24), let Φ^λ be the value defined on Γ such that for every game $(M, v) \in \Gamma$ the i-th component of the value vector $(\Phi_i^\lambda(T, v))_{i \in T}$ is of the following form:

$$\Phi_i^\lambda(T, v) = \frac{v(M)}{m}[1 - \beta_m^\lambda] + \sum_{S \subset T \setminus i} \frac{s!(t - s - 1)!}{t!}[\beta_{s+1}^\lambda v(S \cup i) - \beta_s^\lambda v(S)].$$

(26)

Further, let us define the following class of values on Γ:

$$\mathcal{F}^* := \{\Phi^\lambda : 0 \leq \lambda \leq 1\}.$$

(27)

The first two corollaries are immediate consequences of Theorem 4.

Corollary 10 *A value Φ of the form (22) is probabilistically consistent if and only if $\Phi \in \mathcal{F}^*$. Moreover, every value in $\mathcal{F}^*$ satisfies the fair treatment and monotonicity properties.*

Corollary 11 *For any $0 \leq \lambda \leq 1$ the value Φ^λ, determined by (26) and (24), is the only value on Γ which is λ-standard for two person games and probabilistically consistent with respect to the reduced games $v_{M,\lambda}^{\Phi,i}$ of the form*

$$v_{M,\lambda}^{\Phi,i}(S) = \frac{(1-\lambda)s + \lambda}{(1-\lambda)(m-1) + \lambda}\left[v(S \cup i) - \Phi_i^\lambda(S \cup i, v)\right] + \frac{(1-\lambda)(m-1-s)}{(1-\lambda)(m-1) + \lambda}v(S)$$

(28)

for $S \subset M \setminus i$. If $0 < \lambda \leq 1$, $v_{M,\lambda}^{\Phi,i}$ are the only reduced games with respect to which the value Φ^λ is probabilistically consistent.

The next corollary shows that the Shapley value, the equal split value and the solidarity value belong to the class $\mathcal{F}^*$.

Corollary 12 *(i) $\Phi^{Sh} = \Phi^1$; (ii) $\Phi^{Eq} = \Phi^0$; (iii) $\Phi^{So} = \Phi^{1/2}$.*

Proof It is enough to verify (with the help of (24)) that the formula (26), with $\lambda = 1$, $\lambda = 0$ and $\lambda = \frac{1}{2}$, coincides with the formulas (2), (1) and (5), respectively. $\square$

Remark 10 In view of (27) and Corollaries 10 and 12, it is rather surprising that such classical values as the Shapley value and the equal split value are two endpoints of the class $\mathcal{F}^*$. However, it is much more surprising that the solidarity value is the central element of $\mathcal{F}^*$ and "lies" exactly in the middle between the Shapley value and the equal split value. On the other hand, for $0 < \alpha < 1$ the α-egalitarian Shapley value $\widetilde{\Phi}^\alpha = \alpha \Phi^{Eq} + (1-\alpha)\Phi^{Sh}$ does not belong to $\mathcal{F}^*$, and thereby it is not probabilistically consistent. To see this, note that $\widetilde{\Phi}^\alpha$ is of the form (22) with $\beta_s = 1 - \alpha$ for $s > 0$, but then the system of equations (23), (24) has no solutions in λ (because $0 < 1 - \alpha < 1$). In the same way we can check that for $0 \leq \alpha < 1$, the α-egalitarian per-capita value $\Psi^\alpha = \alpha \Phi^{Eq} + (1 - \alpha)\Phi^{Pc}$ is not probabilistically consistent either.

The next corollary gives a characterization of the equal split value in terms of λ-standardness and probabilistic consistency.

Corollary 13 *The equal split value Φ^{Eq} is the only value which is 0-standard for two-person games and probabilistically consistent with respect to the reduced games $v_M^{\Phi,i} = v_M^{Eq,i}$ of the form*

$$v_M^{Eq,i}(S) = v(S \cup i) - \Phi_i^{Eq}(S \cup i, v) \quad for \quad S \subset M \setminus i. \tag{29}$$

Proof It is an immediate consequence of Theorem 3 with $\delta = 0$.   □

Remark 11 Obviously, the equal split value Φ^{Eq} is 0-standard for two-person games. It is interesting that for this value Corollary 11 gives also (after putting $\lambda = 0$ in (28)) other reduced games $v_M^{\Phi,i} = v_{M,0}^{Eq,i}$ of the form

$$v_{M,0}^{Eq,i}(S) = \frac{s}{m-1}\left[v(S \cup i) - \Phi_i^{Eq}(S \cup i, v)\right] + \frac{m-s-1}{m-1}v(S).$$

So reduced game for Φ^{Eq} is not unique. Nevertheless this does not contradict Theorem 4 because $\lambda = 0$ for Φ^{Eq}. Moreover, it is shown in Remark 8 that for the value Φ^{Eq}, arbitrary constants $A_{m,s}$ and $B_{m,s}$ satisfy the system of equations (18). Therefore, by Theorem 2, the equal split value is probabilistically consistent with respect to any reduced games of the form (15) with arbitrary constants $0 \leq A_{m,s} \leq 1$ and $B_{m,s} = 1 - A_{m,s}$ satisfying (17).

The next two corollaries give characterizations of the Shapley value and the solidarity value. These two results were obtained in Driessen and Radzik (2003) (see Examples 1 and 2 there) without the "uniqueness" in their formulations.

Corollary 14 *The Shapley value Φ^{Sh} is the only value which is 1-standard for two-person games, and probabilistically consistent with respect to the reduced games $v_M^{Sh,i}$ of the form*

$$v_M^{Sh,i}(S) = v(S \cup i) - \Phi_i^{Sh}(S \cup i, v) \quad for \quad S \subset M \setminus i. \tag{30}$$

The games $v_M^{Sh,i}$ are the only reduced games with respect to which the value Φ^{Sh} is probabilistically consistent.

Proof One can easily check that the reduced games $v_{M,\lambda}^{\Phi,i}$ of the form (28) with $\lambda = 1$ coincide with the ones of the form (30). Hence the statement (i) of Corollary 12 and Corollary 11 immediately complete the proof.   □

Remark 12 The result of Corollary 14 in a slightly weaker form was earlier proved in Hart and Mas-Colell (1989). Namely, the uniqueness of the Shapley value was shown there in the class of values which are relatively invariant under strategic equivalence and have the equal treatment property instead of 1-standardness. Their result is a simple consequence of a very nice theory of potential for cooperative games, constructed in

that paper. It is also worth mentioning here that two other extensions of reduced games (30) in terms of consistency were discussed in the literature. Namely Joosten et al. (1994) considered some α-reduced games, while Derks and Peters (1997) applied the consistency property to TU-games with restricted coalitions.

The next corollary strengthens the results presented in Example 2 in Driessen and Radzik (2003).

Corollary 15 *The solidarity value Φ^{So} is the only value which is $\frac{1}{2}$-standard for two-person games and probabilistically consistent with respect to the reduced games $v_M^{\Phi,i} = v_M^{So,i}$ of the form*

$$v_M^{So,i}(S) = \frac{s+1}{m}\left[v(S \cup i) - \Phi_i^{So}(S \cup i, v)\right] + \frac{m-s-1}{m}v(S) \quad for \ \ S \subset M \setminus i. \tag{31}$$

The games $v_M^{So,i}$ are the only reduced games with respect to which the value Φ^{So} is probabilistically consistent.

Proof It is easy to verify that the reduced games $v_{M,\lambda}^{\Phi,i}$ of the form (28) with $\lambda = \frac{1}{2}$ coincide with the ones of the form (31). Hence the statement (ii) of Corollary 12 and Corollary 11 immediately complete the proof. $\quad\square$

Remark 13 It is worth mentioning that Sobolev (1973) considered a slightly different consistency property, and showed that the Shapley value is the only one which is 1-standard for two person games and consistent with respect to the reduced games of the form

$$v_M^{\Phi,i}(S) = \frac{s}{m-1}\left[v(S \cup i) - \Phi_i^{Sh}(M, v)\right] + \frac{m-s-1}{m-1}v(S),$$

surprisingly similar to the reduced games $v_M^{So,i}$ of the form (31), related to the solidarity value, and to the reduced game $v_{M,0}^{Eq,i}$ (from Remark 11), implied by Theorem (4) for the equal split value. Sobolev's result was recently generalized by van den Brink et al. (2013) who showed that for any $0 \leq \alpha \leq 1$, the α-egalitarian Shapley value $\widetilde{\Phi}^\alpha = \alpha\Phi^{Eq} + (1 - \alpha)\Phi^{Sh}$ is the unique value which is α-standard for two person games and consistent with respect to the same reduced games $v_M^{\Phi,i}$. It is also worth mentioning that Joosten (1996) showed that for $0 \leq \delta \leq 1$, δ-standardness combined with Hart and Mas-Colell consistency (with respect to reduced games of the form (30)) characterize the δ-discounted Shapley value Φ^{Sh^δ} of the form (6).

6 Proofs of the theorems

In this section we give the proofs of Theorems 1–4 formulated in Sects. 3–5.

Proof of Theorem 1 ($\Leftarrow$) Let $N = \{1, 2, \cdots, n\}$ with $n \geq 2$, and let $\Phi = (\Phi_i(v))_{i \in N}$ be a value on Γ_N of the form (12). We can easily check that Φ coincides with the value of the form (9) with $b_n = 1$ and with $b_s = \frac{1-\alpha}{\beta_n}\beta_s$ for $1 \leq s < n$. Therefore, by

Proposition 2, Φ is an ESL-value on Γ_N. The fact that Φ also satisfies the $(\bar{\beta}, \alpha)$-null player axiom follows immediately from (12).

($\Rightarrow$) Assume now that Φ is an ESL-value on Γ_N satisfying the $(\bar{\beta}, \alpha)$-null player property. By Proposition 2, there exists a unique collection of real constants $\{b_s\}_{s=0,1,\ldots,n}$ with $b_n = 1$ and $b_0 = 0$ such that for all $i \in N$, $\Phi_i(v)$ is of the form (9). But then $\Phi_i(v)$ can be easily rewritten in the form

$$\Phi_i(v) = \frac{v(N)}{n}(1 - b'_n) + \sum_{S \subset N \setminus i} \frac{s!(n - s - 1)!}{n!}[b'_{s+1}v(S \cup i) - b'_s v(S)] \qquad (32)$$

for all $i \in N$ and $v \in \Gamma_N$, where $b'_s = b_s$ for $s = 0, 1, \ldots, n - 1$ and b'_n is arbitralily chosen. Therefore we can take $b'_n = 1 + \alpha,$.

Let us define another value $\Phi' = (\Phi'_i(v))_{i \in N}$ on Γ_N as in (12), that is by

$$\Phi'_i(v) = \alpha \frac{v(N)}{n} + \frac{1 - \alpha}{\beta_n} \sum_{S \subset N \setminus i} \frac{s!(n - s - 1)!}{n!}[\beta_{s+1}v(S \cup i) - \beta_s v(S)] \qquad (33)$$

for all $i \in N$ and $v \in \Gamma_N$. It is clear that $\Phi = \Phi'$ if $b'_s = \frac{1-\alpha}{\beta_n}\beta_s$ for $s = 1, 2, \ldots, n-1$. This is thus what we show in the remainder of the proof.

Let us denote

$$\alpha_n = 0, \quad \alpha_0 = 0 \quad \text{and} \quad \alpha_s = b'_s - \frac{1 - \alpha}{\beta_n}\beta_s \quad \text{for } s = 1, \ldots, n - 1. \qquad (34)$$

Now consider the vector function $E(v) = \{E_i(v)\}_{i \in N}$ on Γ_N defined by

$$E_i(v) = \sum_{S \subset N \setminus i} \frac{s!(n - s - 1)!}{n!}[\alpha_{s+1}v(S \cup i) - \alpha_s v(S)], \quad i \in N. \qquad (35)$$

One can easily check that $E(v) = \Phi_i(v) - \Phi'_i(v)$ for $v \in \Gamma_N$. Hence, we need only to show that

$$\alpha_s = 0 \quad \text{for} \quad s = 1, \ldots, n - 1. \qquad (36)$$

By assumption, Φ is an ESL-value on Γ_N and has the $(\bar{\beta}, \alpha)$-null player property. By the part ($\Leftarrow$), the value Φ' has the same properties. Consequently, for every game $v \in \Gamma_N$

$$E_i(v) = 0 \quad \text{if } i \quad \text{is a } \bar{\beta} \text{ -null player in } v. \qquad (37)$$

Let us consider the class of games $V = \{v_K : \emptyset \neq K \subset N\}$, defined on N by the following: $v_K(S) = \frac{1}{\beta_s}$ if $K \subset S \subset N$ and $v_K(S) = 0$, otherwise.

Let $K_s = N \setminus \{1, 2, \ldots, s\}$ for $s = 1, \ldots, n - 1$. One can easily state that player 1 is a $\bar{\beta}$-null player in all the games $v_{K_1}, v_{K_2}, \ldots, v_{K_{n-1}}$. Therefore by (37)

$$E_1(v_{K_s}) = 0 \quad \text{for} \quad s = 1, \ldots, n - 1. \qquad (38)$$

But (35) leads to the equality

$$E_1(v_{K_1}) = \frac{1}{n} \left[\frac{\alpha_n}{\beta_n} - \frac{\alpha_{n-1}}{\beta_{n-1}} \right], \tag{39}$$

whence, by (34) and (38) with $s = 1$, we get

$$\alpha_{n-1} = 0. \tag{40}$$

Further, (35) implies the following:

$$E_1(v_{K_2}) = \frac{1}{n} \left[\frac{\alpha_n}{\beta_n} - \frac{\alpha_{n-1}}{\beta_{n-1}} \right] + \frac{2}{n(n-1)} \left[\frac{\alpha_{n-1}}{\beta_{n-1}} - \frac{\alpha_{n-2}}{\beta_{n-2}} \right]. \tag{41}$$

But this, in view of (34), (40) and (38) with $s = 2$, immediately gives

$$\alpha_{n-2} = 0. \tag{42}$$

We can continue these calculations successively for $E_1(v_{K_3}), \ldots, E_1(v_{K_{n-1}})$ to get

$$\alpha_{n-s} = 0 \quad \text{for} \quad s = 3, \ldots, n - 1. \tag{43}$$

Therefore we have proved (36) and thereby the proof of the theorem is completed.
$\square$

Proof of Proposition 5 ($\Rightarrow$) Let Φ be an efficient value satisfying the assumption. We need only to show that (17) holds.

Let us fix any finite set M with $|M| = m \geq 3$ and a game $v \in \Gamma_M$. Choose $i \in M$ and consider the reduced game $v_M^{\Phi,i}$ on the set $M \setminus i$ of the form (15). By the assumption, Φ is efficient on the class of games $\Gamma_{M \setminus i}$. Therefore, by (15) and (16),

$$\sum_{j \in M \setminus i} \Phi_j(M, v) = \sum_{j \in M \setminus i} \Phi_j(M \setminus i, v_M^{\Phi,i}) = v_M^{\Phi,i}(M \setminus i)$$
$$= A_{m-1,m-1} [v(M) - \Phi_i(M, v)] + B_{m-1,m-1} v(M \setminus i).$$

for $i \in M$. Hence, using the efficiency of Φ, we get the following sequence of equalities:

$$(m - 1)v(M) = mv(M) - \sum_{i \in M} \Phi_i(M, v) = \sum_{i \in M} [v(M) - \Phi_i(M, v)]$$
$$= \sum_{i \in M} \left[\sum_{j \in M} \Phi_j(M, v) - \Phi_i(M, v) \right] = \sum_{i \in M} \sum_{j \in M \setminus i} \Phi_j(M, v)$$
$$= \sum_{i \in M} \left[A_{m-1,m-1} [v(M) - \Phi_i(M, v)] + B_{m-1,m-1} v(M \setminus i) \right]$$

$$= A_{m-1,m-1} \left[mv(M) - \sum_{i \in M} \Phi_i(M, v) \right] + B_{m-1,m-1} \sum_{i \in M} v(M \setminus i)$$

$$= A_{m-1,m-1}(m-1)v(M) + B_{m-1,m-1} \sum_{i \in M} v(M \setminus i).$$

Therefore

$$(m-1)(1 - A_{m-1,m-1})v(M) = B_{m-1,m-1} \sum_{i \in M} v(M \setminus i).$$

Hence, since $m \geq 3$ and the game (M, v) is arbitrarily fixed, we easily deduce that both sides of the last equality must be equal to 0. Consequently, (17) holds because of the arbitrarity of m.

($\Longleftarrow$) To prove this part we can repeat the proof of Lemma 1 in Driessen and Radzik (2003), with the exception that the λ-standardness for Φ should be changed by efficiency of Φ for two person games. This ends the proof of Proposition 5. $\quad\square$

Proof of Proposition 6 Let Φ be a value satisfying the assumption. It is easily seen from (14) that Φ is efficient for two person games. Hence, in view of the part ($\Longleftarrow$) of Proposition 5, Φ is efficient on Γ, Therefore, to prove the theorem it suffices only to show for every finite $M \subset \{1, 2, \ldots\}$ with $|M| \geq 2$, that Φ is uniquely determined, linear and symmetric on Γ_M. We will show it by induction with respect to the number $m = |M|, m = 2, 3, \ldots$.

When $m = 2$, Φ is uniquely defined on every Γ_M with $|M| = 2$, because of the λ-standardness of Φ on the set of all two-person games, and it is obviously linear and symmetric on Γ_M (see (14) in Definition 5). Now, by the induction hypothesis, assume that for some $m \geq 3$, the value Φ is uniquely determined, linear and symmetric on every set Γ_T with $|T| < m$.

By Proposition 1, there is a unique collection of constants $\{\lambda_s^t \,|\, t = 2, 3, \ldots, m - 1, s = 1, 2, \ldots, t - 1\}$ such that for any $|T| < m$ and $v \in \Gamma_T$

$$\Phi_i(T, v) = \frac{v(T)}{t} + \sum_{\substack{S \subseteq T \\ S \ni i}} \frac{\lambda_s^t \, v}{s} v(S) - \sum_{\substack{S \subseteq T \\ \emptyset \neq S \not\ni i}} \frac{\lambda_s^t \, v}{t - s} v(S), \quad i \in T. \qquad (44)$$

Now, let M be any finite set with $|M| = m$. We will show that our inductive assumption implies uniqueness, as well as linearity and symmetry of Φ on Γ_M.

Let us arbitrarily fix a game $v \in \Gamma_M$, and denote

$$\Phi_j(M, v) = x_j, \quad j \in M. \qquad (45)$$

Choose $i \in M$ and consider the reduced game $v_M^{\Phi,i}$ on the set $M \setminus i$ of the form (15). By the inductive assumption, the value Φ is uniquely determined on the class of games $\Gamma_{M \setminus i}$, and is an ESL-value on this set. Hence, by (44),

$$\Phi_j(M \setminus i, v_M^{\Phi,i}) = \frac{1}{m-1} v_M^{\Phi,i}(M \setminus i) + \sum_{\substack{S \subsetneq M \setminus i \\ S \ni j}} \frac{\lambda_s^{m-1}}{s} v_M^{\Phi,i}(S)$$

$$- \sum_{\substack{S \subsetneq M \setminus i \\ S \not\ni j}} \frac{\lambda_s^{m-1}}{m-1-s} v_M^{\Phi,i}(S). \tag{46}$$

By the efficiency of Φ, we have

$$\sum_{i \in N} \Phi_i(M, v) = v(M). \tag{47}$$

Further, by (16), (17) and (15), we have:

$$\Phi_j(M \setminus i, v_M^{\Phi,i}) = \Phi_j(M, v) \quad \text{for} \quad j \in M \setminus i, \tag{48}$$

$$v_M^{\Phi,i}(M \setminus i) = v(M) - \Phi_i(M, v) \tag{49}$$

and

$$v_M^{\Phi,i}(S) = A_{m-1,s}[v(S \cup i) - \Phi_i(S \cup i, v)] + B_{m-1,s} v(S) \quad \text{for} \quad S \subsetneq M \setminus i. \tag{50}$$

Besides (44) implies that

$$\Phi_i(S \cup i, v) = \frac{1}{s+1} v(S \cup i) + \sum_{\substack{U \subsetneq S \cup i \\ U \ni i}} \frac{\lambda_u^{s+1}}{u} v(U) - \sum_{\substack{U \subsetneq S \cup i \\ U \not\ni i}} \frac{\lambda_u^{s+1}}{s+1-u} v(U) \tag{51}$$

for all $S \subsetneq M \setminus i$.

Now, putting (51) in (50), and next, putting (48), (49) and (50) in (46), we get

$$\Phi_j(M, v) + \frac{1}{m-1} \Phi_i(M, v) = a^{ij}(v) \tag{52}$$

for $i \in M$ and $j \in M \setminus i$, where $a^{ij}(v)$ is the uniquely determined linear function of v of the form

$$a^{ij}(v) = \sum_{\substack{S \subset M \\ S \ni i, S \ni j}} \beta_s v(S) + \sum_{\substack{S \subset M \\ S \ni i, S \not\ni j}} \gamma_s v(S) + \sum_{\substack{S \subset M \\ S \not\ni i, S \ni j}} \delta_s v(S) + \sum_{\substack{S \subset M \\ S \not\ni i, S \not\ni j}} \tau_s v(S) \tag{53}$$

with some constants β_s, γ_s, δ_s and τ_s, dependent only on the cardinality $s = |S|$ of sets $S \subset M$.

Hence, after using (45), equalities (52) lead to the system of $m(m-1)$ linear equations with m variables $(x_i)_{i \in M}$ of the following form:

$$\frac{1}{m-1} x_i + x_j = a^{ij}(v) \quad i \in M, \quad j \in M \setminus i. \tag{54}$$

Without loss of generality we can assume that $M = \{1, 2, \ldots, m\}$. Consider now the following $(m \times m)$-subsystem of (54) consisting of the following equations:

$$
\begin{cases}
\frac{1}{m-1}x_1 + x_2 = a^{12}(v) \\
\frac{1}{m-1}x_1 + x_3 = a^{13}(v) \\
\ldots\ldots\ldots\ldots\ldots \quad \ldots\ldots \\
\frac{1}{m-1}x_1 + x_m = a^{1m}(v) \\
\frac{1}{m-1}x_2 + x_1 = a^{21}(v)
\end{cases}
\tag{55}
$$

Let A be its coefficient matrix, and let A_k be the matrix arising from A by changing its k-th column with column $[a^{12}(v), a^{13}(v), \ldots, a^{1m}(v), a^{21}(v)]^T$, $k = 1, 2, \ldots, m$. It is not difficult to check that $\det A = (-1)^m \frac{m(m-2)}{(m-1)^2} \neq 0$ because $m \geq 3$. On the other hand, we easily see that each $\det A_k$ is a linear function of v because all $a^{ij}(v)$ are linear functions of v. Hence, using (45) and Cramer's theorem applied to the system (55), we conclude that all $\Phi_i(M, v)$, $i \in M$, are uniquely determined and are linear functions of v. Consequently, Φ is a linear function on the set of games Γ_M. Therefore, to end the proof we only need to show that Φ is symmetric on Γ_M.

By (52), for arbitrary $i, j \in M$, $i \neq j$, we have $\frac{1}{m-1}\Phi_i(M, v) + \Phi_j(M, v) = a^{ij}(v)$ and $\frac{1}{m-1}\Phi_j(M, v) + \Phi_i(M, v) = a^{ji}(v)$. Hence, we easily get the equation

$$
\Phi_i(M, v) = \frac{(m-1)a^{ij}(v) - (m-1)^2 a^{ji}(v)}{m(2-m)} \quad \text{for all} \quad i, j \in M, i \neq j .
$$

Therefore, for any permutation π of M,

$$
\Phi_{\pi i}(M, \pi v) = \frac{(m-1)a^{\pi(i),\pi(j)}(\pi v) - (m-1)^2 a^{\pi(j),\pi(i)}(\pi v)}{m(2-m)} = \Phi_i(M, v),
$$

because of the equality, $a^{\pi(i),\pi(j)}(\pi v) = a^{ij}(v)$, easily seen from (53). Thus, by the induction principle, value Φ is uniquely determined and satisfies efficiency, linearity and symmetry, completing the proof. □

Proof of Theorem 2 ($\Rightarrow$) Then, by Proposition 5, the equalities (17) hold. Consequently, Proposition 6 implies that Φ is an ESL-value on Γ. Now the fact that (18) and (19) hold is implied by the part ($\Rightarrow$) of Theorem 4 in Driessen and Radzik (2003). The uniqueness of Φ follows from Proposition 6.

($\Leftarrow$) Assume now that Φ is an ESL-value, and (17), (18) and (19) hold. Hence, by Proposition 4, Φ is of the form (13). Then we can again apply Theorem 4 in Driessen and Radzik (2003) to get that the value Φ possesses $\mathcal{D}$-reduced property on Γ. Therefore it is left to show that Φ is λ standard on the set of two person games.

One can easily check that any value of the form (13) is $b_{2,1}$-standard for two person games (see also Remark 7 in Sect. 3), but $b_{2,1} = \lambda$ according to (18). □

Proof of Theorem 3 ($\Rightarrow$) Assume that an ESL-value Φ of the form (13) is δ-standard for two person games, and consistent with respect to the reduced games $v_M^{\Phi,i}$ of the

form (21). Hence, Φ is consistent with respect to the reduced games of the form (15) with all the constants $A_{k,s} = 1$ and $B_{k,s} = 0$ for $1 \leq s \leq k$, $k \geq 2$. Therefore Theorem 2 implies that the constants $b_{m,s}$ in the representation formula (13) satisfy (18) and (19) with $\lambda = \delta$ and all the constants $A_{k,s} = 1$ and $B_{k,s} = 0$. However, it is straightforward to check that it uniquely determines the solution of the form $b_{m,s} = \delta^{m-s}$ for all $1 \leq s \leq k$, $k \geq 2$. But then (13) coincides with (6), and thereby Φ is the δ-discounted Shapley value $\Phi^{Sh^{\delta}}$.

($\Longleftarrow$) The δ-discounted Shapley value $\Phi^{Sh^{\delta}}$ is of the form (13) with $b_{m,s} = \delta^{m-s}$ for all $1 \leq s \leq k$, $k \geq 2$, as it was stated before. Therefore $b_{2,1} = \delta$ and one can verify that this value is δ-standard for two person games. Next note that (18) and (19) with $\lambda = \delta$ are satisfied by $A_{k,s} = 1$ and $B_{k,s} = 0$ for $1 \leq s \leq k$, $k \geq 2$. This implies that the δ-discounted Shapley value is consistent with respect to the reduced games $v_M^{\Phi,i}$ of the form (21).

The fact that under $0 \leq \delta \leq 1$, the value $\Phi^{Sh^{\delta}}$ satisfies the fair treatment and monotonicity properties was justified in Remark 3. $\qquad\square$

Proof of Theorem 4 First we will prove the part ($\Rightarrow$) of the statement (i).

By the assumption, Φ is a value of the form (22) with the λ-standardness property for two-person games. But (22) with $M = \{i, j\}$ is equivalent to (14) for $\beta_1 = \lambda$.

Now assume that a value Φ on Γ of the form (22) with $\beta_1 = \lambda$, is probabilistically consistent. Therefore Φ is an ESL-value (because of (22)) and possesses the $\mathcal{D}$-reduced game property for some collection of pairs of nonnegative constants $\mathcal{D} = \{(A_{k,s}, B_{k,s})\}_{s=1,\ldots,k}^{k=2,3,\ldots}$ satisfying (20). The constants in $\mathcal{D}$ must also satisfy (17) which is a consequence of Proposition 5. Besides, by Theorem 2, the constants $A_{k,s}$ and $B_{k,s}$ satisfy the system of equations (18) - (19) with $b_{k,s} = \beta_s$ for all s.

One can easily verify that (19) implies the following:

$$x_{k,s} = \beta_{s+1} \prod_{i=s+2}^{k} \left[\frac{i-1+\beta_i}{i} \right] \quad \text{for} \quad 1 \leq s < k. \tag{56}$$

(Here and in the sequel $\prod_{i=k+1}^{k}(\cdot) \equiv 1$.) Hence, by (18), the constants $A_{k,s}$ and $B_{k,s}$ for $1 \leq s < k$ satisfy the system of equations:

$$\begin{cases} \beta_1 = \lambda \\ \beta_s A_{k,s} = \beta_{s+1} \prod_{i=s+2}^{k}[\frac{i-1+\beta_i}{i}] \\ 0 = \beta_s \left[1 - B_{k,s} - \prod_{i=s+1}^{k}[\frac{i-1+\beta_i}{i}] \right]. \end{cases} \tag{57}$$

But this system implies that

$$\beta_s A_{s+1,s} = \beta_{s+1} \quad \text{for} \quad s = 1, 2, \ldots \tag{58}$$

and

$$\beta_s \left[1 - B_{s+1,s} - \frac{s + \beta_{s+1}}{s+1} \right] = 0 \quad \text{for} \quad s = 1, 2, \ldots. \tag{59}$$

Now, we consider two cases.

Case 1: $\lambda = 0$.

Then $\beta_1 = \lambda = 0$. Hence, by (58), $\beta_s = 0$ for all $s \geq 1$, and thereby (24) holds.

Case 2: $\lambda \neq 0$.

Then $\beta_1 \neq 0$, and suppose that for some $u \geq 1$, $\beta_u \neq 0$ and $\beta_{u+1} = 0$. Then, by (58) and (59), $A_{u+1,u} = 0$ and $B_{u+1,u} = \frac{1}{u+1}$, contradicting (20). Therefore, for all $s \geq 1$ $\beta_s \neq 0$, and consequently, the only solution of the system (57) is

$$A_{k,s} = \frac{\beta_{s+1}}{\beta_s} \prod_{i=s+2}^{k} [\frac{i-1+\beta_i}{i}] \quad \text{and} \quad B_{k,s} = 1 - \prod_{i=s+1}^{k} [\frac{i-1+\beta_i}{i}] \qquad (60)$$

for $k \geq 2$ and $1 \leq s < k$. Hence, it follows that for $1 \leq s < k$

$$B_{k,s} = 1 - \frac{(s+\beta_{s+1})\beta_s}{(s+1)\beta_{s+1}} A_{k,s} \quad \text{and} \quad B_{k,s+1} = 1 - \frac{\beta_s}{\beta_{s+1}} A_{k,s}. \qquad (61)$$

On the other hand, for all $1 \leq s \leq k$ we also have

$$B_{k,s} = 1 - A_{k,s}, \qquad (62)$$

because Φ is probabilistically consistent. Therefore, (61) and (62) imply that

$$A_{k,s} \left[1 - \frac{(s+\beta_{s+1})\beta_s}{(s+1)\beta_{s+1}} \right] = 0 \quad \text{for} \quad 1 \leq s < k. \qquad (63)$$

Suppose now that $A_{k,u} = 0$ for some $1 \leq u < k$ with u chosen maximal. Since $A_{k,k} = 1$ (because of (17)), we have $A_{k,u+1} \neq 0$. Therefore, by the second equality in (61), it follows that $B_{k,u+1} = 1$, contradicting (62) for $s = u + 1$. Consequently, $A_{k,s} \neq 0$ for all $1 \leq s < k$, whence, by (63), we easily get

$$\beta_{s+1} = \frac{s\beta_s}{(s+1) - \beta_s} \quad \text{for} \quad s = 1, 2, \dots, \qquad (64)$$

together with $\beta_1 = \lambda$. One can check that the formula for the sequence described by (64) is the following:

$$\beta_s = \frac{\lambda}{s(1-\lambda) + \lambda} \quad \text{for} \quad s = 1, 2, \dots, \qquad (65)$$

which proves (24) also in the second case ($\lambda \neq 0$). Now, one can directly verify that the equalities in (60) with $k = m - 1$ and β_s of the form (65) imply (25). Therefore the numbers $A_{k,s}$ and $B_{k,s}$ are uniquely determined in the considered case $\lambda \neq 0$. However this implies that then $v_{M,\lambda}^{\Phi,i}$ are the only reduced games with respect to which the value Φ given by (22)–(24) is probabilistically consistent.

Further, since the value Φ is probabilistically consistent, $0 \leq A_{2,1} \leq 1$, which, by (25), is equivalent to $0 \leq \lambda \leq 1$. Thus we have proved the part ($\Rightarrow$) of the statement (i) and the statement (ii) of Theorem 4.

Now, assume that the constants β_s are of the form (23) and (24) for some $0 \leq \lambda \leq 1$. Then, by the statement (ii), Φ is a probabilistically consistent value, which proves the part ($\Leftarrow$) of the statement (i), completing the proof of Theorem 4. $\square$

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