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The vector k-constrained KP hierarchy and Sato's Grassmannian

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Abstract

We use the representation theory of the infinite matrix group to show that (in the polynomial case) the n-vector k-constrained KP hierarchy has a natural geometrical interpretation on Sato's infinite Grassmannian. This description generalizes the k-reduced KP or Gelfand–Dickey hierarchies.

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1. Introduction

It is well known that the kth Gelfand–Dickey hierarchy, which generalizes the Korteweg–de Vries (KdV) hierarchy, can be obtained as a reduction of the Kadomtsev–Petviashvili (KP) hierarchy. The latter is defined as the set of deformation equations

$$\frac{\partial L}{\partial t_k} = [(L^k)_+, L]$$

for the first-order pseudo-differential operator

$$L \equiv L(t, \partial) = \partial + u_1(t)\partial^{-1} + u_2(t)\partial^{-2} + \cdots,$$

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here $\partial = \partial/\partial t_1$, $t = (t_1, t_2, \ldots)$ and $(L^k)_+$ stands for the differential part of L^k. Now L dresses as $L = P\partial P^{-1}$ with

$$P \equiv P(t, \partial) = 1 + a_1(t)\partial^{-1} + a_2(t)\partial^{-2} + \cdots$$

One can choose P in such a way that

$$P(t, z) = \frac{\tau(t - [z^{-1}])}{\tau(t)},$$

where $\tau(t) = \tau(t_1, t_2, t_3, \ldots)$ is the famous τ-function, introduced by the Kyoto group [DJKM1–3] and $[z] = (z, \frac{1}{2}z^2, \frac{1}{3}z^3, \ldots)$. Sato [S] showed that such a τ-function corresponds to a point of some infinite Grassmannian Gr (see e.g. [S,SW]). Let H be the space of formal Laurent series $\sum a_n t^n$ such that $a_n = 0$ for $n \gg 0$. The points of Gr are those linear subspaces $W \subset H$ for which the projection π_+ of W into $H_+ = \{\sum a_n t^n \in H \,|a_n = 0 \text{ for all } n < 0\}$ is a Fredholm operator. The kth reduction or kth Gelfand–Dickey hierarchy is obtained by assuming that

$$L^k = (L^k)_+,$$

which corresponds to a τ-function for which

$$\frac{\partial \tau}{\partial t_k} = \lambda\tau \quad \text{for some } \lambda \in \mathbb{C}.$$

In the polynomial case, i.e. τ is a polynomial, clearly $\lambda = 0$. The point in the Grassmannian that corresponds to such a reduced τ-function satisfies

$$t^k W \subset W.$$

In recent years a lot of attention has been drawn to a new kind of reduction of the KP hierarchy, viz. the so-called k-constrained KP hierarchies [AFGZ,C,CSZ,CZ,D,DS,OS,Z] (and references therein). Here one assumes that

$$L^k = (L^k)_+ + q\partial^{-1}r, \tag{1.1}$$

$q = q(t), r = r(t)$ being functions. Under this condition the KP hierarchy is constrained to

$$\frac{\partial L}{\partial t_k} = [(L^k)_+, L], \qquad \frac{\partial q}{\partial t_k} = (L^k)_+ q, \qquad \frac{\partial r}{\partial t_k} = -(L^k)_+^* r. \tag{1.2}$$

Here A^* stands for the adjoined operator of A (see e.g. [KV] for more details about pseudo-differential operators). The AKNS, Yajima–Oikawa and Melnikov hierarchies are some of the examples that appear amongst these constrained KP families.

In this paper we consider the generalization of this k-constrained KP hierarchy, which was introduced by Sidorenko and Strampp [SS], the n-vector k-constrained hierarchy. We assume that

$$L^k = (L^k)_+ + \sum_{j=1}^{n} q_j \partial^{-1} r_j, \tag{1.3}$$

then one obtains the following integrable system:

$$\frac{\partial L}{\partial t_k} = [(L^k)_+, L], \qquad \frac{\partial q_j}{\partial t_k} = (L^k)_+ q_j, \quad \frac{\partial r_j}{\partial t_k} = -(L^k)_+^* r_j \qquad \text{for } 1 \le j \le n. \tag{1.4}$$

For $k = 1$ this hierarchy contains the coupled vector non-linear Schrödinger. Zhang and Cheng showed [ZC] that if one assumes that

$$q_j(t) = \frac{\rho_j(t)}{\tau(t)} \quad \text{and} \quad r_j(t) = \frac{\sigma_j(t)}{\tau(t)}, \tag{1.5}$$

then L, q_j and r_j, $1 \le j \le n$ satisfy the n-vector k-constrained hierarchy if and only if $\tau(t)$, $\rho_j(t)$ and $\sigma_j(t)$ satisfy the following set of equations:

$$\mathrm{Res}_{z=0} \mathrm{e}^{-\eta(t,z)} \tau(t) \mathrm{e}^{\xi(t,z)} \mathrm{e}^{\eta(t',z)} \tau(t') \mathrm{e}^{-\xi(t',z)} = 0, \tag{1.6}$$

$$\mathrm{Res}_{z=0} z^k \mathrm{e}^{-\eta(t,z)} \tau(t) \mathrm{e}^{\xi(t,z)} \mathrm{e}^{\eta(t',z)} \tau(t') \mathrm{e}^{-\xi(t',z)} = \sum_{j=1}^{n} \rho_j(t) \sigma_j(t'), \tag{1.7}$$

$$\mathrm{Res}_{z=0} z^{-1} \mathrm{e}^{-\eta(t,z)} \tau(t) \mathrm{e}^{\xi(t,z)} \mathrm{e}^{\eta(t',z)} \rho_j(t') \mathrm{e}^{-\xi(t',z)} = \rho_j(t) \tau(t'), \tag{1.8}$$

$$\mathrm{Res}_{z=0} z^{-1} \mathrm{e}^{-\eta(t,z)} \sigma_j(t) \mathrm{e}^{\xi(t,z)} \mathrm{e}^{\eta(t',z)} \tau(t') \mathrm{e}^{-\xi(t',z)} = \tau(t) \sigma_j(t'), \tag{1.9}$$

where

$$\eta(t, z) = \sum_{i=1}^{\infty} \frac{\partial}{\partial t_i} \frac{z^{-i}}{i}, \qquad \xi(t, z) = \sum_{i=1}^{\infty} t_i z^i \tag{1.10}$$

and $\mathrm{Res}_{z=0} \sum_i a_i z^i = a_{-1}$.

In the case that $n = 1$, Loris and Willox [LW] show that one can deduce some additional bilinear identities, but now involving $\partial\tau/\partial t_k$. It is unclear if this is possible for $n > 1$, but we will not need these extra bilinear identities.

We will show in this paper that in fact L satisfies the n-vector k-constrained KP hierarchy, (1.3) and (1.4), if and only if the corresponding point W in Gr has a linear subspace $W' \subset W$ of codimension n such that

$$t^k W' \subset W. \tag{1.11}$$

We will prove this only in the polynomial case, i.e. polynomial τ,ρ_j and σ_j, but Gerand Helminck and the author recently obtained the same result in the Segal–Wilson scattering [HV]. We use the representation theory of the infinite-dimensional matrix group GL_∞, developed by Kac and Peterson [KP1,KP2] (see also [KR]), to achieve this result.

Notice that in this way we get a filtration of hierarchies, i.e. the n-vector k-constrained hierarchy is a subsystem of the $(n + 1)$-vector k-constrained hierarchy, $n = 0$ being the k-reduced KP or Gelfand–Dickey hierarchies.

Finally we want to mention that recently Aratyn et al. [ANP] have related these n-vector k-constrained KP hierarchies to: (1) the general rational reductions of the KP hierarchy as considered by Krichever [K] and (2) matrix models that generalize the familiar ones with standard polynomial matrix potentials.

2. The semi-infinite wedge representation of the group GL_∞ and Sato's Grassmannian

Consider the infinite complex matrix group

$$GL_\infty = \{A = (a_{ij})_{i,j\in\mathbb{Z}+1/2} | A \text{ is invertible and all but a finite number of } a_{ij} - \delta_{ij} \text{ are } 0$$

and its Lie algebra

$$gl_\infty = \{a = (a_{ij})_{i,j\in\mathbb{Z}+1/2} | \text{ all but a finite number of } a_{ij} \text{ are } 0\}$$

with bracket $[a, b] = ab - ba$. The Lie algebra gl_∞ has a basis consisting of matrices $E_{ij}, i, j \in \mathbb{Z} + \frac{1}{2}$, where E_{ij} is the matrix with a 1 on the (i, j)th entry and zeros elsewhere. Let $\mathbb{C}^\infty = \bigoplus_{j\in\mathbb{Z}+1/2} \mathbb{C}v_j$ be an infinite-dimensional complex vector space with fixed basis $\{v_j\}_{j\in\mathbb{Z}+1/2}$. Both the group GL_∞ and its Lie algebra gl_∞ act linearly on $\mathbb{C}^\infty$ via the usual formula:

$$E_{ij}(v_k) = \delta_{jk} v_i.$$

The well-known semi-infinite wedge representation is constructed as follows [KP2] (see also [KR,KV]). The semi-infinite wedge space $F = \Lambda^{1/2\infty}\mathbb{C}^\infty$ is the vector space with a basis consisting of all semi-infinite monomials of the form $v_{i_1} \wedge v_{i_2} \wedge v_{i_3} \cdots$, where $i_1 > i_2 > i_3 > \cdots$ and $i_{l+1} = i_l - 1$ for $l \gg 0$. We can now define representations R of GL_∞ and r of gl_∞ on F by

$$\begin{aligned} R(A)(v_{i_1} \wedge v_{i_2} \wedge v_{i_3} \wedge \cdots) &= Av_{i_1} \wedge Av_{i_2} \wedge Av_{i_3} \wedge \cdots, \\ r(a)(v_{i_1} \wedge v_{i_2} \wedge v_{i_3} \wedge \cdots) &= \sum_k v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_{k-1}} \wedge av_{i_k} \wedge v_{i_{k+1}} \wedge \cdots \end{aligned} \tag{2.1}$$

These equations are related by the usual formula

$$\exp(r(a)) = R(\exp a) \quad \text{for } a \in gl_\infty.$$

In order to perform calculations later on, it is convenient to introduce a larger group

$$\overline{GL_\infty} = \{A = (a_{ij})_{i,j\in\mathbb{Z}+1/2} | A \text{ is invertible and all but a finite number of } a_{ij} - \delta_{ij} \text{ with } i \geq j \text{ are } 0\}$$

and its Lie algebra

$$\overline{gl_\infty} = \{a = (a_{ij})_{i,j\in\mathbb{Z}+1/2} | \text{ all but a finite number of } a_{ij} \text{ with } i \geq j \text{ are } 0\}.$$

Both $\overline{GL_\infty}$ and $\overline{gl_\infty}$ act on a completion $\overline{\mathbb{C}^\infty}$ of the space $\mathbb{C}^\infty$, where

$$\overline{\mathbb{C}^\infty} = \left\{ \sum_j c_j v_j | c_j = 0 \text{ for } j \gg 0 \right\}.$$

It is easy to see that the representations R and r extend to representations of $\overline{GL_\infty}$ and $\overline{gl_\infty}$ on the space F.

The representation r of gl_∞ and $\overline{gl_\infty}$ can be described in terms of wedging and contracting operators in F (see e.g. [KP2,KR]). Let v_j^* be the linear functional on $\mathbb{C}^\infty$ defined by $\langle v_i^*, v_j\rangle := v_i^*(v_j) = \delta_{ij}$ and let $\mathbb{C}^{\infty *} = \bigoplus_{j\in\mathbb{Z}+1/2}\mathbb{C}v_j^*$ be the restricted dual of $\mathbb{C}^\infty$, then for any $w \in \mathbb{C}^\infty$, we define a wedging operator $\psi^+(w)$ on F by

$$\psi^+(w)(v_{i_1}\wedge v_{i_2}\wedge\cdots) = w\wedge v_{i_1}\wedge v_{i_2}\cdots \tag{2.2}$$

Let $w^* \in \mathbb{C}^{\infty *}$, we define a contracting operator

$$\begin{aligned}&\psi^-(w^*)(v_{i_1}\wedge v_{i_2}\wedge\cdots)\\ &\quad=\sum_{s=1}^{\infty}(-1)^{s+1}\langle w^*, v_{i_s}\rangle v_{i_1}\wedge v_{i_2}\wedge\cdots\wedge v_{i_{s-1}}\wedge v_{i_{s+1}}\wedge\cdots\end{aligned} \tag{2.3}$$

For simplicity we write

$$\psi_j^+ = \psi^+(v_{-j}),\quad \psi_j^- = \psi^-(v_j^*)\qquad \text{for } j\in\mathbb{Z}+\tfrac{1}{2}. \tag{2.4}$$

These operators satisfy the following relations ($i, j \in \mathbb{Z}+\frac{1}{2}, \lambda, \mu = +, -$):

$$\psi_i^\lambda\psi_j^\mu + \psi_j^\mu\psi_i^\lambda = \delta_{\lambda,-\mu}\delta_{i,-j},$$

hence they generate a Clifford algebra, which we denote by $\mathcal{C}\ell$.

Introduce the following elements of F ($m \in \mathbb{Z}$):

$$|m\rangle = v_{m-1/2}\wedge v_{m-3/2}\wedge v_{m-5/2}\wedge\cdots$$

It is clear that F is an irreducible $\mathcal{C}\ell$-module generated by the vacuum $|0\rangle$ such that

$$\psi_j^\pm|0\rangle = 0 \quad \text{for} j > 0.$$

It is straightforward that the representation r is given by the following formula:

$$r(E_{ij}) = \psi_{-i}^+\psi_j^-. \tag{2.5}$$

Define the *charge decomposition*

$$F = \bigoplus_{m\in\mathbb{Z}} F^{(m)} \tag{2.6}$$

by letting

$$\text{charge}(|0\rangle) = 0 \quad\text{and}\quad \text{charge}(\psi_j^\pm) = \pm 1. \tag{2.7}$$

It is clear that the charge decomposition is invariant with respect to $r(gl_\infty)$ (and hence with respect to $R(GL_\infty)$). Moreover, it is easy to see that each $F^{(m)}$ is irreducible with respect to gl_∞ (and GL_∞). Note that $|m\rangle$ is its highest weight vector, i.e.

$$\begin{aligned}&r(E_{ij})|m\rangle = 0 \quad \text{for } i < j,\\ &r(E_{ii})|m\rangle = 0 \quad (\text{resp.} = |m\rangle) \quad \text{if } i > m \text{ (resp. if } i < m).\end{aligned}$$

Let $w \in F$, we define the Annihilator space $\mathrm{Ann}(w)$ of w as follows:

$$\mathrm{Ann}(w) = \{v \in \mathbb{C}^\infty \mid v \wedge w = 0\}. \tag{2.8}$$

Notice that $\mathrm{Ann}(w) \neq 0$, since $v_j \in \mathrm{Ann}(w)$ for $j \ll 0$. This Annihilator space for perfect (semi-infinite) wedges $w \in F^{(m)}$ is related to the GL_∞-orbit

$$\mathcal{O}_m = R(GL_\infty)|m\rangle \subset F^{(m)}$$

of the highest weight vector $|m\rangle$ as follows. Let $A = (A_{ij})_{i,j\in\mathbb{Z}} \in GL_\infty$, denote by $A_j = \sum_{i\in\mathbb{Z}} A_{ij} v_i$ then by (2.8)

$$\tau_m = R(A)|m\rangle = A_{m-1/2} \wedge A_{m-3/2} \wedge A_{m-5/2} \wedge \cdots \tag{2.9}$$

with $A_{-j} = v_{-j}$ for $j \gg 0$. Notice that since τ_m is a perfect (semi-infinite) wedge

$$\mathrm{Ann}(\tau_m) = \sum_{j<m} \mathbb{C}A_j \subset \mathbb{C}^\infty.$$

By identifying $v_i = t^{-i-1/2}$ for $i \in \mathbb{Z}+\frac{1}{2}$, we can write $A_j = A_j(t) = \sum_{i\in\mathbb{Z}+1/2} A_{ij} t^{-i-1/2}$ as a Laurent polynomial in t. In this way we can identify $\mathrm{Ann}(\tau_m)$ with a subspace $W_{\tau_m} = \sum_{j<m} \mathbb{C}A_j(t)$ of the space H of all Laurent polynomials. Notice that this space H differs from the one described in Section 1. So from now on let Gr consist of all linear subspaces of H which contain

$$H_j := \sum_{i=-j}^{\infty} \mathbb{C}t^i$$

for $j \gg 0$ and let $\mathrm{Gr} = \bigcup_{m\in\mathbb{Z}} \mathrm{Gr}_m$ (disjoint union) with

$$\mathrm{Gr}_m = \{W \in \mathrm{Gr} \mid H_j \subset W \text{ and } \dim W/H_j = m - j \text{ for } j \ll 0\},$$

then we can construct a cannonical map

$$\phi : \mathcal{O}_m \to \mathrm{Gr}_m, \qquad \phi(\tau_m) = W_{\tau_m} := \sum_{i<m} \mathbb{C}A_i(t).$$

It is clear that $\phi(|m\rangle) = H_m$ and that ϕ is surjective with fibers $\mathbb{C}^\times$. This construction is due to Sato [S], we call Gr the polynomial Grassmannian. From now on we will call a perfect wedge also a τ-function (note that $\tau = 0$ is also a τ-function).

3. The boson–fermion correspondence

Introduce the fermionic fields ($z \in \mathbb{C}^\times$):

$$\psi^\pm[z] \overset{\text{def}}{=} \sum_{k\in\mathbb{Z}+1/2} \psi_k^\pm z^{-k-1/2}. \tag{3.1}$$

Next we introduce bosonic fields:

$$\alpha[z] \equiv \sum_{k\in\mathbb{Z}} \alpha_k z^{-k-1} \stackrel{\text{def}}{=} :\psi^+[z]\psi^-[z]:, \tag{3.2}$$

where : : stands for the *normal ordered product* defined in the usual way ($\lambda, \mu = +$ or $-$):

$$:\psi_k^\lambda \psi_l^\mu: = \begin{cases} \psi_k^\lambda \psi_l^\mu & \text{if } l \geq k, \\ -\psi_l^\mu \psi_k^\lambda & \text{if } l < k. \end{cases}$$

One checks (using e.g. Wick's formula) that the operators α_k satisfy the commutation relations of the associative oscillator algebra, one has

$$[\alpha_k, \alpha_l] = k\delta_{k,-l} \quad \text{and} \quad \alpha_k|m\rangle = 0 \quad \text{if } k > 0. \tag{3.3}$$

In order to express the fermionic fields $\psi^\pm(z)$ in terms of the bosonic operators α_l, we need some additional operator Q. This operator is uniquely defined as follows:

$$Q(v_{i_1} \wedge v_{i_2} \wedge \cdots) = (v_{i_1+1} \wedge v_{i_2+1} \wedge \cdots). \tag{3.4}$$

So

$$Q|0\rangle = |1\rangle, \qquad Q\psi_k^\pm = \psi_{k\mp 1}^\pm Q$$

and Q satisfies the following commutation relations with the α's:

$$[\alpha_k, Q] = \delta_{k0} Q.$$

In this paper the operator Q^{-k} will play an important role. If $w_{m-1/2} \wedge w_{m-3/2} \wedge \cdots$ is a perfect wedge then

$$Q^{-k}(w_{m-1/2} \wedge w_{m-3/2} \wedge \cdots) = \Lambda^k w_{m-1/2} \wedge \Lambda^k w_{m-3/2} \wedge \cdots, \tag{3.5}$$

where $\Lambda = \sum_{j\in\mathbb{Z}+1/2} E_{j,j+1}$.

Theorem 3.1. [DJKM1,JM]

$$\psi^\pm[z] = Q^{\pm 1} z^{\pm\alpha_0} \exp\left(\mp \sum_{k<0} \frac{1}{k} \alpha_k z^{-k}\right) \exp\left(\mp \sum_{k>0} \frac{1}{k} \alpha_k z^{-k}\right). \tag{3.6}$$

Proof. See [TV].

The operators on the right-hand side of (3.6) are called vertex operators. They made their first appearance in string theory (cf. [FK]).

We now describe the boson–fermion correspondence. Let $\mathbb{C}[t]$ be the space of polynomials in indeterminates $t = (t_1, t_2, t_3, \ldots)$. Let $B = \mathbb{C}[q, q^{-1}, t] = \mathbb{C}[t] \otimes_{\mathbb{C}} \mathbb{C}[q, q^{-1}]$ be the tensor product of algebras. Then the boson–fermion correspondence is the vector space isomorphism

$$\sigma : F \xrightarrow{\sim} B,$$

given by

$$\sigma(\alpha_{-m_1}\cdots\alpha_{-m_s}|k\rangle) = m_1\cdots m_s t_{m_1}\cdots t_{m_s} q^k.$$

Notice that the power of q is the value of the charge. The transported action of the operators α_m and Q looks as follows:

$$\sigma Q\sigma^{-1} = q, \qquad \sigma\alpha_m\sigma^{-1} = \begin{cases} -mt_m & \text{if } m < 0, \\ \dfrac{\partial}{\partial t_m} & \text{if } m > 0, \\ q\dfrac{\partial}{\partial q} & \text{if } m = 0. \end{cases} \tag{3.7}$$

Hence

$$\sigma\psi^{\pm}[z]\sigma^{-1} = q^{\pm 1} z^{\pm q\partial/\partial q} \mathrm{e}^{\pm\xi(t,z)} \mathrm{e}^{\mp\eta(t,z)} \tag{3.8}$$

with $\eta(t,z)$ and $\xi(t,z)$ given by (1.10).

4. Identification of the bilinear identities

From now on we assume that $\tau \in F^{(m)}$, hence that τ is the inverse image under σ. Using the boson–fermion correspondence of Section 3, we rewrite the bilinear identities (1.6)–(1.9), of Zhang and Cheng now as equations in $F \otimes F$. Notice first the following equality of operators on $F \otimes F$:

$$\mathrm{Res}_{z=0}\psi^{+}[z] \otimes \psi^{-}[z] = \sum_{i\in\mathbb{Z}+1/2} \psi_i^{+} \otimes \psi_{-i}^{-}.$$

Now (1.6)–(1.9) turn into the following equations:

$$\sum_{i\in\mathbb{Z}+1/2} \psi_i^{+}\tau \otimes \psi_{-i}^{-}\tau = 0, \tag{4.1}$$

$$\sum_{i\in\mathbb{Z}+1/2} \psi_i^{+}\tau \otimes \psi_{-i}^{-}Q^{-k}\tau = \sum_{j=1}^{n} \rho_j \otimes \sigma_j, \tag{4.2}$$

$$\sum_{i\in\mathbb{Z}+1/2} \psi_i^{+}\tau \otimes \psi_{-i}^{-}\rho_j = \rho_j \otimes \tau, \tag{4.3}$$

$$\sum_{i\in\mathbb{Z}+1/2} \psi_i^{+}\sigma_j \otimes \psi_{-i}^{-}Q^{-k}\tau = Q^{-k}\tau \otimes \sigma_j. \tag{4.4}$$

Here $Q^{-k}\tau \in F^{(m-k)}$, $\rho_j \in F^{(m+1)}$ and $\sigma_j \in F^{(m-k-1)}$ for all $1 \le j \le n$. Eq. (4.1) is called the KP hierarchy in the fermionic picture, it characterizes the GL_∞-orbit $\mathcal{O}_m$, i.e.:

Proposition 4.1. [KP2] *A non-zero element τ of $F^{(m)}$ lies in $\mathcal{O}_m$ if and only if τ satisfies Eq.* (4.1).

If $\tau \in \mathcal{O}_m$, then we can write τ as a perfect wedge

$$\tau = w_{m-1/2} \wedge w_{m-3/2} \wedge w_{m-5/2} \wedge w_{m-7/2} \wedge \cdots, \tag{4.5}$$

such that $w_{-l} = v_{-l}$ for $l \gg 0$. The corresponding point $W_\tau \in \mathrm{Gr}_m$ is then given by

$$W_\tau = \langle w_{m-1/2}, w_{m-3/2}, w_{m-5/2}, w_{m-7/2}, \cdots \rangle. \tag{4.6}$$

The geometrical interpretation of (4.3)–(4.4) is given by the following proposition.

Proposition 4.2. *Let $\tau \in \mathcal{O}_m$, $\rho \in F^{(m+1)}$ and $\sigma \in F^{(m-1)}$, then*
(1) *τ and ρ satisfy*

$$\sum_{i \in \mathbb{Z}+1/2} \psi_i^+ \tau \otimes \psi_{-i}^- \rho = \rho \otimes \tau \tag{4.7}$$

if and only if $\rho \in \mathcal{O}_{m+1}$ and $W_\tau \subset W_\rho$,
(2) *τ and σ satisfy*

$$\sum_{i \in \mathbb{Z}+1/2} \psi_i^+ \sigma \otimes \psi_{-i}^- \tau = \tau \otimes \sigma \tag{4.8}$$

if and only if $\sigma \in \mathcal{O}_{m-1}$ and $W_\sigma \subset W_\tau$.

Proof. Without loss of generality we may assume (since the operator $\sum_i \psi_i^+ \otimes \psi_{-i}^-$ commutes with the action of $R(GL_\infty) \otimes R(GL_\infty)$) that $\tau = |m\rangle$. Then (4.7) is equivalent to

$$\sum_{i>m} v_i \wedge |m\rangle \otimes \psi_i^- \rho = \rho \otimes |m\rangle.$$

Since all elements $v_i \wedge |m\rangle$, for $i > m$, are linearly independent, we deduce that $\psi_i^- \rho = \lambda_i |m\rangle$ and that $\rho \in \langle v_i \wedge |m\rangle | i > m\rangle$. Hence $\rho = w \wedge |m\rangle$ for some $w \in \mathbb{C}^\infty$ and thus $\rho \in \mathcal{O}_{m+1}$ and $W_\tau \subset W_\rho$.

The converse, since $W_\tau \subset W_\rho$, $\rho = w \wedge |m\rangle$ for some $w \in \mathbb{C}^\infty$. Then

$$\begin{aligned}\sum_{i \in \mathbb{Z}+1/2} \psi_i^+ \tau \otimes \psi_{-i}^-(w \wedge \tau) &= (w \wedge \tau) \otimes \tau - (1 \otimes \psi^+(w)) \left(\sum_{i \in \mathbb{Z}+1/2} \psi_i^+ \tau \otimes \psi_{-i}^- \tau \right) \\ &= (w \wedge \tau) \otimes \tau\end{aligned}$$

For $\tau = |m\rangle$, (4.8) is equivalent to

$$\sum_{i<m} (v_i \wedge \sigma) \otimes \psi_i^- |m\rangle = |m\rangle \otimes \sigma.$$

Since the elements $\psi_i^- |m\rangle$ for $i < m$ are all linearly independent, we conclude that $v_i \wedge \sigma = \lambda_i |m\rangle$ and that $\sigma \in \langle \psi_i^- |m\rangle | i < m\rangle$. Hence $\sigma = \sum_{i=-\infty}^{m-1/2} a_i \psi_i^- |m\rangle$. Since $\sigma \in F^{(m-1)}$, $a_i = 0$ for all $i < -N \ll 0$. We now calculate $\mathrm{Ann}(\sigma)$. Clearly $\mathrm{Ann}(\sigma) \subset \langle v_i | i < m\rangle = \mathrm{Ann}(|m\rangle)$, so let $v = \sum_{i<m} (-)^i b_i v_i$, then $\sum_{i=-N+1/2}^{m-1/2} a_i b_i = 0$. Hence, if $\sigma \neq 0$, we only

find one restriction for the collection of b_i's, from which we conclude that σ is a perfect wedge. The converse of this statement follows immediately by writing $\tau = w \wedge \sigma$. □

We next prove the following.

Proposition 4.3. *Let $\tau \in \mathcal{O}_m$, $\rho_j \in \mathcal{O}_{(m+1)}$ and $\sigma_j \in \mathcal{O}_{(m-k-1)}$, $1 \le j \le n$, be related by*

$$W_\tau \subset W_{\rho_j}, \qquad W_{\sigma_j} \subset \Lambda^k W_\tau, \tag{4.9}$$

then τ satisfies Eq. (4.2) *if and only if there exists a subspace $W' \subset W_\tau$ of codimension n such that $\Lambda^k W' \subset W_\tau$.*

Proof. Notice first that $\Lambda^k W_\tau = W_{Q^{-k}\tau}$. We assume that n is minimal, so that all σ_j and ρ_j are non-zero perfect wedges, and that τ is of the form (4.5). Then

$$\begin{aligned}
&\sum_{i\in\mathbb{Z}+1/2} \psi_i^+\tau \otimes \psi_{-i}^- Q^{-k}\tau \\
&= \sum_{l=0}^{\infty}(-)^l \Lambda^k w_{m-l-1/2} \wedge \tau \otimes \Lambda^k w_{m-1/2} \wedge \cdots \\
&\qquad \wedge \Lambda^k w_{m-l+1/2} \wedge \Lambda^k w_{m-l-3/2} \wedge \cdots \\
&= \sum_{j=1}^{n} u_j \wedge \tau \otimes \sigma_j,
\end{aligned}$$

where $\rho_j = u_j \wedge \tau$. Since all vectors $\Lambda^k w_{m-1/2} \wedge \cdots \wedge \Lambda^k w_{m-l+1/2} \wedge \Lambda^k w_{m-l-3/2} \wedge \cdots$ are linearly independent, we deduce that

$$\Lambda^k w_{m-l-1/2} \wedge u_1 \wedge u_2 \wedge \cdots \wedge u_n \wedge \tau = 0$$

for all $l = 0, 1, 2, \ldots$ Since we have assumed that n is minimal, also all u_j's are linearly independent and moreover $u_1 \wedge u_2 \wedge \cdots \wedge u_n \wedge \tau \neq 0$, hence

$$\Lambda^k w_{m-l-1/2} \in \langle u_1, u_2, \ldots, u_n, w_{m-1/2}, w_{m-3/2}, \ldots \rangle,$$

so there exists a subspace $W' \subset W_\tau$ of codimension n such that $\Lambda^k W' \subset W_\tau$.

For the converse, choose a basis $w_{m-n-1/2}, w_{m-n-3/2}, \ldots$ of W' and extend it to a basis $w_{m-1/2}, w_{m-3/2}, \ldots, w_{m-n+1/2}, w_{m-n-1/2}, w_{m-n-3/2}, \ldots$ of W_τ, then

$$\begin{aligned}
&\sum_{i\in\mathbb{Z}+1/2} \psi_i^+\tau \otimes \psi_{-i}^- Q^{-k}\tau \\
&= \sum_{l=0}^{\infty}(-)^l \Lambda^k w_{m-l-1/2} \wedge \tau \otimes \Lambda^k w_{m-1/2} \wedge \cdots \\
&\qquad \wedge \Lambda^k w_{m-l+1/2} \wedge \Lambda^k w_{m-l-3/2} \wedge \cdots \\
&= \sum_{l=0}^{n-1}(-)^l \Lambda^k w_{m-l-1/2} \wedge \tau \otimes \Lambda^k w_{m-1/2} \wedge \cdots \\
&\qquad \wedge \Lambda^k w_{m-l+1/2} \wedge \Lambda^k w_{m-l-3/2} \wedge \cdots
\end{aligned}$$

So choose

$$\rho_j = \Lambda^k w_{m-j+1/2} \wedge \tau,$$
$$\sigma_j = \Lambda^k w_{m-1/2} \wedge \cdots \wedge \Lambda^k w_{m-j+3/2} \wedge \Lambda^k w_{m-j-1/2} \wedge \cdots,$$

then W_τ, $\Lambda^k W_\tau$, W_{σ_j} and W_{ρ_j} clearly satisfy Eq. (4.9). □

From this proposition we deduce the main theorem of this paper.

Theorem 4.4. *The pseudo-differential operator*

$$L = \partial + u_1 \partial^{-1} + u_2 \partial^{-2} + \cdots$$

satisfies the n-vector k-constrained KP hierarchy if and only if the corresponding point $W \in Gr_m$ has a subspace W' of codimension n such that $t^k W' \subset W$.

As an easy consquence we obtain

Corollary 4.5. *Let τ be a polynomial τ-function of the n-vector k-constrained KP hierarchy, then $\partial \tau / \partial t_k = \sum_{l=1}^{n} \tau_l$ where every τ_l satisfies the KP hierarchy, i.e. Eq.* (4.1).

Proof. Follows immediately by taking the same basis for W_τ as in the converse part of the proof of Proposition 4.3. □

If $n = 1$, one can prove [V] that every polynomial τ-function τ, for which $\partial\tau/\partial t_k$ is again τ-function, is a solution of the k-constrained KP hierarchy.

Notice that we have constructed a natural filtration on the space Gr_m, which is determined by the n-vector k-constrained KP hierarchy for $n = 0, 1, 2, \ldots$ Let

$$\mathrm{Gr}_m^{(n,k)} = \{W \in \mathrm{Gr}_m \mid \text{there exists a subspace } W' \subset W \text{ of codimension } n \text{ such that } t^k W' \subset W\}, \tag{4.10}$$

then

$$\mathrm{Gr}_m^{(0,k)} \subset \mathrm{Gr}_m^{(1,k)} \subset \cdots \subset \mathrm{Gr}_m^{(n,k)} \subset \mathrm{Gr}_m^{(n+1,k)} \subset \cdots \tag{4.11}$$

It is obvious that every point $W \in \mathrm{Gr}_m$ (in this polynomial case) is contained in $\mathrm{Gr}_m^{(n,k)}$ for $n \gg 0$, in other words

$$\mathrm{Gr}_m = \bigcup_{n \in \mathbb{Z}_+} \mathrm{Gr}_m^{(n,k)}. \tag{4.12}$$

So for every τ-function of the KP hierarchy there exists a non-negative integer n such that for all $m \geq n$, τ is also a τ-function of the m-vector k-constrained KP hierarchy. In other words, for every L, corresponding to a polynomial τ-function, one can find a non-negative integer n such that L satisfies (1.3).

5. Polynomial solutions of the n-vector k-constrained KP hierarchy

We will now state an immediate consequence of the boson–fermion correspondence, viz., we calculate the image under σ of a perfect wedge of the form (2.9). One finds the following result.

Proposition 5.1. *Let S_i be the elementary Schur functions, defined by* $\exp \sum_{i=1}^{\infty} t_i z^i = \sum_{i\in\mathbb{Z}}^{\infty} S_i(t) z^i$ *($S_i = 0$ for $i < 0$) and let $\tau_m \in \mathcal{O}_m$ be of the form* (2.9), *i.e.,*

$$\tau_m = A_{m-1/2} \wedge A_{m-3/2} \wedge A_{m-5/2} \wedge \cdots$$

with $A_j = \sum_{i\in\mathbb{Z}+1/2} A_{ij} v_i$ and $A_{-k} = v_{-k}$ for all $k > N \gg 0$. Set $A = (A_{ij})_{i\in\mathbb{Z}+1/2, m>j\in\mathbb{Z}+1/2}$ and let $\Lambda = \sum_{i\in\mathbb{Z}=1/2} E_{i,i+1} \in \overline{gl_\infty}$. Then

$$\sigma(\tau_m) = \det\left(\sum_{i,j=-n+1/2}^{m-1/2} \left(\sum_{l=-N+1/2}^{\infty} S_{l-i} A_{lj} \right) E_{ij} \right) q^m. \tag{5.1}$$

Proof. The proof of this proposition is the same as the proof of Theorem 6.1 of [KR]. One computes

$$\sigma\left(\exp\left(\sum_{i=1}^{\infty} t_i \Lambda^i \right) \tau_m \right)$$

and takes the coefficient of q^m. One thus obtains (see also [DJKM1,M]):

$$\sigma(\tau_m) = \det\left(\left(\exp\left(\sum_{i=1}^{\infty} t_i \Lambda^i \right) A \right)_{<m} \right) q^m, \tag{5.2}$$

where $B_{<m}$ denotes the submatrix of B where one only takes the rows $j \in \mathbb{Z} + \frac{1}{2}$ with $j < m$. Notice that $\sum_i t_i \Lambda^i \in \overline{gl_\infty}$ and $\exp(\sum_i t_i \Lambda^i) \in \overline{GL_\infty}$. Here we calculate the determinant of an infinite matrix. However, there is no problem since the matrix is of the form $(B_{ij})_{m>i,j\in\mathbb{Z}+1/2}$ with all but a finite number of $B_{ij} - \delta_{ij}$ with $i \geq j$ are zero.

It is clear that one can subtract $\sum_{i<-N} A_{ij} v_i$ from every A_j, with $j > -N$, in τ_m, this will not change τ_m. Then the new A is of the form

$$A = \sum_{-N<i, -N<j<m} A_{ij} E_{ij} + \sum_{i<-N} E_{ii},$$

it is then straightforward, using the elementary Schur functions, to calculate the right-hand side of (5.2). One finds formula (5.1). □

We will use this proposition to obtain all polynomial solutions of the n-vector k-constrained KP hierarchy. Notice that our approach is different from the one in [ZC]. Instead of taking τ_m of the form (2.9), we may choose another basis of W_{τ_m} and construct the corresponding perfect wedge, it is clear that this will be a multiple of τ_m. We can choose this basis in such a way

$$W_{\tau_m} = \langle A_{m-1/2}, A_{m-3/2}, A_{m-5/2}, \ldots, A_{-N+1/2}, v_{-N-1/2}, v_{-N-3/2}, \ldots \rangle$$

such that $A_j = \sum_{i=-N+1/2}^{\infty} A_{ij} v_i$ and that, except for at most n vectors A_j, all A_j satisfy the following condition:

$$A^k A_j \begin{cases} = A_l & \text{for some } -N+\frac{1}{2} \le l \le m-\frac{1}{2}, \text{ or} \\ \in \langle v_{-N-1/2}, v_{-N-3/2}, \ldots \rangle. \end{cases}$$

Of course every A_j is bounded, i.e. there exists an integer M such that all $A_j = \sum_{i=-N+1/2}^{M-1/2} A_{ij} v_i$. Now making a shift in the index and permuting the columns we obtain the following result.

Proposition 5.2. *Let $M, N \in \mathbb{Z}$ such that $M > N > 0$ and let e_j, $1 \le j \le M$, be an orthonormal basis of $\mathbb{C}^M$. Let R be the $M \times M$-matrix $R = \sum_{i=1}^{M-k} E_{i,i+k}$ and let $A = (A_{ij})_{1\le i\le M, 1\le j\le N}$ be an $M \times N$-matrix of rank N. Denote by $A_j = \sum_{i=1}^{M} A_{ij} e_i$. If all A_j satisfy the condition that $RA_j \ne A_i$ for all $1 \le i < j$ and if all A_j, except for at most n, satisfy the condition that*

$$RA_j = \begin{cases} A_{j+1} & \text{or} \\ 0, \end{cases}$$

then

$$\tau = \det\left(\sum_{i,j=1}^{N} \left(\sum_{l=1}^{M} S_{l-i} A_{lj} \right) E_{ij} \right) \tag{5.3}$$

is a τ-function of the n-vector k-constrained KP hierarchy. All polynomial solutions can be obtained in this way.

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